Cryptography Meets Algorithms (15893) Lecture Notes

Lecture 1: Private Information Retrieval

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The Private Information Retrieval problem was first introduced by Chor, Kushilevitz, Goldreich and Sudan [CKGS98]. In this setting, we will have a client and one or more server(s). The servers each have a public database indexed from 1 to n (e.g., the DNS repository, a repository of webpages, a leaked password database, etc).

A client wants to fetch an entry indexed $i \in [n]$ from this database but does not want to leak its query to the server(s). More formally, we define a single-server PIR scheme as follows.

Definition 1 (Single-server PIR)**.** A single-server PIR, parametrized by a security parameter $\lambda \in \mathbb{N}$, is a protocol between a client and a server with the following syntax:

- The client's input is a desired index $i \in [n]$, and the server's input is a database $\mathsf{DB} \in \{0,1\}^n$. Both the client and server also obtain 1^λ as input.
- At the end of the protocol, the client outputs a bit $b \in \{0,1\}$.

We want the scheme to satisfy the following properties.

- **Correctness**: for all λ, n, for any $\mathsf{DB} \in \{0,1\}^n, i \in [n]$, under honest execution,
$$\Pr[b = \mathsf{DB}[i]] = 1$$
- **Privacy**: For any λ, any n polynomially bounded in λ, any $i, j \in [n], \mathsf{DB} \in \{0,1\}^n$, it holds that
$$\mathsf{view}_s(1^\lambda, \mathsf{DB}, i) \approx \mathsf{view}_s(1^\lambda, \mathsf{DB}, j)$$
where $\mathsf{view}_s(1^\lambda, \mathsf{DB}, i)$ is a random variable representing the view of the server if we execute the PIR protocol over client input $(1^\lambda, i)$ and server input $(1^\lambda, \mathsf{DB})$, and $\approx$ stands for statistical or computational indistinguishability.

Remark 1 (Honest-server vs. malicious-server privacy)**.** *The above privacy definition assumes an honest server. It is also possible to define privacy against a malicious server. In today's lecture, all the PIR constructions will only have a single round-trip — in this special case, honest-server privacy and malicious-server privacy are equivalent. So we will simply define honest-server privacy here.*

This definition is naturally extended to a setting with two or more servers that do not communicate, where privacy should hold for any individual server's view. In a setting with more than two servers, it also makes sense to define t-out-of-n security, where we want privacy to hold for the union of any combination of t servers' views. For example, a 2-server PIR scheme is defined as follows.

Definition 2 (Two-server PIR)**.** A two-server PIR, parametrized with some security parameter λ, is a protocol between a client c and two servers s_1, s_2 with the following syntax:

- The client's input is 1^λ and a desired index $i \in [n]$, and each server's input is a database $\mathsf{DB} \in \{0,1\}^n$.
- At the end of the protocol, the client outputs a bit $b \in \{0,1\}$.

with properties,

- **Correctness**: for all λ, n, for all $\mathsf{DB} \in \{0,1\}^n, i \in [n]$, under honest execution,
$$\Pr[b = \mathsf{DB}[i]] = 1$$
- **Privacy**: for any λ, any n that is polynomially bounded in λ, any $i, j \in [n]$, any $\mathsf{DB} \in \{0,1\}^n$, it holds that
$$\mathsf{view}_{s_1}(1^\lambda, \mathsf{DB}, i) \approx \mathsf{view}_{s_1}(1^\lambda, \mathsf{DB}, j)$$
$$\mathsf{view}_{s_2}(1^\lambda, \mathsf{DB}, i) \approx \mathsf{view}_{s_2}(1^\lambda, \mathsf{DB}, j)$$
where $\mathsf{view}_{s_1}(1^\lambda, \mathsf{DB}, i)$ and $\mathsf{view}_{s_2}(1^\lambda, \mathsf{DB}, i)$ are random variables representing the view of the first and the second server, respectively, in a protocol execution with client input $(1^\lambda, i)$ and server input $(1^\lambda, \mathsf{DB})$.

