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HW2

Columbia University
IEOR4703 – Monte Carlo Simulation (Hirsa)
Assignment 2 – Due 17:40 on Tuesday Feb 20th, 2018

Problem 1 (CI based on Chebyshev's Inequality): Chebyshev's Inequality states that for a random variable X with expectation $\mathbb{E}(X) = \theta$ and for any $\alpha > 0$,

$$P(|X - \theta| \geq \alpha) \leq \frac{\text{Var}(X)}{\alpha^2}$$

In our case we would write it as

$$P(|\hat{\theta}_n - \theta| \geq \alpha) \leq \frac{\text{Var}(\hat{\theta}_n)}{\alpha^2}$$

where $\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} h(X_i)$ is the sample mean. Use the inequality to construct confidence intervals for θ. Show that it generally yields to conservative confidence intervals.

Problem 2 (CLT vs. Bootstrap): Consider a European call option under Black-Merton-Scholes model with risk-free rate $r = 0.05$, continuous divided rate of $q = 0.01$, volatility of $\sigma = 0.30$, maturity of $T = 1$ year, spot price of $S_0 = \$100$, and strike price of $K = \$90$.

(a) Use standard simulation to estimate the option price and compare it with the exact solution (use 10,000 simulation paths

(b) Find approximate 95% confidence interval based on Central limit theorem.

(c) Find approximate 95% confidence interval based on bootstrap approach.

(You should include your codes and print out the results)

Problem 3 (Antithetic Variates): Consider the following average price call

$$\theta = e^{-rT}\mathbb{E}[(\overline{S} - K)^+]$$

with $\overline{S} = \frac{1}{m}\sum_{i=1}^{m} S_{t_i}$ for $t_i = i\Delta t$ where $\Delta t = T/m$. Assume S_t follows a risk-neutral probability measure $GBM(r - q, \sigma)$ with risk-free rate $r = 0.04$, continuous divided rate of $q = 0.015$, and volatility of $\sigma = 0.30$. Assume maturity of $T = 1$ year, monitoring times of $m = 12$, spot price of $S_0 = \$1,000$, and strike price of $K = \$1,100$.

(a) Use standard simulation to estimate the option price and find approximate 99% confidence interval.

(b) Use simulation w/ antithetic variates to estimate the option price and find approximate 99% confidence interval.

Use 50,000 simulation paths in your estimations (You should include your codes and print out the results).

Problem 4 (Control Variates): Consider the following average price call

$$\theta = e^{-rT}\mathbb{E}[(\overline{S} - K)^+]$$

with $\overline{S} = \frac{1}{m}\sum_{i=1}^{m} S_{t_i}$ for $t_i = i\Delta t$ where $\Delta t = T/m$. Assume S_t follows a risk-neutral probability measure $GBM(r - q, \sigma)$ with risk-free rate $r = 0.04$, continuous divided rate of $q = 0.015$, and volatility of $\sigma = 0.30$. Assume maturity of $T = 1$ year, monitoring times of $m = 12$, spot price of $S_0 = \$1,000$, and strike price of $K = \$1,100$.

(a) Use standard simulation to estimate the option price and find approximate 99% confidence interval.

(b) Set $Z = S_T$ as a control variate to estimate the option price and find approximate 99% confidence interval.

(c) Set $Z = \exp(-rT)\max(S_T - K, 0)$ as a control variate to estimate the option price and find approximate 99% confidence interval.

(d) Set $Z = \exp(-rT)\max(\sqrt[m]{\prod_{i=1}^{m} S_{t_i}} - K, 0)$ as a control variate to to estimate the option price and find approximate 99% confidence interval. Note that for the geometric average price the expectation is known (i.e. $\mathbb{E}(Z)$

Use 50,000 simulation paths in your estimations. From your results conclude whine one results in a substantial variance reduction (You should include your codes and print out the results).

