

## Lecture 2: Sub-poly Communication 2-server Classical PIR

Scribe: Ashrujit Ghoshal

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### 1 Sub-poly Communication 2-server Classical PIR

As introduced in the earlier lecture, Private Information Retrieval (PIR) [CKGS98] is a cryptographic mechanism which allows a user holding an index  $i \in [n]$  to retrieve the  $i$ -th bit from a  $n$ -bit database, copies of which are held by one or more (non-colluding) servers, such that the servers do not learn anything about  $i$ . We saw a construction of two-server information-theoretic PIR that has bandwidth  $O(n^{1/3})$  in the last lecture. The ultimate goal in this lecture is to see at a PIR construction that has bandwidth sub-polynomial in  $n$ . As a first step, we will see a 2-server PIR construction by Woodruff and Yekhanin [WY05] that has bandwidth  $O(n^{1/3})$ , based on an interpolation approach.

#### 1.1 An Interpolation Approach to 2-Server PIR

As a warm-up, we will first start with a naïve *four-server* construction. Let  $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \{0, 1\}^n$  be the database. Define  $E : [n] \rightarrow \{0, 1\}^m$  such that  $E(1), E(2), \dots, E(n)$  are  $n$  distinct points of Hamming weight 3. Note that such a mapping  $E$  exists as long as  $\binom{m}{3} \geq n$ . Hence, we set  $m = O(n^{1/3})$ . Define the multivariate polynomial  $F(z_1, z_2, \dots, z_m)$  over  $\mathbb{F}_5$  as

$$F(z_1, z_2, \dots, z_m) = \sum_{i=1}^n x_i \prod_{E(i)_\ell=1} z_\ell.$$

First off, observe that since each  $E(i)$  has hamming weight 3,  $F$  has degree 3. Furthermore, for each  $i$ ,  $F(E(i)) = x_i$ .

Therefore, in the PIR protocol a user that has index  $i$ , needs to retrieve the value of  $F(\mathbf{p})$  for  $\mathbf{p} = E(i)$ . Let  $\mathbf{v}$  be some randomly chosen element of  $\mathbb{F}_5^m$ . Suppose, the user learns the values  $F(\mathbf{p} + \mathbf{v}), F(\mathbf{p} + 2\mathbf{v}), F(\mathbf{p} + 3\mathbf{v}), F(\mathbf{p} + 4\mathbf{v})$ . Let  $f(\lambda) = F(\mathbf{p} + \lambda\mathbf{v})$ . Since the degree of  $f$  is 3, and the user knows  $f(1), f(2), f(3), f(4)$ , they can interpolate  $f$  to find  $f(0) = F(\mathbf{p}) = x_i$ . This observation gives us the following protocol.

1. User picks  $\mathbf{v} \in \mathbb{F}_5^m$  uniformly at random
2. To server  $j \in [4]$ , user sends  $\mathbf{p} + j\mathbf{v}$
3. Server  $j$  returns  $F(\mathbf{p} + j\mathbf{v})$
4. User computes  $F(\mathbf{p})$  using interpolation

The correctness of the protocol follows directly from the observation above, while privacy follows because  $(\mathbf{p} + j\mathbf{v})$  is distributed uniformly over  $\mathbb{F}_5^m$  and does not reveal  $\mathbf{p}$ . Observe that the

communication is somewhat asymmetric: the user sends an element of  $\mathbb{F}_5^m$  to the servers while the servers respond with an element of  $\mathbb{F}_5$ . We will make this symmetric and reduce the number of servers next. Towards that, we prove the following simple lemma. Note that the derivative of a function  $f$  at  $x$  is denoted  $f'(x)$ .

