Cryptography Meets Algorithms (15893) Lecture Notes

Lecture 9: Oblivious Sorting

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1 Oblivious Sorting

Definition 1. An algorithm ALG is **oblivious** if for all pairs of inputs $\mathcal{I}, \mathcal{I}'$ with equal lengths, we have:

$$\text{ACCESSPATTERN}(\text{ALG}, \mathcal{I}) \approx \text{ACCESSPATTERN}(\text{ALG}, \mathcal{I}'),$$

where $\text{ACCESSPATTERN}(\text{ALG}, \mathcal{I})$ denotes the observed access pattern when executing ALG on input $\mathcal{I}$, and $\approx$ denotes two distributions are indistinguishable (either statistically or computationally, depending on the specific setting.)

Non-Example 2. QuickSort is **not** oblivious because it reads and writes array elements depending on the result of comparing with a pivot. These instructions leak information about the content of the input array.

Consider the following two arrays:

$$Arr_1 := \boxed{7 \mid 6 \mid 5 \mid \mathbf{4} \mid 3 \mid 2 \mid 1}$$

$$Arr_2 := \boxed{1 \mid 2 \mid 3 \mid \mathbf{4} \mid 5 \mid 6 \mid 7}$$

If in both Arr_1 and Arr_2, $\mathbf{4}$ is picked to be the pivot, then QuickSort would move $Arr_1[0:2]$ to the right of $\mathbf{4}$ while keeping $Arr_2[0:2]$ at the left of $\mathbf{4}$. Therefore, one can distinguish the two inputs Arr_1 and Arr_2 by observing the different access patterns.

We can naïvely compile QuickSort to be oblivious using PATHORAM. However, this incurs a log-squared overhead. Since the runtime of QuickSort is $\mathcal{O}(n \log n)$ where n is the input length, the resulting oblivious version of QuickSort runs in $\mathcal{O}(n \log^3 n)$.

In this lecture, we will see two direct constructions of oblivious sorting that achieve faster runtime than compiled QuickSort. Our ultimate goal is to match the runtime of non-oblivious QuickSort, i.e. $\mathcal{O}(n \log n)$.

Remark 1. The first construction that achieves the desired $\mathcal{O}(n \log n)$ performance is the AKS sorting network [AKS83]. However, the AKS network is not used in practice due to the large constant factor hidden by the Big-O notation.

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2 Bitonic Sort [Bat68]

Definition 3. A sequence $A = [A_0, A_1, \cdots, A_n]$ is **bitonic** if and only if:

1. There exists some $i \in [n]$ such that $A_j \leq A_{j+1}$ for all $j < i$, and $A_j \geq A_{j+1}$ for all $j \geq i$, i.e.,

$$A_0 \leq A_1 \leq \cdots \leq A_i \geq A_{i+1} \geq \cdots \geq A_n,$$

or

2. There exists a cyclic shift π on A such that $\pi(A)$ satisfies the bullet point 1.

We say a bitonic sequence is in the canonical form if it satisfies bullet point 1.

In the following, we focus our discussion on the canonical form of the bitonic sequences. This is without loss of generality because any bitonic sequence can be converted into its canonical form in linear time by applying the corresponding cyclic shift π. So the performance of the sorting algorithm is asymptotically the same.

We specify the BitonicSort algorithm as multiple components. We start with introducing the following BITONICSPLIT operation:

Algorithm 1: BITONICSPLIT

Input: A, a bitonic sequence of length n.
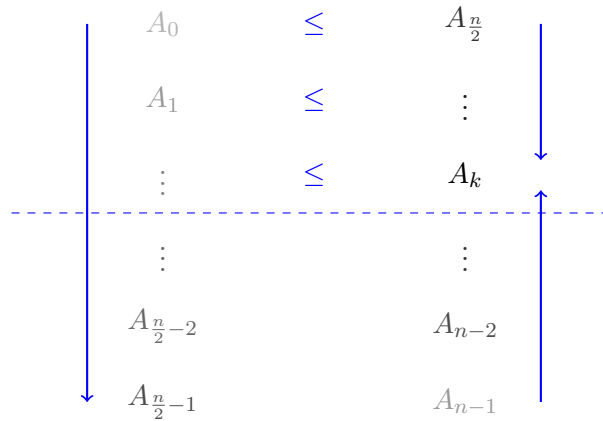**Output:** (L_A, R_A), a pair of two sequences each of length $n/2$.

1 $L_A \leftarrow [\min(A_0, A_{\frac{n}{2}}), \min(A_1, A_{\frac{n}{2}+1}), \cdots, \min(A_{\frac{n}{2}-1}, A_{n-1})]$;
2 $R_A \leftarrow [\max(A_0, A_{\frac{n}{2}}), \max(A_1, A_{\frac{n}{2}+1}), \cdots, \max(A_{\frac{n}{2}-1}, A_{n-1})]$;
3 return (L_A, R_A);

Claim 4. L_A and R_A are bitonic, and $\max_i(L_A[i]) \leq \min_j(R_A[j])$.