Note that PIR schemes can be extended for retrieving records containing multiple bits, rather than just 1-bit records.

Naïve approach. The naïve approach is for the client to download the entire database. However, this approach suffers from linear bandwidth, and linear server/client computation.

We will now show some PIR constructions with sublinear bandwidth.

1 Single-Server PIR Construction based on Fully Homorphic Encryption

It is easy to obtain a bandwidth-efficient PIR scheme if we assume a Fully Homomorphic Encryption (FHE) scheme. An FHE scheme allows us to perform addition and multiplication operations in the ciphertext space. An FHE scheme supports the following operations:

- $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{Gen}(1^\lambda)$: samples a public key pk and a secret key sk;
- $c \leftarrow \mathsf{Enc}(\mathsf{pk}, m)$: encrypts a message m from some message space using the public key pk, and outputs the ciphertext c;
- $m \leftarrow \mathsf{Dec}(\mathsf{sk}, c)$: decrypts a ciphertext c using the secret key sk, and outputs a plaintext message m;
- $c' \leftarrow \mathsf{Eval}(\mathsf{pk}, \mathsf{Circ}, c)$: given the public key pk, some circuit Circ, and a ciphertext c, output a transformed ciphertext c'. Correctness requires that c' be a valid FHE encryption of $\mathsf{Circ}(c)$.

PIR from FHE. We can construct a single-server PIR scheme from an FHE scheme as follows.

1. The client samples $(\mathsf{pk}, \mathsf{sk}) \leftarrow \mathsf{FHE.Gen}(1^\lambda)$, and encrypts their query as $q \leftarrow \mathsf{FHE.Enc}(i)$. The client then sends q to the server.
2. The server homomorphically evaluates the selection circuit S_i and computes $c = \mathsf{FHE.Eval}(\mathsf{pk}, S_i, q)$, where $S_i(\mathsf{DB})$ is the circuit that selects the i-th bit from the input DB, and sends the resulting ciphertext c back to the client.

3. The client decrypts $\mathsf{FHE.Dec}(\mathsf{sk}, c)$.

The correctness of the PIR scheme is easy to see given correctness of the FHE scheme. The scheme has $\widetilde{O}(1)$ bandwidth and client computation, and $\widetilde{O}(n)$ server computation[1] where $\widetilde{O}(\cdot)$ hides $\mathsf{poly}(\lambda, \log n)$ factors.

Some notable PIR schemes based on FHE include Spiral [MW22] and SimplePIR [HHCG+22].

Question: Can we get sub-linear bandwidth PIR without any cryptographic assumptions?

In fact, this is possible in the two-server setting (we will see this next), and later we will prove that it is impossible in the single-server setting.

2 Two-Server PIR Constructions

2.1 $\sqrt{n}$-Bandwidth 2-Server PIR

We now introduce a 2-server $\sqrt{n}$-bandwidth PIR scheme with information theoretic security, i.e., the scheme does not rely on any cryptographic assumptions, and achieves perfect privacy and correctness.

The key idea is the view of database $DB \in \{0,1\}^n$ as a $S \in \{0,1\}^{\sqrt{n}\times\sqrt{n}}$ matrix. Say the client wants to query the (i,j)'th entry. To do this, the server can simply return the entire column j, which we can afford to do via the square-root bandwidth. To provide security, the client also supplies a one-hot-vector q_j, split into two random shares v_1, v_2 via a 2-share secret sharing scheme (that is, sample random v_1, v_2 conditioned on $v_1 \oplus v_2 = q_j$).

The scheme is as follows:

1. The client creates v_1, v_2 as described above. The client sends these shares to servers s_1, s_2 respectively.
2. Each server s_i, on receiving v_i computes the matrix-vector product,
$$r_i \leftarrow Sv_i \bmod 2$$
and sends the $\sqrt{n}$-size vector r_i to the client.
3. The client computes and outputs $r_1 \oplus r_2$.

