Vector Algebra and Calculus

1. Revision of vector algebra, scalar product, vector product
2. Triple products, multiple products, applications to geometry
3. Differentiation of vector functions, applications to mechanics
4. Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates
5. **Vector operators — grad, div and curl**
6. Vector Identities, curvilinear co-ordinate systems
7. Gauss' and Stokes' Theorems and extensions
8. Engineering Applications

6. Vector Operators: Grad, Div and Curl

- We introduce three field operators which reveal interesting collective field properties, viz.
  - the **gradient** of a scalar field,
  - the **divergence** of a vector field, and
  - the **curl** of a vector field.
- There are two points to get over about each:
  - The mechanics of taking the grad, div or curl, for which you will need to brush up your calculus of several variables.
  - The underlying physical meaning — that is, why they are worth bothering about.

The gradient of a scalar field

- Recall the discussion of temperature distribution, where we wondered how a scalar would vary as we moved off in an arbitrary direction ...
- If $U(\mathbf{r})$ is a scalar field, its **gradient** is defined in Cartesians coords by

$$\mathrm{grad}U = \frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} \ .$$

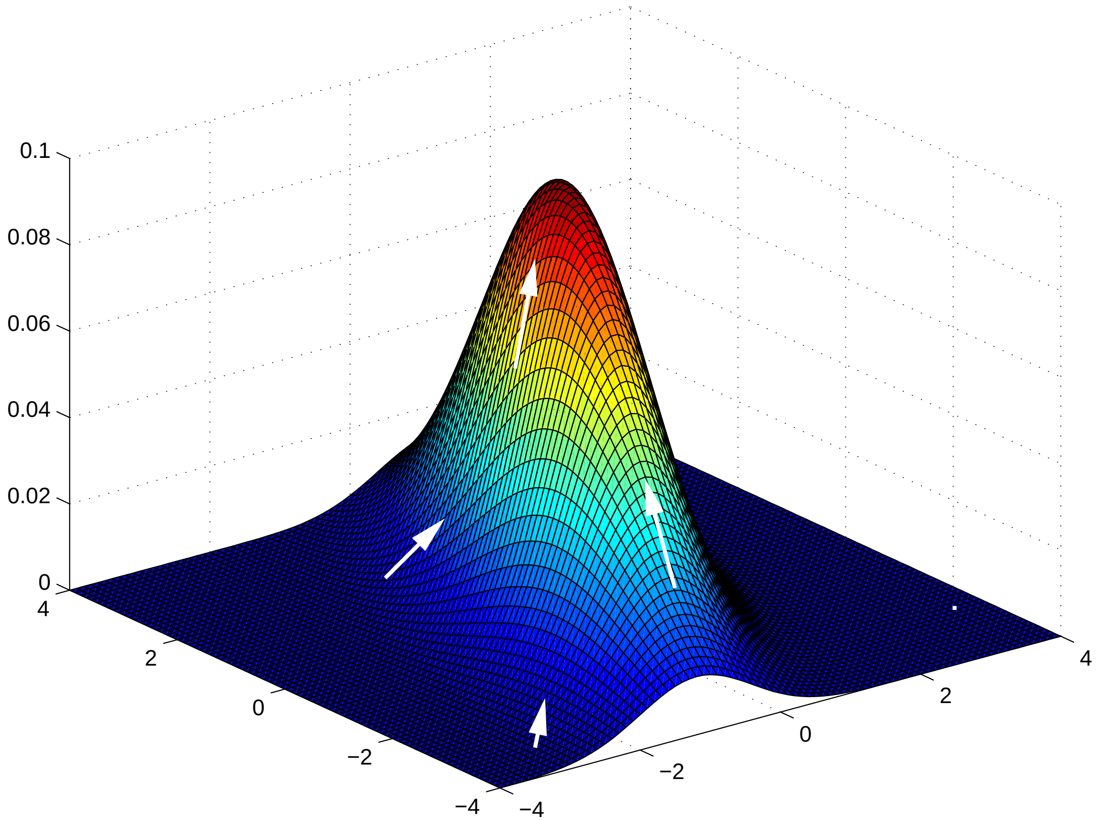
- It is usual to define the **vector operator** $\boldsymbol{\nabla}$

$$\boldsymbol{\nabla} = \left[\hat{\mathbf{i}}\,\frac{\partial}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right]$$

  which is called "del" or "nabla". We can write $\mathrm{grad}U \equiv \boldsymbol{\nabla} U$

  NB: $\mathrm{grad}U$ or $\boldsymbol{\nabla} U$ is a **vector** field!

- Without thinking too hard, notice that $\mathrm{grad}U$ tends to point in the direction of greatest change of the scalar field U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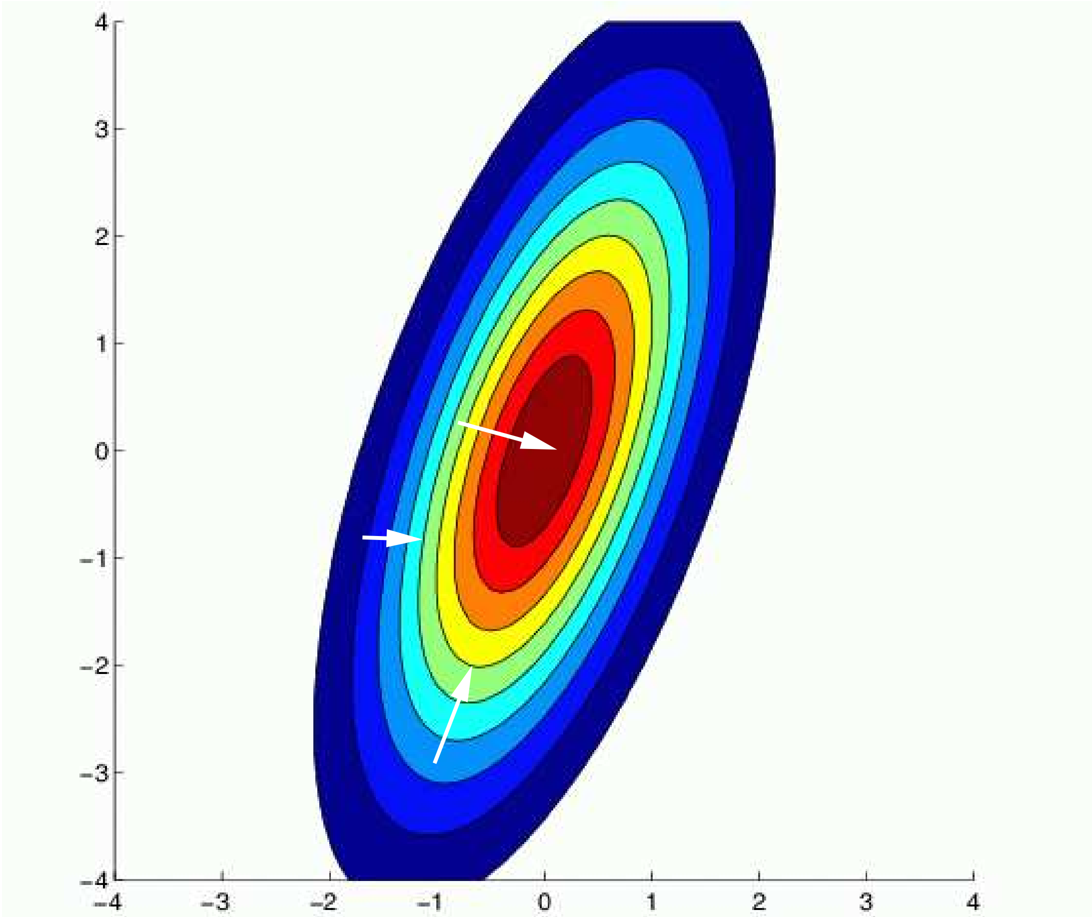
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1. $U = x^2$

