

## Lecture 6: 2-Server PIR from Distributed Point Functions Batch PIR

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### 1 Distributed Point Function and 2-server PIR with Logarithmic Communication

In earlier lectures, we learned an information-theoretic 2-server PIR scheme by Dvir and Gopi [DG16], which achieves  $n^{O(\sqrt{\log \log n / \log n})}$  communication per query. In this lecture, we will show how to get a 2-server PIR with logarithmic communication relying only on a pseudorandom generator (PRG). This scheme is also interesting from a practical perspective since in practice, we can use AES to realize the PRG, and modern CPUs have hardware acceleration for evaluating AES.

Recall that from our undergraduate cryptography course, we know that the existence of a PRG is equivalent to the existence of a one-way function (OWF) [HILL99]. Also, from an earlier lecture, we learned that any 1-server classical PIR scheme with non-trivial bandwidth implies Oblivious Transfer which cannot be constructed in a blackbox manner from OWF [IR89]. Therefore, the scheme we will talk about today is in the 2-server setting.

#### 1.1 Preliminary: Pseudorandom Generator

We will rely on a pseudorandom generator (PRG), which takes in a short random seed and expands the seed to a longer pseudorandom string.

**Definition 1** (PRG). Let  $\ell(\cdot)$  be a polynomial and let  $G : \{0, 1\}^n \rightarrow \{0, 1\}^{\ell(n)}$  be a deterministic polynomial-time algorithm.  $G$  is a PRG if it has the following properties:

- **Expansion:**  $\forall n, \ell(n) > n$ .
- **Pseudorandomness:** for any probabilistic polynomial-time distinguisher  $D$ , there exists a negligible function  $\text{negl}(\cdot)$ , such that

$$\left| \Pr_{r \xleftarrow{\$} \{0,1\}^{\ell(n)}} [D(r) = 1] - \Pr_{s \xleftarrow{\$} \{0,1\}^n} [D(G(s)) = 1] \right| \leq \text{negl}(n)$$

#### 1.2 Distributed Point Function

**Definition 2** (Point function). A point function parametrized by some point  $x \in \{0, 1\}^\ell$  is a function that evaluates to 1 at  $x$ , and evaluates to 0 everywhere else. We will henceforth use the notation  $P_x : \{0, 1\}^\ell \rightarrow \{0, 1\}$  to denote a point function. By definition,  $P_x(x) = 1$  and  $P_x(x') = 0$  for  $x' \neq x$ .

Boyle, Gilboa and Ishai [BGI16] introduced the concept of distributed point function. A distributed point function is a functional secret-sharing of a point function. In this lecture, we will specifically focus on a 2-way secret sharing of a point function. Essentially, given some point function  $P_{x^*}$ , one can “secretly share” the function to two keys  $k_L, k_R$ . Then, for party  $t \in \{L, R\}$  which receives the key  $k_t$ , it can evaluate the function on any point  $x$  and get a share of the outcome denoted  $\text{Eval}(k_t, x)$ . It is guaranteed that in every point  $x$ ,  $\text{Eval}(k_L, x) \oplus \text{Eval}(k_R, x) = P_{x^*}(x)$ . In other words, it is possible to combine the two outcome-shares to reconstruct the evaluation of the point function at any point. Finally, security of the DPF requires that each party  $t \in \{L, R\}$  does not learn the “special point” (i.e.,  $x^*$ ) given its individual key  $k_t$ .

**Definition 3** (2-share DPF). *A DPF is a pair of possibly randomized algorithms  $(\text{Gen}, \text{Eval})$  with the following syntax:*

- $\text{Gen}(1^\lambda, x^*)$ : *Outputs a pair of keys  $k_L, k_R$ .*
- $\text{Eval}(1^\lambda, k, x)$ : *Outputs the evaluation outcome  $y \in \{0, 1\}$ .*

**Correctness.** *Correctness requires that for any  $\lambda$ , any  $\ell$ , any  $x^*, x \in \{0, 1\}^\ell$ ,*

$$\Pr \left[ k_L, k_R \leftarrow \text{Gen}(1^\lambda, x^*) : \text{Eval}(k_L, x) \oplus \text{Eval}(k_R, x) = P_{x^*}(x) \right] = 1$$

**Security.** *Security requires that there exists a probabilistic polynomial-time simulator  $\text{Sim}$ , such that for any  $\ell = \ell(\lambda)$  that is a polynomial function in  $\lambda$ , and any  $x^* \in \{0, 1\}^{\ell(\lambda)}$ , the following experiments are computationally indistinguishable for both  $t = L$  and  $t = R$ :*

- $\text{Real}(1^\lambda, x^*)$ :  $k_L, k_R \leftarrow \text{Gen}(1^\lambda, x^*)$  and output  $k_t$ ;
- $\text{Ideal}(1^\lambda)$ : *Output  $\text{Sim}(1^\lambda, \ell)$ .*

Intuitively, security requires that the any individual key  $k_L$  or  $k_R$  can be simulated without knowledge of the special point  $x^*$ .