Proof. As mentioned above, we assume that A is in canonical form. We denote the index of the "turning point" as k, such that

$$A_0 \leq A_1 \leq \cdots \leq A_k \geq A_{k+1} \geq \cdots \geq A_n.$$

Assuming without loss of generality that $k \geq n/2$ (the case of $k < n/2$ is symmetric), we can visualize the sequence as the following two columns:

$$
\begin{array}{ccc}
A_0 & \leq & A_{\frac{n}{2}} \\
A_1 & \leq & \vdots \\
\vdots & \leq & A_k \\
\vdots & & \vdots \\
A_{\frac{n}{2}-2} & & A_{n-2} \\
A_{\frac{n}{2}-1} & & A_{n-1}
\end{array}
$$

where deeper colors represent larger values, and the arrows point to the larger values. Note that in the top $k - n/2$ rows (until the dashed line), the value on the right column must be greater than or equal to that on the left column.

Moreover, starting from the $(k - n/2)^{\text{th}}$ row, if we gradually move down the dashed line, we will find a unique row r such that:

- Above this row, the value on the right column is greater than or equal to that on the left column, and

- Below this row, the value on the right column is less than or equal to that on the left column.
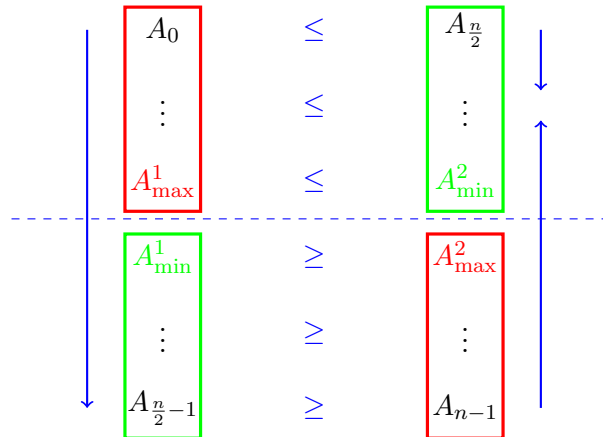
This is visualized as follows:
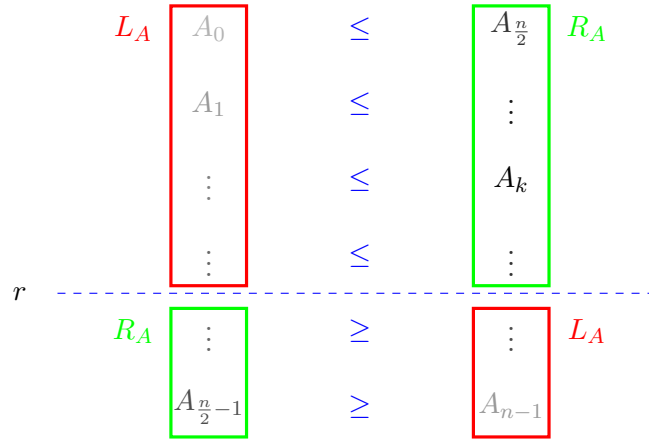
After BITONICSPLIT, L_A is composed of the elements in the red brackets, and R_A is composed of the elements in the green brackets. Therefore, both L_A and R_A are in fact **subsequences** of the original bitonic sequence A, which must be bitonic.

Lastly, we argue that $\max_i(L_A[i]) \le \min_j(R_A[j])$. Observe that

$$\max_i(L_A[i]) = \max(A_{\max}^1, A_{\max}^2) \text{ and } \min_j(R_A[j]) = \max(A_{\min}^1, A_{\min}^2)$$

in the following figure.

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We have that:

$$A_{\max}^1 \leq A_{\min}^1 \text{ and } A_{\max}^2 \leq A_{\min}^2 \qquad \text{as shown by the directions of arrows}$$
$$A_{\max}^1 \leq A_{\min}^2 \text{ and } A_{\max}^2 \leq A_{\min}^1 \qquad \text{as shown by the signs between columns}$$

Therefore $\max_i(L_A[i]) = \max(A_{\max}^1, A_{\max}^2) \leq \min_j(R_A[j]) = \max(A_{\min}^1, A_{\min}^2)$.

$\square$

With BITONICSPLIT as a subroutine, we can now specify the following recursive merging step and the general BITONICSORT algorithm. We use $\|$ to denote the concatenation of two sequences and use $\neg$ to denote flipping the direction, i.e., $\neg dec = inc$ and $\neg inc = dec$.

