Cryptography Meets Algorithms (15893) Lecture Notes

Lecture 6: 2-Server PIR from Distributed Point Functions Batch PIR

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1 Distributed Point Function and 2-server PIR with Logarithmic Communication

In earlier lectures, we learned an information-theoretic 2-server PIR scheme by Dvir and Gopi [DG16], which achieves $n^{O(\sqrt{\log\log n/\log n})}$ communication per query. In this lecture, we will show how to get a 2-server PIR with logarithmic communication relying only on a pseudorandom generator (PRG). This scheme is also interesting from a practical perspective since in practice, we can use AES to realize the PRG, and modern CPUs have hardware acceleration for evaluating AES.

Recall that from our undergraduate cryptography course, we know that the existence of a PRG is equivalent to the existence of a one-way function (OWF) [HILL99]. Also, from an earlier lecture, we learned that any 1-server classical PIR scheme with non-trivial bandwidth implies Oblivious Transfer which cannot be constructed in a blackbox manner from OWF [IR89]. Therefore, the scheme we will talk about today is in the 2-server setting.

1.1 Preliminary: Pseudorandom Generator

We will rely on a pseudorandom generator (PRG), which takes in a short random seed and expands the seed to a longer pseudorandom string.

Definition 1 (PRG)**.** *Let $\ell(\cdot)$ be a polynomial and let $G : \{0,1\}^n \to \{0,1\}^{\ell(n)}$ be a deterministic polynomial-time algorithm. G is a* PRG *if it has the following properties:*

- ***Expansion:*** $\forall n, \ell(n) > n$.
- ***Pseudorandomness:*** *for any probabilistic polynomial-time distinguisher D, there exists a negligible function* $\mathsf{negl}(\cdot)$*, such that*

$$\left| \Pr_{r \xleftarrow{\$} \{0,1\}^{\ell(n)}}[D(r) = 1] - \Pr_{s \xleftarrow{\$} \{0,1\}^{n}}[D(G(s)) = 1] \right| \leq \mathsf{negl}(n)$$

1.2 Distributed Point Function

Definition 2 (Point function)**.** *A point function parametrized by some point $x \in \{0,1\}^\ell$ is a function that evaluates to 1 at x, and evaluates to 0 everywhere else. We will henceforth use the notation $P_x\{0,1\}^\ell \to \{0,1\}$ to denote a point function. By definition, $P_x(x) = 1$ and $P_x(x') = 0$ for $x' \neq x$.*

Boyle, Gilboa and Ishai [BGI16] introduced the concept of a distributed point function. A distributed point function is a functional secret-sharing of a point function. In this lecture, we will specifically focus on a *2-way* secret sharing of a point function. Essentially, given some point function P_{x^*}, one can "secretly share" the function to two keys k_L, k_R. Then, for party $t \in \{L, R\}$ which receives the key k_t, it can evaluate the function on any point x and get a share of the outcome denoted $\mathsf{Eval}(k_t, x)$. It is guaranteed that in every point x, $\mathsf{Eval}(k_L, x) \oplus \mathsf{Eval}(k_R, x) = P_{x^*}(x)$. In other words, it is possible to combine the two outcome-shares to reconstruct the evaluation of the point function at any point. Finally, security of the DPF requires that each party $t \in \{L, R\}$ does not learn the "special point" (i.e., x^*) given its individual key k_t.

Definition 3 (2-share DPF)**.** *A DPF is a pair of possibly randomized algorithms (*Gen,Eval*) with the following syntax:*

- $\mathsf{Gen}(1^\lambda, x^*)$*: Outputs a pair of keys* k_L, k_R*.*
- $\mathsf{Eval}(1^\lambda, k, x)$*: Outputs the evaluation outcome* $y \in \{0, 1\}$*.*

Correctness. *Correctness requires that for any* λ*, any* ℓ*, any* $x^*, x \in \{0,1\}^\ell$*,*

$$\Pr\left[k_L, k_R \leftarrow \mathsf{Gen}(1^\lambda, x^*) : \mathsf{Eval}(k_L, x) \oplus \mathsf{Eval}(k_R, x) = P_{x^*}(x)\right] = 1$$

Security. *Security requires that there exists a probabilistic polynomial-time simulator* Sim*, such that for any* $\ell = \ell(\lambda)$ *that is a polynomial function in* λ*, and any* $x^* \in \{0,1\}^{\ell(\lambda)}$*, the following experiments are computationally indistinguishable for both* $t = L$ *and* $t = R$*:*

