Columbia University
IEOR4703 – Monte Carlo Simulation (Hirsa)
Assignment 1 – Due 17:40 on Tuesday Feb 6th, 2018

Problem 1 (Sampling from tail distributions via Inverse Transform): Assume we can easily sample from $\mathbf{U}(0,1)$. Also assume that we can easily calculate both $\Phi(x)$, the cumulative distribution function of $\mathcal{N}(0,1)$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du,$$

and its inverse $\Phi^{-1}(x)$. Show how to sample from the tail of the normal distribution on $[-\infty, a]$ using inverse transform:

$$g(x) = \frac{1}{\sqrt{2\pi}A\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

where $A = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$.

Problem 2 (Sampling via Inverse Transform & Acceptance-Rejection): Suppose the following is the probability density function of the random variable X.

$$f(x) = \begin{cases} 0 & : \quad x < -10 \\ \kappa(x+10) & : \quad -10 \le x \le 0 \\ \kappa(10-x) & : \quad 0 \le x \le 10 \\ 0 & : \quad x > 10 \end{cases}$$

(a) What is the value of κ?

(b) Describe in detail the inverse transform method to generate a sample of X given a uniform $U(0,1)$ random number generator.

(c) Describe in detail an *acceptance-rejection* algorithm for generating a sample of X given a uniform $U(0,1)$ random number generator. How many uniform random variables on average will be required to generate one sample of X?

Problem 3 (Sampling via Inverse Transform): Suppose the following is the probability density function of the random variable X.

$$f(x) = \begin{cases} \kappa e^{x} & : \quad x \le 0 \\ \kappa e^{-x} & : \quad x > 0 \end{cases}$$

(a) What is the value of κ?

(b) Describe in detail the inverse transform method to generate a sample of X given a uniform $U(0,1)$ random number generator.

Problem 4 (Monte Carlo Simulation): Estimate the following integral

$$\theta = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

via Monte-Carlo integration. Hint: First convert it to a definite integral by change of a variable. Compare your estimate with the exact solution.

Problem 5 (Sampling from non-homogeneous Poisson distribution): There are many algorithms for generating a sample from non-homogeneous Poisson distribution. Give a detailed pseudo-code for one of them. One could be the extension of the method explained for the homogeneous case.

Problem 6 (Cost of delta hedging shorting a call option in Black-Merton-Scholes Model: In the Black-Merton-Scholes model for $r = q = 0$ we show the initial call price is the same as cost of continuous delta hedging plus selling the stock and payoff if any at maturity i.e.

$$C = \Delta_{t_0} \times S_0 + \sum_{i=0}^{n-1} (\Delta_{t_i} - \Delta_{t_{i-1}}) S_{t_i} + (S_T - K)^+ - \Delta_{t_{n-1}} \times S_T$$

where Δ_{t_i} and S_{t_i} are the delta of the call option and stock price at time t_i respectively for $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$. Extend this argument for the case that $r > 0$ and $q > 0$ by looking into investing and borrowing for each transaction and also dividends would be received on the stock positions holding in each time interval $[t_{i-1}\ t_i]$.