Reinforcement Learning: Homework #3

Due on April 16, 2020 at 11:59pm

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Problem 1

Solution

a) Because the initial state X_0 follows the stationary distribution, for every state X_i, the marginal distribution is s. i.e. $\forall n \in N, P(X_n = i) = s_i$. In this case, there are 10 states, and $P(X_n = 3) = s_3$. So, on average, the number of states in $X_0, \ldots, X_9$ equals to 3 is $10s_3$.

b) In this case, it's obviously that if $X_n = 1$ or $X_n = 2$, then $Y_n = 0$, and if $X_n = 3$, then $Y_n = 2$. It means, we have 'merged' state space 1 and 2 to 0. So if we make the transition matrix Q be:

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

The Markov property holds. But if we put Q be the following form, Y_n is not Markov.

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \end{bmatrix}$$

Problem 2

Solution

a) It's obviously that:

$$P(X_{n+1} = 1 | X_n = 1) = p,\ P(X_{n+1} = 2 | X_n = 2) = p$$
$$P(X_{n+1} = 1 | X_n = 2) = p,\ P(X_{n+1} = 2 | X_n = 1) = 1 - p$$

So teh matrix is:

$$Q = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

b) Let the stationary distribution be: $\pi = \binom{\pi_1}{\pi_2}$, then we can find π_1 and π_2 by solving $\pi = \pi Q$.

$$\begin{aligned} & \pi = \pi Q \\ \Rightarrow\ & (I - Q)\pi = 0 \\ \Rightarrow\ & \begin{bmatrix} p-1 & 1-p \\ 1-p & p-1 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} = 0 \\ \Rightarrow\ & \pi_1 - \pi_2 = 0 \end{aligned}$$

Notice that $\pi_1 + \pi_2 = 1$. Hence, $\pi_1 = \pi_2 = \frac{1}{2}$, the stationary distribution is $\pi = \binom{\frac{1}{2}}{\frac{1}{2}}$

c) Suppose

$$Q^n = \begin{bmatrix} p_n & 1-p_n \\ 1-p_n & p_n \end{bmatrix}$$

for all n, then we have:

$$Q^n = Q^{n-1}Q = \begin{bmatrix} p_{n-1} & 1-p_{n-1} \\ 1-p_{n-1} & p_{n-1} \end{bmatrix} \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Problem 2 continued on next page...

Notice that $p_1 = p$, hence we have this recursive equation:

$$p_n - (2p-1)p_{n-1} + p - 1 = 0$$

Solve this equation by characteristic root, we can obtain:

$$p_n = \frac{1}{2}(2p-1)^n + \frac{1}{2}$$

Because of $2p - 1 < 1$,

$$\lim_{n\to\infty} p_n = \frac{1}{2} \cdot 0 + \frac{1}{2} = \frac{1}{2}$$

Thus,

$$\lim_{n\to\infty} Q^n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Problem 3

Solution

a) Similar like problem 2, notice that $P(X_n = 0) = s_0$, because there are 25 states in this problem, each state has a probability s_0, by the linearity of expectation, the expected number of $X_0, \ldots, X_{25}$ are 0 is $25s_0$.

b) Yes, it is a Markov chain. we can take any $x_{n+1}, y_{n+1}, z_{n+1} \in M$, and let $T_X = \{X_0 = x_0, \ldots, X_n = x_n\}$, $T_Y = \{Y_0 = y_0, \ldots, Y_n = y_n\}$, $T_Z = \{Z_0 = z_0, \ldots, Z_n = z_n\}$, and $T_{all} = T_X \cap T_Y \cap T_Z$. We have that:

$$\begin{aligned}
&P(X_{n+1} = x_{n+1}, Y_{n+1} = y_{n+1}, Z_{n+1} = z_{n+1}|T_{all}) \\
=&P(X_{n+1} = x_{n+1}|T_{all})P(Y_{n+1} = y_{n+1}|X_{n+1} = x_{n+1}, T_{all}) \\
&P(Z_{n+1} = z_{n+1}|X_{n+1} = x_{n+1}, Y_{n+1} = y_{n+1}, T_{all}) \\
=&P(X_{n+1} = x_{n+1}|T_X)P(Y_{n+1} = y_{n+1}|T_Y)P(Z_{n+1} = z_{n+1}|T_Z) \\
=&P(X_{n+1} = x_{n+1}|X_n = x_n)P(Y_{n+1} = y_{n+1}|Y_n = y_n)P(Z_{n+1} = z_{n+1}|Z_n = z_n)
\end{aligned}$$