P1.

$$P(|\hat{\theta}_n - \theta| \geq k) \leq \frac{Var(\hat{\theta}_n)}{k^2} = \alpha \quad , \quad k = \frac{\sqrt{Var(\hat{\theta}_n)}}{\sqrt{\alpha}}$$

$$P(-k \leq |\hat{\theta}_n - \theta| \leq k) = 1-\alpha$$

$$\Rightarrow \left[-\sqrt{\frac{Var(\hat{\theta}_n)}{\alpha}} + \hat{\theta}_n \ , \ \sqrt{\frac{Var(\hat{\theta}_n)}{\alpha}} + \hat{\theta}_n \right]$$

is the $100(1-\alpha)\%$ confidence interval for θ

According to Central Limit Theorem : assume $Var(X_i) = \sigma^2$, $Var(\hat{\theta}_n) = \frac{\sigma^2}{n}$

$$\frac{\hat{\theta}_n - \theta}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1) \quad \text{as} \quad n \to \infty$$

$$\Rightarrow \quad P(-z_{1-\frac{\alpha}{2}} \leq Z \leq z_{1-\frac{\alpha}{2}}) = 1-\alpha \quad \text{where } Z \sim N(0,1)$$

$$P\left(-z_{1-\frac{\alpha}{2}} \leq \frac{\hat{\theta}_n - \theta}{\sigma/\sqrt{n}} \leq z_{1-\frac{\alpha}{2}}\right) = 1-\alpha$$

$$P\left(-\frac{\sigma z_{1-\frac{\alpha}{2}}}{\sqrt{n}} + \hat{\theta}_n \leq \theta \leq \frac{\sigma \cdot z_{1-\frac{\alpha}{2}}}{\sqrt{n}} + \hat{\theta}_n\right) = 1-\alpha$$

$$\left[-\frac{\sigma z_{1-\frac{\alpha}{2}}}{\sqrt{n}} + \hat{\theta}_n \ , \ \frac{\sigma \cdot z_{1-\frac{\alpha}{2}}}{\sqrt{n}} + \hat{\theta}_n \right]$$

is the $100(1-\alpha)\%$ confidence interval for θ when n is large

comparing $\sqrt{\frac{1}{\alpha}}$ & $z_{1-\frac{\alpha}{2}}$ from normal table

$$\sqrt{\frac{1}{\alpha}} > z_{1-\frac{\alpha}{2}} \quad \forall \alpha \in [0,1]$$

$\Rightarrow$ CI based on Chebyshe's Inequality is conservative CI

P2.

```
>> hw2_q2

ans =

    "Standard simulation price is 19.227"


ans =

    "Exact solution price is 18.959"


ans =

    "Confidence Interval based on Central Limit Theorem is [18.740, 19.713]"


ans =

    "Confidence Interval based on Bootstrap is [18.791, 19.663]"
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(See attached code)

P3

```
>> hw2_q3

ans =

    "Standard simulation price is 40.721"


ans =

    "Standard simulation Confidence Interval is [39.656, 41.787]"


ans =

    "Simulation with antithetic variates price is 40.648"


ans =

    "Simulation with antithetic variates Confidence Interval is [39.691, 41.605]"
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(See attached code)

P_r

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ans =

    "Standard simulation price is 41.480"


ans =

    "Standard simulation Confidence Interval is [40.397, 42.562]"


ans =

    "(b) Simulation with Control Variates price is 41.330"


ans =

    "(b) Simulation with Control Variates Confidence Interval is [40.587, 42.074]"


ans =

    "(c) Simulation with Control Variates price is 41.005"


ans =

    "(c) Simulation with Control Variates Confidence Interval is [40.411, 41.599]"


ans =

    "(d) Simulation with Control Variates price is 40.695"


