

# THE SQUARE-ROOT UNSCENTED KALMAN FILTER FOR STATE AND PARAMETER-ESTIMATION

*Rudolph van der Merwe and Eric A. Wan*

Oregon Graduate Institute of Science and Technology  
20000 NW Walker Road, Beaverton, Oregon 97006, USA  
{rvdmerwe,ericwan}@ece.ogi.edu

## ABSTRACT

Over the last 20-30 years, the *extended Kalman filter* (EKF) has become the algorithm of choice in numerous nonlinear estimation and machine learning applications. These include estimating the state of a nonlinear dynamic system as well estimating parameters for nonlinear system identification (e.g., learning the weights of a neural network). The EKF applies the standard linear Kalman filter methodology to a linearization of the true nonlinear system. This approach is sub-optimal, and can easily lead to divergence. Julier et al. [1] proposed the *unscented Kalman filter* (UKF) as a derivative-free alternative to the extended Kalman filter in the framework of state-estimation. This was extended to parameter-estimation by Wan and van der Merwe [2, 3]. The UKF consistently outperforms the EKF in terms of prediction and estimation error, at an equal computational complexity of  $\mathcal{O}(L^3)^1$  for general state-space problems. When the EKF is applied to parameter-estimation, the special form of the state-space equations allows for an  $\mathcal{O}(L^2)$  implementation. This paper introduces the *square-root unscented Kalman filter* (SR-UKF) which is also  $\mathcal{O}(L^3)$  for general state-estimation and  $\mathcal{O}(L^2)$  for parameter estimation (note the original formulation of the UKF for parameter-estimation was  $\mathcal{O}(L^3)$ ). In addition, the square-root forms have the added benefit of numerical stability and guaranteed positive semi-definiteness of the state covariances.

## 1. INTRODUCTION

The EKF has been applied extensively to the field of nonlinear estimation for both *state-estimation* and *parameter-estimation*. The basic framework for the EKF (and the UKF) involves estimation of the state of a discrete-time nonlinear dynamic system,

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{v}_k \quad (1)$$

$$\mathbf{y}_k = \mathbf{H}(\mathbf{x}_k) + \mathbf{n}_k, \quad (2)$$

where  $\mathbf{x}_k$  represent the unobserved state of the system,  $\mathbf{u}_k$  is a known exogenous input, and  $\mathbf{y}_k$  is the observed measurement signal. The *process* noise  $\mathbf{v}_k$  drives the dynamic system, and the *observation* noise is given by  $\mathbf{n}_k$ . The EKF involves the recursive estimation of the mean and covariance of the state under a Gaussian assumption.

In contrast, parameter-estimation, sometimes referred to as system identification, involves determining a nonlinear mapping

$\mathbf{y}_k = \mathbf{G}(\mathbf{x}_k, \mathbf{w})$ , where  $\mathbf{x}_k$  is the input,  $\mathbf{y}_k$  is the output, and the nonlinear map,  $\mathbf{G}(\cdot)$ , is parameterized by the vector  $\mathbf{w}$ . Typically, a training set is provided with sample pairs consisting of known input and desired outputs,  $\{\mathbf{x}_k, \mathbf{d}_k\}$ . The error of the machine is defined as  $\mathbf{e}_k = \mathbf{d}_k - \mathbf{G}(\mathbf{x}_k, \mathbf{w})$ , and the goal of learning involves solving for the parameters  $\mathbf{w}$  in order to minimize the expectation of some given function of the error. While a number of optimization approaches exist (e.g., gradient descent and Quasi-Newton methods), parameters can be efficiently estimated on-line by writing a new state-space representation

$$\mathbf{w}_{k+1} = \mathbf{w}_k + \mathbf{r}_k \quad (3)$$

$$\mathbf{d}_k = \mathbf{G}(\mathbf{x}_k, \mathbf{w}_k) + \mathbf{e}_k, \quad (4)$$

where the parameters  $\mathbf{w}_k$  correspond to a stationary process with identity state transition matrix, driven by process noise  $\mathbf{r}_k$  (the choice of variance determines convergence and tracking performance). The output  $\mathbf{d}_k$  corresponds to a nonlinear observation on  $\mathbf{w}_k$ . The EKF can then be applied directly as an efficient “second-order” technique for learning the parameters [4].