### 1.3 2-Server PIR Based on DPF

Let  $\text{DB} \in \{0, 1\}^n, i \in [n]$  and  $\lambda$  be the security parameter. Suppose we have an efficient 2-share binary DPF scheme  $\text{DPF.Gen}$  and  $\text{DPF.Eval}$ , such that the key length is  $O_\lambda(\log n)$  (where  $n$  is the size of the input domain), and the evaluation algorithm is efficient. Then, we can have the following 2-server PIR scheme.

1. Given the query  $i$ , Client computes  $(k_0, k_1) \leftarrow \text{DPF.Gen}(1^\lambda, i)$
2. Client sends  $(k_0, k_1)$  to Server 0 and Server 1 respectively.
3. For  $t \in \{0, 1\}$ , Server  $t$  responds with  $y_i \leftarrow \bigoplus_{j \in [n]} \text{DPF.Eval}(k_t, j) \cdot \text{DB}[j]$ ;
4. Client computes the answer as  $y_0 \oplus y_1$ .

## Correctness

$$\begin{aligned}
y_0 \oplus y_1 &= \left( \bigoplus_j \text{DB}[j] \cdot \text{DPF.Eval}(k_0, j) \right) \oplus \left( \bigoplus_j \text{DB}[j] \cdot \text{DPF.Eval}(k_1, j) \right) \\
&= \left( \bigoplus_j \text{DB}[j] \cdot (\text{DPF.Eval}(k_0, j) \oplus \text{DPF.Eval}(k_1, j)) \right) \\
&= \left( \bigoplus_j \text{DB}[j] \cdot P_i(j) \right) \\
&= \text{DB}[i].
\end{aligned}$$

**Security** Security follows from the security of the DPF.

### 1.3.1 DPF construction using PRG

Now we see how to construct this efficient 2-share binary DPF, based on a PRG  $G$ :

$$G : \{0, 1\}^\lambda \rightarrow \{0, 1\}^{2\lambda+2}.$$

The algorithm is based on a binary tree expansion idea.

**The key structure.** The **Gen** algorithm outputs two keys,  $k_0, k_1$ . For each key  $k_t$  where  $t \in \{0, 1\}$ , it contains the “root” information as a  $\lambda$ -bit random string  $s$  and a bit  $b \in \{0, 1\}$ . It also contains  $\log n$  “correction vectors”  $\text{CV}_1, \dots, \text{CV}_{\log n}$ , each of size  $2\lambda + 2$  bit.  $k_0, k_1$  will share the same sets of correction vectors. We will see how they are constructed later.

**Evaluation algorithm.** Given a key containing  $s, b$  and  $\text{CV}_1, \dots, \text{CV}_{\log n}$ , the evaluation algorithm is as follows. We consider a binary tree with  $\log n + 1$  levels and  $n$  leaves (assume  $n$  is a power of 2). The root is on the 0-th level and the leaves are on the  $\log n$ -th level. Each node contains a  $\lambda$ -bit (pseudo)random string  $s \in \{0, 1\}^\lambda$  and a bit  $b \in \{0, 1\}$ . The root’s string and bit are included in the description of the key. For all other nodes, their strings and bits will be decided by their parents. Given a node on level  $(i - 1)$ , suppose now we want to decide the bits and the strings of its children (on level  $i$ ). We only need the following one-line algorithm:

$$s_L || b_L || s_R || b_R \leftarrow G(s) \oplus \begin{cases} 0 & b = 0; \\ \text{CV}_i & b = 1. \end{cases}$$

That is,  $s_L, b_L$  are the string and bit on its left child, and  $s_R, b_R$  are the string and bit on its right child. Finally, the bit on the  $i$ -th leaf will be  $\text{Eval}(k, i)$ .