$\nabla U = \left[\hat{\mathbf{i}}\,\frac{\partial}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right] x^2$

Only $\partial/\partial x$ exists so

$$\nabla U = 2x\hat{\mathbf{i}}\ .$$

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2. $U = r^2 = x^2 + y^2 + z^2$, so

$$\begin{aligned} \nabla U &= \left[\hat{\mathbf{i}}\,\frac{\partial}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right](x^2 + y^2 + z^2) \\ &= 2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}} \\ &= 2\ \mathbf{r} \end{aligned}$$

.

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3. $U = \mathbf{c} \cdot \mathbf{r}$, where $\mathbf{c}$ is constant.

$$\begin{aligned} \nabla U &= \left[\hat{\mathbf{i}}\,\frac{\partial}{\partial x}\ +\ \hat{\mathbf{j}}\,\frac{\partial}{\partial y}\ +\ \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right](c_1 x + c_2 y + c_3 z) \\ &= c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}} = \mathbf{c}\quad . \end{aligned}$$

4. $U = f(r)$, where $r = \sqrt{(x^2 + y^2 + z^2)}$
U is a function of r alone so df/dr exists. As $U = f(x, y, z)$ also,

$$\frac{\partial f}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x} \qquad \frac{\partial f}{\partial y} = \frac{df}{dr}\frac{\partial r}{\partial y} \qquad \frac{\partial f}{\partial z} = \frac{df}{dr}\frac{\partial r}{\partial z} \quad .$$

$$\Rightarrow \boldsymbol{\nabla} U = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}} \qquad = \frac{df}{dr}\left(\frac{\partial r}{\partial x}\hat{\mathbf{i}} + \frac{\partial r}{\partial y}\hat{\mathbf{j}} + \frac{\partial r}{\partial z}\hat{\mathbf{k}}\right)$$

But $r = \sqrt{x^2 + y^2 + z^2}$, so $\partial r/\partial x = x/r$ and similarly for y, z.

$$\Rightarrow \boldsymbol{\nabla} U = \frac{df}{dr}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{r}\right) = \frac{df}{dr}\left(\frac{\mathbf{r}}{r}\right) \quad .$$

Note that $f(r)$ is spherically symmetrical and the resultant vector field is radial out of a sphere.

The significance of grad

- We know that the **total differential** and **grad** are defined as

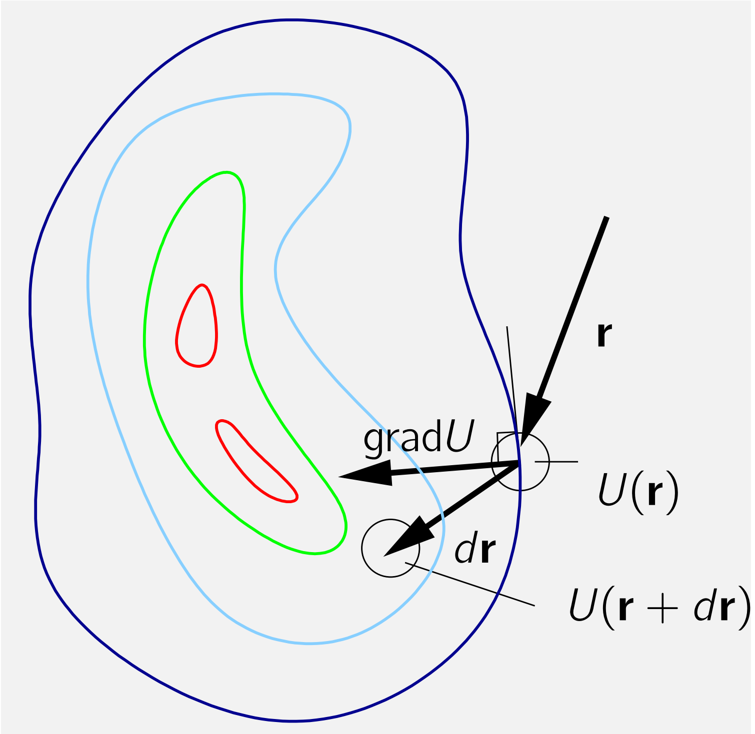
$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \quad \& \quad \nabla U = \frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}}$$

- So, we can rewrite the change in U as

$$dU = \nabla U \cdot (dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}) = \nabla U \cdot d\mathbf{r}$$

- Conclude that

$$\boxed{\nabla U \cdot d\mathbf{r} \text{ is the small change in } U \text{ when we move by } d\mathbf{r}}$$



- We also know (Lecture 3) that $d\mathbf{r}$ has magnitude ds.
- So divide by ds

$$\Rightarrow \frac{dU}{ds} = \boldsymbol{\nabla} U \cdot \left[\frac{d\mathbf{r}}{ds}\right]$$

- But $d\mathbf{r}/ds$ is a unit vector in the direction of $d\mathbf{r}$.
- Conclude that

> grad$U$ has the property that the rate of change of U wrt distance in any direction $\hat{\mathbf{d}}$ is the projection of grad$U$ onto that direction $\hat{\mathbf{d}}$

- That is

$$\frac{dU}{ds}(\text{in direction of } \hat{\mathbf{d}}) = \boldsymbol{\nabla} U \cdot \hat{\mathbf{d}}$$