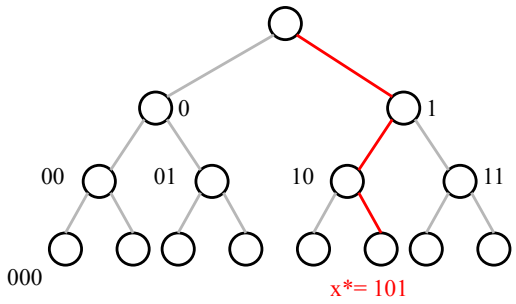
- $\mathsf{Real}(1^\lambda, x^*)$*:* $k_L, k_R \leftarrow \mathsf{Gen}(1^\lambda, x^*)$ *and output* k_t*;*
- $\mathsf{Ideal}(1^\lambda)$*: Output* $\mathsf{Sim}(1^\lambda, \ell)$*.*

Intuitively, security requires that the any individual key k_L or k_R can be simulated without knowledge of the special point x^*.

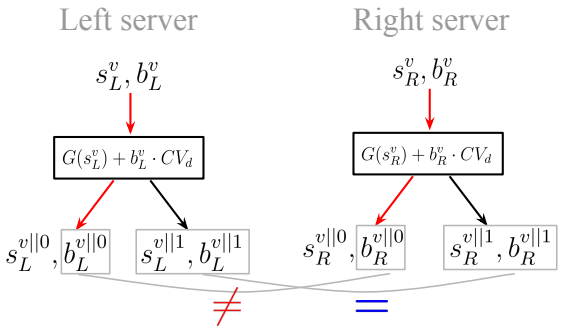
1.3 DPF $\Longrightarrow$ 2-Server PIR

Given a DPF scheme henceforth denoted $(\mathsf{DPF.Gen}, \mathsf{DPF.Eval})$, we can construct a 2-server PIR scheme as follows. Henceforth we use $\mathsf{DB} \in \{0,1\}^n$ to denote the database.

1. Given the query i, the client computes $(k_L, k_R) \leftarrow \mathsf{DPF.Gen}(1^\lambda, i)$
2. The client sends k_L to the left server and sends k_R to the right server.
3. Each server $t \in \{L, R\}$ receives k_t, and replies $y_t := \bigoplus_{j \in [n]} \mathsf{DPF.Eval}(k_t, j) \cdot \mathsf{DB}[j]$;
4. the client receives y_L and y_R from the two servers respectively, and outputs $y_L \oplus y_R$.



(a) The binary tree structure used in DPF. Each node has a unique name. The path from the root to the leaf node x^* is the "special path".



(b) Expansion during the evaluation algorithm. The key generation computes the correction vector CV in a way that guarantees the following: for any u not on the special path, $(s_L^u, b_L^u) = (s_R^u, b_R^u)$; and for any u on the special path, $b_L^u \neq b_R^u$ and moreover, the pair (s_L^u, s_R^u) is indistinguishable from two independent random strings.

Figure 1: Illustration of the DPF construction.

Correctness. It is not hard to check that the answer output by the client is correct by the DPF's correctness:

$$\begin{aligned} y_0 \oplus y_1 &= \left(\bigoplus_j \mathsf{DB}[j] \cdot \mathsf{DPF.Eval}(k_0, j) \right) \oplus \left(\bigoplus_j \mathsf{DB}[j] \cdot \mathsf{DPF.Eval}(k_1, j) \right) \\ &= \left(\bigoplus_j \mathsf{DB}[j] \cdot (\mathsf{DPF.Eval}(k_0, j) \oplus \mathsf{DPF.Eval}(k_1, j)) \right) \\ &= \left(\bigoplus_j \mathsf{DB}[j] \cdot P_i(j) \right) \\ &= \mathsf{DB}[i]. \end{aligned}$$

Security. Security of the PIR follows directly from the security of the DPF.

1.4 DPF Construction

We now show how to construct an efficient DPF based on a pseudorandom generator G:

$$G : \{0,1\}^\lambda \to \{0,1\}^{2\lambda+2}.$$

The algorithm is based on a binary tree expansion idea.

Binary tree structure. Suppose we want to evaluate the DPF at n points denoted $0, 1, \ldots, n-1$, and we assume that n is a power of 2.

