SI114H: Homework #2

Due on November 24, 2020 at 11:59pm

Professor Qifeng Liao

Tianyuan Wu
63305667

Problem 1

Solution

$$
\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \\
&= \int_{0}^{\infty} e^{-(ik+a)x}dx + \int_{-\infty}^{0} e^{(a-ik)x}dx && (1) \\
&= \frac{1}{a+ik} - \frac{1}{a-ik} && (2) \\
&= \frac{-2ik}{a^2+k^2} && (3)
\end{aligned}
$$

So, the decay rate of $\hat{f}(k)$ is $\frac{1}{k^2}$, and there is a jump discontinuity in $g(x)$.

Problem 2

(a)

$$
\begin{aligned}
\hat{f}(k) &= \int_{0}^{L} 1 \cdot e^{-ikx}dx \\
&= -\frac{1}{ik}(e^{-ikL} - 1) && (4) \\
&= \frac{1}{ik} && (5)
\end{aligned}
$$

(b) We can calculate this fourier transformation by calculate the limit of $a = 0$ of problem1

$$
\begin{aligned}
\hat{f}(k) &= \lim_{a \to 0} \frac{-2ik}{a^2+k^2} \\
&= \frac{-2i}{k} && (6)
\end{aligned}
$$

(c) Notice that $f(x) = \int_0^1 e^{ikx}dx$ is the inverse fourier transform of the following function:

$$
g(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}
$$

Hence, the fourier transform of $f(x)$ is

$$
\hat{f}(k) = \begin{cases} 1, & 0 \leq k \leq 1 \\ 0, & \text{otherwise} \end{cases}
$$

(d)

$$
\begin{aligned}
\hat{f}(k) &= \int_{0}^{4\pi} sin(x)e^{-ikx}dx \\
&= \int_{0}^{4\pi} \frac{e^{ix} - e^{-ix}}{2i} e^{-ikx}dx && (7) \\
&= \frac{1}{2i}\frac{1}{(1-k)i}(e^{(1-k)4\pi i} - 1) - \frac{1}{2i}\frac{1}{(1+k)i}(1 - e^{-(1+k)4\pi i}) && (8) \\
&= \frac{1}{1-k^2} && (9)
\end{aligned}
$$

Problem 3

(a) By the property of delta function $\delta(x)$, we know

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \delta(k) e^{ikx} dk \\ &= 1 \end{aligned} \tag{10}$$

(b)

$$\begin{aligned} f(x) &= \int_{-\infty}^{0} e^{k} e^{ikx} dk + \int_{0}^{\infty} e^{-k} e^{ikx} dk \\ &= \lim_{w \to 0} \left(\int_{-w}^{0} e^{k} e^{ikx} dk + \int_{0}^{w} e^{-k} e^{ikx} dk \right) && (11) \\ &= \lim_{w \to 0} \left(\frac{2}{x^2+1} + e^{-w}[ix(e^{-ixw} - e^{ixw}) - (e^{-ixw} + e^{ixw})] \right) && (12) \\ &= \frac{2}{x^2+1} && (13) \end{aligned}$$