Notice that if we only want to learn  $\text{Eval}(k, i)$  for a particular  $i$ , the computation cost will be  $O(\log n)$  calls to the PRG, because we can focus on the path from the root to the  $i$ -th leaf and ignore other nodes. However, if we want to evaluate  $\text{Eval}(k, i)$  for all  $i \in [n]$ , the computation cost will be  $O(n)$  because we can simply do the expansion on the whole tree.

**Generation Algorithm.** The generation algorithm needs to generate two correlated keys, such that when we look at the bit at the  $x$ -th leaves (say  $b_x^0$  and  $b_x^1$ ) on the trees expanded by  $k_0$  and  $k_1$ , it must that  $b_x^0 \oplus b_x^1 = P_{x^*}(x)$ . In fact, we want to ensure the following stronger properties. Say on a particular node, we denote the information expanded by  $k_0$  and  $k_1$  as  $(s_0, b_0)$  and  $(s_1, b_1)$ , respectively.

**(New Version) Generation Algorithm:  $\text{Gen}(1^\lambda, x^*)$**

**Initialization:**

- Sample  $s_0, s_1$  as two  $\lambda$ -bit random strings.
- Sample a random bit  $b_0$ . Let  $b_1 = b_0 \oplus 1$ .
- Let  $\{x^*[1], \dots, x^*[\log n]\}$  be  $x^*$ 's binary representation.
- Store the roots: let  $(\text{root}.s_0, \text{root}.b_0) = (s_0, b_0)$  and  $(\text{root}.s_1, \text{root}.b_1) = (s_1, b_1)$ .

**Constructing correction vectors:** For  $i \in \{1, \dots, \log n\}$ :

- If  $x^*[i] = 0$ , set  $\text{Keep} = L$ . Otherwise, set  $\text{Keep} = R$ .
- Sample a random string  $r \xleftarrow{\$} \{0, 1\}^\lambda$ .
- Set  $\text{CV}_i \in \{0, 1\}^{2\lambda+2}$  as an unknown variable and build the equation as follows.
  - Let  $s_L^0 || b_L^0 || s_R^0 || b_R^0 \leftarrow G(s_0) \oplus (b_0 \cdot \text{CV}_i)$ ; *Expanding the first key*
  - Let  $s_L^1 || b_L^1 || s_R^1 || b_R^1 \leftarrow G(s_1) \oplus (b_1 \cdot \text{CV}_i)$ ; *Expanding the second key*
  - If  $\text{Keep} = L$ : set the invariant as

$$(s_L^0 || b_L^0 || s_R^0 || b_R^0) \oplus (s_L^1 || b_L^1 || s_R^1 || b_R^1) = (r || 1 || 0^\lambda || 0).$$

- If  $\text{Keep} = R$ : set the invariant as

$$(s_L^0 || b_L^0 || s_R^0 || b_R^0) \oplus (s_L^1 || b_L^1 || s_R^1 || b_R^1) = (0^\lambda || 0 || r || 1).$$

*(Force the Keep direction to be different and the other direction to be identical.)*

- Solve the equation based on the invariant to get  $\text{CV}_i$ . The equation is always solvable.
- Update  $(s_0, b_0) \leftarrow (s_{\text{Keep}}^0, b_{\text{Keep}}^0)$ ;
- Update  $(s_1, b_1) \leftarrow (s_{\text{Keep}}^1, b_{\text{Keep}}^1)$ ;

**Output:** Output  $k_0, k_1$ :

$$k_0 = (\text{root}.s_0, \text{root}.b_0, \text{CV}_1, \dots, \text{CV}_{\log n})$$

$$k_1 = (\text{root}.s_1, \text{root}.b_1, \text{CV}_1, \dots, \text{CV}_{\log n})$$

Figure 1: The 2-share binary DPF generation algorithm.

**Generation Algorithm:  $\text{Gen}(1^\lambda, x^*)$**

**Initialization:**

- Sample  $s_0, s_1$  as two  $\lambda$ -bit random strings.
- Sample a random bit  $b_0$ . Let  $b_1 = b_0 \oplus 1$ .
- Let  $k_0 = (s_0, b_0)$  and  $k_1 = (s_1, b_1)$ ;
- Let  $\{x^*[1], \dots, x^*[\log n]\}$  be  $x^*$ 's binary representation.