Imagine that there is a binary tree as depicted in Figure 1a. Each node in the tree has a name: the root's name is empty, and the two children of a node v are named $v||0$ and $v||1$, respectively. Henceforth, we say that the root is at *depth* 0, the leaves are at depth $\log n$, and so on. Each leaf node corresponds to a point in $\{0, 1, \ldots, n-1\}$. In particular, it helps to express each point in a binary representation. For a point function P_{x^*}, the path from the root to the leaf node x^* is called the special path highlighted in red in the figure.

Key structure and evaluation algorithm. The DPF's Gen algorithm outputs two keys $k_L = ((s_L, b_L), \mathsf{CV})$ and $k_R = ((s_R, b_R), \mathsf{CV})$, where

- s_L and s_R are both λ-bit PRG seeds;
- $b_L, b_R \in \{0,1\}$ are flags indicating whether correction is necessary during the key expansion (see use of the correction vector later in the algorithm). It is guaranteed that $b_L \neq b_R$; and
- $\mathsf{CV} = (\mathsf{CV}_1, \ldots, \mathsf{CV}_{\log n})$ is a correction vector.

We will focus on the left server's perspective for describing the evaluation algorithm. The right server's algorithm is the same except that its input is k_R instead of k_L. Imagine that initially, the root of the tree is associated with the pair (s_L, b_L) which comes from k_L. Starting from the root, we will perform a key expansion to compute a pair (s_L^v, b_L^v) for every node v in the tree. Each coordinate of the correction vector $\mathsf{CV}_1, \ldots, \mathsf{CV}_{\log n}$ will be consumed at each different level of the tree during the key expansion process.

More specifically, suppose some node v has the pair (s_L^v, b_L^v), we can compute the corresponding values at its two children $v||0$ and $v||1$ as follows where d denotes the depth of v's children:

$$(s_L^{v||0}, b_L^{v||0}), (s_L^{v||1}, b_L^{v||1}) \leftarrow G(s_L^v) \oplus \begin{cases} \mathbf{0} & \text{if } b_L^v = 0; \\ \mathsf{CV}_d & \text{if } b_L^v = 1. \end{cases}$$

Notice that the correction component CV_d is only applied if $b_L^v = 1$.

At the end of the expansion process, for each leaf node x, let b_L^x and b_R^x be the two bits associated with the leaf x output by the left and right servers, respectively. The outcome of the DPF at point x is then $b_L^x \oplus b_R^x$.

Key generation algorithm. The key generation algorithm samples random $s_L \xleftarrow{\$} \{0,1\}^\lambda$ and $s_R \xleftarrow{\$} \{0,1\}^\lambda$ at the root nodes for the left and right servers, respectively. It chooses a random bit $b_L \xleftarrow{\$} \{0,1\}$ and sets $b_R = b_L \oplus 1$. Then, it will choose the correction vector $\mathsf{CV}_1, \ldots, \mathsf{CV}_{\log n}$ in a way such that the following **invariants** are guaranteed for each tree node v:

- For every tree node v that is not on the special path leading to x^*, it must be that $(s_L^v, b_L^v) = (s_R^v, b_R^v)$.
- For every tree node v that is on the special path leading to x^*, it must be that $b_L^v \neq b_R^v$, and moreover, the pair (s_L^v, s_R^v) is indistinguishable from two independent λ-bit random strings.

Note that for some node v, if $(s_L^v, b_L^v) = (s_R^v, b_R^v)$ is already guaranteed, then for any node u that is in the subtree of v, $(s_L^u, b_L^u) = (s_R^u, b_R^u)$ is automatically guaranteed because the left and right servers will behave identically when the apply the same expansion algorithm to compute all values in the subtree of v. Therefore, in the key generation algorithm, we can compute the correction vector CV using only the special path, to maintain the aforementioned invariants.

The detailed key generation algorithm is specified in Figure 2. It is not hard to see that the key size is $\lambda + 1 + \log n \cdot (2\lambda + 2) = O_\lambda(\log n)$.

Analysis. The correctness can be verified by inductively checking that the aforementioned invariants hold at every level.