## 2. THE UNSCENTED KALMAN FILTER

The inherent flaws of the EKF are due to its linearization approach for calculating the mean and covariance of a random variable which undergoes a nonlinear transformation. As shown in shown in [1, 2, 3], the UKF addresses these flaws by utilizing a deterministic “sampling” approach to calculate mean and covariance terms. Essentially,  $2L + 1$ , *sigma* points ( $L$  is the state dimension), are chosen based on a square-root decomposition of the prior covariance. These sigma points are propagated through the true nonlinearity, without approximation, and then a weighted mean and covariance is taken. A simple illustration of the approach is shown in Figure 1 for a 2-dimensional system: the left plot shows the true mean and covariance propagation using Monte-Carlo sampling; the center plots show the results using a linearization approach as would be done in the EKF; the right plots show the performance of the new “sampling” approach (note only 5 sigma points are required). This approach results in approximations that are accurate to the third order (Taylor series expansion) for Gaussian inputs for all nonlinearities. For non-Gaussian inputs, approximations are accurate to at least the second-order [1]. In contrast, the linearization approach of the EKF results only in first order accuracy.

The full UKF involves the recursive application of this “sampling” approach to the state-space equations. The standard UKF implementation is given in Algorithm 2.1 for state-estimation, and uses the following variable definitions:  $\{W_i\}$  is a set of scalar weights ( $W_0^{(m)} = \lambda/(L + \lambda)$ ,  $W_0^{(c)} = \lambda/(L + \lambda) + (1 - \alpha^2 + \beta)$ )

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<sup>1</sup>  $L$  is the dimension of the state variable.

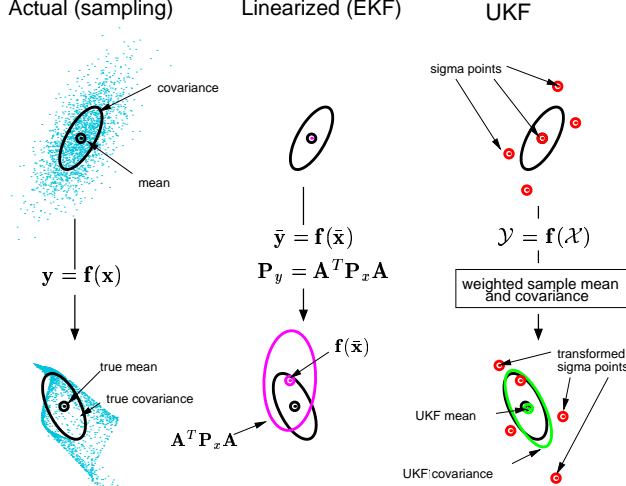


Figure 1: Example of mean and covariance propagation. a) actual, b) first-order linearization (EKF), c) new “sampling” approach (UKF).

,  $W_i^{(m)} = W_i^{(c)} = 1/\{2(L + \lambda)\}$   $i = 1, \dots, 2L$ .  $\lambda = L(\alpha^2 - 1)$  and  $\eta = \sqrt{L + \lambda}$  are scaling parameters. The constant  $\alpha$  determines the spread of the sigma points around  $\hat{\mathbf{x}}$  and is usually set to  $1e - 4 \leq \alpha \leq 1$ .  $\beta$  is used to incorporate prior knowledge of the distribution of  $\mathbf{x}$  (for Gaussian distributions,  $\beta = 2$  is optimal). Also note that we define the linear algebra operation of adding a column vector to a matrix, i.e.  $\mathbf{A} \pm \mathbf{u}$  as the addition of the vector to each column of the matrix. The superior performance of the UKF over the EKF has been demonstrated in a number of applications [1, 2, 3]. Furthermore, unlike the EKF, no explicit derivatives (i.e., Jacobians or Hessians) need to be calculated.