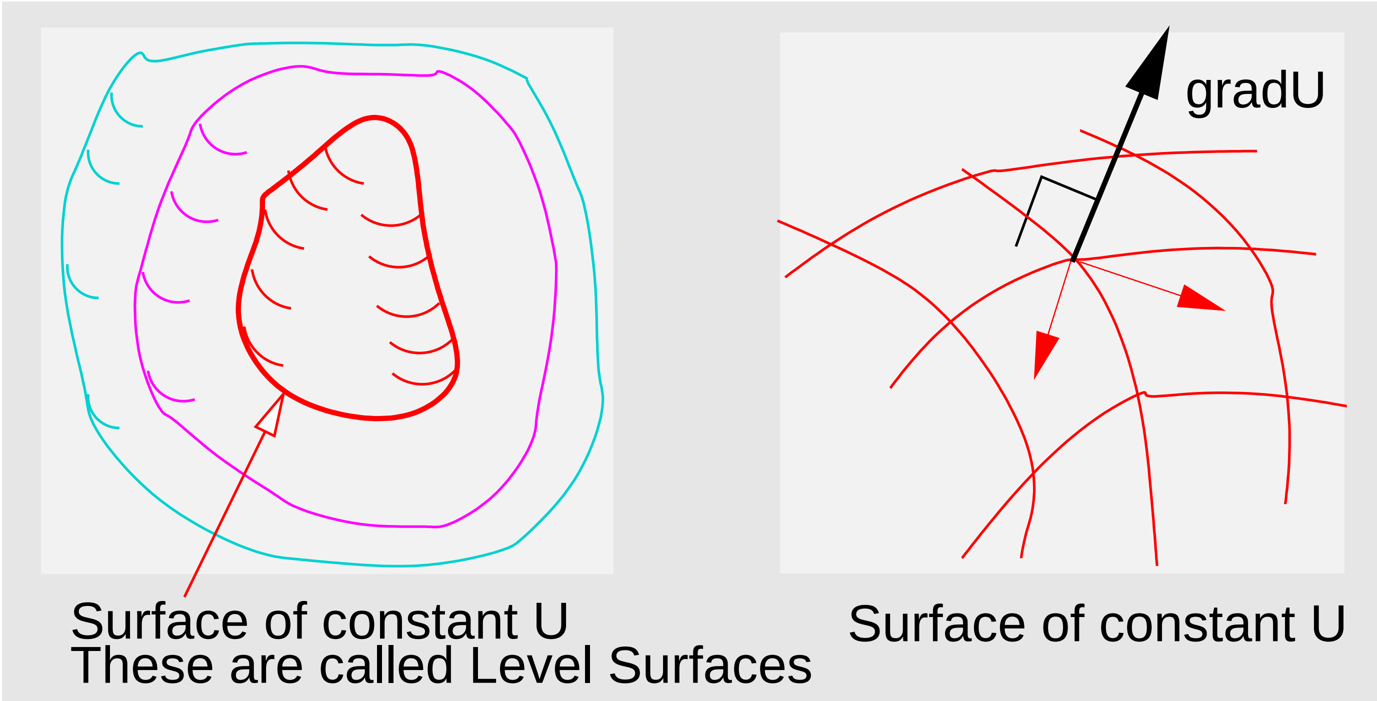
- The quantity dU/ds is called a **directional derivative**.
- In general, a directional derivative
  - had a different value for each direction,
  - has no meaning until you specify the direction.
- We could also say that

> At any point P, grad$U$
>
> * points in the direction of greatest rate of change of U wrt distance at P, and
>
> * has magnitude equal to the rate of change of U wrt distance in that direction.

- Think of a surface of constant U — the locus (x, y, z) for $U(x, y, z) = \text{const}$
- If we move a tiny amount **within** the surface, that is in any tangential direction, there is no change in U, so $dU/ds = 0$. So for any $d\mathbf{r}/ds$ in the surface

$$\boldsymbol{\nabla} U \cdot \frac{d\mathbf{r}}{ds} = 0 \; .$$

Conclusion is that:
gradU is NORMAL to a surface of constant U



The divergence of a vector field

- Let $\mathbf{a}$ be a vector field:

$$\mathbf{a}(x, y, z) = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$$

- The divergence of $\mathbf{a}$ at any point is defined in Cartesian co-ordinates by

$$\text{div } \mathbf{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

- The divergence of a vector field is a scalar field.

- We can write div as a scalar product with the $\boldsymbol{\nabla}$ vector differential operator:

$$\text{div } \mathbf{a} \equiv \left[\hat{\mathbf{i}}\,\frac{\partial}{\partial x} + \hat{\mathbf{j}}\,\frac{\partial}{\partial y} + \hat{\mathbf{k}}\,\frac{\partial}{\partial z}\right] \cdot \mathbf{a} \equiv \boldsymbol{\nabla} \cdot \mathbf{a}$$

$\mathbf{a}$	div$\mathbf{a}$
$x\hat{\mathbf{i}}$	1
$\mathbf{r}(= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$	3
$\mathbf{r}/r^3$	0
$r\mathbf{c}$	$(\mathbf{r}\cdot\mathbf{c})/r$ where $\mathbf{c}$ is constant