Security can also be proven inductively. We prove left-server security below since right-server security is symmetric. Henceforth, we use $x^*[:d]$ to denote the first d bits of the binary representation of x^*. For the base case, observe that the terms (s_L, b_L) are random and independent of s_R. Now, suppose that the joint distribution of $(s_L, b_L, \mathsf{CV}_1, \ldots, \mathsf{CV}_{d-1})$ and $s_R^{x^*[:d-1]}$

Generation Algorithm: $\mathsf{Gen}(1^\lambda, x^*)$

Initialization:

- Sample s_L, s_R as two λ-bit random strings.
- Sample a random bit b_L. Let $b_R = b_L \oplus 1$.
- Let $\{x^*[1], \ldots, x^*[\log n]\}$ be x^*'s binary representation.

Constructing correction vectors:

Initialize v to be an empty string.

For $d \in \{1, \ldots, \log n\}$:

– Sample a random string $r \xleftarrow{\$} \{0,1\}^\lambda$.

– Solve for $\mathsf{CV}_d \in \{0,1\}^{2\lambda+2}$ such that the following constraints are satisfied:

(1) Let $(s_L^{v||0}, b_L^{v||0}), (s_L^{v||1}, b_L^{v||1}) \leftarrow G(s_L^v) \oplus (b_L^v \cdot \mathsf{CV}_d)$; *// Expansion on the LHS*

(2) Let $(s_R^{v||0}, b_R^{v||0}), (s_R^{v||1}, b_R^{v||1}) \leftarrow G(s_R^v) \oplus (b_R^v \cdot \mathsf{CV}_d)$; *// Expansion on the RHS*

(3a) If $x^*[d] = 0$: add the following constraint:

$$(s_L^{v||0}, b_L^{v||0}, s_L^{v||1}, b_L^{v||1}) \oplus (s_R^{v||0}, b_R^{v||0}, s_R^{v||1}, b_R^{v||1}) = (r, 1, 0^\lambda, 0).$$

(3b) If $x^*[d] = 1$: add the following constraint:

$$(s_L^{v||0}, b_L^{v||0}, s_L^{v||1}, b_L^{v||1}) \oplus (s_R^{v||0}, b_R^{v||0}, s_R^{v||1}, b_R^{v||1}) = (0^\lambda, 0, r, 1).$$

– Let $v \leftarrow v||x^*[d]$.

Output: Output the following k_L, k_R:

$$k_L = (s_L, b_L, \mathsf{CV}_1, \ldots, \mathsf{CV}_{\log n})$$
$$k_R = (s_R, b_R, \mathsf{CV}_1, \ldots, \mathsf{CV}_{\log n})$$

Figure 2: DPF key generation algorithm.

are indistinguishable from random. We want to inductively prove that the joint distribution of $(s_L, b_L, \mathsf{CV}_1, \ldots, \mathsf{CV}_d)$ and $s_R^{x^*[:d]}$ are indistinguishable from random. Without loss of generality, assume that $x^*[d] = 0$ since the other direction is symmetric. Let $v = x^*[:d-1]$, observe that

$$\mathsf{CV}_d = (s_L^{v||0}, b_L^{v||0}, s_L^{v||1}, b_L^{v||1}) \oplus (s_R^{v||0}, b_R^{v||0}, s_R^{v||1}, b_R^{v||1}) \oplus (r, 1, 0, 0)$$

By our induction hypothesis, given $(s_L, b_L, \mathsf{CV}_1, \ldots, \mathsf{CV}_{d-1})$, s_R^v is indistinguishable from random. Due to the security of the PRG, $(s_R^{v||0}, b_R^{v||0}, s_R^{v||1}, b_R^{v||1}) = G(s_R^v)$ is indistinguishable from random given $(s_L, b_L, \mathsf{CV}_1, \ldots, \mathsf{CV}_{d-1})$. Therefore, given $(s_L, b_L, \mathsf{CV}_1, \ldots, \mathsf{CV}_{d-1})$, CV_d is indistinguishable from random. Further, note that $s_R^{v||0} = \mathsf{CV}_d[:\lambda] \oplus r$ where $r \xleftarrow{\$} \{0,1\}^\lambda$ is a fresh random string of length λ. This means that given $(s_L, b_L, \mathsf{CV}_1, \ldots, \mathsf{CV}_{d-1})$, both CV_d and $s_R^{v||0}$ are indistinguishable from fresh random strings.

2 Batch PIR

2.1 Motivation

So far, every PIR scheme we have seen only retrieves 1 bit at a time. In many use cases, the client may want to fetch up to Q database entries at the same time. The navie solution is just to repeat the single-query algorithm Q times. If each instance requires $O(n)$ computation for example, then together all Q instances would require $O(n \cdot Q)$ computation. Batch PIR aims to handle a batch of queries at the same time, reducing the amortized cost.