Algorithm 2: BITONICMERGE

Input: $(A, \text{direction})$
A is a bitonic sequence of length n and direction $\in \{inc, dec\}$.
Output: *result*, a sorted sequence.

1 **if** $|A| = 1$ **then**
2 $\quad$ **return** A;
3 $(L_A, R_A) \leftarrow$ BITONICSPLIT(A);
4 **if** direction $= dec$ **then**
5 $\quad$ Swap (L_A, R_A);
6 **return** BITONICMERGE$(L_A, \text{direction}) \|$ BITONICMERGE$(R_A, \text{direction})$;

In summary, BITONICMERGE sorts a sequence under the assumption that it is bitonic. Claim 4 guarantees that the assumption is maintained during the recursions. Finally, in order to sort a general array, we need to firstly preprocess it to be bitonic. This is again done by recursion.

Algorithm 3: BITONICSORT

Input: $(A, \text{direction})$
A is a sequence of length n and direction $\in \{inc, dec\}$.
Output: *result*, a sorted sequence.

1 **if** $|A| = 1$ **then**
2 $\quad$ **return** A;
3 $S_1 \leftarrow$ BITONICSORT$(A[0, n/2 - 1], \text{direction})$;
4 $S_2 \leftarrow$ BITONICSORT$(A[n/2, n - 1], \neg\text{direction})$;
5 **return** BITONICMERGE$(S_1 \| S_2, \text{direction})$;

A concrete example of the recursion is given as follows:

Analysis. BITONICSORT is oblivious because the behavior of the algorithm does not depend on the content of the input, i.e., the sequence of comparisons is deterministic regardless of the comparison results. At each layer of the recursion, the total number of comparisons made by all calls in this layer is upper bounded by $n/2$. The depth of the recursion is given by

$$D(n) = D(n/2) + \log n$$

which solves to $\log n(\log n + 1)/2$. Therefore the total number of comparisons is upper bounded by $n/2 \cdot \log n(\log n + 1)/2 = \mathcal{O}(n \log^2 n)$, where the Big-O notation hides a relatively small constant.

3 Bucket Oblivious Sort [ACN$^+$]

We start by introducing the Bucket Oblivious Sorting algorithm that has an $\mathcal{O}(n \log n(\log \log n)^2)$ complexity. Afterward, we will discuss the improvement to speed it up to $\mathcal{O}(n \log n)$. In general, the framework of BucketSort is as follows:

Given a sequence A:

1. Apply an **oblivious random permutation** (ORP) on A to get $A' \leftarrow \mathrm{ORP}(A)$.

2. Sort A' using any (potentially non-oblivious) comparison-based sorting algorithm, e.g. QuickSort.

As the name suggested, the specification of ORP is that given a sequence A, outputs a randomly permuted version of A. The obliviousness guarantee is that the access pattern should reveal nothing about the applied random permutation. Our plan is to construct ORP as a randomized algorithm with a *deterministic access pattern*.

Non-Example 5. The Fisher–Yates shuffle is not an oblivious random permutation. It works as follows: Given $A := [A_0, \cdots, A_{n-1}]$,

- for $i \in [0, n-1)$:
    - $j \xleftarrow{\$} [i, n)$;
    - Swap(A_i, A_j);

Apparently, one can exactly learn the random permutation applied by observing the Swap instructions.

Remark 2. It is crucial that in the second step of the BucketSort framework, the sorting algorithm needs to be comparison-based. For example, we cannot use a counting-based sort because even after the sequence is randomly permuted, applying a counting sort would reveal the frequency of elements in each bin, which is exactly what we want to hide. In contrast, a comparison-based sorting will only reveal the relative order between elements, which is fine after we randomly permuted the positions of elements.

3.1 Oblivious Random Permutation

For the purpose of construction ORP, we are going to assign each element to a random bin and then route the elements through a butterfly network to their assigned random bins. A visualization of the network is given on the next page. The behavior of the algorithm is as follows:

- (Set-up): Let Z be a size of buckets (which we will choose later as a function of n). Let $B := 2n/Z$ be the number of buckets. Each bucket is going to contain $Z/2$ *real* elements from the input sequence and $Z/2$ dummy elements.

- Each element is randomly assigned to one of the B buckets indexed by a key of $\log B$ bits.

- The elements are routed to their respective destinations through a butterfly network, which contains $\log B$ layers that are connected through MERGESORT operators.

- The MERGESORT operator takes two buckets from the ith level as inputs and reorganize their elements into two buckets for the $(i+1)^{\text{th}}$ layer, according to the $(i+1)^{\text{th}}$ most significant bit of the elements' keys. i.e., At the ith level, every $2i$ consecutive buckets are *semi-sorted* by the most significant i bits of the keys.