Eg 3: div $(\mathbf{r}/r^3) = 0$

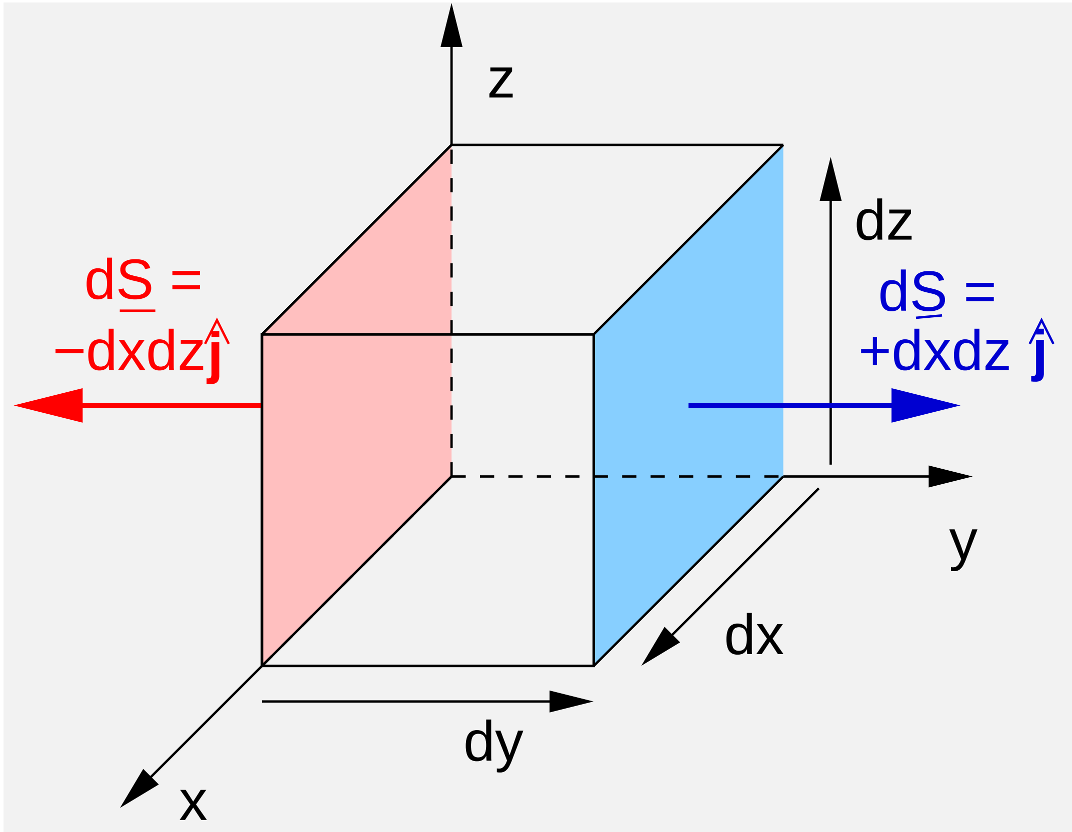
The x component of $\mathbf{r}/r^3$ is $x.(x^2+y^2+z^2)^{-3/2}$

We need to find $\partial/\partial x$ of it ...

$$\begin{aligned}\frac{\partial}{\partial x}x.(x^2+y^2+z^2)^{\frac{-3}{2}} &= 1.(x^2+y^2+z^2)^{\frac{-3}{2}} + x\frac{-3}{2}(x^2+y^2+z^2)^{\frac{-5}{2}}.2x \\ &= r^{-3}\left(1-3x^2r^{-2}\right)\end{aligned}$$

Adding this to similar terms for y and z gives

$$r^{-3}\left(3-3(x^2+y^2+z^2)r^{-2}\right) = r^{-3}\left(3-3\right) = 0$$



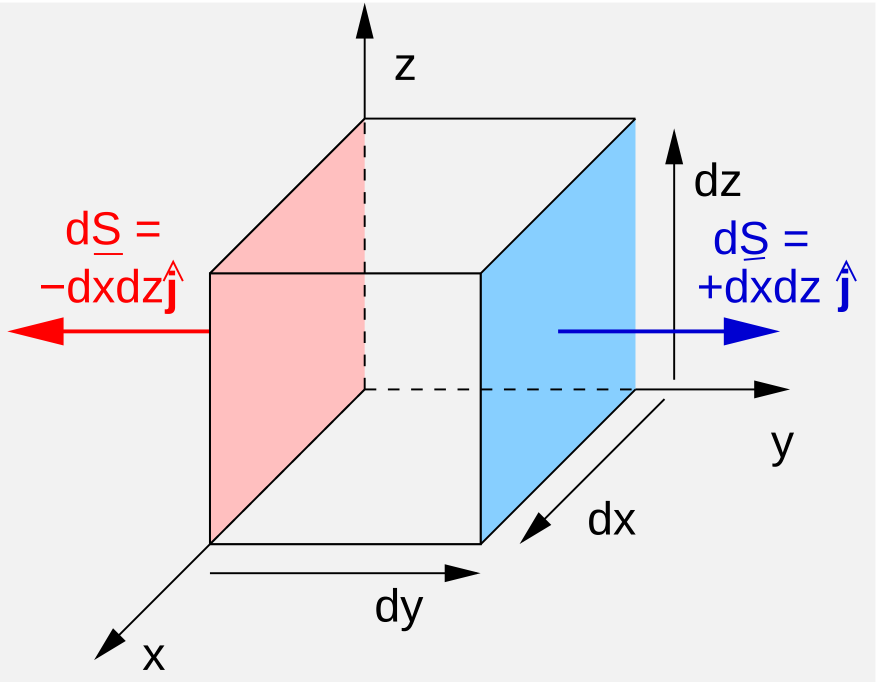
- Consider vector field $\mathbf{f}(\mathbf{r})$ (eg water flow).
  This vector has magnitude equal to the mass of water crossing a unit area perpendicular to the direction of $\mathbf{f}$ per unit time.
- Take volume element dV and compute balance of the flow of $\mathbf{f}$ in and out of dV.
- Look at the shaded face on the left
  The contribution to OUTWARD flux from surface is

$$\mathbf{f}(y) \cdot d\mathbf{S} = [f_x(y)\hat{\mathbf{i}} + f_y(y)\hat{\mathbf{j}} + f_z(y)\hat{\mathbf{k}}] \cdot (-dx\ dz\ \hat{\mathbf{j}}) = -f_y(y)dxdz.$$

Look at the shaded face on the right ...

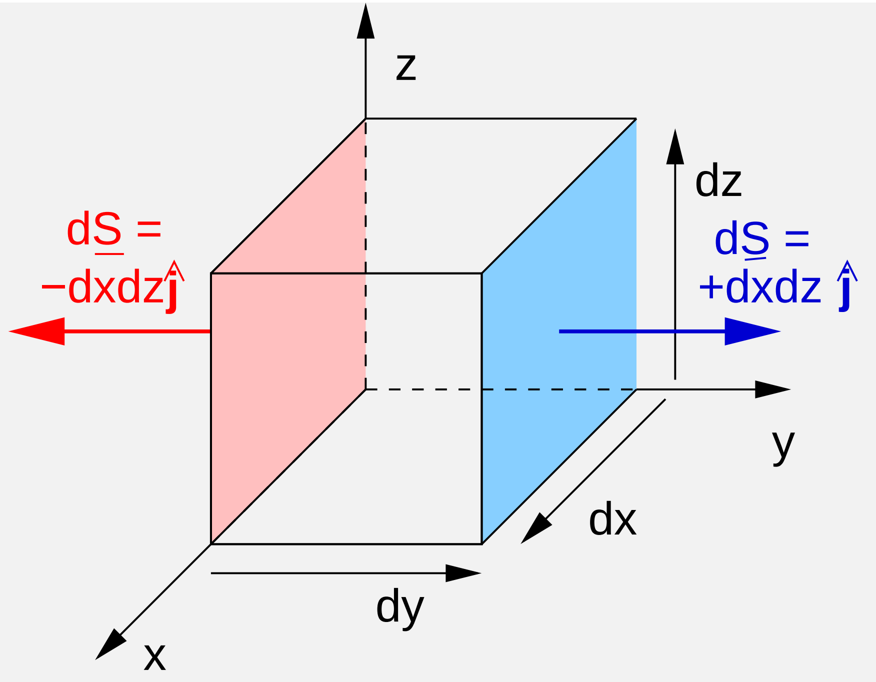
- A similar contribution, but of opposite sign, will arise from the opposite face ...
- BUT! we must remember that we have moved along y by an amount dy.
- So that this OUTWARD amount is