### 3. EFFICIENT SQUARE-ROOT IMPLEMENTATION

The most computationally expensive operation in the UKF corresponds to calculating the new set of sigma points at each time update. This requires taking a matrix square-root of the state covariance matrix<sup>2</sup>,  $\mathbf{P} \in \mathbb{R}^{L \times L}$ , given by  $\mathbf{S}\mathbf{S}^T = \mathbf{P}$ . An efficient implementation using a Cholesky factorization requires in general  $\mathcal{O}(L^3/6)$  computations [5]. While the square-root of  $\mathbf{P}$  is an integral part of the UKF, it is still the full covariance  $\mathbf{P}$  which is recursively updated. In the SR-UKF implementation,  $\mathbf{S}$  will be propagated directly, avoiding the need to refactorize at each time step. The algorithm will in general still be  $\mathcal{O}(L^3)$ , but with improved numerical properties similar to those of standard square-root Kalman filters [6]. Furthermore, for the special state-space formulation of parameter-estimation, an  $\mathcal{O}(L^2)$  implementation becomes possible.

The square-root form of the UKF makes use of three powerful linear algebra techniques<sup>3</sup>, *QR decomposition*, *Cholesky factor updating* and *efficient least squares*, which we briefly review below:

- *QR decomposition*. The QR decomposition or factorization of a matrix  $\mathbf{A} \in \mathbb{R}^{L \times N}$  is given by,  $\mathbf{A}^T = \mathbf{Q}\mathbf{R}$ , where  $\mathbf{Q} \in \mathbb{R}^{N \times N}$  is orthogonal,  $\mathbf{R} \in \mathbb{R}^{N \times L}$  is upper triangular and  $N \geq L$ . The upper triangular part of  $\mathbf{R}$ ,  $\bar{\mathbf{R}}$ , is the transpose of the Cholesky factor of  $\mathbf{P} = \mathbf{A}\mathbf{A}^T$ , i.e.,

<sup>2</sup>For notational clarity, the time index  $k$  has been omitted.

<sup>3</sup>See [5] for theoretical and implementation details.

Initialize with:

$$\hat{\mathbf{x}}_0 = \mathbb{E}[\mathbf{x}_0] \quad \mathbf{P}_0 = \mathbb{E}[(\mathbf{x}_0 - \hat{\mathbf{x}}_0)(\mathbf{x}_0 - \hat{\mathbf{x}}_0)^T] \quad (5)$$

For  $k \in \{1, \dots, \infty\}$ ,

Calculate sigma points:

$$\mathcal{X}_{k-1} = [\hat{\mathbf{x}}_{k-1} \quad \hat{\mathbf{x}}_{k-1} + \eta\sqrt{\mathbf{P}_{k-1}} \quad \hat{\mathbf{x}}_{k-1} - \eta\sqrt{\mathbf{P}_{k-1}}] \quad (6)$$

Time update:

$$\mathcal{X}_{k|k-1} = \mathbf{F}[\mathcal{X}_{k-1}, \mathbf{u}_{k-1}] \quad (7)$$

$$\hat{\mathbf{x}}_k^- = \sum_{i=0}^{2L} W_i^{(m)} \mathcal{X}_{i,k|k-1} \quad (8)$$

$$\mathbf{P}_k^- = \sum_{i=0}^{2L} W_i^{(c)} [\mathcal{X}_{i,k|k-1} - \hat{\mathbf{x}}_k^-][\mathcal{X}_{i,k|k-1} - \hat{\mathbf{x}}_k^-]^T + \mathbf{R}^v \quad (9)$$

$$\mathcal{Y}_{k|k-1} = \mathbf{H}[\mathcal{X}_{k|k-1}]$$

$$\hat{\mathbf{y}}_k^- = \sum_{i=0}^{2L} W_i^{(m)} \mathcal{Y}_{i,k|k-1} \quad (10)$$

