Columbia University
IEOR4703 – Monte Carlo Simulation (Hirsa)
Assignment 3 – Due 17:40 on Tuesday March 6th, 2018

Problem 1 (Conditional Monte Carlo): Consider the following probability

$$\theta = P(X_1 + X_2 > 8)$$

where $X_1 \sim \text{Exp}(2)$ and $X_2 \sim \text{Exp}(1)$. For $X \sim \text{Exp}(\lambda)$, the corresponding CDF is given by $F(x) = 1 - e^{-\lambda x}$.

(a) Use standard simulation to estimate it.

(b) Use Conditional Monte Carlo to estimate it.

(c) Find approximate 99% confidence intervals in both cases and compare.

Use 10,000 draws/samples (You should include your codes and print out the results).

Problem 2 (Conditional Monte Carlo): Assume that the stock price follows $GBM(r-q,\sigma)$. We would like to calculate the price of an option that has the following payoff:

$$h(X) = \begin{cases} (\lambda_1 S_{\frac{T}{2}} - S_T)^+ & : \text{ if } S_{\frac{T}{2}} < H \\ (S_T - \lambda_2 S_{\frac{T}{2}})^+ & : \text{ if } S_{\frac{T}{2}} \geq H \end{cases}$$

where $X = (S_{\frac{T}{2}}, S_T)$. The price of the option can be written as

$$\begin{aligned} P(S_0, t=0; \lambda_1, \lambda_2, H, T) &= e^{-rT}\mathbb{E}\left(h(X)\right) \\ &= e^{-rT}\mathbb{E}\left(\mathbf{1}_{S_{\frac{T}{2}}<H}\{(\lambda_1 S_{\frac{T}{2}} - S_T)^+\} + \mathbf{1}_{S_{\frac{T}{2}}\geq H}\{(S_T - \lambda_2 S_{\frac{T}{2}})^+\}\right) \end{aligned}$$

For the following parameter set: spot price of $S_0 = 100$, risk-free rate $r = 0.04$, continuous divided rate of $q = 0.015$, volatility of $\sigma = 0.30$, maturity of $T = 1$ year, level $H = 110$, $\lambda_1 = 0.9$, and $\lambda_2 = 1.1$ price the option by utilizing

(a) standard simulation to estimate the option price and find approximate 99% confidence interval.

(b) conditional Monte-Carlo simulation to estimate the option price and find approximate 99% confidence interval.

Use 20,000 simulation paths in your estimations (You should include your codes and print out the results).

Problem 3 (Importance Sampling): Assume that the stock price follows $GBM(r-q,\sigma)$. We would like to calculate the price of deep-out-of-money (European) put for the following parameter set: spot price of $S_0 = \$200$, strike price of $K = 120$, risk-free rate $r = 0.05$, continuous divided rate of $q = 0.025$, volatility of $\sigma = 0.35$, and maturity of $T = 1$ year.

(a) Use standard simulation to estimate the option price and find approximate 99% confidence interval.

(b) Use importance sampling by specifying g to estimate the option price and find approximate 99% confidence interval.

Use 20,000 simulation paths in your estimations. Compare your results with the exact solution (You should include your codes and print out the results).

Problem 4 (Stratified Sampling): Assume that the stock price follows $GBM(r-q,\sigma)$. We would like to calculate the price of deep-out-of-money (European) put for the following parameter set: spot price of $S_0 = \$200$, strike price of $K = 120$, risk-free rate $r = 0.05$, continuous divided rate of $q = 0.025$, volatility of $\sigma = 0.35$, and maturity of $T = 1$ year.

(a) Use standard simulation to estimate the option price and find approximate 99% confidence interval and compare it with the exact solution.

(b) Use stratified sampling using sub-optimal choice for n_i to estimate the option price and find approximate 99% confidence interval.

(c) Use stratified sampling using optimal choice for n_i to estimate the option price and find approximate 99% confidence interval.

Use 20,000 simulation paths in your estimations. Compare your results with the exact solution (You should include your codes and print out the results).

