Material Derivative.

Consider the temperature of a fluid in 3D:

$$T(x, y, z; t)$$

The total time derivative is:

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} \cdot \overbrace{\frac{\partial x}{\partial t}}^{v_x} + \frac{\partial T}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial T}{\partial z} \cdot \frac{\partial z}{\partial t}$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \left[v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right]$$

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + (\vec{v} \cdot \vec{\nabla}) T$$

$$\therefore \boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v} \cdot \vec{\nabla})}$$

total time derivative

explicit time dependence

convective term

often written as.

$$\frac{D}{Dt} \equiv \text{material derivative.}$$

What is the time-dependent acceleration field of the fluid?

$$\vec{a}(x,y,z,t) \equiv \frac{D\vec{v}}{Dt}$$

$$\boxed{\vec{a}_{(x,y,z,t)} = \frac{\partial \vec{v}}{dt} + \underline{(\vec{v}\cdot\vec{\nabla})\vec{v}}}$$

know that

From Newton's Laws, we ...

the acceleration is proportional to the net force on the fluid.

In general, the forces on the fluid come in two forms:

① pressure forces
→ gravity (body force)
→ external pressure from humans.

② frictional forces
→ viscous forces.

<u>Gravity</u>: $\vec{a}_g = -g\hat{j} \equiv \vec{g}$

<u>External Pressures</u>: $\vec{a}_{Ext.P} = -\frac{\vec{\nabla} P(x,y,z)}{\rho}$

$P_1 > P_2$

$\therefore \frac{\partial P}{\partial x} < 0$

$[N/m^2]/m \,/\, kg/m^3 = N/kg$

Viscous Forces $\qquad a_{viscous} = \frac{\quad}{\rho}$

$\overset{\leftrightarrow}{\tau} \equiv$ deviatoric stress tensor

$[\tau] = Pa$

→ this is the nasty part!

($\vec{F} = m\vec{a}$ for fluids)

$$\boxed{\underset{m}{\rho} \underbrace{\frac{D\vec{v}}{Dt}}_{a} = -\vec{\nabla} p + \vec{\nabla} \cdot \overset{\leftrightarrow}{\tau} + \rho \vec{g}}$$

This is known as the Cauchy momentum equation. Various hypotheses for the form of $\overset{\leftrightarrow}{\tau}$ lead to the Navier-Stokes equations etc.

Conservation of mass: $\qquad \frac{D}{Dt}(\rho V) = 0$ (volume)

(no sources or sinks)

$$\boxed{\frac{D\rho}{Dt} = 0} \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + (\vec{v} \cdot \vec{\nabla})\rho =}$$

Continuity Equation.

$$\boxed{\cancel{\frac{\partial \rho}{\partial t}} + \vec{\nabla}\cdot(\rho\vec{v}) = 0}$$

[Note: for an incompressible fluid, ρ = constant, and hence $\vec{\nabla}\cdot\vec{v} = 0$]

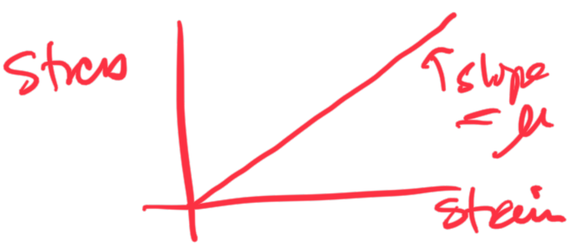
Navier-Stokes Equations:

Assumptions:

① Stress tensor is a linear function of the strain rate tensor, or equivalently the velocity gradient.

$\vec{\tau} \propto \vec{\nabla}\cdot\vec{u}$

Newtonian Fluid.



