IEOR 4703 - Monte Carlo Simulation
Solutions for Assignment 2

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1 Problem 1

The $1-\lambda$ confidence interval given by Chebyshev's Inequality is $\left[\hat{\theta}_n - \sqrt{\frac{\mathrm{var}(\hat{\theta}_n)}{\lambda}}, \hat{\theta}_n + \sqrt{\frac{\mathrm{var}(\hat{\theta}_n)}{\lambda}}\right]$.

The confidence interval by CLT is $\left[\hat{\theta}_n - z_{1-\frac{\lambda}{2}}\sqrt{\mathrm{var}(\hat{\theta}_n)}, \hat{\theta}_n + z_{1-\frac{\lambda}{2}}\sqrt{\mathrm{var}(\hat{\theta}_n)}\right]$. $z_{1-\frac{\lambda}{2}}$ is smaller than $\frac{1}{\sqrt{\lambda}}$. So Chebyshev's Inequality usually gives a conservative confidence interval. (Any sensible answer is OK.)

2 Problem 2

(a) The exact solution given by BS formula is approximately 18.9586. The steps of simulation is following:

Set the parameters as the problem says
$n = 10000$
for $i = 1$ **to** n
  Generate a standard normal variable R_i
  $C_i = \exp(-rT)\max\{S_0 \exp((r - q - \frac{\sigma^2}{2})T + \sigma\sqrt{T}R_i) - K, 0\}$
$\hat{C}_n = \frac{1}{n}\sum_{i=1}^{n} C_i$

Delta hedging is also acceptable in this question. It just replaces the calculation of C_i in the algorithm. The details can be found in the answer to homework 1.

(b) $\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(C_i - \hat{C}_n)^2}{n-1}}$. The confidence interval is $\left[\hat{C}_n - z_{0.975}\frac{\hat{\sigma}}{\sqrt{n}}, \hat{C}_n + z_{0.975}\frac{\hat{\sigma}}{\sqrt{n}}\right]$.

(c) Let $\hat{F}(x)$ be the empirical distribution function of $R_i, 1 \le i \le 10000$
$n = 10000$
$B = 1000$(for example)
for $k = 1$ **to** B
  for $i = 1$ **to** n
    Generate $R_{i,k}$ from $\hat{F}(x)$
    $C_{i,k} = \exp(-rT)\max\{S_0 \exp((r - q - \frac{\sigma^2}{2})T + \sigma\sqrt{T}R_{i,k}) - K, 0\}$
  $\tilde{C}_k = \frac{1}{n}\sum_{i=1}^{n} C_{i,k}$
$\alpha = 0.05$
Let $\tilde{C}_{(\alpha/2)}$ and $\tilde{C}_{(1-\alpha/2)}$ be the $\alpha/2$ and $1-\alpha/2$ percentiles of $\tilde{C}_k$.

The confidence interval is $\left[2\hat{C}_n - \tilde{C}_{(1-\alpha/2)}, 2\hat{C}_n - \tilde{C}_{(\alpha/2)}\right]$.

3 Problem 3

(a) Set the parameters as the problem says

$n = 50000$

for $i = 1$ **to** n

  $S_{i,0} = S_0$

  for $j = 1$ **to** m

    Generate a standard normal variable $R_{i,j}$

    $S_{i,j} = S_{i,j-1} \exp((r - q - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t} R_{i,j})$

  $\bar{S}_i = \frac{1}{m} \sum_{j=1}^{m} S_{i,j}$

  $\theta_i = \exp(-rT) \max\{\bar{S}_i - K, 0\}$

$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} \theta_i$

$\hat{\sigma} = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (\theta_i - \hat{\theta}_n)^2}$

The confidence interval is $\left[\hat{\theta}_n - z_{0.995} \frac{\hat{\sigma}}{\sqrt{n}}, \hat{\theta}_n + z_{0.995} \frac{\hat{\sigma}}{\sqrt{n}}\right]$

(b) Set the parameters as the problem says

$n = 50000$

for $i = 1$ **to** $n/2$

  $S_{2i-1,0} = S_0$

  $S_{2i,0} = S_0$

  for $j = 1$ **to** m

    Generate a standard normal variable $R_{i,j}$

    $S_{2i-1,j} = S_{2i-1,j-1} \exp((r - q - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t} R_{i,j})$

    $S_{2i,j} = S_{2i,j-1} \exp((r - q - \frac{\sigma^2}{2})\Delta t - \sigma\sqrt{\Delta t} R_{i,j})$

  $\bar{S}_{2i-1} = \frac{1}{m} \sum_{j=1}^{m} S_{2i-1,j}$

  $\bar{S}_{2i} = \frac{1}{m} \sum_{j=1}^{m} S_{2i,j}$

  $\theta_i = \frac{1}{2} \exp(-rT) \left(\max\{\bar{S}_{2i-1} - K, 0\} + \max\{\bar{S}_{2i} - K, 0\}\right)$

$\hat{\theta}_n = \frac{2}{n} \sum_{i=1}^{n/2} \theta_i$

$\hat{\sigma}_A = \sqrt{\frac{1}{n/2-1} \sum_{i=1}^{n/2} (\theta_i - \hat{\theta}_n)^2}$

The confidence interval is $\left[\hat{\theta}_n - z_{0.995} \frac{\hat{\sigma}_A}{\sqrt{n/2}}, \hat{\theta}_n + z_{0.995} \frac{\hat{\sigma}_A}{\sqrt{n/2}}\right]$.