HW3

You Wu
yw3027

Q1. For $i = 1, \cdots, n$

generate u_1, u_2 from $U(0,1)$

$x_1 = -\frac{1}{2}\log(1-u_1)$

$x_2 = -\log(1-u_2)$

if $x_1 + x_2 > 8$

$X_i = 1$

else

$X_i = 0$

$\hat{\theta} = \bar{X}$

```
>> hw3_q1

ans =

    "Standard simulation estimation: 0.000700"


ans =

    "Standard simulation estimation Confidence Interval: [0.000019, 0.001381]"


ans =

    "Conditional Monte Carlo simulation estimation: 0.000670"


ans =

    "Conditional Monte Carlo simulation estimation Confidence Interval: [0.000649, 0.000691]"
```

Q2. let $V = E[h(x) \mid S_{\frac{T}{2}}] \cdot e^{-rT}$

$$\Rightarrow \quad V = \begin{cases} e^{-r\frac{T}{2}} \cdot E[(\lambda_1 S_{\frac{T}{2}} - S_T)^+] & , S_{\frac{T}{2}} < H \\ e^{r\frac{T}{2}} \cdot E[(S_T - \lambda_2 \cdot S_{\frac{T}{2}})^+] & , S_{\frac{T}{2}} \geq H \end{cases}$$

$$= \begin{cases} e^{-r\frac{T}{2}} \cdot \text{Put}(S_{\frac{T}{2}}, \lambda_1 \cdot S_{\frac{T}{2}}, r, \frac{T}{2}, \sigma, q) \\ e^{-r\frac{T}{2}} \cdot \text{Call}(S_{\frac{T}{2}}, \lambda_2 \cdot S_{\frac{T}{2}}, r, \frac{T}{2}, \sigma, q) \end{cases}$$

```
>> hw3_q2

ans =

    "Standard simulation estimation: 4.189814"


ans =

    "Standard simulation estimation Confidence Interval: [4.016237, 4.363390]"


ans =

    "Conditional Monte Carlo simulation estimation: 4.158923"


ans =

    "Conditional Monte Carlo simulation estimation Confidence Interval: [4.130098, 4.187747]"
```

Q3. $\hat{\theta} = e^{-rT} E[\mathbb{1}_{\{S_T < K\}} (K - S_T)^+]$

$$S_T = S_0 \cdot \exp((r - q - \frac{\sigma^2}{2})T + \sigma\sqrt{T} \cdot Z)$$

$$Z \sim N(0,1)$$

$$\mathbb{1}(S_T < K) \Rightarrow \mathbb{1}\left(Z < \frac{\log(\frac{K}{S_0}) - (r - q - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right)$$

$$\Rightarrow h(Z) = \mathbb{1}_{\left\{Z < \frac{\log(\frac{K}{S_0}) - (r - q - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right\}} \cdot (K - S_T)$$

$\Rightarrow g \sim N(\mu, 1)$

let $\mu = \dfrac{\log(\frac{K}{S_0}) - (r - q - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$

$$\Rightarrow \hat{\theta} = e^{-rT} E_g[h(Z) \cdot e^{-\mu Z + \frac{\mu^2}{2}}]$$

```
>> hw3_q3

ans =

    "Standard simulation estimation: 1.439030"


ans =

    "Standard simulation estimation Confidence Interval: [1.330179, 1.547881]"


ans =

    "Importance Sampling Estimation: 1.400408"


ans =

    "Importance Sampling Estimation Confidence Interval: [1.371677, 1.429138]"
```

```
>> hw3_q4

ans =

    "Exact price: 1.410054"


ans =

    "Standard simulation estimation: 1.446294"


ans =

    "Standard simulation estimation Confidence Interval: [1.338866, 1.553722]"
```

```
ans =

    "Sub-optimal Stratified Sampling: 1.419257"


ans =

    "Sub-optimal Stratified Sampling Confidence Interval: [1.368329, 1.470185]"


ans =

    "Optimal Stratified Sampling: 1.415342"


ans =

    "Optimal Stratified Sampling Confidence Interval: [1.395145, 1.435539]"
```