**Lemma 1.** *Let  $p$  be a prime. Suppose  $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{F}_p$  are such that  $x_1 \neq x_2$ . Then there exists at most one polynomial  $f(\lambda) \in \mathbb{F}_p(\lambda)$  of degree  $\leq 3$  such that  $f(x_1) = y_1, f(x_2) = y_2, f'(x_1) = z_1, f'(x_2) = z_2$ .*

*Proof.* Assume there exist two such polynomials  $f_1, f_2$ . Consider the polynomial  $f = f_1 - f_2$ . Clearly,  $f(x_1) = f(x_2) = 0 = f'(x_1) = f'(x_2)$ . Therefore  $(\lambda - x_1)^2(\lambda - x_2)^2$  divides  $f(\lambda)$ . Since the degree of  $f(\lambda)$  is at most 3, this implies  $f(\lambda) = 0$ .  $\square$

Therefore, if the user were given the value of  $f(1), f(2), f'(1), f'(2)$  then they can compute  $f(0)$ . This observation allows us to give the following 2-server protocol and even reduce the finite field to  $\mathbb{F}_3$ . (Recall  $\mathbf{p}$  is such that  $\mathbf{p} = E(i)$  where  $i$  is the index whose value the user wants to retrieve, and  $f(\lambda) = F(\mathbf{p} + \lambda \mathbf{v})$ ).

1. User picks  $\mathbf{v} \in \mathbb{F}_3^m$  uniformly at random
2. To server  $j \in [2]$ , user sends  $\mathbf{p} + j\mathbf{v}$
3. Server  $j$  returns  $F(\mathbf{p} + j\mathbf{v}), \frac{\partial F}{\partial z_1}|_{(\mathbf{p}+j\mathbf{v})}, \dots, \frac{\partial F}{\partial z_m}|_{(\mathbf{p}+j\mathbf{v})}$
4. For  $j \in [2]$ , server computes  $f'(j) = \sum_{\ell=1}^m \frac{\partial F}{\partial z_\ell}|_{(\mathbf{p}+j\mathbf{v})} \mathbf{v}_\ell$  (where  $\mathbf{v}_\ell$  is the value at the  $\ell$ -th index of  $\mathbf{v}$ ). Use  $f(1), f(2), f'(1), f'(2)$  to compute  $f(0)$  and output the answer.

Correctness follows since  $f'(\lambda)|_j = \sum_{\ell=1}^m \frac{\partial F}{\partial z_\ell}|_{(\mathbf{p}+j\mathbf{v})} \mathbf{v}_\ell$  using the chain rule. Privacy follows for the same reason as in the previous protocol. Note that the communication here is  $O(m) = O(n^{1/3})$  but symmetric. Next, we will see a protocol where the communication is significantly reduced using ideas from coding theory.

## 1.2 2-Server PIR using Matching Vector Families

In this section, we will see the 2-server PIR construction by Dvir and Gopi [DG16]. We first define what a matching vector family is.

**Definition 2.** *Let  $S \in \mathbb{Z}_m \setminus \{0\}$  and let  $\mathcal{F} = (\mathcal{U}, \mathcal{V})$  where  $\mathcal{U} = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n)$ ,  $\mathcal{V} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$  and for all  $i \in [n]$ ,  $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{Z}_m^k$ . Then  $\mathcal{F}$  is called an  $S$ -matching vector family of size  $n$  and dimension  $k$  if for all  $i, j \in [n]$ ,*

$$\langle \mathbf{u}_i, \mathbf{v}_j \rangle \begin{cases} = 0 & \text{if } i = j \\ \in S & \text{if } i \neq j \end{cases}$$

Matching vector family constructions will not be the focus of this class (if you want to learn more about matching vector families and their applications to coding theory, refer to [DGY11] and chapter 4 of [Y<sup>+</sup>12]), and we will directly use the following result from [Gro00] in our PIR construction.