**Constructing correction vectors:** For  $i \in \{1, \dots, \log n\}$ :

- Let  $s_L^0 || b_L^0 || s_R^0 || b_R^0 \leftarrow G(s_0)$ ;
- Let  $s_L^1 || b_L^1 || s_R^1 || b_R^1 \leftarrow G(s_1)$ ;
- Let  $s_L || b_L || s_R || b_R = (s_L^0 || b_L^0 || s_R^0 || b_R^0) \oplus (s_L^1 || b_L^1 || s_R^1 || b_R^1)$ ;
- Sample  $r$  as a random  $\lambda$ -bit random string;
- If  $x^*[i] = 0$ :
  - Let  $\text{CV}_i = r || (b_L \oplus 1) || s_R || b_R$ ;
  - Update  $(s_0 || b_0) \leftarrow (s_L^0 || b_L^0) \oplus (b_0 \cdot (r || (b_L \oplus 1)))$ ;
  - Update  $(s_1 || b_1) \leftarrow (s_L^1 || b_L^1) \oplus (b_1 \cdot (r || (b_L \oplus 1)))$ ;
- If  $x^*[i] = 1$ :
  - Let  $\text{CV}_i = s_L || b_L || r || (b_R \oplus 1)$ ;
  - Update  $(s_0 || b_0) \leftarrow (s_R^0 || b_L^0) \oplus (b_0 \cdot (r || (b_R \oplus 1)))$ ;
  - Update  $(s_1 || b_1) \leftarrow (s_R^1 || b_L^1) \oplus (b_1 \cdot (r || (b_R \oplus 1)))$ ;

**Output:**

- Attach  $\text{CV}_1, \dots, \text{CV}_{\log n}$  to  $k_0$  and  $k_1$ .
- Output  $k_0, k_1$ .

Figure 2: The 2-share binary DPF generation algorithm.

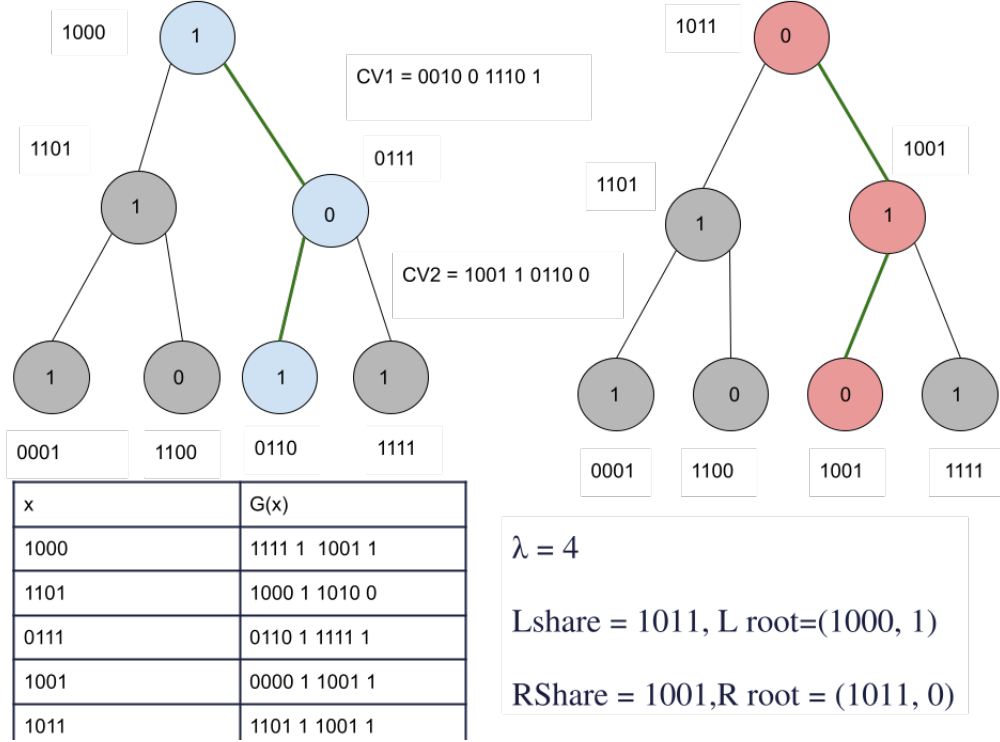


Figure 3: An example of the DPF construction.

1. If the node is on the “special path” (i.e., the path from the root to the  $x^*$ -th leaf), then  $b_0 \neq b_1$ , and  $s_0$  and  $s_1$  are independent.
2. Otherwise,  $b_0 = b_1$  and  $s_0 = s_1$ .

