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hw6

Columbia University
IEOR4703 – Monte Carlo Simulation (Hirsa)
Assignment 6 – Due 17:40 on Tuesday April 17th, 2018

Problem 1 (Simulation of Continuous-Time Markov Chain): Consider a Continuous-Time Markov Chain with finite states $S = \{0, 1, .., N\}$ and some given transition matrix. Extend the Matlab code *exampleMarkovChain.m* for this Continuous-Time Markov Chain. Assume the waiting time, H_i has an exponential distribution at rate λ_i. The input to the code should be: (a) initial state, (b) T, (c) transition probability matrix P which is an $(N+1)\times(N+1)$ matrix and (d) λ_i for $i = 0, \ldots, N$. Output would be (a) t_j switching times and (b) states at those times.

Problem 2 (Markov Chain & Betting): Mr. Johnson is in jail and has \$5. He can get out on bail if he has \$10. A guard agrees to make a series of bets with him. If Mr. Johnson bets A dollars, he wins A dollars with 35% probability and loses A dollars with 65% probability. Find the probability that he wins \$10 before losing all of his money if

(a) he bets \$1 each time (conservative strategy).

(b) each time he bets as much as possible but never more than necessary to bring his fortune up to \$10 (aggressive strategy).

(c) Which strategy gives Mr. Johnson a better chance of getting out of jail?

Problem 3 (Markov Chain): Consider a Markov chain with states $S = \{0, 1, .., N\}$ and transition probabilities $p_{i,i+1} = p$, $p_{i,i-1} = 1-p$, for $1 \le i \le N-1$ and $0 < p < 1$. Assume $p_{0,1} = 1$, $p_{N,N-1} = 1$.

(a) Draw the transition diagram

(b) Is the Markov chain irreducible?

(c) Is it aperiodic?

(d) Find the stationary distribution

(e) Is it reversible?

Problem 4 (Markov Chain & Patterns): A fair coin is tossed repeatedly and independently. Find the expected number of tosses until the pattern THHT appears.

P1. Please see the attached code.

P2. Let τ_x be the time Johnson has x dollars

P_i be the probability that $\tau_{10} < \tau_0$ if Johnson has i dollars now

(a) $\Rightarrow$ $P_i = \underset{\text{win } (i \text{ to } i+1)}{0.35 \cdot P_{i+1}} + \underset{\text{lose } (i \text{ to } i-1)}{0.65\, P_{i-1}}$

$P_{10} = 1$, $P_0 = 0$, find P_5

$$P_{i+1} - P_i = \frac{0.65}{0.35}(P_i - P_{i-1})$$

$$\begin{cases} P_{10} - P_9 = \left(\frac{0.65}{0.35}\right)^9 P_1 \\ \vdots \\ P_5 - P_4 = \frac{0.65}{0.35}(P_4 - P_3) = \left(\frac{0.65}{0.35}\right)^2 (P_3 - P_2) = \left(\frac{0.65}{0.35}\right)^3 (P_2 - P_1) = \left(\frac{0.65}{0.35}\right)^4 P_1 \\ P_1 - P_0 = P_1 \end{cases}$$

$$\Rightarrow P_{10} = \left(\left(\frac{0.65}{0.35}\right)^0 + \cdots + \left(\frac{0.65}{0.35}\right)^9\right) P_1 = \frac{1 - \left(\frac{0.65}{0.35}\right)^{10}}{1 - \frac{0.65}{0.35}} P_1$$

$$\Rightarrow P_1 = 0.001760$$

$$\Rightarrow P_5 = \frac{1 - \left(\frac{0.65}{0.35}\right)^5}{1 - \frac{0.65}{0.35}} P_1 = 0.04331$$

(b) When Johnson has 5 he will bet 5 to get to \$10

$\Rightarrow P_5 = 0.35$

(c) aggressive givens better chance $P_{(b)} = 0.35 > P_{(a)} = 0.04331$

P3. (a)

$$\begin{bmatrix} 0 & 1 & 0 & & & \cdots & 0 \\ 1-p & 0 & p & 0 & & \cdots & 0 \\ 0 & 1-p & 0 & p & 0 & \cdots & 0 \\ 0 & 0 & 1-p & 0 & p & 0 \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & & 0 & 1-p & 0 & p \\ 0 & \cdots & & & 0 & 1 & 0 \end{bmatrix} = P$$

(b) since $\forall i, j \in S$, it is possible for i to go to j within finite step / time

$\exists n < \infty$, $P_{ij}^{(n)} > 0$

$\Rightarrow$ $\exists m < \infty$, $P(X_{n+m} = j \mid X_n = i) > 0$ $\Rightarrow$ irreducible

(c) $\forall i \in S$, $P_{ii}^{(n)} > 0$ when n is even, $P_{ii}^{(n)} = 0$ when n is odd