**Proposition 3** ([Gro00]). *There is an explicitly constructible  $S$ -matching vector family in  $\mathbb{Z}_6^k$  of size  $n \geq \left( \Omega \left( \frac{(\log k)^2}{\log \log k} \right) \right)$  where  $S = \{1, 3, 4\} \subset \mathbb{Z}_6$ .*

Note that  $n = \left( \Omega \left( \frac{(\log k)^2}{\log \log k} \right) \right)$  implies  $k = \exp(O(\sqrt{\log n \log \log n}))$ .

We will work with polynomials over the ring  $\mathcal{R} = \mathcal{R}_{6,6} = \mathbb{Z}_6[\gamma]/(\gamma^6 - 1)$ . We will denote the vector  $(\gamma^{z_1}, \gamma^{z_2}, \dots, \gamma^{z_k})$  by  $\gamma^{\mathbf{z}}$  where  $\mathbf{z} = (z_1, z_2, \dots, z_k) \in \mathbb{Z}_6^k$ . Further for  $\mathbf{y} = (y_1, y_2, \dots, y_k)$ ,  $\mathbf{z} = (z_1, z_2, \dots, z_k)$ , we denote by  $\mathbf{y}^{\mathbf{z}}$  the monomial  $\prod_{i=1}^k y_i^{z_i}$ .

Let  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  be the database and  $(\mathcal{U}, \mathcal{V})$  be a  $\{1, 3, 4\}$ -matching vector family of dimension  $k$  and size  $n$ . Define  $F(\mathbf{y}) \in \mathcal{R}[\mathbf{y}] = \mathcal{R}[y_1, y_2, \dots, y_k]$  given by

$$F(\mathbf{y}) = F(y_1, y_2, \dots, y_k) = \sum_{\ell=1}^n x_{\ell} \mathbf{y}^{\mathbf{u}_{\ell}}.$$

Here is the protocol. Let  $i \in [n]$  be the index the user wants to read.

1. The user picks  $\mathbf{z}$  uniformly at random from  $\mathbb{Z}_6^k$
2. For  $j = 1, 2$ , the user sends  $\mathbf{z} + (j-1)\mathbf{v}_i$  to server  $j$
3. Server  $j$  sends back  $F(\gamma^{\mathbf{z}+(j-1)\mathbf{v}_i})$  and

$$F^{(1)}(\gamma^{\mathbf{z}+(j-1)\mathbf{v}_i}) := \begin{bmatrix} y_1 \frac{\partial F}{\partial y_1} \Big|_{\gamma^{\mathbf{z}+(j-1)\mathbf{v}_i}} \\ y_2 \frac{\partial F}{\partial y_2} \Big|_{\gamma^{\mathbf{z}+(j-1)\mathbf{v}_i}} \\ \vdots \\ y_k \frac{\partial F}{\partial y_k} \Big|_{\gamma^{\mathbf{z}+(j-1)\mathbf{v}_i}} \end{bmatrix}$$

4. Let

$$M := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 1 & \gamma & \gamma^3 & \gamma^4 \\ 0 & \gamma & 3\gamma^3 & 4\gamma^4 \end{bmatrix}$$

Compute the first entry of the matrix

$$M^{-1} \begin{bmatrix} F(\gamma^{\mathbf{z}}) \\ \langle F^{(1)}(\gamma^{\mathbf{z}}, \mathbf{v}_i) \rangle \\ F(\gamma^{\mathbf{z}+\mathbf{v}_i}) \\ \langle F^{(1)}(\gamma^{\mathbf{z}+\mathbf{v}_i}, \mathbf{v}_i) \rangle \end{bmatrix}.$$

Return 0 if it is 0 and 1 otherwise.