Hence, we've proved $P(X_{n+1}, Y_{n+1}, Z_{n+1}|T_{all}) = P(X_{n+1}|X_n)P(Y_{n+1}|Y_n)P(Z_{n+1}|Z_n)$, so $W_0, \ldots, W_n$ is a Markov chain.

c) For each dragon, it will take $\frac{1}{s_0}$ time to go back home again on average. Beacuse 3 dragons are independent, the average time of 3 dragons back home together is $1/s_0^3$

Problem 4

Solution

a) Yes, $(|X_n|)_n$ is a Markov chain. It has state space $0, 1, 2, 3$, and if $X_n \neq 0$ and $X_n \neq 3$, it moves left or right with probability . And if $X_n = 0$, it will transfer to 1 with probability 1, and if $X_n = 3$, it will transfer to 2 with probability 1. Beacuse if $|X_n| = k$, then we can get $X_n = k$ or $X_n = -k$. So without knowing $|X_{n-1}|, |X_{n-2}|, \ldots$, we can get $|X_{n+1}|$.

b) No, $(sgn(X_n))_n$ is not a Markov chain. Notice that:

$$P(sgn(X_2) = 1|sgn(X_1) = 1) > P(sgn(X_2) = 1|sgn(X_1) = 1, sgn(X_0) = 0)$$

since RHS implies $X_1 = 1$, for if $X_0 = 0$, the next state with positive value can only be 1. But LHS implies X_1 may be 1, 2 or 3, for no more information are known.

Problem 4 continued on next page. . .

c) By using the property of random walks, the stationary distribution is the degree of some node divides sum of degrees of all nodes. Hence, the stationary distribution is:

$$s = [\frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}]$$

d) Connect -3 and 3, to make the chain be symmetric. That is, to ensure that if $X_n = 3$, it will have $P = \frac{1}{2}$ to go to -3, and if $X_n = -3$, it will have $P = \frac{1}{2}$ to go to 3. Thus, the stationary distribution will be:

$$s = [\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}]$$

Which is a uniform distribution.

Problem 5

Solution

a) If there doesn't exist an edge between vertex i and j, then $q_{ij} = 0$.
For all $i \neq j$, and there exists an edge between i and j, then

$$q_{ij} = \frac{1}{d_i} min(1, \frac{d_i}{d_j}) = \begin{cases} \frac{1}{d_i}, & d_i > d_j \\ \frac{1}{d_j}, & d_i \leq d_j \end{cases}$$

For q_{ii}, it follows

$$q_{ii} = 1 - \sum_{i \neq j} q_{ij}$$

b) Notice that the matrix is symmetric. That is, $q_{ij} = q_{ji}$, for all i, j. For no edges case, it's obviously that $q_{ij} = q_{ji} = 0$. For other cases, notice that:

$$q_{ji} = \frac{1}{d_j} min(1, \frac{d_j}{d_i}) = \begin{cases} \frac{1}{d_i}, & d_i > d_j \\ \frac{1}{d_j}, & d_i \leq d_j \end{cases} = q_{ij}$$

Hence, the stationary distribution is a uniform distribution: $s = [\frac{1}{M}, \ldots, \frac{1}{M}]$

Problem 6

Solution

a) First, notice 2 marginal cases:
(1) $P(X_{n+1} = 1 | X_n = 0) = 1$, because if $X_n = 0$, all black balls are in urn 2, an exchange must be put a black ball from urn 2 to urn 1.
(2) $P(X_{n+1} = N - 1 | X_n = N) = 1$, because if $X_n = N$, all black balls are in urn 1, an exchange must be put a black ball from urn 1 to urn 2.

Second, for all $i \in \{1, 2, \ldots, N - 1\}$, the next state of i can only be $i - 1$, i and $i + 1$.
(1) If $P(X_{n+1} = i + 1$, we can only choose white ball from urn 1 and balck ball from urn 2, hence

$$P(X_{n+1} = i + 1 | X_n = i) = \frac{N - i}{N} \cdot \frac{N - i}{N} = \frac{(N - i)^2}{N^2}$$

Problem 6 continued on next page. . .

