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HW1

Columbia University
IEOR4703 – Monte Carlo Simulation (Hirsa)
Assignment 1 – Due 17:40 on Tuesday Feb 6th, 2018

$G(x) = U \qquad G^{-1}(U) = x$

Problem 1 (Sampling from tail distributions via Inverse Transform): Assume we can easily sample from $\mathbf{U}(0,1)$. Also assume that we can easily calculate both $\Phi(x)$, the cumulative distribution function of $\mathcal{N}(0,1)$

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du,$$

and its inverse $\Phi^{-1}(x)$. Show how to sample from the tail of the normal distribution on $[-\infty, a]$ using inverse transform:

$$g(x) = \frac{1}{\sqrt{2\pi}A\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

where $A = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx$.

Problem 2 (Sampling via Inverse Transform & Acceptance-Rejection): Suppose the following is the probability density function of the random variable X.

$50k - 100k \qquad 100k - 50k \qquad U < 0.5 . \ \frac{k}{2}x^2 + 10kx = U$

integrate must find the right constant term

$$\left(\frac{kx^2}{2} + 10kx\right)\Big|_{-10}^{0} + \left(10kx - \frac{k}{2}x^2\right)\Big|_{0}^{10} \qquad f(x) = \begin{cases} 0 & : \quad x < -10 \\ \kappa(x+10) & : \quad -10 \le x \le 0 \\ \kappa(10-x) & : \quad 0 \le x \le 10 \\ 0 & : \quad x > 10 \end{cases} \qquad \frac{-10k \pm \sqrt{100k^2 + 2kU}}{k}$$

$50k - 100k \qquad -e^{-x}\Big|_0^{\infty}$

(a) What is the value of κ? 0.01 $\quad = 50k + 50k = 1$

(b) Describe in detail the inverse transform method to generate a sample of X given a uniform $U(0,1)$ random number generator.

(c) Describe in detail an *acceptance-rejection* algorithm for generating a sample of X given a uniform $U(0,1)$ random number generator. How many uniform random variables on average will be required to generate one sample of X?

Problem 3 (Sampling via Inverse Transform): Suppose the following is the probability density function of the random variable X.

$$f(x) = \begin{cases} \kappa e^{x} & : \quad x \le 0 \\ \kappa e^{-x} & : \quad x > 0 \end{cases} \qquad \int_{-\infty}^{0} \kappa e^{x}\, dx + \int_{0}^{\infty} \kappa e^{-x}\, dx$$

$$= k + k = 1$$

(a) What is the value of κ? 0.5

(b) Describe in detail the inverse transform method to generate a sample of X given a uniform $U(0,1)$ random number generator.

Problem 4 (Monte Carlo Simulation): Estimate the following integral

$$\theta = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

change variable to change bounds to use U

via Monte-Carlo integration. Hint: First convert it to a definite integral by change of a variable. Compare your estimate with the exact solution.

Problem 5 (Sampling from non-homogeneous Poisson distribution): There are many algorithms for generating a sample from non-homogeneous Poisson distribution. Give a detailed pseudo-code for one of them. One could be the extension of the method explained for the homogeneous case.

Problem 6 (Cost of delta hedging shorting a call option in Black-Merton-Scholes Model: In the Black-Merton-Scholes model for $r = q = 0$ we show the initial call price is the same as cost of continuous delta hedging plus selling the stock and payoff if any at maturity i.e.

$$C = \Delta_{t_0} \times S_0 + \sum_{i=0}^{n-1} (\Delta_{t_i} - \Delta_{t_{i-1}}) S_{t_i} + (S_T - K)^+ - \Delta_{t_{n-1}} \times S_T$$

where Δ_{t_i} and S_{t_i} are the delta of the call option and stock price at time t_i respectively for $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$. Extend this argument for the case that $r > 0$ and $q > 0$ by looking into investing and borrowing for each transaction and also dividends would be received on the stock positions holding in each time interval $[t_{i-1}\ t_i]$.