It is straightforward to verify correctness by seeing that

$$r_1 \oplus r_2 = Sr_1 \oplus Sr_2 \bmod 2 = S(r_1 \oplus r_2) \bmod 2$$

Lastly, this scheme is private, since v_1, v_2 are uniformly random in each server's view, respectively.

2.2 $n^{\frac{1}{3}}$-BW 2-server Scheme[CKGS98]

We will first motivate this construction with a two server scheme with expected $\frac{n}{2}$ bandwidth and a eight server scheme with $n^{\frac{1}{3}}$ bandwidth.

1. **$\mathbb{E}[\frac{n}{2}]$-BW 2-server Scheme**

   The client samples $S_1 \subseteq [n]$ uniformly as follows: for each $i \in [n]$, add i to S_1 with probability $\frac{1}{2}$. Then, the client computes
$$S_2 = S_1 \Delta \{i\}$$

[1] We assume a compact FHE scheme with the following performance bounds: the ciphertext size is $\widetilde{O}(1)$ for encrypting a plaintext of $\widetilde{O}(1)$ bits, and the encryption and decryption times are also $\widetilde{O}(1)$, and the homomorphic evaluation time is $\widetilde{O}(|\mathsf{Circ}|)$ for a circuit Circ.

where Δ is the symmetric difference operator. That is, if i is included in S_1, we remove it from the set, otherwise we add it to the set. S_1, S_2 are sent to each server respectively, where server i computes and sends back,

$$r_i \leftarrow \bigoplus_{j \in S_i} \mathsf{DB}[j]$$

The client on receiving r_1, r_2 outputs $r_1 \oplus r_2$.

Correctness follows from the fact that i is the only database index that appears once, with every other index appearing twice (hence XOR'ing to 0).

Privacy: First, S_1 is uniformly random. Second, to see why S_2 is uniformly random to server 2, just consider the following distribution – "tossing n random coins, and flip the i-th coin afterwards, regardless of the original result". This distribution is uniformly random even when i is known.

Remark 2. *Note that if we run the above $n/2$-BW scheme not on bits, but on blocks of $\sqrt{n}$ size (i.e., treat the n-bit database as $\sqrt{n}$ blocks each of size $\sqrt{n}$), the scheme is equivalent to the earlier $\sqrt{n}$-BW scheme.*

2. $n^{\frac{1}{3}}$**–BW 8-server Scheme**

   The idea is to view the database as a $n^{\frac{1}{3}} \times n^{\frac{1}{3}} \times n^{\frac{1}{3}}$ cube. Then, each index $i \in \{0, 1, \ldots, n-1\}$ can be expressed as a thruple $(x^\star, y^\star, z^\star)$ (note we start at 0 for natural base 3 representation.

   The client samples $X, Y, Z \subseteq \{0, \ldots, n^{\frac{1}{3}} - 1\}$ independently as follows: for each $x \in \{0, \ldots, n^{\frac{1}{3}}\}$ add it to X with probability $\frac{1}{2}$. Do the same for Y, Z.

   Then, we compute 8 different sets by a 3-wise cartesian products with symmetric differences, enumerated as

$$\begin{aligned}
S_{000} &= X \times Y \times Z \\
S_{001} &= X \times Y \times (Z \Delta \{z^\star\}) \\
S_{010} &= X \times (Y \Delta \{y^\star\}) \times Z \\
S_{011} &= X \times (Y \Delta \{y^\star\}) \times (Z \Delta \{z^\star\}) \\
S_{100} &= (X \Delta \{x^\star\}) \times Y \times Z \\
S_{101} &= (X \Delta \{x^\star\}) \times Y \times (Z \Delta \{z^\star\}) \\
S_{110} &= (X \Delta \{x^\star\}) \times (Y \Delta \{y^\star\}) \times Z \\
S_{111} &= (X \Delta \{x^\star\}) \times (Y \Delta \{y^\star\}) \times (Z \Delta \{z^\star\})
\end{aligned}$$

   See that each of these sets have their sizes concentrated around $\left(\frac{n^{\frac{1}{3}}}{2}\right)^3 = \frac{n}{8}$. Then, the client sends a succinct description of each set (i.e., sends the three "marginal" vectors instead of the full set) of size $O(n^{\frac{1}{3}})$ to each server.