② The fluid is isotropic

③ for a fluid at rest, $\vec{\nabla}\cdot\vec{\tau} = 0$

(i.e. we get the hydrostatic pressu

MATH

$$\vec{\tau} = \mu\left(\vec{\nabla}\cdot\vec{u} + (\vec{\nabla}\cdot\vec{u})^T\right) + \lambda(\vec{\nabla}\cdot\vec{u})\mathbb{I}$$

idea

$$\boxed{\tau_{ij}} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + \delta_{ij}\lambda\frac{\partial v_k}{\partial x_k}$$

first coefficient of viscosity (The normal one)

second coefficient of viscosity (volume or bulk viscosity)

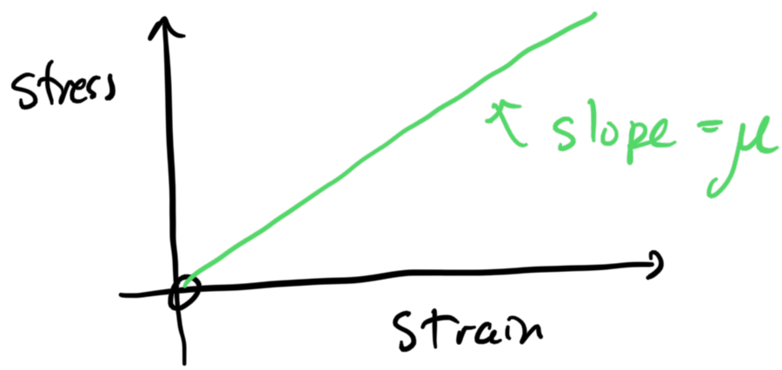
$$\boxed{\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right) = -\vec{\nabla}p + \underbrace{\vec{\nabla}\cdot\left[\mu\left[\nabla\vec{v} + (\nabla\vec{v})^T\right]\right] + \vec{\nabla}\cdot\left[\lambda(\vec{\nabla}\cdot\vec{v})\mathbb{I}\right]}}$$

$+ \rho \vec{g}$

Navier-Stokes Equations

<u>Example</u>: Incompressible Newtonian Fluid

(i) viscosity is a constant



(ii) second viscosity $= \lambda = 0$

(iii) continuity equation

$$\vec{\nabla} \cdot \vec{v} = 0$$

$\left(\partial \vec{v} \quad . \quad (\vec{v} \cdot \vec{\nabla}) \vec{v} \right|$

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right)$$

$$\rho\frac{D\vec{V}}{Dt} = -\vec{\nabla}p + \vec{\nabla}\cdot\left[\mu\left(\vec{\nabla}\vec{V} + (\vec{\nabla}\vec{V})^T\right)\right] + \rho\vec{g}$$

Let's look at the form of the viscous terms, in the x-direction for example.

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad (+\phi$$

$$\tau_{11} = \mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right)$$

u, v, w
x, y, z

(τ_{xx})

$$\tau_{12} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{13} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\left(\vec{\nabla}\cdot\vec{\tau}\right)_x = \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right)$$

$$\hat{i}\frac{\partial}{\partial x}\cdot\tau_{11}\hat{i} \qquad + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)$$

$$\hat{j}\frac{\partial}{\partial y}\cdot\tau_{12}\hat{j}$$

$$\hat{k}\frac{\partial}{\partial z}\cdot\tau_{13}\hat{k} \qquad + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)$$

$$= 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 v}{\partial x\partial y}$$

$$+ \mu\frac{\partial^2 u}{\partial z^2} + \mu\frac{\partial^2 w}{\partial x\partial z}$$

$$= \boxed{\mu\nabla^2 u} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x\partial y} + \frac{\partial^2 w}{\partial x\partial z}\right)$$

$$= \mu\nabla^2\mu + \mu\frac{\partial}{\partial x}\cancel{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}$$

$$\boxed{\vec{\nabla}\cdot\vec{V}} = 0$$

$$\boxed{\nabla \cdot V} = 0$$

$$= \boxed{\mu \nabla^2 u}$$

In the y- and z- directions, we get $\mu \nabla^2 v$ and $\mu \nabla^2 w$, respectively.

$$\rho \left(\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \vec{V} \right) = -\vec{\nabla} P + \boxed{\mu \nabla^2 \vec{V}} + \rho \vec{g}$$

Diffusion

Navier-Stokes Equations for an incompressible Newtonian Fluid.

