

# Cryptography Meets Algorithms (15893) Lecture Notes

## Lecture 1: Private Information Retrieval

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The Private Information Retrieval problem was first introduced by Chor, Kushilevitz, Goldreich and Sudan [CKGS98]. In this setting, we will have a client and one or more server(s). The servers each have a public database indexed from 1 to  $n$  (e.g., the DNS repository, a repository of webpages, a leaked password database, etc).

A client wants to fetch an entry indexed  $i \in [n]$  from this database but does not want to leak its query to the server(s). More formally, we define a single-server PIR scheme as follows.

**Definition 1** (Single-server PIR). A single-server PIR, parametrized by a security parameter  $\lambda \in \mathbb{N}$ , is a protocol between a client and a server with the following syntax:

- The client's input is a desired index  $i \in [n]$ , and the server's input is a database  $DB \in \{0, 1\}^n$ . Both the client and server also obtain  $1^\lambda$  as input.
- At the end of the protocol, the client outputs a bit  $b \in \{0, 1\}$ .

We want the scheme to satisfy the following properties.

- **Correctness:** for all  $\lambda, n$ , for any  $DB \in \{0, 1\}^n, i \in [n]$ , under honest execution,

$$\Pr[b = DB[i]] = 1$$

- **Privacy:** For any  $\lambda$ , any  $n$  polynomially bounded in  $\lambda$ , any  $i, j \in [n], DB \in \{0, 1\}^n$ , it holds that

$$\text{view}_S(1^\lambda, DB, i) \approx \text{view}_S(1^\lambda, DB, j)$$

where  $\text{view}_S(1^\lambda, DB, i)$  is a random variable representing the view of the server if we execute the PIR protocol over client input  $(1^\lambda, i)$  and server input  $(1^\lambda, DB)$ , and  $\approx$  stands for statistical or computational indistinguishability.

**Remark 1** (Honest-server vs. malicious-server privacy). *The above privacy definition assumes an honest server. It is also possible to define privacy against a malicious server. In today's lecture, all the PIR constructions will only have a single round-trip — in this special case, honest-server privacy and malicious-server privacy are equivalent. So we will simply define honest-server privacy here.*

This definition is naturally extended to a setting with two or more servers that do not communicate, where privacy should hold for any individual server's view. In a setting with more than two servers, it also makes sense to define  $t$ -out-of- $n$  security, where we want privacy to hold for the union of any combination of  $t$  servers' views. For example, a 2-server PIR scheme is defined as follows.

**Definition 2** (Two-server PIR). A two-server PIR, parametrized with some security parameter  $\lambda$ , is a protocol between a client  $c$  and two servers  $S_1, S_2$  with the following syntax:

- The client's input is  $1^\lambda$  and a desired index  $i \in [n]$ , and each server's input is a database  $\text{DB} \in \{0, 1\}^n$ .
- At the end of the protocol, the client outputs a bit  $b \in \{0, 1\}$ .

with properties,

- **Correctness:** for all  $\lambda, n$ , for all  $\text{DB} \in \{0, 1\}^n, i \in [n]$ , under honest execution,

$$\Pr[b = \text{DB}[i]] = 1$$

- **Privacy:** for any  $\lambda$ , any  $n$  that is polynomially bounded in  $\lambda$ , any  $i, j \in [n]$ , any  $\text{DB} \in \{0, 1\}^n$ , it holds that

$$\text{view}_{S_1}(1^\lambda, \text{DB}, i) \approx \text{view}_{S_1}(1^\lambda, \text{DB}, j)$$

$$\text{view}_{S_2}(1^\lambda, \text{DB}, i) \approx \text{view}_{S_2}(1^\lambda, \text{DB}, j)$$

where  $\text{view}_{S_1}(1^\lambda, \text{DB}, i)$  and  $\text{view}_{S_2}(1^\lambda, \text{DB}, i)$  are random variables representing the view of the first and the second server, respectively, in a protocol execution with client input  $(1^\lambda, i)$  and server input  $(1^\lambda, \text{DB})$ .

