1 Problem Formulation

Consider an LTI system $\dot{x} = Ax$ where A is partitioned into n-by-n blocks (the diagonal blocks are all square but not necessarily of the same size):

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix},$$

under what condition does there exist a block-diagonally structured

$$P = \begin{bmatrix} P_1 & 0 & \dots & 0 \\ 0 & P_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_n \end{bmatrix}$$

that satisfies the Lyapunov inequality $AP + PA' \prec 0$?

2 A Necessary and Sufficient Condition

The exists a P described in Section 1 if and only if there exists a sequence of $\boldsymbol{P_i} \succ 0$, and $\boldsymbol{\gamma_{ij}} > 0$, $i, j = 1, 2, \dots, n, i \neq j$ such that

$$\boldsymbol{\gamma_{ij}\gamma_{ji}} = 1 \tag{1a}$$
$$A_{ii}\boldsymbol{P_i} + \boldsymbol{P_i}A'_{ii} + \sum_{j\neq i} \boldsymbol{\gamma_{ij}}(A_{ij}A'_{ij} + \boldsymbol{P_i}^2) \prec 0 \tag{1b}$$

are feasible.The matrix power ${P_i}^2 = P_i * P'_i$

Out of the very computational consideration that motivates this work, we would like to point out that the members in constraint (1a) are codependet on each other through the nonconvex constraint (1b). One way to mitigate this non-convexity is to fix $\gamma_{ij} = 1 \forall i \neq j$, and this comes with additional computational gain of they can be equivalently cast as decoupled LMIs, and better yet, solved by a method based on Riccati equations. We'll discuss these details in Section 4.

On the other hand, from a control theoretic viewpoint, we would also like to emphasize that this is a unifying theorem that unveils the structure of P from fully-parameterized one all the way to a diagnoal one, and it gives a explicit procedure to construct one. In particular, when A is not partitioned, the summation part in (1b) goes away, and it reduces to the ordinary Lyapunov inequality; on the other extreme, when A is partitioned into scalar blocks, the constructed P would be a pure diagnoal one.

3 Proof

3.1 Notations

Let us denote $AP + PA'$ as M, so

$$M = \begin{bmatrix} A_{11}P_1 + P_1A'_{11} & A_{12}P_2 + P_1A'_{21} & \dots & A_{1n}P_n + P_1A'_{n1} \\ A_{21}P_1 + P_2A'_{12} & A_{22}P_2 + P_2A'_{22} & \dots & A_{2n}P_n + P_2A'_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}P_1 + P_nA'_{1n} & A_{n2}P_2 + P_nA'_{2n} & \dots & A_{nn}P_n + P_nA'_{nn} \end{bmatrix},$$

and let M_k be the k-th leading principal submatrix of M in the blocks sense. That is, for example, M_1 equals $A_{11}P_1 + P_1A'_{11}$ instead of the first scalar element in M. Also let $\tilde{M}_k$ be the last column-blocks of M_k with its last block element deleted, and then $\tilde{M}'_k$ is obviously last column-blocks of M_k with its last block element deleted.

Additionally, define a sequence of matrices:

$$N_k = \begin{bmatrix} \sum_{j=k+1}^{n} \gamma_{1j}(A_{1j}A'_{1j} + {P_1}^2) & 0 & \dots & 0 \\ 0 & \sum_{j=k+1}^{n} \gamma_{2j}(A_{2j}A'_{2j} + {P_2}^2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sum_{j=k+1}^{n} \gamma_{kj}(A_{kj}A'_{kj} + {P_k}^2) \end{bmatrix}$$

for $k = 1, 2, \dots, n-1$, and $N_k = 0$ for $k = n$. Let $\bar{N}_k$ be N_k with its last row-blocks and last column-blocks deleted (in other words, the biggest pricipal minor in the block sense). It's obvious then by construction that $N_k \succeq 0, \forall k$. Finally, denote the sequence of identity matrices as $\{I_i\}$, where $dim(I_i) = dim(P_i)$.

3.2 Proof of Sufficiency

We will use induction to show that $M_k + N_k \prec 0, \forall k$, so then in the terminal case $k = n$, we would arrive at the desired Lyapunov inequality $M = M_n + N_n \prec 0$. For $k = 1$, $M_1 + N_1 \prec 0$ is trivially guaranteed by taking $i = 1$ in (1b). Suppose $M_k + N_k \prec 0$ for a particular $k \le n-1$, let us now show that $M_{k+1} + N_{k+1} \prec 0$.