We assume that $Q = o(n)$ since otherwise the server can just send the entire database to the client. We also assume that $Q \geq \log^c n$ for some constant $c > 1$, since if the batch size is too small, it is not too expensive to just do Q-fold repetition of a PIR scheme. It might help your understanding if you think of the special case when $Q = \sqrt{n}$.

2.2 Batch PIR Based on Balls-and-Bins Hashing

Fact 1. *If we throw n balls into n/k bins where $k = \omega(\log n)$, then except with negligible in n probability, no bin will receive more than $2k$ balls.*

The above fact can be proven through a simple application of Chernoff bound, and then taking union bound over all bins.

Batch PIR based on balls-and-bins hashing. Given an n-bit database, imagine that we randomly throw each element into $\frac{Q}{k}$ bins where $k = \omega(\log n)$. The expected number of elements per bin is nk/Q. By Fact 1, except with negligible in n probability, no bin will exceed $2nk/Q$ elements. In practice, we can use a pseudorandom function $\mathsf{PRF}_k(i)$ to decide the random bin choice for each database index $i \in [n]$. The key k of the PRF is known to both the client and the server.

We shall assume that the Q queried indices are independent of the coins that determine the bin choices for each database index. In this case, among the Q queries, the expected number of queries that land in each bin is k. By Fact 1, no bin will receive more than $2k$ queries.

The idea of the batch PIR scheme is to perform $2k$ PIR queries for each bin. If a bin receives strictly less than $2k$ queries, we can always pad with fake queries such that each bin has exactly $2k$ queries. We need this padding for security, because the client cannot reveal to the server how many queries land in each bin, otherwise it will leak some partial information about the distribution of the queries[1].

[1] On the other hand, the number of database elements that land in each bin may be publicly known.

Henceforth, let $\mathsf{C}(m)$ and $\mathsf{T}(m)$ denote the per-query bandwidth and server-computation of a PIR scheme on an m-bit database, respectively. The cost of the above batch PIR scheme for making Q queries is upper bounded the following

$$\begin{aligned} &\text{Bandwidth}: && \mathsf{C}(2nk/Q) \cdot 2k \cdot \tfrac{Q}{k} = O(Q) \cdot \mathsf{C}(2nk/Q) \\ &\text{Server compute}: && \mathsf{T}(2nk/Q) \cdot 2k \cdot \tfrac{Q}{k} = O(Q) \cdot \mathsf{T}(2nk/Q) \end{aligned}$$

As a special case, imagine that $k = \log^2 n$, and $Q = \sqrt{n}$. Further, suppose that $\mathsf{C}(m) = O(1)$, and $\mathsf{T}(m) = O(m)$, e.g., imagine an underlying PIR scheme based on fully homomorphic encryption. In this case, the batch PIR consumes only $\widetilde{O}(n)$ server computation for answering all $Q = \sqrt{n}$ queries where $\widetilde{O}(\cdot)$ hides polylogarithmic factors. It is almost as if we answered all Q queries with the computation overhead of just a single query (ignoring polylogarithmic factors). Further, the total bandwidth is only $\widetilde{O}(n)$ for answering all $Q = \sqrt{n}$ queries. The bandwidth is not too much worse than a simple Q-fold repetition of the underlying PIR.

2.3 Cuckoo Hashing based scheme [ACLS18]

Angel et al. [ACLS18] proposed SealPIR that uses cuckoo hashing to do batch PIR.

Cuckoo Hashing

Definition 4 (Cuckoo hashing)**.** *Given n balls, b buckets, and w independent hash functions $h_0, \cdots, h_w$ that map a ball to a random bucket, compute w candidate buckets for each ball by applying the w hash functions. For each ball x, place x in any empty candidate bucket. If none of the w candidate buckets are empty, select one at random, remove the ball currently in that bucket (x_{old}), place x in the bucket, and re-insert x_{old}. If re-inserting x_{old} causes another ball to be removed, this process continues recursively until we finish the insertion or a maximum number of iterations is achieved.*

Batch PIR based on Cuckoo Hashing. The scheme is as follows.