1D

u,

→ non-linear

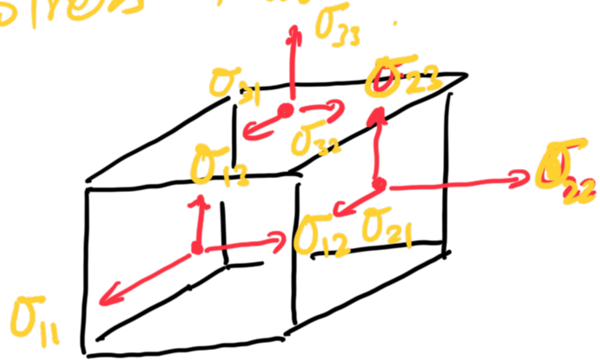
→ coupled

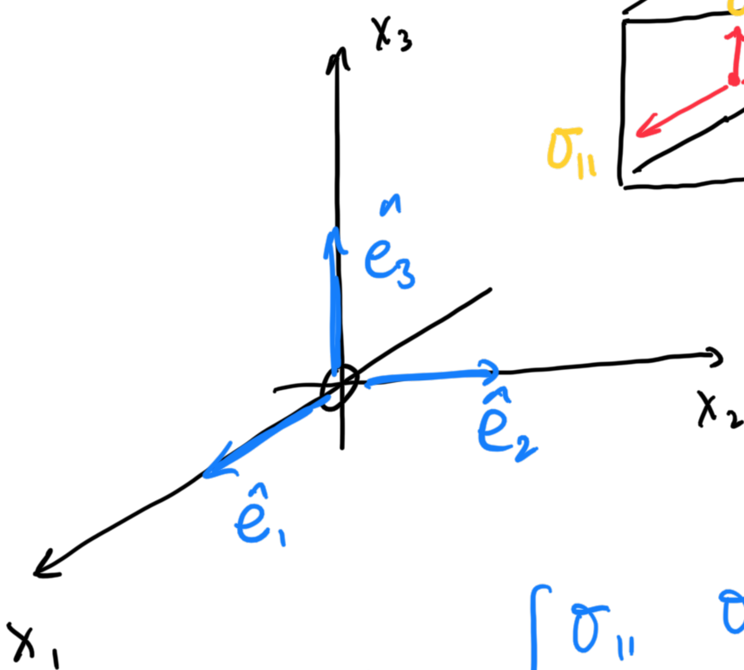
$\mu = 0$

Inviscid Fluid $\quad \vec{g} = 0$

Inviscid Fluid

More on the stress tensor!





$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$[\sigma] = \frac{N}{m^2} = Pa$$

$$\vec{T} = \sigma \hat{n} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{e}_3 \end{bmatrix}$$

→ Normal stress $(\sigma_{11}, \sigma_{22}, \sigma_{33}) \equiv \sigma$

→ Shearing stress (σ_{ij} , $i \neq j$) $\equiv \tau_{ij}$

$$\sigma = \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_2 & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_3 \end{bmatrix}$$

Because Presure is so imputat, we separte that out of the diagonal elements.

$$\sigma_1 = p + \tau_{11}$$

$$\sigma_2 = p + \tau_{22}$$

$$\sigma_3 = p + \tau_{33}$$

$$\therefore \quad \sigma = p + \overbrace{\begin{bmatrix} \tau_{11} & \tau_{22} & \tau_{23} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}}^{\tilde{\tau}}$$

$$\tilde{T} = \underbrace{p\tilde{\mathrm{I}}}_{} + \underbrace{\tilde{\tau}}_{}$$

tends to change the volume of the element

tends to distort the element