First, notice that for $k \le n-1$, the sequence of N_k satisfies this recursive update:

$$N_k = n_k + \bar{N}_{k+1}, \tag{2}$$

where

$$n_k = \begin{bmatrix} \gamma_{1(k+1)}(A_{1(k+1)}A'_{1(k+1)} + {P_1}^2) & 0 & \dots & 0 \\ 0 & \gamma_{2(k+1)}(A_{2(k+1)}A'_{2(k+1)} + {P_2}^2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_{k(k+1)}(A_{k(k+1)}A'_{k(k+1)} + {P_k}^2) \end{bmatrix} \tag{3}$$

Notice also that n_k can be expanded as $n_k = L_k * S_{k+1} * L'_k$ where

$$L_k = \begin{bmatrix} P_1 & 0 & \dots & 0 & A_{1(k+1)} & 0 & \dots & 0 \\ 0 & P_2 & \dots & 0 & 0 & A_{2(k+1)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_k & 0 & 0 & \dots & A_{k(k+1)} \end{bmatrix} \tag{4}$$

and

$$S_{k+1} = blkdiag(\gamma_{1(k+1)}I_1, \gamma_{2(k+1)}I_2, \dots, \gamma_{k(k+1)}I_k, \underbrace{\gamma_{1(k+1)}I_{k+1}, \gamma_{2(k+1)}I_{k+1}, \dots, \gamma_{k(k+1)}I_{k+1}}_{k \text{ such scaled identity matrices}}). \tag{5}$$

From the assumption that $M_k + N_k \prec 0$, we then have $M_k + N_k = M_k + L_k * S_{k+1} * L'_k + \bar{N}_{k+1} \prec 0$. Rearrange the terms:

$$-(M_k + \bar{N}_{k+1}) \succ L_k * S_{k+1} * L'_k \tag{6}$$

Next, by Schur complement, constraint (1b) with $i = k+1$ is equivalent to:

$$\begin{bmatrix} \overline{Cstr}_{k+1} & T_{k+1} \\ \\ T'_{k+1} & -S_{k+1} \end{bmatrix} \prec 0 \tag{7}$$

where $\overline{Cstr}_{k+1} = A_{(k+1)(k+1)}P_{k+1} + P_{k+1}A'_{(k+1)(k+1)} + \sum_{j=k+2}^{n}(A_{(k+1)j}A'_{(k+1)j} + P^2_{k+1})$ is the left hand side of constraint (1b) at $i = k+1$ with the summation truncated to be only over indicies $k+2$ to n (as opposed to be over all indices other than $k+1$), $T_{k+1} = [A_{(k+1)1}, A_{(k+1)2}, \ldots, A_{(k+1)k}, \underbrace{P_{k+1}, P_{k+1}, \ldots, P_{k+1}}_{\text{repeat k times}}]$, and S_{k+1} is as defined in (5) by invoking $\gamma_{ij}\gamma_{ji} = 1$

Use Schur complement on (7) again, this time from the opposite direction, it is also equivalent to:

$$S_{k+1} \succ -T'_{k+1} * \overline{Cstr}^{-1}_{k+1} * T_{k+1} \tag{8}$$

Because $\{P_i\} \succ 0$, L_k has full row-rank. Therefore, pre- and post-multiplying (8) with L_k and L'_k preserves the positive definite order, i.e.:

$$L_k * S_{k+1} * L'_k \succ -L_k * (T'_{k+1} * \overline{Cstr}^{-1}_{k+1} * T_{k+1}) * L'_k \tag{9}$$

Realize that

$$L_k * T'_{k+1} = \begin{bmatrix} A_{1(k+1)}P_{k+1} + P_1A'_{(k+1)1} \\ A_{2(k+1)}P_{k+1} + P_2A'_{(k+1)2} \\ \vdots \\ A_{k(k+1)}P_{k+1} + P_kA'_{(k+1)k} \end{bmatrix}$$

is precisely $\tilde{M}_{k+1}$, and $T_{k+1} * L'_k = \tilde{M}'_{k+1}$.

Finally, combining (6) and (9), we have

$$-(M_k + \bar{N}_{k+1}) \succ -\tilde{M}_{k+1} * \overline{Cstr}_{k+1} * \tilde{M}'_{k+1} \tag{10}$$

By again taking Schur complement, this is equivalent to:

$$\begin{bmatrix} M_k + \bar{N}_{k+1} & \tilde{M}_{k+1} \\ \\ \tilde{M}'_{k+1} & \overline{Cstr}_{k+1} \end{bmatrix} = M_{k+1} + N_{k+1} \prec 0 \tag{11}$$

□

3.3 Proof of Necessity

The proof of necessity is essentially the reverse arguing the sufficiency proof. Specifically, for $k = n, n-1, \ldots, 1$, we'll show the sequence of the set of matrix inequalities

$$\begin{cases} A_{kk}P_k + P_kA'_{kk} + \sum_{j\neq k}\gamma_{kj}(A_{kj}A'_{kj} + P^2_k) \prec 0 \\ A_{ii}P_i + P_iA'_{ii} + \sum_{j\geq k, j\neq i}^{n}\gamma_{ij}(A_{ij}A'_{ij} + P^2_i) \prec 0, \quad \forall i < k \end{cases} \tag{12}$$

hold under the additional constraint $\gamma_{ij}\gamma_{ji} = 1$. Then at the terminal $k = 1$, we'll recover the full set of matrix inequalities (1b).