Measurement update equations:

$$\mathbf{P}_{\tilde{\mathbf{y}}_k \tilde{\mathbf{y}}_k} = \sum_{i=0}^{2L} W_i^{(c)} [\mathcal{Y}_{i,k|k-1} - \hat{\mathbf{y}}_k^-][\mathcal{Y}_{i,k|k-1} - \hat{\mathbf{y}}_k^-]^T + \mathbf{R}^n$$

$$\mathbf{P}_{\mathbf{x}_k \mathbf{y}_k} = \sum_{i=0}^{2L} W_i^{(c)} [\mathcal{X}_{i,k|k-1} - \hat{\mathbf{x}}_k^-][\mathcal{Y}_{i,k|k-1} - \hat{\mathbf{y}}_k^-]^T \quad (11)$$

$$\mathcal{K}_k = \mathbf{P}_{\mathbf{x}_k \mathbf{y}_k} \mathbf{P}_{\tilde{\mathbf{y}}_k \tilde{\mathbf{y}}_k}^{-1} \quad (12)$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathcal{K}_k (\mathbf{y}_k - \hat{\mathbf{y}}_k^-) \quad (13)$$

$$\mathbf{P}_k = \mathbf{P}_k^- - \mathcal{K}_k \mathbf{P}_{\tilde{\mathbf{y}}_k \tilde{\mathbf{y}}_k} \mathcal{K}_k^T \quad (14)$$

where  $\mathbf{R}^v$ =process noise cov.,  $\mathbf{R}^n$ =measurement noise cov.

#### Algorithm 2.1: Standard UKF algorithm.

$\bar{\mathbf{R}} = \mathbf{S}^T$ , such that  $\bar{\mathbf{R}}^T \bar{\mathbf{R}} = \mathbf{A}\mathbf{A}^T$ . We use the shorthand notation  $\text{qr}\{\cdot\}$  to denote a QR decomposition of a matrix where only  $\bar{\mathbf{R}}$  is returned. The computational complexity of a QR decomposition is  $\mathcal{O}(NL^2)$ . Note that performing a Cholesky factorization directly on  $\mathbf{P} = \mathbf{A}\mathbf{A}^T$  is  $\mathcal{O}(L^3/6)$  plus  $\mathcal{O}(NL^2)$  to form  $\mathbf{A}\mathbf{A}^T$ .

- *Cholesky factor updating*. If  $\mathbf{S}$  is the original Cholesky factor of  $\mathbf{P} = \mathbf{A}\mathbf{A}^T$ , then the Cholesky factor of the rank-1 update (or downdate)  $\mathbf{P} \pm \sqrt{\nu}\mathbf{u}\mathbf{u}^T$  is denoted as  $\mathbf{S} = \text{cholupdate}\{\mathbf{S}, \mathbf{u}, \pm\nu\}$ . If  $\mathbf{u}$  is a matrix and not a vector, then the result is  $M$  consecutive updates of the Cholesky factor using the  $M$  columns of  $\mathbf{u}$ . This algorithm (available in Matlab as `cholupdate`) is only  $\mathcal{O}(L^2)$  per update.
- *Efficient least squares*. The solution to the equation  $(\mathbf{A}\mathbf{A}^T)\mathbf{x} = \mathbf{A}^T\mathbf{b}$  also corresponds to the solution of the overdetermined least squares problem  $\mathbf{A}\mathbf{x} = \mathbf{b}$ . This can be solved efficiently using a QR decomposition with pivoting (implemented in Matlab's `\backslash` operator).

The complete specification of the new square-root filters is given in Algorithm 3.1 for state-estimation and 3.2 for parameter-

estimation. Below we describe the key parts of the square-root algorithms, and how they contrast with the standard implementations.