Proof of correctness: Define

$$G(j) := F(\gamma^{\mathbf{z}+(j-1)\mathbf{v}_i}) = \sum_{\ell=1}^n x_{\ell} \gamma^{\langle \mathbf{z}, \mathbf{u}_{\ell} \rangle + (j-1)\langle \mathbf{v}_i, \mathbf{u}_{\ell} \rangle}.$$

Using the fact that  $\gamma^6 = 1$  we can write

$$G(j) = \sum_{m=0}^5 c_m \gamma^{(j-1)m},$$

where each  $c_m \in \mathcal{R}$  is given by

$$c_m = \sum_{\ell: \langle \mathbf{u}_{\ell}, \mathbf{v}_i \rangle = m} x_{\ell} \gamma^{\langle \mathbf{z}, \mathbf{u}_{\ell} \rangle}.$$

Since

$$\langle \mathbf{u}_{\ell}, \mathbf{v}_i \rangle \bmod 6 \begin{cases} = 0 & \text{if } \ell = i \\ \in \{1, 3, 4\} & \text{if } \ell \neq i \end{cases}$$

we can conclude that  $c_0 = x_i \gamma^{\langle \mathbf{u}_i, \mathbf{z} \rangle}$  and  $c_2 = c_5 = 0$ . Therefore,

$$G(j) = c_0 + c_1 \gamma^{(j-1)} + c_3 \gamma^{3(j-1)} + c_4 \gamma^{4(j-1)} .$$

Next, consider the polynomial

$$g(T) = c_0 + c_1 T + c_3 T^3 + c_4 T^4 \in \mathcal{R}[T] .$$

By definition,

$$g(\gamma^{j-1}) = G(j) = F(\gamma^{\mathbf{z}+(j-1)\mathbf{v}_i})$$

Further, consider this inner product:

$$\left\langle F^{(1)}(\gamma^{\mathbf{z}+(j-1)\mathbf{v}_i}), \mathbf{v}_i \right\rangle$$

This is equal to

$$\begin{aligned} & \sum_{\ell=1}^n x_\ell \langle \mathbf{u}_\ell, \mathbf{v}_i \rangle \gamma^{\langle \mathbf{z}, \mathbf{u}_\ell \rangle + (j-1) \langle \mathbf{v}_i, \mathbf{u}_\ell \rangle} \\ &= \sum_{m=0}^5 m \left( \sum_{\ell: \langle \mathbf{u}_\ell, \mathbf{v}_i \rangle = m \bmod 6} x_\ell \gamma^{\langle \mathbf{z}, \mathbf{u}_\ell \rangle} \right) \gamma^{(j-1)m} = \sum_{m=0}^5 m c_m \gamma^{(j-1)m} \end{aligned}$$

Therefore

$$\begin{bmatrix} F(\gamma^{\mathbf{z}}) \\ \langle F^{(1)}(\gamma^{\mathbf{z}}), \mathbf{v}_i \rangle \\ F(\gamma^{\mathbf{z}+\mathbf{v}_i}) \\ \langle F^{(1)}(\gamma^{\mathbf{z}+\mathbf{v}_i}), \mathbf{v}_i \rangle \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \\ 1 & \gamma & \gamma^3 & \gamma^4 \\ 0 & \gamma & 3\gamma^3 & 4\gamma^4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_3 \\ c_4 \end{bmatrix} = M \begin{bmatrix} c_0 \\ c_1 \\ c_3 \\ c_4 \end{bmatrix}$$

Note that determinant of  $M$  is non-zero. Further since  $c_0 = x_i \gamma^{\mathbf{u}_i, \mathbf{z}}$  and  $x_i \in \{0, 1\}$ , we have that  $c_0 = 0$  if and only if  $x_i = 0$ . Therefore the first entry of

$$M^{-1} \begin{bmatrix} F(\gamma^{\mathbf{z}}) \\ \langle F^{(1)}(\gamma^{\mathbf{z}}), \mathbf{v}_i \rangle \\ F(\gamma^{\mathbf{z}+\mathbf{v}_i}) \\ \langle F^{(1)}(\gamma^{\mathbf{z}+\mathbf{v}_i}), \mathbf{v}_i \rangle \end{bmatrix}$$

is 0 if and only if  $x_i = 0$ .

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