4 Problem 4

(a) This is the same as the question (a) in Problem 3.

(b) Set the parameters as the problem says

$n = 50000$

$p = 1000$(for example)

// pilot simulation

for $i = 1$ **to** p

  $S_{i,0} = S_0$

  for $j = 1$ **to** m

    Generate a standard normal variable $R_{i,j}$

    $S_{i,j} = S_{i,j-1} \exp((r - q - \frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t} R_{i,j})$

  $\bar{S}_i = \frac{1}{m} \sum_{j=1}^{m} S_{i,j}$

  $\theta_i = \exp(-rT) \max\{\bar{S}_i - K, 0\}$

  $Z_i = S_{i,m}$

$\hat{\theta}_p = \frac{1}{p}\sum_{i=1}^{p}\theta_i$
$E(Z) = S_0\exp((r-q)T)$
$\mathrm{var}(Z) = \frac{1}{p}\sum_{i=1}^{p}(Z_i - E(Z))^2$
$\mathrm{cov}(Z,\theta) = \frac{1}{p-1}\sum_{i=1}^{p}(Z_i - E(Z))(\theta_i - \hat{\theta}_p)$
$c = -\frac{\mathrm{cov}(Z,\theta)}{\mathrm{var}(Z)}$
// main simulation
for $i = 1$ **to** n
    $S_{i,0} = S_0$
    for $j = 1$ **to** m
        Generate a standard normal variable $R_{i,j}$
        $S_{i,j} = S_{i,j-1}\exp((r-q-\frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}R_{i,j})$
    $\bar{S}_i = \frac{1}{m}\sum_{j=1}^{m}S_{i,j}$
    $\theta_i = \exp(-rT)\max\{\bar{S}_i - K, 0\}$
    $Z_i = S_{i,m}$
$\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n}\theta_i$
$\mathrm{var}(\theta) = \frac{1}{n-1}\sum_{i=1}^{n}(\theta_i - \hat{\theta}_n)^2$
$\tilde{\theta} = \hat{\theta}_n + \frac{c}{n}\sum_{i=1}^{n}(Z_i - E(Z))$
$\hat{\sigma} = \sqrt{\mathrm{var}(\theta) - \frac{\mathrm{cov}(Z,\theta)^2}{\mathrm{var}(Z)}}$
(Or define $\tilde{\theta}_i = \theta_i + c(Z_i - E(Z))$ and $\hat{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(\tilde{\theta}_i - \tilde{\theta})^2}{n-1}}$)

The confidence interval is $\left[\tilde{\theta} - z_{0.995}\frac{\hat{\sigma}}{\sqrt{n}}, \tilde{\theta} + z_{0.995}\frac{\hat{\sigma}}{\sqrt{n}}\right]$.

(c) Replace $Z_i = \exp(-rT)\max\{S_{i,m} - K, 0\}$ and $E(Z) = \mathrm{BSFormula}(S_0, K, r, q, T, \sigma)$ in the former algorithm.

(d) Replace $Z_i = \exp(-rT)\max\{\sqrt[m]{\prod_{j=1}^{m}S_{i,j}} - K, 0\}$ in the former algoritm. The key to calculate $E(Z)$ is that $\sqrt[m]{\prod_{j=1}^{m}S_{i,j}}$ is lognormal. Let

$$G = \log(\sqrt[m]{\prod_{j=1}^{m}S_{i,j}}) = \log S_0 + \sum_{j=1}^{m}\frac{m+1-j}{m}((r-q-\frac{\sigma^2}{2})\Delta t + \sigma\sqrt{\Delta t}R_j)$$

Its expectation is $\mu_G = \log S_0 + \frac{m+1}{2m}(r-q-\frac{\sigma^2}{2})T$ and its variance is $\sigma_G^2 = \frac{(m+1)(2m+1)}{6m^2}\sigma^2 T$. In the same way as calculating BS Formula, we get

$$E(Z) = e^{\mu_G + \frac{1}{2}\sigma_G^2 - rT}\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

where $d_1 = \frac{\mu_G - \log K + \sigma_G^2}{\sigma_G}$ and $d_2 = \frac{\mu_G - \log K}{\sigma_G}$. There are other ways to calculate $E(Z)$.

- Conclusion: The control variate in (d) is the best.