Note that PIR schemes can be extended for retrieving records containing multiple bits, rather than just 1-bit records.

**Naïve approach.** The naïve approach is for the client to download the entire database. However, this approach suffers from linear bandwidth, and linear server/client computation.

We will now show some PIR constructions with sublinear bandwidth.

## 1 Single-Server PIR Based on Fully Homomorphic Encryption

It is easy to obtain a bandwidth-efficient PIR scheme if we assume a Fully Homomorphic Encryption (FHE) scheme. An FHE scheme allows us to perform addition and multiplication operations in the ciphertext space. An FHE scheme supports the following operations:

- $(\text{pk}, \text{sk}) \leftarrow \text{Gen}(1^\lambda)$ : samples a public key  $\text{pk}$  and a secret key  $\text{sk}$ ;
- $c \leftarrow \text{Enc}(\text{pk}, m)$ : encrypts a message  $m$  from some message space using the public key  $\text{pk}$ , and outputs the ciphertext  $c$ ;
- $m \leftarrow \text{Dec}(\text{sk}, c)$ : decrypts a ciphertext  $c$  using the secret key  $\text{sk}$ , and outputs a plaintext message  $m$ ;
- $c' \leftarrow \text{Eval}(\text{pk}, \text{Circ}, c)$ : given the public key  $\text{pk}$ , some circuit  $\text{Circ}$ , and a ciphertext  $c$ , output a transformed ciphertext  $c'$ . Correctness requires that  $c'$  be a valid FHE encryption of  $\text{Circ}(c)$ .

**PIR from FHE.** We can construct a single-server PIR scheme from an FHE scheme as follows.

1. The client samples  $(\text{pk}, \text{sk}) \leftarrow \text{FHE.Gen}(1^\lambda)$ , and encrypts its query as  $q \leftarrow \text{FHE.Enc}(\text{pk}, i)$ . The client then sends  $q$  to the server.
2. The server homomorphically evaluates the selection circuit  $S_{\text{DB}}$  by calling  $c \leftarrow \text{FHE.Eval}(\text{pk}, S_{\text{DB}}, q)$ , where  $S_{\text{DB}}(i)$  is the circuit that selects the  $i$ -th bit from the input  $\text{DB}$ , and sends the resulting ciphertext  $c$  back to the client.

3. The client decrypts  $b \leftarrow \text{FHE.Dec}(\text{sk}, c)$ .

The correctness of the PIR scheme is easy to see given correctness of the FHE scheme. The scheme has  $\tilde{O}(1)$  bandwidth and client computation, and  $\tilde{O}(n)$  server computation<sup>1</sup> where  $\tilde{O}(\cdot)$  hides  $\text{poly}(\lambda, \log n)$  factors. Note that the server computation is at least linear because the selection circuit  $S_{\text{DB}}(\cdot)$  must encode the entire database.

Many recent works optimized FHE-based PIR schemes, e.g., Spiral [MW22] and SimplePIR [HHCG<sup>+</sup>22].

**Question:** Can we get sub-linear bandwidth PIR without any cryptographic assumptions?

In fact, this is possible in the two-server setting (we will see this next), and later we will prove that it is impossible in the single-server setting.

## 2 Two-Server PIR Constructions

### 2.1 $\sqrt{n}$ -Bandwidth 2-Server PIR

We now introduce a 2-server  $\sqrt{n}$ -bandwidth PIR scheme with information theoretic security, i.e., the scheme does not rely on any cryptographic assumptions.