$\Rightarrow$ period $d_i = 2$

$\Rightarrow$ not aperiodic

(d) $\pi = \pi \cdot P$, $\sum_{i=0}^{N} \pi_i = 1$

$\pi_0 = \pi_0 \cdot 0 + \pi_1 (1-p)$

$\pi_1 = \pi_0 \cdot 1 + \pi_2 (1-p)$

$\vdots$

$\pi_i = \pi_{i-1} \cdot p + \pi_{i+1} (1-p)$

$\vdots$

$\pi_{N-1} = \pi_{N-2} \cdot p + \pi_N \cdot 1$

$\pi_1 = \frac{1}{1-p} \pi_0$

$\pi_2 = \frac{1}{1-p} (\pi_1 - \pi_0) = \frac{p}{(1-p)^2} \pi_0$

$\pi_3 = \frac{p^2}{(1-p)^3} \pi_0$

$\pi_i = \frac{p^{i-1}}{(1-p)^i} \pi_0 \quad \forall i = 2, \cdots, N-1$

$$\pi_N = \pi_{N-1} \cdot p \qquad \Rightarrow \qquad \pi_N = \frac{p^{N-1}}{(1-p)^{N-1}} \pi_0$$

$$\Rightarrow \left(1 + \frac{1}{1-p} + \left(\frac{p}{(1-p)^2} + \cdots + \frac{p^{N-2}}{(1-p)^{N-1}} \right) + \frac{p^{N-1}}{(1-p)^{N-1}} \right) \pi_0 = 1$$

$$\text{if } p \neq 0.5 \Rightarrow \left(1 + \frac{1}{(1-p)} \left(\frac{1 - \left(\frac{p}{1-p}\right)^{N-1}}{1 - \frac{p}{1-p}} \right) + \frac{p^{N-1}}{(1-p)^{N-1}} \right) \pi_0 = 1$$

$$p \neq 0.5 \begin{cases} \pi_0 = \dfrac{(1-p)^{N-1}(1-2p)}{2(1-p)^N - 2p^N} \\ \forall i = 1, \cdots, N-1, \quad \pi_i = \dfrac{(1-p)^{N-1}(1-2p)}{2(1-p)^N - 2p^N} \cdot \dfrac{(p)^{i-1}}{(1-p)^i} \\ \pi_N = \dfrac{(1-p)^{N-1}(1-2p)}{2(1-p)^N - 2p^N} \cdot \dfrac{(p)^{N-1}}{(1-p)^{N-1}} \end{cases}$$

$$\text{if } p = 0.5 \Rightarrow (1 + 2 + 2 + \cdots + 2 + 1)\, \pi_0 = 1$$

$$p = 0.5 \begin{cases} \pi_0 = \dfrac{1}{2N} \\ \forall i = 1, \cdots, N-1, \quad \pi_i = \dfrac{1}{N} \\ \pi_N = \dfrac{1}{2N} \end{cases}$$

(e) Reversible $\Rightarrow \quad \pi_i P_{ij} = \pi_j P_{ji} \quad \forall i, j$

① $\forall i = j$. $P_{ij} = 0 \Rightarrow \pi_i P_{ij} = \pi_j P_{ji}$

② $\forall |i-j| \geq 2$, $P_{ij} = 0 \Rightarrow \pi_i P_{ij} = \pi_j P_{ji}$

③ $\forall |i-j| = 1$, (1) $i = 0, j = 1$, $P_{ij} = 1, P_{ji} = 1-p \Rightarrow \pi_i P_{ij} = \pi_j P_{ji}$

(2) $i = 1 \cdots, N-1$, $j = i+1$ $\quad P_{ij} = p$, $P_{ji} = 1-p$, $\pi_j = \pi_i \cdot \frac{p}{1-p}$

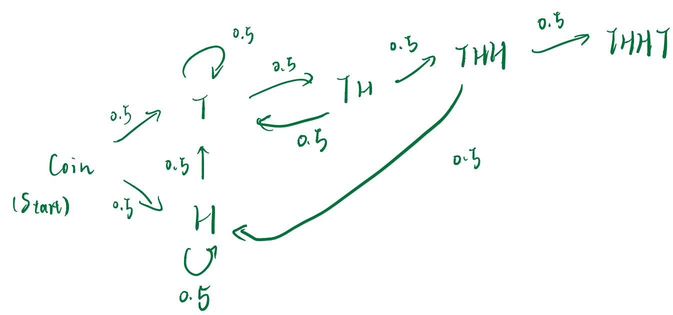
(Because of symmetry, $j = i-1$ is also covered in (2)) $\Rightarrow \pi_i P_{ij} = \pi_j P_{ji}$

(3) $i = N$, $j = N-1$ $\quad P_{ij} = 1$, $P_{ji} = p$, $\pi_N = p\,\pi_{N-1}$

$\Rightarrow \pi_i P_{ij} = \pi_j P_{ji}$

so $\Rightarrow \pi_i P_{ij} = \pi_j P_{ji} \Rightarrow$ reversible

Pr.



let $u_i = E[\text{THHT} \mid \text{current state} = i]$: expected number of toss until THHT if current state is i

$u_S = 0.5(1 + u_T) + 0.5(1 + u_H)$

$u_T = 0.5(1 + u_T) + 0.5(1 + u_{TH}) \Rightarrow u_T = 2 + u_{TH} \Rightarrow u_{TH} = u_T - 2$

$u_H = 0.5(1 + u_H) + 0.5(1 + u_T) \Rightarrow u_H = 2 + u_T$

$u_{TH} = 0.5(1 + u_T) + 0.5(1 + u_{THH}) \Rightarrow u_{THH} = u_T - 6$

$u_{THH} = 0.5(1 + u_H) + 0.5(1 + u_{THHT}) \Rightarrow u_H = 2u_T - 14$

$u_{THHT} = 0$ $\qquad u_T = 16$, $u_H = 18$

$\Rightarrow u_s = 0.5(16+1) + 0.5(18+1) = 18 \Rightarrow$ expected # of tosses