The properties also hold for the leaf nodes, which is sufficient to prove the correctness of the DPF. We now see how to ensure these properties. We have the first lemma, which can be proved by a simple induction argument.

**Lemma 4.** *If on some particular node,  $b_0 = b_1$  and  $s_0 = s_1$ , then the subtrees expanded based on  $(s_0, b_0)$  and  $(s_1, b_1)$  will be identical, regardless of  $CV_1, \dots, CV_{\log n}$ .*

This lemma shows that we only need to focus on the “special path”. We can sample the roots first by sampling two random strings  $s_0, s_1$  and let  $b_0$  and  $b_1$  be two different bits. Then, we can just “simulate” the expansion process on the special path, and then set up the correction vector according to the target properties. Since on the  $i$ -th level of the special path, there will be only one side affected by the correction vector (because the bits are different). Therefore, we can always set the correction vector to ensure that after applying the correction vector, the children on the special path still have different bits and independent random strings, while the children deviating from the path will become identical. The algorithm is presented in Figure 2 and an example is presented in Figure 3.

**Analysis.** The correctness can be verified by doing an induction proof from level 1 to level  $\log n$ . The security analysis is referred to [BGI16]. It is also clear to see that the key size is  $\lambda + 1 + \log n \cdot (2\lambda + 2) = O_\lambda(\log n)$ .

## 2 Batch-PIR

### 2.1 Motivation

So far, every PIR scheme we have seen only retrieves 1 bit at a time. In many use cases, the client may want to fetch up to  $Q$  database entries at the same time. The naive solution is just to repeat the single-query algorithm  $Q$  times. For instance, to compute  $Q$  queries using a single-query PIR scheme with  $O(n)$  computation, it requires  $O(Qn)$  computation. Batch-PIR aims to handle multiple parallel queries at the same time, reducing the amortized cost.

### 2.2 A simple load-balancing scheme

Given an  $n$  bit long database and  $Q = o(n)$  queries, we load-balance the database into  $\frac{Q}{\log Q}$  buckets – we use a hash function to hash each database entry to a bucket (by hashing their index). Then, given the  $Q$  queries, we again use the hash function to place the queries to the buckets. In expectation, each bucket will have  $\log Q$  queries. Based on the balls-into-bins argument, each bucket will have no more than  $\lambda \log Q$  queries with  $1 - \text{negl}(\lambda)$  probability. Therefore, we just do  $\lambda \log Q$  PIR queries in each bucket (possibly dummy queries) to retrieve the target entries. This ensures the success probability is at least  $1 - \text{negl}(\lambda)$ .

**Security.** We are always making fix number of queries ( $\lambda \log Q$  PIR queries in each bucket), so the scheme is secure by a reduction to the security of the underlying PIR scheme.

**Cost.** Moreover, say the underlying single-query PIR computation cost is linear in the size of the database. We are making  $\lambda \log Q$  queries to each bucket, and the total size of the buckets is just  $n$ . Therefore, the total computation cost is  $O(\lambda \log Q n)$ . So if  $Q > \lambda \log Q$ , this simple scheme saves computation.

### 2.3 Cuckoo Hashing based scheme [ACLS18]

Angel et al. [ACLS18] proposed SealPIR that uses cuckoo hashing to do batch PIR.

#### Cuckoo Hashing

**Definition 5** (Cuckoo hashing). *Given  $n$  balls,  $b$  buckets, and  $w$  independent hash functions  $h_0, \dots, h_w$  that map a ball to a random bucket, compute  $w$  candidate buckets for each ball by applying the  $w$  hash functions. For each ball  $x$ , place  $x$  in any empty candidate bucket. If none of the  $w$  candidate buckets are empty, select one at random, remove the ball currently in that bucket ( $x_{old}$ ), place  $x$  in the bucket, and re-insert  $x_{old}$ . If re-inserting  $x_{old}$  causes another ball to be removed, this process continues recursively until we finish the insertion or a maximum number of iterations is achieved.*

**Batch PIR based on Cuckoo Hashing.** The scheme is as follows.

- **Server encoding.** Given an  $n$  bit database,  $b$  buckets, and  $w$  hash functions, we hash each entry in the database (using their index as the key) to all  $w$  candidate buckets and store it there. This results in a encoded database that each original entry is replicated  $w$  times. The server will share the hash functions to the client.
- **Client scheduling.** Given the  $Q$  queries, the client use the cuckoo hashing method to insert (or say, schedule) the queries to the buckets (again, using the indices as the keys). Our target is that each bucket has at most one query, and all query can be inserted in one of its candidate bucket.