The key idea is to view the database  $\text{DB} \in \{0, 1\}^n$  as a  $\sqrt{n}$  by  $\sqrt{n}$  matrix<sup>2</sup> henceforth denoted  $M \in \{0, 1\}^{\sqrt{n} \times \sqrt{n}}$ . Say the client wants to query the bit  $M[i, j]$ . To achieve this, the client will perform a 2-server PIR protocol at the end of which it retrieves the entire column  $M[:, j]$ . The scheme is as follows:

1. The client samples random column vectors  $v_1, v_2 \in \{0, 1\}^{\sqrt{n}}$  such that  $v_1 \oplus v_2 = e_j$  where  $e_j$  is the vector that is 1 at position  $j$ , and 0 everywhere else, and  $\oplus$  denotes bitwise XOR (i.e., addition mod 2). The client sends  $v_1$  and  $v_2$  to the two servers  $S_1$  and  $S_2$  respectively.
2. Each server  $S_i$ , on receiving  $v_i$ , computes the matrix-vector product,

$$r_i \leftarrow Mv_i \text{ mod } 2$$

and sends the  $\sqrt{n}$ -sized vector  $r_i$  to the client.

3. The client computes and outputs  $r_1 \oplus r_2$ .

It is straightforward to verify correctness by seeing that

$$r_1 \oplus r_2 = Mv_1 + Mv_2 \text{ mod } 2 = M(v_1 + v_2) \text{ mod } 2 = M \cdot e_j$$

Lastly, this scheme is private, since each individual vector  $v_1$  or  $v_2$  is uniformly random.

### 2.2 $n^{\frac{1}{3}}$ -BW 2-server Scheme[CKGS98]

We will first motivate this construction with a two server scheme with expected  $\frac{n}{2}$  bandwidth and a eight server scheme with  $n^{\frac{1}{3}}$  bandwidth.

<sup>1</sup>We assume a compact FHE scheme with the following performance bounds: the ciphertext size is  $\tilde{O}(1)$  for encrypting a plaintext of  $\tilde{O}(1)$  bits, and the encryption and decryption times are also  $\tilde{O}(1)$ , and the homomorphic evaluation time is  $\tilde{O}(|\text{Circ}|)$  for a circuit  $\text{Circ}$ .

<sup>2</sup>Without loss of generality, we may assume that  $n$  is a perfect square — if not, we can always round it up to a perfect square incurring only constant blowup.

### 1. $\mathbb{E}[\frac{n}{2}]$ -BW 2-server Scheme

The client samples  $S_1 \subseteq [n]$  uniformly as follows: for each  $i \in [n]$ , add  $i$  to  $S_1$  with probability  $\frac{1}{2}$ . Then, the client computes

$$S_2 = S_1 \Delta \{i\}$$

where  $\Delta$  is the symmetric difference operator. That is, if  $i$  is included in  $S_1$ , we remove it from the set, otherwise we add it to the set.  $S_1, S_2$  are sent to each server respectively, where server  $i$  computes and sends back,

$$r_i \leftarrow \bigoplus_{j \in S_i} \text{DB}[j]$$

The client on receiving  $r_1, r_2$  outputs  $r_1 \oplus r_2$ .

**Correctness** follows from the fact that  $i$  is the only database index that appears once, with every other index appearing twice (hence XOR'ing to 0).

**Privacy:** First,  $S_1$  is uniformly random. Second, to see why  $S_2$  is uniformly random to server 2, just consider the following distribution – “tossing  $n$  random coins, and flip the  $i$ -th coin afterwards, regardless of the original result”. This distribution is uniformly random even when  $i$  is known.

**Remark 2.** Note that if we run the above  $n/2$ -BW scheme not on bits, but on blocks of  $\sqrt{n}$  size (i.e., treat the  $n$ -bit database as  $\sqrt{n}$  blocks each of size  $\sqrt{n}$ ), the scheme is equivalent to the earlier  $\sqrt{n}$ -BW scheme.

### 2. $n^{\frac{1}{3}}$ -BW 8-server Scheme

The idea is to view the database as a  $n^{\frac{1}{3}} \times n^{\frac{1}{3}} \times n^{\frac{1}{3}}$  cube. Then, each index  $i \in \{0, 1, \dots, n-1\}$  can be expressed as a thruple  $(x^*, y^*, z^*)$  (note we start at 0 for natural base 3 representation).