**Square-Root State-Estimation:** As in the original UKF, the filter is initialized by calculating the matrix square-root of the state covariance once via a Cholesky factorization (Eqn. 16). However, the propagated and updated Cholesky factor is then used in subsequent iterations to directly form the sigma points. In Eqn. 20 the *time-update* of the Cholesky factor,  $\mathbf{S}^-$ , is calculated using a QR decomposition of the compound matrix containing the weighted propagated sigma points and the matrix square-root of the additive process noise covariance. The subsequent Cholesky update (or downdate) in Eqn. 21 is necessary since the zero'th weight,  $W_0^{(c)}$ , may be negative. These two steps replace the *time-update* of  $\mathbf{P}^-$  in Eqn. 9, and is also  $\mathcal{O}(L^3)$ .

The same two-step approach is applied to the calculation of the Cholesky factor,  $\mathbf{S}_{\tilde{\mathbf{y}}}$ , of the observation-error covariance in Eqns. 24 and 25. This step is  $\mathcal{O}(LM^2)$ , where  $M$  is the observation dimension. In contrast to the way the Kalman gain is calculated in the standard UKF (see Eqn. 12), we now use two nested inverse (or *least squares*) solutions to the following expansion of Eqn. 12,  $\mathcal{K}_k(\mathbf{S}_{\tilde{\mathbf{y}}}_k \mathbf{S}_{\tilde{\mathbf{y}}}_k^T) = \mathbf{P}_{\mathbf{x}_k \mathbf{y}_k}$ . Since  $\mathbf{S}_{\tilde{\mathbf{y}}}$  is square and triangular, efficient “back-substitutions” can be used to solve for  $\mathcal{K}_k$  directly without the need for a matrix inversion.

Finally, the posterior measurement update of the Cholesky factor of the state covariance is calculated in Eqn. 29 by applying  $M$  sequential Cholesky downdates to  $\mathbf{S}_k^-$ . The downdate vectors are the columns of  $\mathbf{U} = \mathcal{K}_k \mathbf{S}_{\tilde{\mathbf{y}}}_k$ . This replaces the posterior update of  $\mathbf{P}_k$  in Eqn. 14, and is also  $\mathcal{O}(LM^2)$ .

**Square-Root Parameter-Estimation:** The parameter-estimation algorithm follows a similar framework as that of the state-estimation square-root UKF. However, an  $\mathcal{O}(ML^2)$  algorithm, as opposed to  $\mathcal{O}(L^3)$ , is possible by taking advantage of the *linear* state transition function. Specifically, the time-update of the state covariance is given simply by  $\mathbf{P}_{\mathbf{w}_k}^- = \mathbf{P}_{\mathbf{w}_{k-1}} + \mathbf{R}_{\mathbf{w}_{k-1}}^T$ . Now, if we apply an exponential weighting on past data<sup>4</sup>, the process noise covariance is given by  $\mathbf{R}_k^r = (\gamma^{-1} - 1)\mathbf{P}_{\mathbf{w}_k}$ , and the time update of the state covariance becomes,

$$\mathbf{P}_{\mathbf{w}_k}^- = \mathbf{P}_{\mathbf{w}_{k-1}} + (\gamma^{-1} - 1)\mathbf{P}_{\mathbf{w}_{k-1}} = \gamma^{-1}\mathbf{P}_{\mathbf{w}_{k-1}}. \quad (15)$$

This translates readily into the factored form,  $\mathbf{S}_{\mathbf{w}_k}^- = \gamma^{-1/2}\mathbf{S}_{\mathbf{w}_{k-1}}$  (see Eqn. 32), and avoids the costly  $\mathcal{O}(L^3)$  QR and Cholesky based updates necessary in the state-estimation filter.

#### 4. EXPERIMENTAL RESULTS

The improvement in error performance of the UKF over that of the EKF for both state and parameter-estimation is well documented [1, 2, 3]. The focus of this section will be to simply verify the equivalent error performance of the UKF and SR-UKF, and show the reduction in computational cost achieved by the SR-UKF for parameter-estimation. Figure 2 shows the superior performance of UKF and SR-UKF compared to that of the EKF on estimating the Mackey-Glass-30 chaotic time series corrupted by additive white noise (3dB SNR). The error performance of the SR-UKF and UKF are indistinguishable and are both superior to the EKF. The computational complexity of all three filters are of the same order but

<sup>4</sup>This is identical to the approach used in weighted recursive least squares (W-RLS).  $\gamma$  is a scalar weighting factor chosen to be slightly less than 1, i.e.  $\gamma = 0.9995$ .

