SI114H: Homework #1

Due on November 22, 2020 at 11:59pm

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Problem 1

Show the Gibbs phenomenon

Solution

In this problem, I use `python` to show the fourier series of $step(x)$ in range $[0, 1]$. We know that

$$s(x) = \sum_{k=1}^{\infty} \frac{2}{\pi} \cdot h_k \cdot sin(kx)$$

where

$$h_k = \begin{cases} 0, & \text{k is even} \\ \frac{2}{k}, & \text{k is odd} \end{cases}$$

and for delta function ($\delta(x)$):

$$\delta(x) = \sum_{k=1}^{\infty} \frac{1}{\pi} \cdot sin(kx)$$

The following diagram shows the fourier sums when $k = 1$, $k = 10$, $k = 100$ and $k = 500$ of step function and delta function. We can observe that when $x = 0$ and $x = \pi$, the Fourier series has large oscillations near the jump (Gibbs phenomenon).

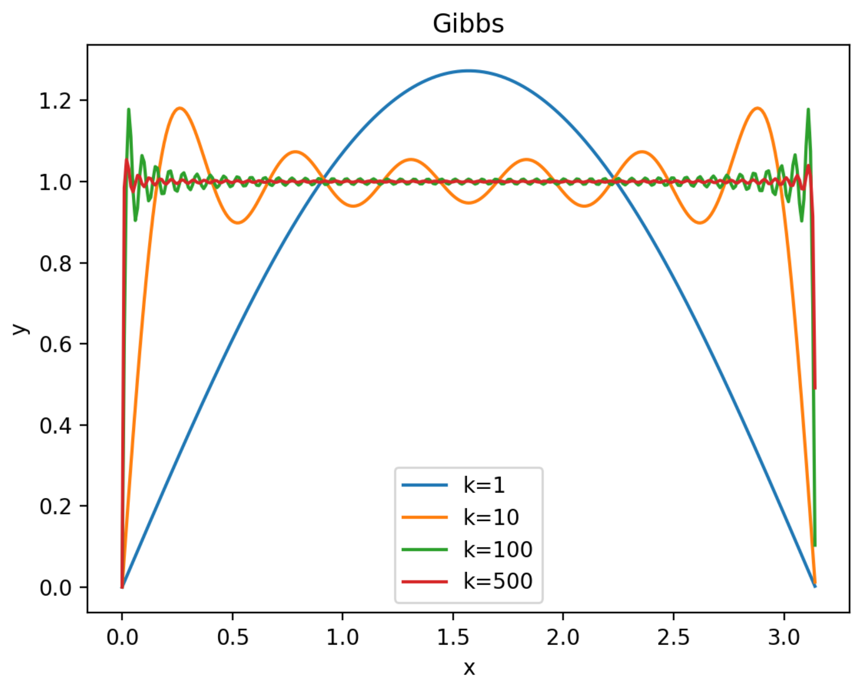

Figure 1: Gibbs phenomenon - step function

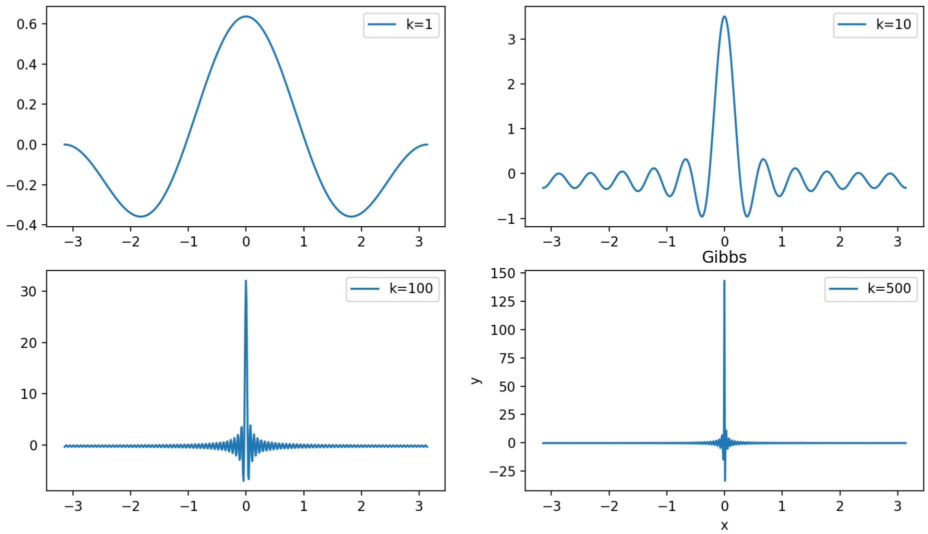

Figure 2: Gibbs phenomenon - delta function

Problem 2

Solution

(a) $f(x) = sin^3(x)$
We know that $sin^3(x)$ is an odd function, so all a_n equals to 0.

$$b_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} sin^3(x)dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(nx)sin^3(x)dx$$

Notice that

$$sin^3(x) = \frac{3}{4}sin(x) - frac14sin(3x)$$

Hence,

$$b_1 = \frac{3}{4}$$

$$b_3 = \frac{1}{4}$$

And all other $b_n = 0$

(b) $f(x) = |sin(x)|$
We know that $sin^3(x)$ is an even function, so all b_n equals to 0.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |sin(x)|dx = \frac{4}{\pi}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} cos(nx)|sin(x)|dx \\
&= \frac{1}{\pi}(\int_{0}^{\pi} cos(nx)sin(x)dx - \int_{-\pi}^{0} cos(nx)sin(x)dx) \\
&= \frac{2}{\pi}\int_{0}^{\pi} \frac{1}{2}(sin(n+1)x) - sin((n-1)x)dx \\
&= \begin{cases} -\frac{4}{\pi(n^2-1)}, & \text{n is even} \\ 0, & \text{n is odd} \end{cases}
\end{aligned} \tag{1}$$

Let $n = 2k$, we have:

$$|sinx| = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k cos(2kx) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{cos(2kx)}{4k^2-1}$$

(c) $f(x) = x$
We know that:

$$a_0 = \int_{-\pi}^{\pi} xdx = 0$$

and

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} xcos(nx)dx \\
&= \frac{1}{n\pi}\int_{-\pi}^{\pi} xd(sin(nx)) \\
&= 0
\end{aligned} \tag{2}$$

$$\begin{aligned}
b_n &= \int_{-\pi}^{\pi} xsin(nx)dx \\
&= -\frac{1}{n\pi}\int_{-\pi}^{\pi} xd(cos(nx)) \\
&= (-1)^{n-1}\cdot\frac{2}{n}
\end{aligned} \tag{3}$$

Hence,

$$f(x) = 2\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}sin(nx)$$

(d) $f(x) = e^x$, complex form

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}dx \\
&= \frac{1}{2\pi}\cdot\frac{1}{1-ni}(e^{(1-ni)\pi} - e^{(ni-1)\pi})
\end{aligned} \tag{4}$$

Hence,

$$f(x) = \sum_{n=-\infty}^{\infty}\frac{1}{2\pi(1-ni)}(e^{(1-ni)\pi} - e^{(ni-1)\pi})e^{inx}$$