(2) If $P(X_{n+1} = i - 1$, we can only choose white ball from urn 2 and balck ball from urn 1, hence

$$P(X_{n+1} = i - 1 | X_n = i) = \frac{i}{N} \cdot \frac{i}{N} = \frac{i^2}{N^2}$$

(3) If $P(X_{n+1} = i | X_n = i)$, we can only choose same color balls from two urns, hence

$$P(X_{n+1} = i | X_n = i) = 2 \cdot \frac{i}{N} \cdot \frac{N-i}{N} = \frac{2i(N-i)}{N^2}$$

b) We need to check $s_i \cdot q_{ij} = s_j \cdot q_{ji}$
(1) If $i = 0$ or $i = N$, then we only need to check $j = 1$:

$$s_0 q_{01} = \frac{\binom{N}{0}\binom{N}{N}}{\binom{2N}{N}} \cdot 1 = \frac{1}{\binom{2N}{N}}$$

$$s_1 q_{10} = \frac{\binom{N}{1}\binom{N}{N-1}}{\binom{2N}{N}} \cdot \frac{1}{N^2} = \frac{N^2}{\binom{2N}{N}} \frac{1}{N^2} = \frac{1}{\binom{2N}{N}}$$

Thus, $s_i \cdot q_{ij} = s_j \cdot q_{ji}$ is correct when $i = 0$, and $i = N$ can also be proved in a similar way.
(2) For all $i \in \{1, 2, \ldots, N-1\}$, we need to prove $j = i - 1$ and $j = i + 1$ cases, we need to prove following equation for $j = i - 1$:

$$\frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}} \cdot \frac{i^2}{N^2} = \frac{\binom{N}{i-1}\binom{N}{N-i+1}}{\binom{2N}{N}} \cdot \frac{(N-i+1)^2}{N^2}$$

Which is to prove:

$$\binom{N}{i}\binom{N}{N-i} i^2 = \binom{N}{i-1}\binom{N}{N-i+1}(N-i+1)^2$$

Notice that $\binom{N}{i} = \binom{N}{N-i}$, we just need to prove:

$$\binom{N}{i}^2 i^2 = \binom{N}{i-1}^2 (N-i+1)^2$$

Equals to prove:

$$\binom{N}{i} i = \binom{N}{i-1}(N-i+1)$$

This is a very simple problem:

$$LHS = \frac{N!}{i!(N-i)!} = (N-i+1)\frac{N!}{(i-1)!(N-i+1)!} = \binom{N}{i-1}(N-i+1) = RHS$$

And case $j = i + 1$ can be provec in a similar way. Thus, we have shown the chain is reversible, hence s is the stationary distribution.

Problem 7

Solution

a) To prove $(v_1, v_2, \ldots, v_n)$ is proportional to s, we need to prove v satisfies $v = vQ$. That is, to prove

$$\forall i \in \{1, 2, \ldots, N\},\ v_i = \sum_{j=1}^{N} v_j P(X_{n+1} = i | X_n = j)$$

Problem 7 continued on next page...

Notice that $P(X_{n+1} = i|X_n = j) = \frac{w_{ij}}{v_j}$

$$v_i = \sum_{j=1}^{N} v_j \frac{w_{ij}}{v_j} = \sum_{j=1}^{N} w_{ij}$$

That is actually the definition of v_i. Hence, we've proved $v = vQ$, so $(v_1, v_2, \ldots, v_n)$ is proportional to the stationary distribution s.

b) Consider a arbitrary markow chain $(X_n)_n$ with transition matrix Q, state space S and stationary distribution s. We have:

$$s_i q_{ij} = s_j q_{ji}$$

Then we define $w_{ij} = s_i q_{ij}$, and by the equation shown above, $w_{ij} = w_{ji}$, so the network is undirected. Then,

$$v_i = \sum_j w_i j = \sum_j s_i q_{ij} = \sum_j s_j q_{ji} = \sum_j P(X_0 = j)P(X_1 = i|X_0 = j) = s_i$$

Hence, we shown that $v = s$. So for any Markov chain, it can be represented by a undirected network.

Problem 8

Solution

a) For the cat,

$$Q_{cat} = \begin{bmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{bmatrix}$$

Notice that it's a symmetric matrix, so $s_{cat} = (0.5,\ 0.5)$
For the mouse,

$$Q_{mouse} = \begin{bmatrix} 0.7 & 0.3 \\ 0.6 & 0.4 \end{bmatrix}$$

By solving $\pi = \pi Q$, we can get $\pi = (\frac{2}{3},\ \frac{1}{3})$ Hence, $s_{mouse} = (\frac{2}{3},\ \frac{1}{3})$

b) Yes, it's a Markov chain. Because the actions of mouse and cat are both Markov chains. The future is independent of the past actions of cat and mouse, given the information of cat and mouse right now. So, $(Z_n)_n$ is a Markov chain, the proof is very similar like problem 3.