ans =

    "(d) Simulation with Control Variates Confidence Interval is [40.640, 40.751]"
```

(d) results in a substantial variance reduction since arithmetic mean has high covarian with geometric mean.

(b) $S_t = S_0 \cdot e^{(r-q-\frac{1}{2}\sigma^2)t + \sigma B_t}$

$$E[S_t] = S_0 \cdot e^{(\mu-q-\frac{1}{2}\sigma^2)t} \, E[e^{\sigma B_t}]$$

$B_t \sim N(0,t)$

$$= S_0 \cdot e^{(\mu-q-\frac{1}{2}\sigma^2)t} \cdot e^{\frac{1}{2}\sigma^2 t} \Leftarrow \text{Moment Generating Function}$$

$$= S_0 \cdot e^{(\mu-q)t}$$

$E(e^{xk}) = e^{\mu k} \cdot e^{\frac{1}{2}\sigma^2 k^2}$

$$\Rightarrow E[Z] = E[S_T] = S_0 \cdot e^{(\mu-q)T}$$

(c) $E[e^{-rT}\max(S_T-K,0)]$ is the price given by Black Scholes formula

where $E[e^{-rT}\max(S_T-K,0)] = S N(d_1) - N(d_2) K e^{-rT}$

$$d_1 = \frac{\log(S/K) + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

(d) $S_{t_i} = S_0 \cdot e^{(r-q-\frac{1}{2}\sigma^2)t_i + \sigma B_{t_i}}$

$$\prod_{i=1}^{m} S_{t_i} = S_0 \cdot e^{(r-q-\frac{1}{2}\sigma^2)\sum_{i=1}^{m} t_i} \cdot e^{\sigma \sum_{i=1}^{m} B_{t_i}}$$

$$\sum_{i=1}^{m} B_{t_i} = m B_{t_1} + (m-1)(B_{t_2}-B_{t_1}) + (m-2)(B_{t_3}-B_{t_2}) + \cdots + (m-(m-1))(B_{t_m}-B_{t_{m-1}})$$

$$B_{t_i} - B_{t_{i-1}} \sim N(0, dt), \quad dt = \frac{T}{m}$$

$$\Rightarrow \sum_{i=1}^{m} B_{t_i} = \sqrt{\frac{T}{m}} \cdot \sum_{i=1}^{m}((m+1-i) Z_i), \quad Z_i \sim N(0,1) \quad \forall i \in [1,m]$$

$$\Rightarrow \left(\prod_{i=1}^{m} S_{t_i}\right)^{\frac{1}{m}} = S_0 \cdot e^{(r-q-\frac{1}{2}\sigma^2)\cdot\frac{(m+1)T}{2m}} \cdot e^{\frac{\sigma\sqrt{T}}{m\sqrt{m}}\cdot\sum_{i=1}^{m}((m+1-i)Z_i)}$$

$$\log\left(\left(\prod_{i=1}^{m} S_{t_i}\right)^{\frac{1}{m}}\right) = \log S_0 + (r-q-\frac{1}{2}\sigma^2)\cdot\frac{(m+1)T}{2m} + \frac{\sigma\sqrt{T}}{m\sqrt{m}}\sum_{i=1}^{m}((m+1-i)Z_i)$$

$(\prod^{m} S_{t_i})^{\frac{1}{m}}$ follow lognormal distribution

$$E\left[\log\left(\left(\prod_{i=1}^{m} S_{t_i}\right)^{\frac{1}{m}}\right)\right] = \log S_0 + (r-q-\frac{1}{2}\sigma^2)\cdot\frac{(m+1)T}{2m}$$

$$Var\left[\log\left(\left(\prod_{i=1}^{m} S_{t_i}\right)^{\frac{1}{m}}\right)\right] = \frac{\sigma^2 T}{m^3}\cdot\sum_{i=1}^{m}(m+1-i)^2 Var(Z_i) = \frac{\sigma^2 T}{m^3}\sum_{i=1}^{m} i^2$$

$$= \frac{\sigma^2 T(m+1)(2m+1)}{6m^2}$$

$$\Rightarrow \left(\prod_{i=1}^{m} S_{t_i}\right)^{\frac{1}{m}} = S_0 \cdot e^{(r-q-\frac{1}{2}\sigma^2)(m+1)T/2m + \frac{\sigma}{m}\sqrt{T(m+1)(2m+1)/6}\cdot Z}$$

$$= S_0 \cdot e^{(r-q-\frac{1}{2}\sigma^2)(m+1)T/2m + \frac{\sigma}{m}\sqrt{(m+1)(2m+1)/6} \cdot B_T}$$

$$\Rightarrow \quad E\left[e^{-rT} \cdot \max\left(\left(\prod_{i=1}^{m} S_{t_i}\right)^{\frac{1}{m}} - K,\ 0\right)\right]$$

can be derived using Black Scholes formula

with

$$\sigma_{new} = \frac{\sigma}{m}\sqrt{(m+1)(2m+1)/6}$$

$$\sigma_{new}^2 = \frac{\sigma^2}{m^2} \cdot \frac{(m+1)(2m+1)}{6}$$

$$q_{new} = \frac{m-1}{2m} \cdot r + \frac{m+1}{2m} \cdot q + \frac{m+1}{4m}\sigma^2 - \frac{1}{2}\sigma_{new}^2$$

$$r_{new} = r$$

$$T_{new} = T$$

$$S_{pot} = 1000$$

$$K = 1100$$