   Server i on receiving $S = X' \times Y' \times Z'$ computes

$$p_i = \bigoplus_{j \in S} \mathsf{DB}[j]$$

   Then the client computes $p_0 \oplus \cdots \oplus p_7$.

   We claim that $p_0 \oplus \cdots \oplus p_7 = \mathsf{DB}[i], i = (x^\star, y^\star, z^\star)$.

Proof. For every (x, y, z) not equal to the query, it will appear an even number times in the summation. We can pair up the sets to see this.

On the other hand, $(x^\star, y^\star, z^\star)$ only appears once in the summation so we are done. $\square$

Privacy follows the argument as the previous case.

Now, we finally compress this scheme from eight servers to two servers. To do this, the client sends S_{000} to server 1 and S_{111} to server 2.

Each server on receiving $X' \times Y' \times Z'$ calculates $3n^{\frac{1}{3}}$ parities to send to the client as follows:

1. For each $x' \in \{0, \ldots, n^{\frac{1}{3}} - 1\}$ we calculate the parity for $(X' \Delta \{x'\}) \times Y' \times Z'$ as in the previous scheme.
2. For each $y' \in \{0, \ldots, n^{\frac{1}{3}} - 1\}$ we calculate the parity for $X' \times (Y' \Delta \{y'\}) \times Z'$ as in the previous scheme.
3. For each $z' \in \{0, \ldots, n^{\frac{1}{3}} - 1\}$ we calculate the parity for $X' \times Y' \times (Z' \Delta \{z'\})$ as in the previous scheme.

Now each server returns $1 + 3n^{1/3}$ parities to the client. That is, the server 1 will actually compute S_{000} and $S_{100}, S_{010}, S_{001}$ will be in those $3n^{1/3}$ parities. Similarly, the server 2 will compute S_{111}, and $S_{011}, S_{101}, S_{110}$ will be in those $3n^{1/3}$ parities. The client will be able to pick out the correct parities corresponding to its actual query.

Thus, this scheme still has $n^{\frac{1}{3}}$ bandwidth, and correctness and security still follow.

3 State of the Art and Open Problem

For the 2-server setting, the best known lower bound states that $5 - o(1) \log n$ bandwidth is necessary ([WdW05]), while the best known upper bound requires $n^{O\left(\sqrt{\lg\lg n / \lg n}\right)} = n^{o(1)}$, i.e., sub-polynomial bandwidth ([DG16]). Closing this gap is a long-standing open problem.

References

[CKGS98] Benny Chor, Eyal Kushilevitz, Oded Goldreich, and Madhu Sudan. Private information retrieval. *Journal of the ACM (JACM)*, 45(6):965–981, 1998.

[DG16] Zeev Dvir and Sivakanth Gopi. 2-server pir with subpolynomial communication. *Journal of the ACM (JACM)*, 63(4):1–15, 2016.

[HHCG+22] Alexandra Henzinger, Matthew M. Hong, Henry Corrigan-Gibbs, Sarah Meiklejohn, and Vinod Vaikuntanathan. One server for the price of two: Simple and fast single-server private information retrieval. Cryptology ePrint Archive, Paper 2022/949, 2022. `https://eprint.iacr.org/2022/949`.

[MW22] Samir Jordan Menon and David J. Wu. Spiral: Fast, high-rate single-server PIR via FHE composition. In *IEEE S&P*, 2022.

[WdW05] Stephanie Wehner and Ronald de Wolf. Improved lower bounds for locally decodable codes and private information retrieval. In Luís Caires, Giuseppe F. Italiano, Luís Monteiro, Catuscia Palamidessi, and Moti Yung, editors, *Automata, Languages and Programming*, pages 1424–1436, Berlin, Heidelberg, 2005. Springer Berlin Heidelberg.