$$
\begin{aligned}
\mathbf{f}(y+dy)\cdot d\mathbf{S} &= f_y(y+dy)dxdz \\
&= \left(f_y + \frac{\partial f_y}{\partial y}dy\right)dxdz
\end{aligned}
$$



- Hence the total outward amount from these two faces is

$$
-f_y dxdz + \left(f_y + \frac{\partial f_y}{\partial y}dy\right)dxdz = \frac{\partial f_y}{\partial y}dydxdz = \frac{\partial f_y}{\partial y}dV
$$



- Repeat: Total efflux from these faces is

$$\frac{\partial f_y}{\partial y} dy\,dx\,dz = \frac{\partial f_y}{\partial y} dV$$

- Summing the other faces gives a total outward flux

$$\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right) dV = (\boldsymbol{\nabla} \cdot \mathbf{f})\ \ dV$$

- **Conclusion:**

$$\boxed{\text{The divergence of a vector field represents the flux generation per unit volume at each point of the field.}}$$

* **Di**vergence because it is an efflux not an influx.

* We also saw that the total efflux from the infinitesimal volume was equal to the flux integrated over the surface of the volume.

The Laplacian: div(grad$U$) of a scalar field

- grad$U$ of *any* scalar field U is a vector field. We can take the div of any vector field. $\Rightarrow$we can certainly compute div(grad$U$)

$$\begin{aligned}
\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}U) &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot\left(\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)U\right)\\
&= \left(\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\right)U\\
&= \left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}\right)
\end{aligned}$$

- The operator ∇^2 (del-squared) is called the **Laplacian**

$$\nabla^2 U=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)U$$

and often appears in engineering in **Laplace's equation** and **Poisson's equation**

$$\nabla^2 U=0 \qquad \text{and} \qquad \nabla^2 U=\rho$$

U	$\nabla^2 U$
$r^2 (= x^2 + y^2 + z^2)$	6
xy^2z^3	$2xz^3 + 6xy^2z$
$1/r$	0

Let's prove the last example
$1/r = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ and so

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}(x^2 + y^2 + z^2)^{-\frac{1}{2}} &= \frac{\partial}{\partial x} - x.(x^2 + y^2 + z^2)^{-3/2} \\
&= -(x^2 + y^2 + z^2)^{-3/2} + 3x.x.(x^2 + y^2 + z^2)^{-5/2} \\
&= \frac{1}{r^3}\left(-1 + 3\frac{x^2}{r^2}\right)
\end{aligned}$$

Adding up similar terms for y and z

$$\nabla^2 \frac{1}{r} = \frac{1}{r^3}\left(-3 + 3\frac{(x^2 + y^2 + x^2)}{r^2}\right) = 0$$

The curl of a vector field

- So far we have seen the operator ∇ ...

(i) Applied to a scalar field ∇U; and (ii) Dotted with a vector field $\nabla \cdot \mathbf{a}$.

- You are now overwhelmed by irrestible urge to ...

(iii) cross it with a vector field: $\nabla \times \mathbf{a}$

- This gives the **curl of a vector field**

$$\nabla \times \mathbf{a} \equiv \mathrm{curl}(\mathbf{a})$$

- We can follow the pseudo-determinant recipe for vector products, so that

$$\nabla \times \mathbf{a} = \begin{vmatrix} \hat{\mathbf{\imath}} & \hat{\mathbf{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) \hat{\mathbf{\imath}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial y} \right) \hat{\mathbf{\jmath}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) \hat{\mathbf{k}}$$