Initialize with:

$$\hat{\mathbf{x}}_0 = \mathbb{E}[\mathbf{x}_0] \quad \mathbf{S}_0 = \text{chol} \left\{ \mathbb{E}[(\mathbf{x}_0 - \hat{\mathbf{x}}_0)(\mathbf{x}_0 - \hat{\mathbf{x}}_0)^T] \right\} \quad (16)$$

For  $k \in \{1, \dots, \infty\}$ ,

Sigma point calculation and time update:

$$\mathcal{X}_{k-1} = [\hat{\mathbf{x}}_{k-1} \quad \hat{\mathbf{x}}_{k-1} + \eta \mathbf{S}_k \quad \hat{\mathbf{x}}_{k-1} - \eta \mathbf{S}_k] \quad (17)$$

$$\mathcal{X}_{k|k-1} = \mathbf{F}[\mathcal{X}_{k-1}, \mathbf{u}_{k-1}] \quad (18)$$

$$\hat{\mathbf{x}}_k^- = \sum_{i=0}^{2L} W_i^{(m)} \mathcal{X}_{i,k|k-1} \quad (19)$$

$$\mathbf{S}_k^- = \text{qr} \left\{ \left[ \sqrt{W_1^{(c)}} (\mathcal{X}_{1:2L,k|k-1} - \hat{\mathbf{x}}_k^-) \quad \sqrt{\mathbf{R}^v} \right] \right\} \quad (20)$$

$$\mathbf{S}_k^- = \text{cholupdate} \left\{ \mathbf{S}_k^-, \mathcal{X}_{0,k} - \hat{\mathbf{x}}_k^-, W_0^{(c)} \right\} \quad (21)$$

$$\mathcal{Y}_{k|k-1} = \mathbf{H}[\mathcal{X}_{k|k-1}] \quad (22)$$

$$\hat{\mathbf{y}}_k^- = \sum_{i=0}^{2L} W_i^{(m)} \mathcal{Y}_{i,k|k-1} \quad (23)$$

Measurement update equations:

$$\mathbf{S}_{\tilde{\mathbf{y}}}_k = \text{qr} \left\{ \left[ \sqrt{W_1^{(c)}} [\mathcal{Y}_{1:2L,k} - \hat{\mathbf{y}}_k^-] \quad \sqrt{\mathbf{R}^n} \right] \right\} \quad (24)$$

$$\mathbf{S}_{\tilde{\mathbf{y}}}_k = \text{cholupdate} \left\{ \mathbf{S}_{\tilde{\mathbf{y}}}_k, \mathcal{Y}_{0,k} - \hat{\mathbf{y}}_k^-, W_0^{(c)} \right\} \quad (25)$$

$$\mathbf{P}_{\mathbf{x}_k \mathbf{y}_k} = \sum_{i=0}^{2L} W_i^{(c)} [\mathcal{X}_{i,k|k-1} - \hat{\mathbf{x}}_k^-][\mathcal{Y}_{i,k|k-1} - \hat{\mathbf{y}}_k^-]^T \quad (26)$$

$$\mathcal{K}_k = (\mathbf{P}_{\mathbf{x}_k \mathbf{y}_k} / \mathbf{S}_{\tilde{\mathbf{y}}}_k^T) / \mathbf{S}_{\tilde{\mathbf{y}}}_k \quad (27)$$

$$\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k^- + \mathcal{K}_k(\mathbf{y}_k - \hat{\mathbf{y}}_k^-) \quad (28)$$

$$\mathbf{U} = \mathcal{K}_k \mathbf{S}_{\tilde{\mathbf{y}}}_k \quad (29)$$

where  $\mathbf{R}^v$ =process noise cov.,  $\mathbf{R}^n$ =measurement noise cov.