Before we continue, we'll present two useful facts:

- Fact 1: If $B, C \in \mathbb{S}^{+n\times n}$, then $B \succ C \iff C^{-1} \succ B^{-1} \succ 0$
- Fact 2: If $B \in \mathbb{S}^{+n\times n}$ and $D \in \mathbb{R}^{m\times n}$, then $[B, D]' * ([B, D] * [B, D]')^{-1} * [B, D] \preceq I$

Both facts can be easily proven by Schur complement argument and will be omitted.

For $k = n$, inequality assossiated with A_{nn} is true since we have the freedom of choosing arbitrary $\gamma_{nj} > 0, j \neq n$. The inquality assossiated with $A_{ii}, i < n$

First, since $M \succ 0$, we have

$$
\begin{aligned}
A_{33}P_3 + P_3A_{33}' &\prec (A_{31}P_1 + P_3A_{13}', A_{32}P_2 + P_3A_{23}')M_2^{-1}(A_{31}P_1 + P_3A_{13}', A_{32}P_2 + P_3A_{23}')' \\
&= \\
\iff M_2^{-1} &\prec \begin{bmatrix} -\gamma_{31}(A_{31}A_{31}' + P_3^2) & 0 \\ 0 & -\gamma_{32}(A_{32}A_{32}' + P_3^2) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -\gamma_{13}(A_{31}A_{31}' + P_3^2) & 0 \\ 0 & -\gamma_{23}(A_{32}A_{32}' + P_3^2) \end{bmatrix}^{-1}
\end{aligned} \tag{13}
$$

It's obvious that $M_k \prec 0, \forall k$ and $A_{ij}A_{ij}' + P_i^2 \succ 0, \forall i, j$, hence, there exists $\gamma_{31}, \gamma_{32} > 0$ such that

$$
\begin{aligned}
M_2 &\succ \begin{bmatrix} -\gamma_{31}(A_{31}A_{31}' + P_3^2) & 0 \\ 0 & -\gamma_{32}(A_{32}A_{32}' + P_3^2) \end{bmatrix} \\
\iff M_2^{-1} &\prec \begin{bmatrix} -\gamma_{31}(A_{31}A_{31}' + P_3^2) & 0 \\ 0 & -\gamma_{32}(A_{32}A_{32}' + P_3^2) \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -\gamma_{13}(A_{31}A_{31}' + P_3^2) & 0 \\ 0 & -\gamma_{23}(A_{32}A_{32}' + P_3^2) \end{bmatrix}^{-1}
\end{aligned} \tag{14}
$$

4 Computational Details

By Schur complement, constraints (1b) can be equivalently cast as decoupled LMIs:

$$
\begin{bmatrix} A_{ii}\boldsymbol{P_i} + \boldsymbol{P_i}A_{ii}' + \sum_{j\neq i} A_{ij}A_{ij}' & \sqrt{n-1}\boldsymbol{P_i} \\ \\ \sqrt{n-1}\boldsymbol{P_i}' & -I_i \end{bmatrix} \prec 0 \tag{15}
$$

This comes at the cost of "inflating" the decision variable, in that if P_i is an m-by-m matrix, (15) is a 2m-by-2m sized LMI.

If no other constraints or cost of the system depend on P_i, a much more efficient alternative exists to check the feasibility of (15). By bounded-real lemma, (15) are feasible if and only if for each i, the Riccati equation

$$
A_{ii}\boldsymbol{P_i} + \boldsymbol{P_i}A_{ii}' + \sum_{j\neq i} A_{ij}A_{ij}' + (n-1)\boldsymbol{P_i}\boldsymbol{P_i} = 0 \tag{16}
$$

has a unique positive definite solutions P_i^{are}. That is, P_i^{are} is on the boundary of the set of solutions P_i to (15) in which $P_i \succ P_i^{are}$. Note that Riccati equations are generally easier to solve than LMIs (when the size are the same), and here, the Riccati (16) are only half the size of the LMIs'. Therefore, the Riccati approach could potentially offer a even more significant computational advantage, which is the very motivation behind this work.

5 Additional Notes

- There is in fact a more general sufficient condition, which is the feasibility of

$$
A_{ii}\boldsymbol{P_i} + \boldsymbol{P_i}A_{ii}' + \sum_{j\neq i} \boldsymbol{\gamma_{ij}}(A_{ij}A_{ij}' + \boldsymbol{P_i}\boldsymbol{P_i}) \prec 0 \tag{17}
$$

subject to $\boldsymbol{P_i} \succ 0$, $\boldsymbol{\gamma_{ij}} > 0$, $\boldsymbol{\gamma_{ij}\gamma_{ji}} = 1$, $i, j = 1, 2, \ldots, n, i \neq j$.

This is more general a condition than (1b) because (1b) is just a special case of (17) with fixed $\gamma_{ji} = 1, \forall i \neq j$, so this offers potentially a larger solution set. The proof for the sufficiency of this "scaled" version (17) is also very similar to that of (1b) in Section 3. The only downside of this setup is that all the quadratic inequalities are co-dependent and the codependency comes as $\boldsymbol{\gamma_{ij}\gamma_{ji}} = 1$ that is non-convex.

- We conjecture that (17) is also a necessary condition.