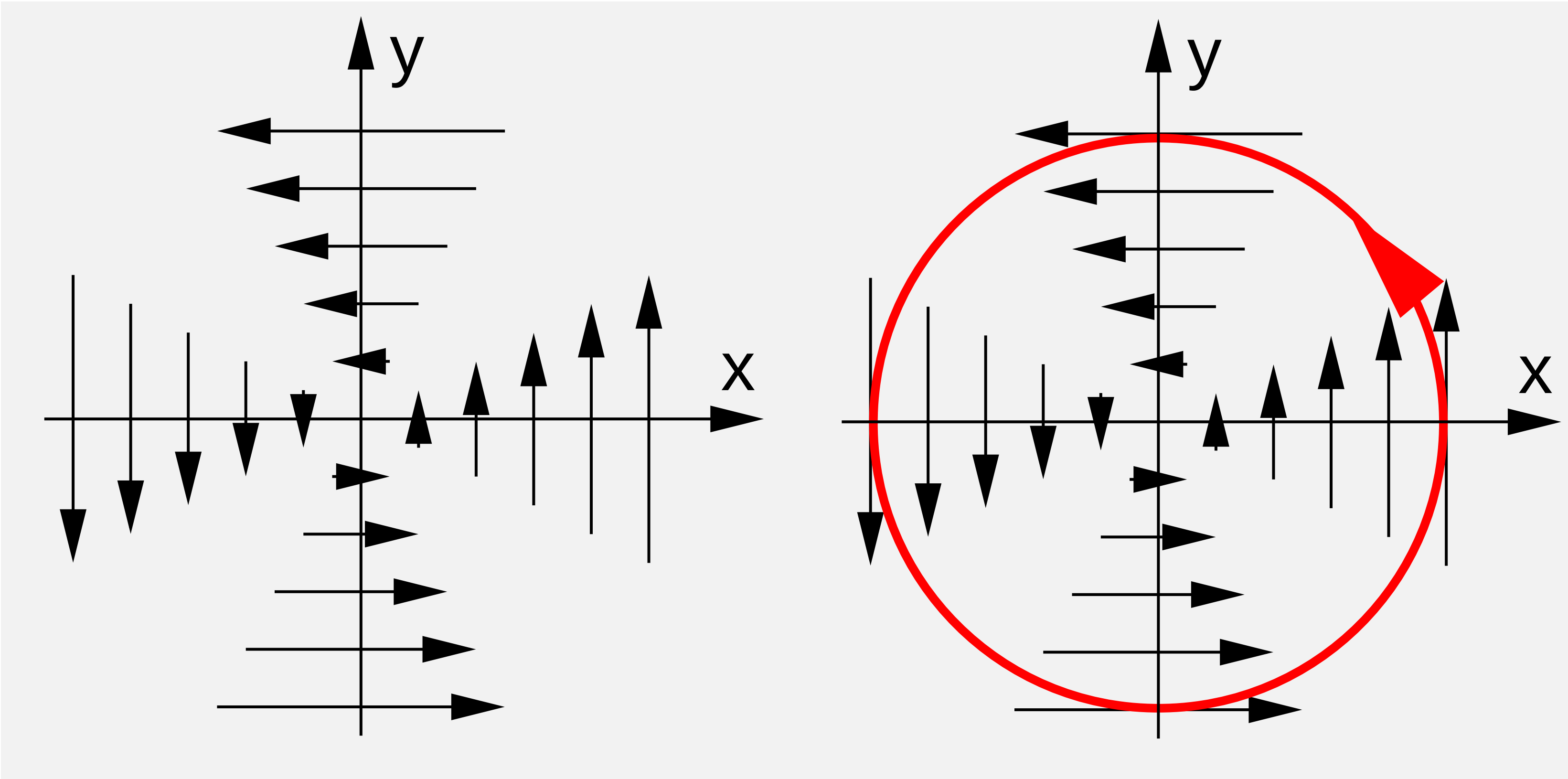
♣ Examples of curl evaluation

$\mathbf{a}$	$\nabla \times \mathbf{a}$
$-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$	$2\hat{\mathbf{k}}$
$x^2y^2\hat{\mathbf{k}}$	$2x^2y\hat{\mathbf{i}} - 2xy^2\hat{\mathbf{j}}$

2nd example:

$$\begin{aligned}
\nabla \times (x^2y^2\hat{\mathbf{k}}) &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & x^2y^2 \end{vmatrix} \\
&= \hat{\mathbf{i}}x^22y - \hat{\mathbf{j}}2xy^2 \\
&= 2x^2y\hat{\mathbf{i}} - 2xy^2\hat{\mathbf{j}}
\end{aligned}$$

- First example gives a clue ... the field $\mathbf{a} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$ is sketched below.
- This field has a curl of $2\hat{\mathbf{k}}$, which is in the r-h screw direction out of the page.
- You can also see that a field like this must give a finite value to the line integral around the complete loop $\oint_C \mathbf{a} \cdot d\mathbf{r}$.