The client samples  $X, Y, Z \subseteq \{0, \dots, n^{\frac{1}{3}} - 1\}$  independently as follows: for each  $x \in \{0, \dots, n^{\frac{1}{3}}\}$  add it to  $X$  with probability  $\frac{1}{2}$ . Do the same for  $Y, Z$ .

Then, we compute 8 different sets by a 3-wise cartesian products with symmetric differences, enumerated as

$$\begin{aligned} S_{000} &= X \times Y \times Z \\ S_{001} &= X \times Y \times (Z \Delta \{z^*\}) \\ S_{010} &= X \times (Y \Delta \{y^*\}) \times Z \\ S_{011} &= X \times (Y \Delta \{y^*\}) \times (Z \Delta \{z^*\}) \\ S_{100} &= (X \Delta \{x^*\}) \times Y \times Z \\ S_{101} &= (X \Delta \{x^*\}) \times Y \times (Z \Delta \{z^*\}) \\ S_{110} &= (X \Delta \{x^*\}) \times (Y \Delta \{y^*\}) \times Z \\ S_{111} &= (X \Delta \{x^*\}) \times (Y \Delta \{y^*\}) \times (Z \Delta \{z^*\}) \end{aligned}$$

See that each of these sets have their sizes concentrated around  $\left(\frac{n^{\frac{1}{3}}}{2}\right)^3 = \frac{n}{8}$ . Then, the client sends a succinct description of each set (i.e., sends the three “marginal” vectors instead of the full set) of size  $O(n^{\frac{1}{3}})$  to each server.

Server  $i$  on receiving  $S = X' \times Y' \times Z'$  computes

$$p_i = \bigoplus_{j \in S} \text{DB}[j]$$

Then the client computes  $p_0 \oplus \dots \oplus p_7$ .

We claim that  $p_0 \oplus \dots \oplus p_7 = \text{DB}[i], i = (x^*, y^*, z^*)$ .

*Proof.* For every  $(x, y, z)$  not equal to the query, it will appear an even number times in the summation. We can pair up the sets to see this.

On the other hand,  $(x^*, y^*, z^*)$  only appears once in the summation so we are done.  $\square$

Privacy follows the argument as the previous case.

Now, we finally compress this scheme from eight servers to two servers. To do this, the client sends  $S_{000}$  to server 1 and  $S_{111}$  to server 2.

Each server on receiving  $X' \times Y' \times Z'$  calculates  $3n^{\frac{1}{3}}$  parities to send to the client as follows:

1. For each  $x' \in \{0, \dots, n^{\frac{1}{3}} - 1\}$  we calculate the parity for  $(X' \Delta \{x'\}) \times Y' \times Z'$  as in the previous scheme.
2. For each  $y' \in \{0, \dots, n^{\frac{1}{3}} - 1\}$  we calculate the parity for  $X' \times (Y' \Delta \{y'\}) \times Z'$  as in the previous scheme.
3. For each  $z' \in \{0, \dots, n^{\frac{1}{3}} - 1\}$  we calculate the parity for  $X' \times Y' \times (Z' \Delta \{z'\})$  as in the previous scheme.

Now each server returns  $1 + 3n^{1/3}$  parities to the client. That is, the server 1 will actually compute  $S_{000}$  and  $S_{100}, S_{010}, S_{001}$  will be in those  $3n^{1/3}$  parities. Similarly, the server 2 will compute  $S_{111}$ , and  $S_{011}, S_{101}, S_{110}$  will be in those  $3n^{1/3}$  parities. The client will be able to pick out the correct parities corresponding to its actual query.

Thus, this scheme still has  $n^{\frac{1}{3}}$  bandwidth, and correctness and security still follow.

### 3 State of the Art and Open Problem

For the 2-server setting, the best known lower bound states that  $5 - o(1) \log n$  bandwidth is necessary ([WdW05]), while the best known upper bound requires  $n^{O(\sqrt{\lg n / \lg n})} = n^{o(1)}$ , i.e., sub-polynomial bandwidth ([DG16]). Closing this gap is a long-standing open problem.

### References

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