**Algorithm 3.1:** Square-Root UKF for state-estimation.

the SR-UKF is about 20% faster than the UKF and about 10% faster than the EKF.

The next experiment shows the reduction in computational cost achieved by the square-root unscented Kalman filters and how that compares to the computational complexity of the EKF for parameter-estimation. For this experiment, we use an EKF, UKF and SR-UKF to train a 2-12-2 MLP neural network on the well known *Mackay-Robot-Arm*<sup>5</sup> benchmark problem of mapping the joint angles of a robot arm to the Cartesian coordinates of the hand. The learning curves (mean square error (MSE) vs. learning epoch) of the different filters are shown in Figure 3. Figure 4 shows how the computational complexity of the different filters scale as a function of the number of parameters (weights in neural network). While the standard UKF is  $\mathcal{O}(L^3)$ , both the EKF and SR-UKF are  $\mathcal{O}(L^2)$ .

<sup>5</sup><http://wol.ra.phy.cam.ac.uk/mackay>

Initialize with:

$$\hat{\mathbf{w}}_0 = E[\mathbf{w}] \quad \mathbf{S}_{\mathbf{w}_0} = \text{chol} \left\{ E[(\mathbf{w} - \hat{\mathbf{w}}_0)(\mathbf{w} - \hat{\mathbf{w}}_0)^T] \right\} \quad (30)$$

For  $k \in \{1, \dots, \infty\}$ ,

Time update and sigma point calculation:

$$\hat{\mathbf{w}}_k^- = \hat{\mathbf{w}}_{k-1} \quad (31)$$

$$\mathbf{S}_{\mathbf{w}_k}^- = \gamma^{-1/2} \mathbf{S}_{\mathbf{w}_{k-1}} \quad (32)$$

$$\mathcal{W}_{k|k-1} = [\hat{\mathbf{w}}_k^- \quad \hat{\mathbf{w}}_k^- + \eta \mathbf{S}_{\mathbf{w}_k}^- \quad \hat{\mathbf{w}}_k^- - \eta \mathbf{S}_{\mathbf{w}_k}^-] \quad (33)$$

$$\mathcal{D}_{k|k-1} = \mathbf{G}[\mathbf{x}_k, \mathcal{W}_{k|k-1}] \quad (34)$$

$$\hat{\mathbf{d}}_k = \sum_{i=0}^{2L} W_i^{(m)} \mathcal{D}_{i,k|k-1} \quad (35)$$

Measurement update equations:

$$\mathbf{S}_{\mathbf{d}_k} = \text{qr} \left\{ \left[ \sqrt{W_1^{(c)}} [\mathcal{D}_{1:2L,k} - \hat{\mathbf{d}}_k] \quad \sqrt{\mathbf{R}^e} \right] \right\} \quad (36)$$

$$\mathbf{S}_{\mathbf{d}_k} = \text{cholupdate} \left\{ \mathbf{S}_{\mathbf{d}_k}, \mathcal{D}_{0,k} - \hat{\mathbf{d}}_k, W_0^{(c)} \right\} \quad (37)$$

$$\mathbf{P}_{\mathbf{w}_k \mathbf{d}_k} = \sum_{i=0}^{2L} W_i^{(c)} [\mathcal{W}_{i,k|k-1} - \hat{\mathbf{w}}_k^-][\mathcal{D}_{i,k|k-1} - \hat{\mathbf{d}}_k]^T \quad (38)$$

$$\mathcal{K}_k = (\mathbf{P}_{\mathbf{w}_k \mathbf{d}_k} / \mathbf{S}_{\mathbf{d}_k}^T) / \mathbf{S}_{\mathbf{d}_k} \quad (39)$$

$$\hat{\mathbf{w}}_k = \hat{\mathbf{w}}_k^- + \mathcal{K}_k (\mathbf{d}_k - \hat{\mathbf{d}}_k) \quad (40)$$

$$\mathbf{U} = \mathcal{K}_k \mathbf{S}_{\mathbf{d}_k} \quad (41)$$

$$\mathbf{S}_{\mathbf{w}_k} = \text{cholupdate} \left\{ \mathbf{S}_{\mathbf{w}_k}^-, \mathbf{U}, -1 \right\} \quad (42)$$

where  $\mathbf{R}^e$ =measurement noise cov (this can be set to an arbitrary value, e.g.,  $.5\mathbf{I}$ .)