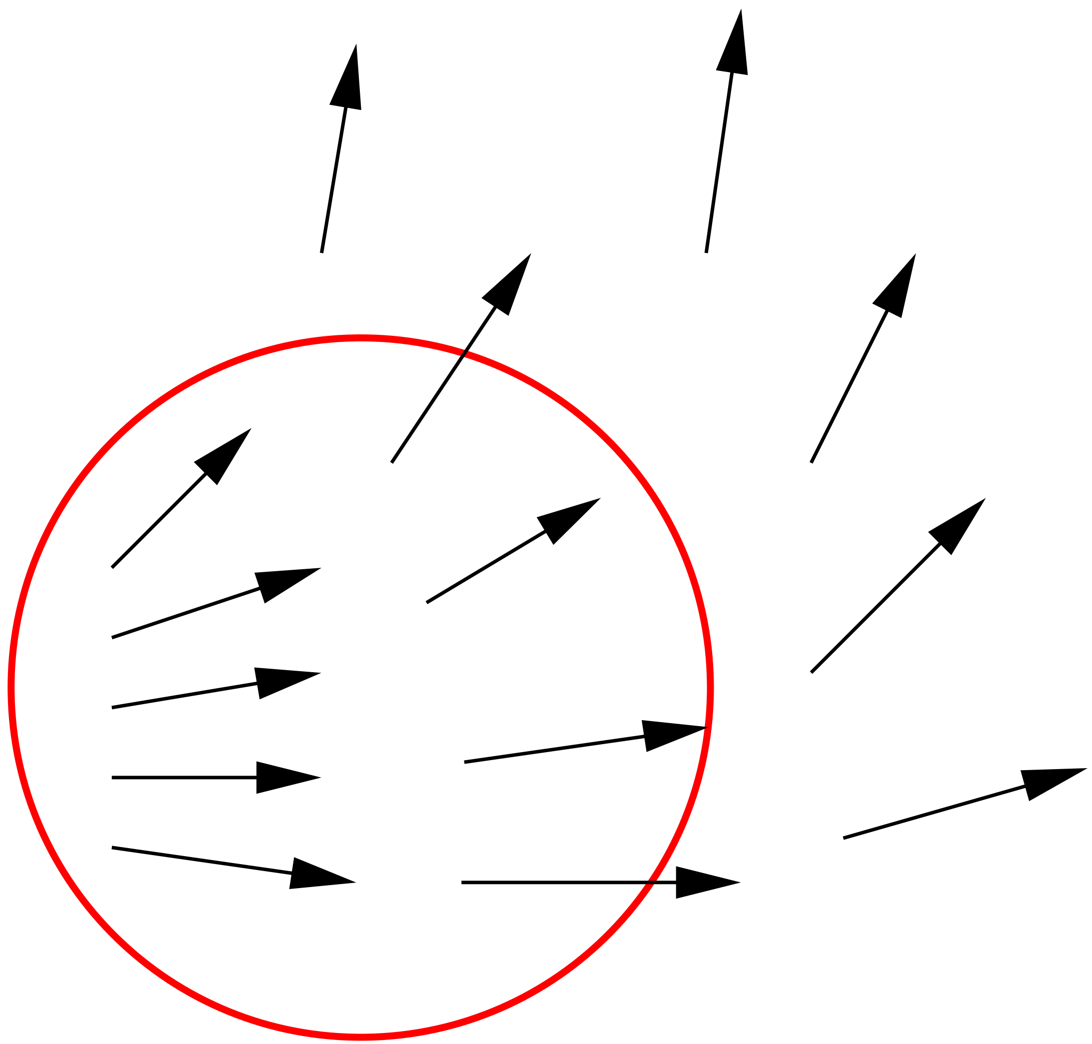
The signficance of curl

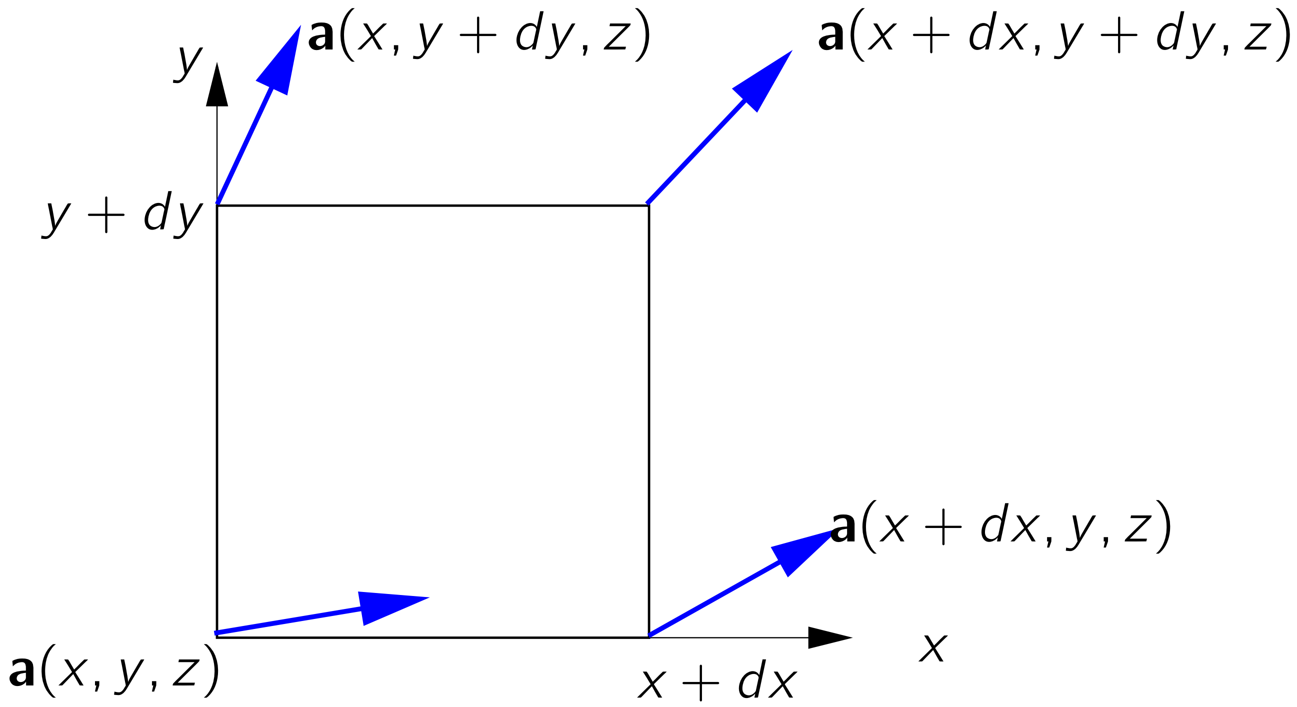
- In fact curl is closely related to the line integral around a loop.
- The **circulation** of a vector field **a** round any closed curve C is defined to be

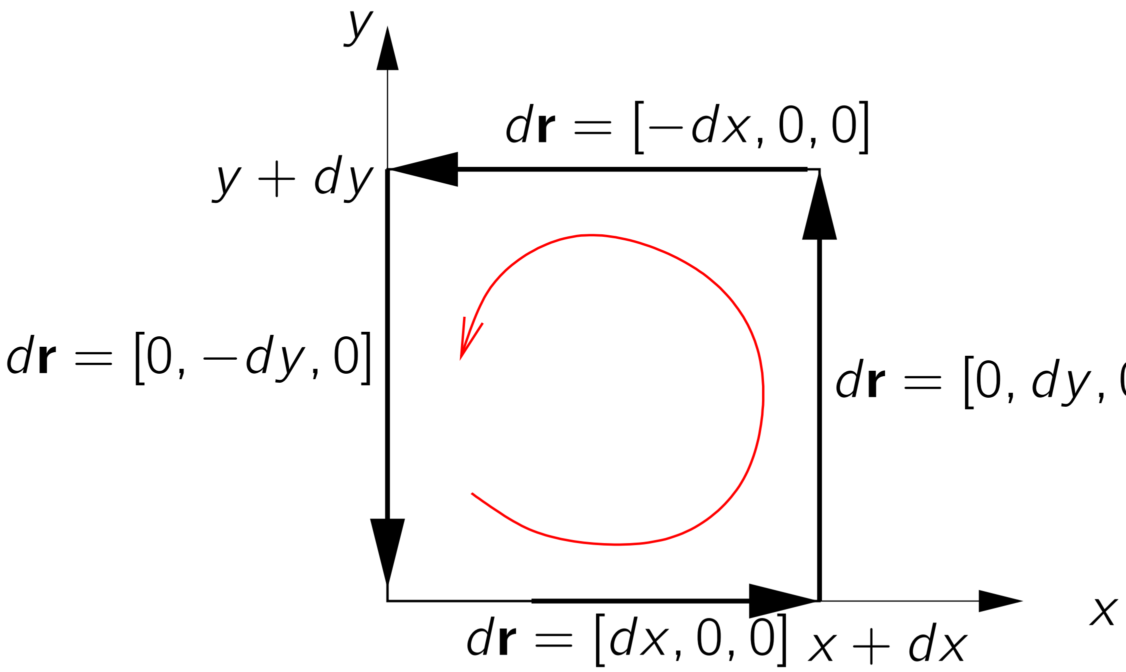
$$\oint_C \mathbf{a} \cdot d\mathbf{r}$$