c) Define x for configuration 1 and y for configuration 2, we have:

$$x = 0.2 \cdot 0.6 + 0.8 \cdot 0.4 + 0.2 \cdot 0.4(1 + x) + 0.8 \cdot 0.6(1 + y)$$

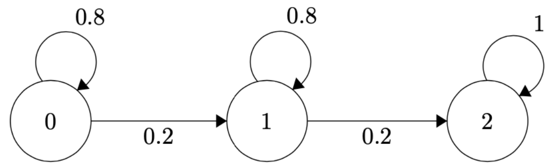
$$y = 0.8 \cdot 0.7 + 0.2 \cdot 0.4 + 0.8 \cdot 0.3(1 + x) + 0.2 \cdot 0.7(1 + y)$$

Hence, $x = \frac{335}{169}$, and $y = \frac{290}{169}$

Problem 9

Solution
The markov chain is shown below:

Problem 9 continued on next page. . .



Notice that at state 0, the chain has $P = 0.8$ to stay the same, $P = 0.2$ to transfer to state 1. So state 0 and state 1 are not recurrent. At state 1, the chain has $P = 0.8$ to stay the same, and $P = 0.2$ to transfer to state 2 and stay in state 2 forever. Notice that state 2 is an absorbing state, and once it leaves 0 or 1, it can not go back.

Consider the following story: One person at state 1, and do some work with $P = 0.8$ fail, and $P = 0.2$ success. Once he successfully done this work, he moves to state 2, and do the same work again, once he succeed again, he will stop. We define t_0 be the time he finished the first work, and t_1 be the time he finished the second work. It's obviously that for each work, the time of his first success follows First Success distribution (i.e. Geometric Distribution), with parameter 0.2 ($t_{0,1} \sim Geom(0.2)$).

Hence,

$$E(T) = E(t_0) + E(t_1) = 2 \cdot \frac{1}{p} = 10$$

and

$$Var(T) = Var(t_0) + Var(t_1) = 2 \cdot \frac{1-p}{p^2} = 40$$

Problem 10

Solution

a) If the chain returns to state 1, it must go to state 2 (with probability 0.5), and then return to state 1 (with probability 0.5), and once it leaves state 1 and 2, it will never return to state 1. So, $P(N = 0) = 0.5 \cdot 0.5 = 0.25$, for it first goes to state 2, then goes to state 3, never returns back. And for all $N = 0, 1, 2, \ldots, k$. Suppose it runs to state 1 and 2 l times in total. From $k + l$ places choose $2l$ of them and say on this steps the chain runs from $state_1$ to $state_2$, other times, it remains on $state_1$, We have:

$$P(N = k) = 0.25 \cdot \sum_{l=0}^{k} \binom{k+l}{2l} (\frac{1}{2})^{k+l}$$

b) In this problem, we only need to consider state 3,4,5 and 6, ignoring state 1 and 2. The new Markov chain is inreducible, and have stationary distribution. Notice that the transition matrix Q is symmetric (i.e. $q_{ij} = q_{ji}$), the stationary distribution is reversible. Hence,

$$s_3 = \frac{1}{4}$$

So, the fraction of the time that the chain spend in state 3 is $\frac{1}{4}$.

Problem 11

Solution

FDM Queue

For this system structure, we consider each subsystem. Define $\rho = \frac{\lambda}{k\mu} < 1$

The mean number of customers in subsystem: $E(N) = \sum_{i=0}^{\infty} i\pi_i = \frac{\rho}{1-\rho}$

The mean number of customers in each queue: $E(N_q) = \sum_{i=0}^{\infty} (i-1)\pi_i = \frac{\rho^2}{1-\rho}$

Mean time in system: $E_t = \frac{1}{\mu-\lambda/k} = \frac{k}{k\mu-\lambda}$

Mean time in queue: $E_{tq} = E_t - \frac{1}{\mu} = \frac{\rho}{\mu-\lambda/k} = \frac{\lambda}{k\mu^2-\lambda\mu}$

- Pros: When input are many independent tasks, it will handles well. The best case is, each queue has the same length, the overload is balanced. In this case, data processing is parallized, the efficiency same as M/M/1 queue.
- Cons: If the overload is not balanced (i.e. One queue is extremely long, while other servers are idle). In this case, this model retrogrades to a single queue with parameter λ and μ/k, the efficiency is very low.