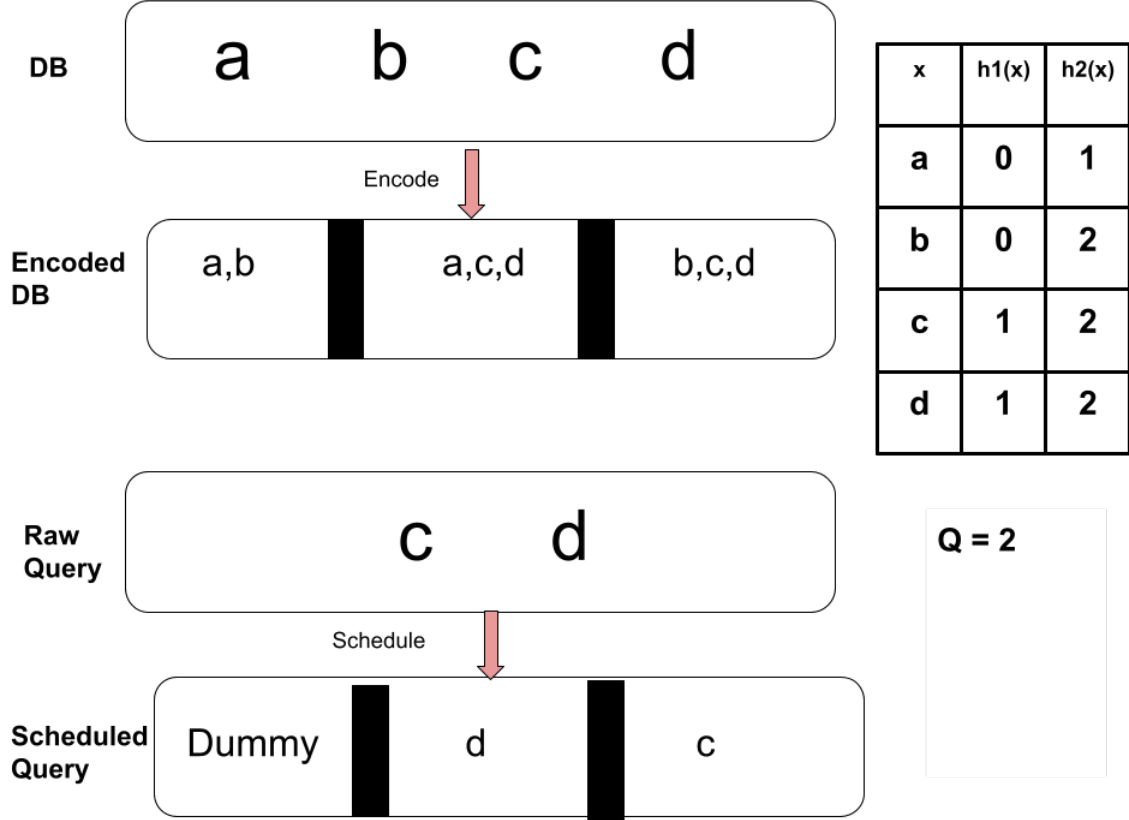


Figure 4: An example of the cuckoo hashing based batch PIR.

- **Client query.** Now the client just makes one query in each bucket. If the client successfully insert all queries earlier, it can then proceed to learn all the target entries because the server has inserted the entries in all the candidate buckets.

An example can be found in Figure 4.

The authors used 3 hash functions for encoding and set the number of buckets  $b = 1.5Q$ . For  $Q \geq 200$ , the author showed that the chance of failure during the scheduling phase is  $\leq 2^{-40}$ . Notice that this is not cryptographically negligible. To enforce negligible failure probability, we can introduce a size  $\lambda$  stash of the cuckoo hash table. That is, the stash stores at most  $\lambda$  elements that fail to be inserted. Then, the client also has to make additional  $\lambda$  PIR queries to the whole database. This ensures the failure probability to be  $\text{negl}(\lambda)$ .

**Cost Analysis.** Assume the underlying single-query PIR scheme is linear. The client will make one query in each bucket and the total bucket size is  $wn$ . Also, the client needs to make  $\lambda$  additional query to the whole database to ensure negligible failure probability. Then, the computation cost for the  $Q$  queries are just  $(w + \lambda)n$ .

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