The **curl** of the vector field **a** represents the

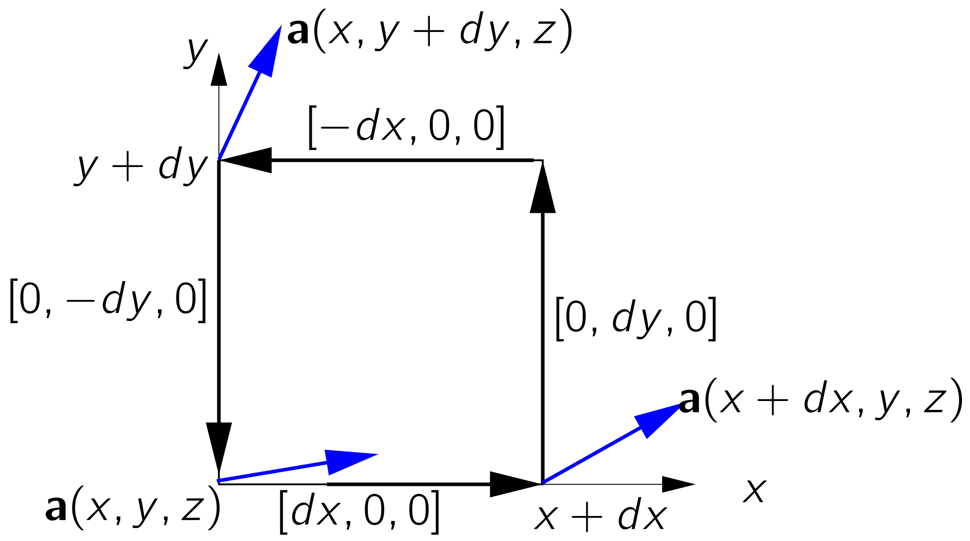
* the **vorticity**, or
* the **circulation per unit area** in the direction of the area's normal




$\mathbf{a}(x, y+dy, z)$
$\mathbf{a}(x+dx, y+dy, z)$
y
$y+dy$
$\mathbf{a}(x+dx, y, z)$
$\mathbf{a}(x, y, z)$
$x+dx$
x



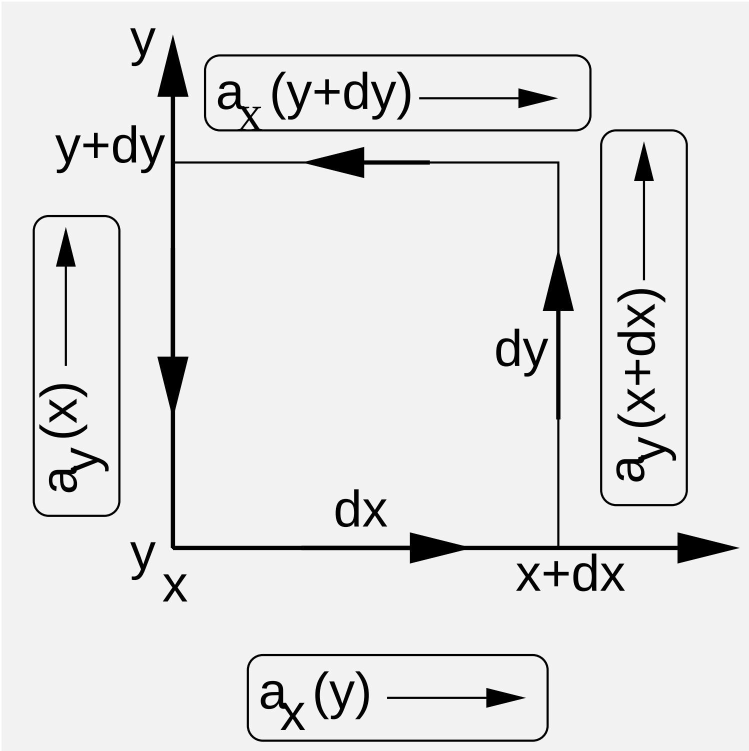
- Consider the circulation round the perimeter of a rectangle dx by dy ...



$$\oint_C \mathbf{a}\cdot d\mathbf{r} = \mathbf{a}(x,y,z).\begin{bmatrix} dx & 0 & 0 \end{bmatrix} + \mathbf{a}(x+dx,y,z).\begin{bmatrix} 0 & dy & 0 \end{bmatrix}$$
$$+\mathbf{a}(x,y+dy,z).\begin{bmatrix} -dx & 0 & 0 \end{bmatrix} + \mathbf{a}(x,y,z).\begin{bmatrix} 0 & -dy & 0 \end{bmatrix}$$

$$
\begin{aligned}
\oint_C \mathbf{a}\cdot d\mathbf{r} &= \mathbf{a}(x,y,z).\begin{bmatrix} dx & 0 & 0\end{bmatrix} + \mathbf{a}(x+dx,y,z).\begin{bmatrix} 0 & dy & 0\end{bmatrix} \\
&\quad + \mathbf{a}(x,y+dy,z).\begin{bmatrix} -dx & 0 & 0\end{bmatrix} + \mathbf{a}(x,y,z).\begin{bmatrix} 0 & -dy & 0\end{bmatrix} \\
&= a_x(x,y,z)dx + a_y(x+dx,y,z) \\
&\quad - a_x(x,y+dy,z)dx - a_y(x,y,z)dy \\
&= a_x dx + a_y dy + \frac{\partial a_y}{\partial x} dx dy \\
&\quad - a_x dx - \frac{\partial a_x}{\partial y} dy dx - a_y dy \\
&= \left[\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right] dx dy \\
&= (\boldsymbol{\nabla}\times\mathbf{a})\cdot dx dy \hat{\mathbf{k}} \\
&= (\boldsymbol{\nabla}\times\mathbf{a})\cdot d\mathbf{S}
\end{aligned}
$$

- Rceapping: consider circulation round the perimeter of a rectangle dx by dy



- The fields in the x-direction at bottom and top are

$$a_x(y) \quad \text{and} \quad a_x(y+dy) = a_x(y) + \frac{\partial a_x}{\partial y} dy$$

- The fields in the y-direction at left and right are

$$a_y(x) \quad \text{and} \quad a_y(x+dx) = a_y(x) + \frac{\partial a_y}{\partial x} dx$$

- Summing around from the bottom in anticlockwise order

$$\begin{aligned} dC &= +[a_x(y)\ dx] + [a_y(x+dx)\ dy] - [a_x(y+dy)\ dx] - [a_y(x)\ dy] \\ &= \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\ dx\ dy = (\boldsymbol{\nabla} \times \mathbf{a}) \cdot dxdy\hat{\mathbf{k}} = (\boldsymbol{\nabla} \times \mathbf{a}) \cdot d\mathbf{S} \end{aligned}$$

- A vector field with zero divergence is said to be

solenoidal.

- A vector field with zero curl is said to be

irrotational.

- A scalar field with zero gradient is said to be

constant.

- **Today we've introduced ...**
  - The gradient of a scalar field
  - The divergence of a vector field
  - The Laplacian
  - The curl of a vector field
- We've described the grunt of working these out in Cartesian coordinates ...

  If your partial differentiation is flaky, sort it.
- We've given some insight into what "physical" aspects of fields they relate too.

  Worth spending time thinking about these. Vector calculus is the natural language of engineering in 3 vector spaces..