**Algorithm 3.2:** Square-Root UKF for parameter-estimation.

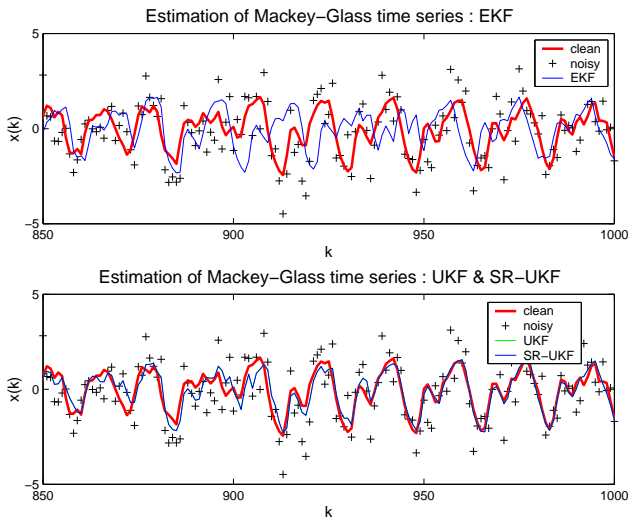


Figure 2: Estimation of the Mackey-Glass chaotic time-series with the EKF, UKF and SR-UKF.

## 5. CONCLUSIONS

The UKF consistently performs better than or equal to the well known EKF, with the added benefit of ease of implementation in

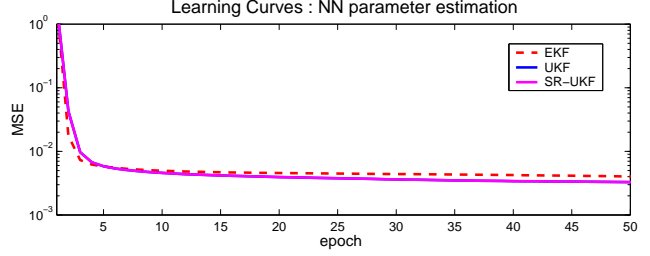


Figure 3: Learning curves for Mackay-Robot-Arm neural network parameter-estimation problem.

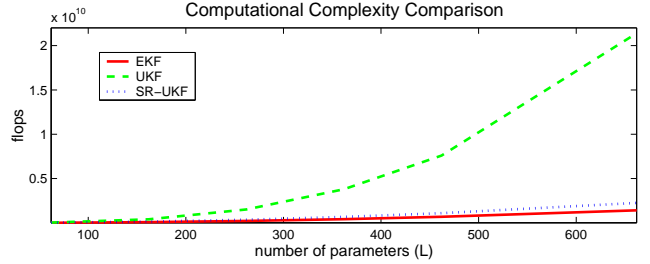


Figure 4: Computational complexity (flops/epoch) of EKF, UKF and SR-UKF for parameter-estimation (Mackay-Robot-Arm problem).

that no analytical derivatives (Jacobians or Hessians) need to be calculated. For state-estimation, the UKF and EKF have equal complexity and are in general  $\mathcal{O}(L^3)$ . In this paper, we introduced square-root forms of the UKF. The square-root UKF has better numerical properties and guarantees positive semi-definiteness of the underlying state covariance. In addition, for parameter-estimation an efficient  $\mathcal{O}(L^2)$  implementation is possible for the square-root form, which is again of the same complexity as efficient EKF parameter-estimation implementations. In this light, the SR-UKF is the logical replacement for the EKF in all state and parameter-estimation applications.

## 6. REFERENCES

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