Finite Element Method for 2D heat equation

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2020.11.26

Overview

1. 2D heat equation
2. FEM for 2D heat equation
3. Mesh generation
4. Performance evaluation & comparison

Goal

1. A FEM solver for 2D heat equation (* implemented in both Python and C++)
2. A mesh generating algorithm for given region Ω
3. Performance evaluation for our solver
4. * Data visualization (rendering) by matplotlib (Python) or OpenGL (C++)

2D heat equation

Formula:

$$\frac{\partial T}{\partial t} = \lambda(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}) + f(x, y, t)$$

where $x, y \in \Omega$, and T is the function of x, y and time t.
Also define the Neumann boundary condition:

$$\frac{\partial T}{\partial \vec{n}}\Big|_{\partial\Omega_n} = T_n$$

And the Dirichlet boundary condition:

$$T\Big|_{\partial\Omega_d} = T_d$$

Time elapse

In this project, we'll use update time variable t by finite difference method:

$$\frac{T(x,y,t_{n+1}) - T(x,y,t_n)}{dt} =$$
$$\lambda(\frac{\partial^2 T(x,y,n+1)}{\partial x^2} + \frac{\partial^2 T(x,y,n+1)}{\partial y^2}) + f(x,y,t_{n+1})$$

Denote $\frac{\partial^2 T(x,y,n)}{\partial x^2} + \frac{\partial^2 T(x,y,n)}{\partial y^2}$ as ΔT_n, and rewrite this equation as:

$$\frac{T_{n+1} - T_n}{dt} = \lambda \Delta T_{n+1} + f_{n+1}$$

Hence:

$$T_{n+1}(1 - \lambda dt \Delta T_{n+1}) = f_{n+1} dt + T_n$$

FEM method for heat equation

Denote:
Nodes: $N = \{n_1, n_2, \ldots, n_k\}$
Triangle Mesh: $M = \{m_1, m_2, \ldots, m_p\}$ Then we discrete this equation as:

$$T_{sol} = T_{sol,d} + T_{rest}$$

where $T_{sol,d}$ is to fit the dirichlet boundary condition.

FEM method for heat equation

Denote $\{\phi_1, \phi_2, \ldots, \phi_k\}$ as the basis The Galerkin discretation of this equation is:

$$\int\limits_{\Omega} T_{rest,n}\phi_j dx + (\int\limits_{\Omega} \nabla T_{rest,n} \cdot \nabla \phi_j dx) \cdot \lambda dt$$

$$= \int\limits_{\Omega} T_{rest,n-1}\phi_j dx + (\int\limits_{\partial\Omega_n} T_{rest,n}^{neum}\phi_j ds) \cdot dt + (\int\limits_{\Omega} F_n \phi_j dx) \cdot dt$$

FEM method for heat equation - Cont'd

$$T_{rest,t} = \sum_{i=1}^{k} \alpha_i \phi_t^i$$

Let Tri denote the triangle mesh, hence we can construct the mass matrix

$$M_{ij} = \sum_{A \in Tri} \int_A \phi_i \phi_j dx$$

, coefficient matrix

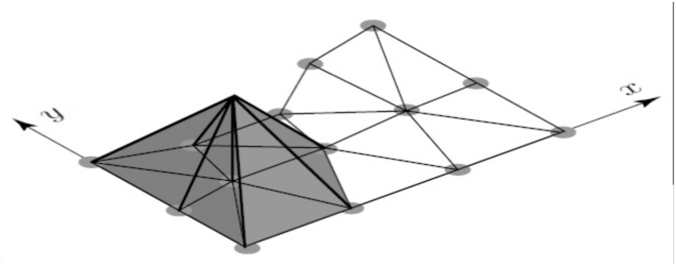
$$A_{ij} = \int_\Omega \nabla \phi_i \cdot \nabla \phi_j dx$$

Hence we can get the matrix form of this equation

$$(Adt + M)T_{rest,n} = MT_{rest,n-1}$$
$$+ \left(\int_{\partial \Omega_n} T_{rest,n}^{neum} \phi_j ds\right) \cdot dt + \left(\int_\Omega F_n \phi_j dx\right) \cdot dt$$

Triangle mesh

The triangle mesh $\phi_i(x, y)$ is shown in the following figure



Hence we can discrete the temperature function T as

$$T_{rest,t} = \sum_{i=1}^{k} \alpha_i \phi_t^i$$

Mesh generation

1. Regular region: Simple to implement
2. Irregular region: Maybe we can refer to Delaunay triangulation, or using KD Tree

Performance evaluation & comparison

1. Evaluation of different mesh size / number of triangles
2. Comparison between different implementations (Python vs. C++)
3. Evaluation of different initial conditions, boundary conditions

Thanks!