- **Serer encoding.** Given an n bit database, b buckets, and w hash functions, we hash each entry in the database (using their index as the key) to all w candidate buckets and store it there. This results in a encoded database that each original entry is replicated w times. The server will share the hash functions to the client.
- **Client scheduling.** Given the Q queries, the client use the cuckoo hashing method to insert (or say, schedule) the queries to the buckets (again, using the indices as the keys). Our target is that each bucket has at most one query, and all query can be inserted in one of its candidate bucket.
- **Client query.**. Now the client just makes one query in each bucket. If the client successfully insert all queries earlier, it can then proceed to learn all the target entries because the server has inserted the entries in all the candidate buckets.

An example can be found in Figure 3.

The authors used 3 hash functions for encoding and set the number of buckets $b = 1.5Q$. For $Q \geq 200$, the author showed that the chance of failure during the scheduling phase is $\leq 2^{-40}$. Notice that this is not cryptographically neligible. To enforce negligible failure probability, we can introduce a size λ stash of the cuckoo hash table. That is, the stash stores at most λ elements that fail to be inserted. Then, the client also has to make additional λ PIR queries to the whole database. This ensures the failure probability to be $\mathsf{negl}(\lambda)$.

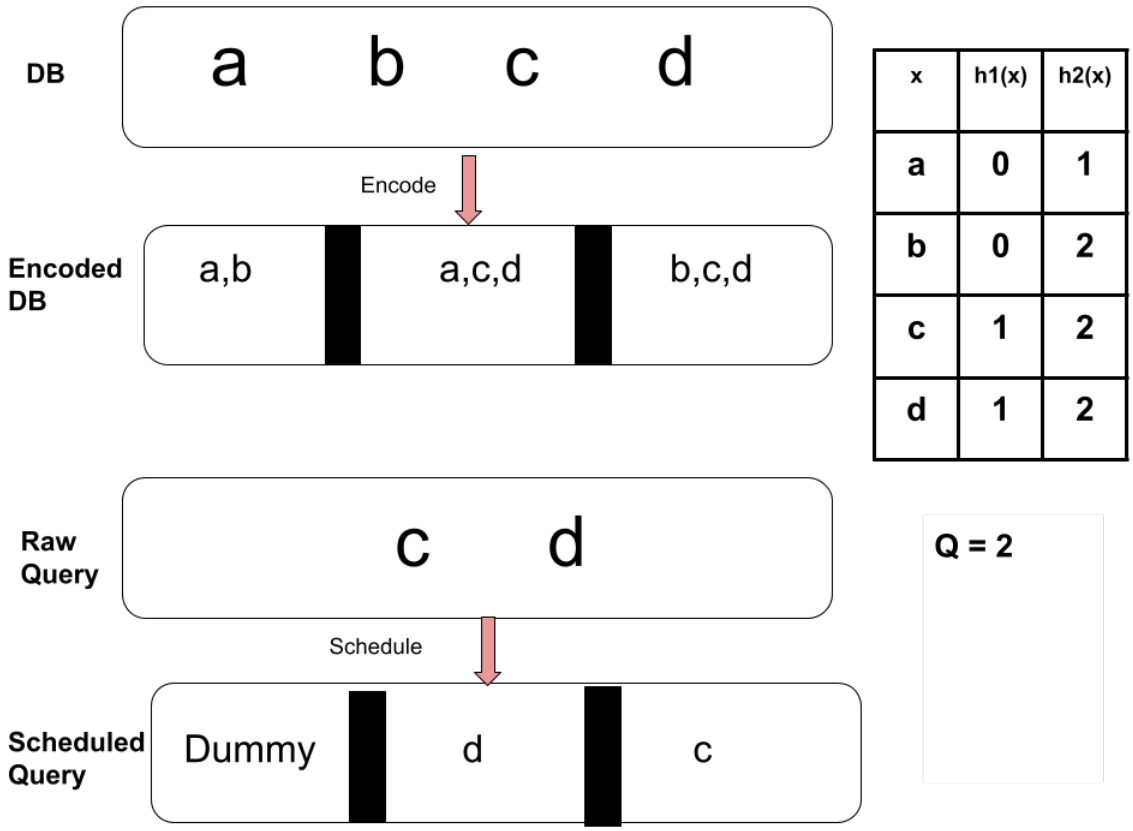


Figure 3: An example of the cuckoo hashing based batch PIR.

Cost Analysis. Assume the underlying single-query PIR scheme is linear. The client will make one query in each bucket and the total bucket size is wn. Also, the client needs to make λ additional query to the whole database to ensure negligible failure probability. Then, the computation cost for the Q queries are just $(w + \lambda)n$.

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