- Finally, to obtain a random permutation from our random bin assignments, we simply remove all dummy elements in each bin and then randomly permute each single bucket at the final layer.
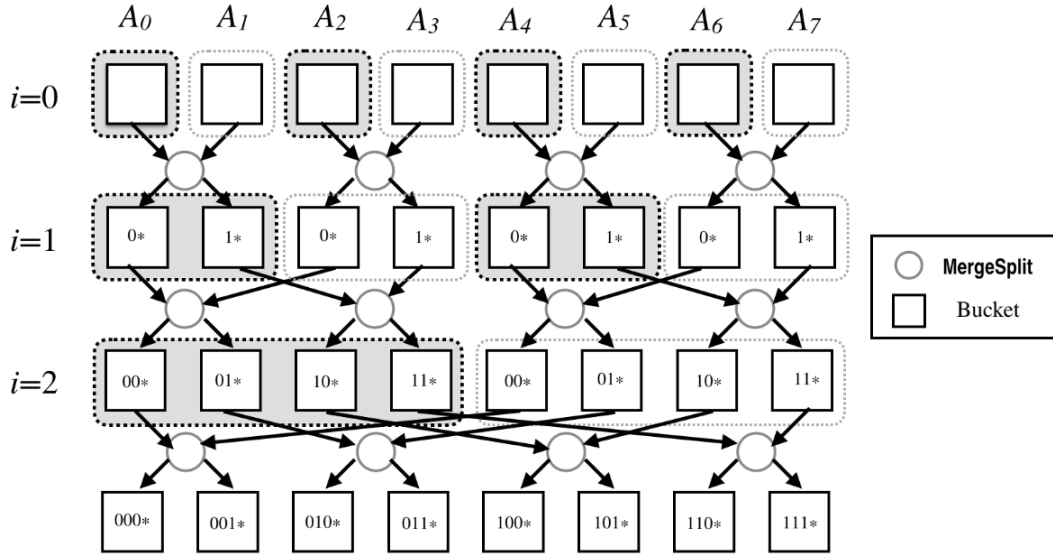
Efficiency for Clients with $2Z$ storage. If the client has a storage size no less than $2Z$, then they can perform MERGESORT and random permute the buckets at the final layer locally. In each of the $\log B = \log \frac{2n}{Z} < \log N$ layers, the $2n$-sized sequence (containing n real elements and n dummy elements) is read and written once. So oblivious bin assignment and bucket ORP run in $4n \log n$ time.

Efficiency for Clients with constant storage. With constant-storage clients, we need to implement oblivious MERGESPLIT using BITONICSORT. This is done by sorting the real elements with respect to the keys and then padding each of the resulting buckets with dummy elements. In the final layer, the bucket-wise permutation is also done with BITONICSORT. Given a bucket, we firstly assign each element with a $\log n$-bit label. We then obliviously sort the elements by their labels using BITONICSORT. In total, at each of the $\log B$ layers, we invoke $B/2$ instances of bitonic sort on $2Z$ elements, thus the runtime is roughly

$$\log B \cdot B/2 \cdot 2Z \log^2(2Z)) \approx 2n \log n \log^2 Z.$$

Success Probability. The above algorithm is guaranteed to output a correct, randomly permuted version of the input sequence if none of the buckets ever overflows (i.e., receives more than Z real elements) during the routing.

Lemma 6. Overflow happens with at most $\epsilon(n, Z) = 2n/Z \cdot \log(2n/Z) \cdot e^{-Z/6}$ probability.

Proof. Consider a bucket B_i at level i which has an associated i-bit long prefix b of keys. B_i receives real elements from 2^i initial buckets in the first layer, each containing $Z/2$ real elements. By the construction, any of these elements reaches B_i only if the most significant i bits of its key match b, which happens with exactly 2^{-i} probability.

A Chernoff bound shows that B_i overflows with less than $e^{-Z/6}$ probability. Hence, taking a union bound over all levels and all buckets, we get that overflow happens with less than $\epsilon(n, Z) := B \cdot \log B \cdot e^{-Z/6}$ probability. $\qquad\square$

References

[ACN+] Gilad Asharov, T-H. Hubert Chan, Kartik Nayak, Rafael Pass, Ling Ren, and Elaine Shi. *Bucket Oblivious Sort: An Extremely Simple Oblivious Sort*, pages 8–14.

[AKS83] M. Ajtai, J. Komlós, and E. Szemerédi. An 0(n log n) sorting network. In *Proceedings of the Fifteenth Annual ACM Symposium on Theory of Computing*, STOC '83, page 1–9, New York, NY, USA, 1983. Association for Computing Machinery.

[Bat68] K. E. Batcher. Sorting networks and their applications. In *Proceedings of the April 30– May 2, 1968, Spring Joint Computer Conference*, AFIPS '68 (Spring), page 307–314, New York, NY, USA, 1968. Association for Computing Machinery.