M/M/1 Queue

Define $\rho = \frac{\lambda}{k\mu} < 1$

The mean number of customers in subsystem: $E(N) = \sum_{i=0}^{\infty} i\pi_i = \frac{\rho}{1-\rho}$

The mean number of customers in each queue: $E(N_q) = \sum_{i=0}^{\infty} (i-1)\pi_i = \frac{\rho^2}{1-\rho}$

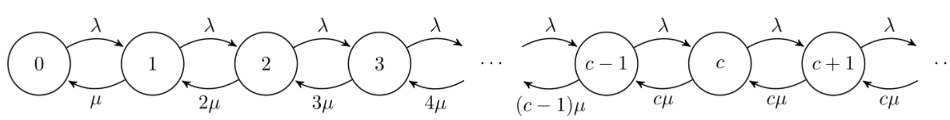
Mean time in system: $E_t = \frac{1}{k\mu-\lambda}$

Mean time in queue: $E_{tq} = E_t - \frac{1}{k\mu} = \frac{\rho}{k\mu-\lambda} = \frac{\lambda}{k^2\mu^2-k\lambda\mu}$

- Pros: Unlike FDM model, the worst case will never happen, it's stable, and we don't need to consider overload balancing. It will always has the same expectation of queuing time and processing time.
- Cons: In real systems, it's hard to implement such a single system with high throughput.

M/M/k Queue

The markov chain of M/M/k model is shown below:



In the M/M/k queue, also define $\rho = \frac{\lambda}{k\mu} < 1$,

$$p_m = \begin{cases} p_0 \prod\limits_{i=0}^{m-1} \frac{\lambda}{(i+1)\mu} = p_0(\frac{\lambda}{\mu})^m \frac{1}{m!}, & m < k \\ p_0 \prod_{i=0}^{k-1} \frac{\lambda}{(i+1)\mu} \prod\limits_{j=0}^{m-1} \frac{\lambda}{k\mu} = p_0(\frac{\lambda}{\mu})^m \frac{1}{k!k^{m-k}}, & m \geq k \end{cases}$$

where $p_0 = [\sum\limits_{m=0}^{k-1} \frac{(k\rho)^m}{m!} + \sum\limits_{m=k}^{\infty} \frac{(k\rho)^m}{k!k^{m-k}}]^{-1} = [\sum\limits_{m=0}^{k-1} \frac{(k\rho)^m}{m!} + \frac{(k\rho)^k}{k!(1-\rho)}]^{-1}$ since $\sum\limits_{m=0}^{\infty} p_m = 1$

Then, we calculate the expected number of customers in the system.

$$E(N) = \sum_{m=0}^{\infty} mp_m = \frac{(k\rho)^k \rho p_0}{k!(1-\rho)^2} + k\rho$$

For we know the total traffic intensity

$$p_i = \frac{p_0(k\rho)^k}{k!}$$

We can obtain:

$$p_q = \sum_{m=k}^{\infty} p_m = \frac{p_i}{1-\rho} = \frac{(k\rho)^k p_0}{(1-\rho)k!}$$

Problem 11 continued on next page. . .

Then, the mean number of customers in queue is

$$E(N_q) = p_0 \frac{(k\rho)^k}{k!} \sum (m-k)\rho^{m-k} = p_0 \frac{(k\rho)^k}{k!} \sum_{n=0}^{\infty} n\rho^n$$

Hence,

$$E(N_q) = \frac{\rho}{1-\rho} p_q$$

By little's law,

$$E_t = \frac{\rho p_q}{\lambda(1-\rho)}$$

$$E_{tq} = E_t + \frac{1}{\mu} = \frac{1}{\mu} + \frac{p_q}{k\mu - \lambda}$$

Since $E_t(M/M/k) = \frac{1}{\mu} + \frac{p_q}{k\mu-\lambda}$, and $E_t(M/M/1) = \frac{1}{k\mu-\lambda}$.
Under light load ($\rho \ll 1$), $p_q \to 0$, we will get $\frac{E_t(M/M/k)}{E_t(M/M/1)} \approx k$
Under heavy load ($\rho \to 1$), $p_q \to 1$ and $\frac{1}{\mu} \ll \frac{1}{k\mu-\lambda}$, we will get $\frac{E_t(M/M/k)}{E_t(M/M/1)} \approx 1$.
So we can analyze the pros and cons:
- Pros: Under heavy load, its performance is the same as the M/M/1 queue, the performance is relative high.
- Cons: Under light load, it will have a pretty poor performance, at most k times slower than M/M/1 queue.