1. $A = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$, $y = \frac{x-\mu}{\sigma}$, $dy = \frac{1}{\sigma} dx$, $x = a = \sigma y + \mu$

$= \int_{-\infty}^{\frac{a-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \phi\left(\frac{a-\mu}{\sigma}\right)$

$g(x) = \frac{1}{\sqrt{2\pi}A\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $y = \frac{x-\mu}{\sigma}$, $dy = \frac{1}{\sigma} dx$, $x = a = \sigma y + \mu$

$G(x) = \int_{-\infty}^{a} g(x)\, dx = \int_{-\infty}^{\frac{a-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}A} e^{-\frac{1}{2}y^2} dy = \frac{1}{A} \cdot A = 1$ $\Big\}\Rightarrow$ c.d.f.

increasing function

$G(y) = \int_{-\infty}^{y_0} \frac{1}{A} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = \frac{1}{A} \cdot \phi(y)$, $\phi^{-1}(AG) = y$

$\Rightarrow$ Inverse Transform

① Generate U from $U(0,1)$

② $y = \phi^{-1}(AU) = \frac{x-\mu}{\sigma}$

set $x = \sigma\phi^{-1}(AU) + \mu$

2. (a). $\left(\frac{kx^2}{2} + 10kx\right)\Big|_{-10}^{0} + \left(10kx - \frac{k}{2}x^2\right)\Big|_{0}^{10} = 1$

$\Rightarrow k = 0.01$

(b).

$$F(x) \begin{cases} 0 & x < -10 \\ \frac{0.01}{2}(x+10)^2 & -10 \le x < 0 \\ 1 - \frac{0.01}{2}(10-x)^2 & 0 \le x \le 10 \\ 1 & x > 10 \end{cases}$$

$F(x) \Rightarrow$ cdf , $F(0) = 0.5$

$$\Rightarrow F^{-1}(x) = \begin{cases} \sqrt{\frac{2x}{0.01}} - 10 , & 0 \le x < 0.5 \\ 10 - \sqrt{\frac{2(1-x)}{0.01}} , & 0.5 \le x \le 1 \end{cases}$$

$\Rightarrow$ Inverse Transform

① Generate U from $U(0,1)$

② set $X = \begin{cases} \sqrt{\frac{2U}{0.01}} - 10 , & 0 \le U < 0.5 \\ 10 - \sqrt{\frac{2(1-X)}{0.01}} , & 0.5 \le U \le 1 \end{cases}$

(c) since $\max(f(x)) = 10k = 0.1$

consider $g(x)$ over $[-10, 10]$, $g(x) = \frac{1}{20}$

$\frac{f(x)}{g(x)} \le 2$ for all x

$\Rightarrow$ acceptance rejection

① generate U from $U(0,1)$

② generate U_2 from $U(0,1)$

③ set $Y = 20U_2 - 10$

④ while $U > \frac{f(Y)}{2g(Y)} = 10f(Y)$

generate U from $U(0,1)$

generate U_2 from $U(0,1)$

set $Y = 20U_2 - 10$

⑤ set $X = Y$

Since $P(U \le 10f(Y)) = \frac{1}{a} = \frac{1}{2}$, let x be the number of uniform r.v. generated

$$X = \frac{1}{2} \times 2 + \frac{1}{2} \times (X+2) \Rightarrow X = 4$$

3. (a). $$\int_{-\infty}^{0} k e^{x} \, dx + \int_{0}^{\infty} k e^{-x} \, dx = 2k = 1$$

$\Rightarrow k = 0.5$

(b). $$F(x) = \begin{cases} 0.5 e^{x}, & x \le 0 \\ 1 - 0.5 e^{-x}, & x > 0 \end{cases}$$

$F(x) \Rightarrow$ cdf, $F(0) = 0.5$

$$F^{-1}(x) = \begin{cases} \log(2x) & , x \le 0.5 \\ -\log(2(1-y)) & , x > 0.5 \end{cases}$$

$\Rightarrow$ Inverse transform

① Generate v from $U(0,1)$

② set $X = F^{-1}(v)$

4. $$\theta = \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{2} + \int_{0}^{1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \frac{1}{2} + \int_{0}^{1} g(x) \, dx$$

Monte Carlo Simulation: $\theta = \frac{1}{2} + E[g(U)]$ where $g(U) = \frac{1}{\sqrt{2\pi}} e^{-\frac{U^2}{2}}$

$\Rightarrow$ ① Generate $U_1, U_2, \cdots, U_n \sim$ IID $U(0,1)$

② Estimate θ with $\hat{\theta}_n = \frac{1}{2} + \frac{g(U_1) + \cdots + g(U_n)}{n}$

$n = 1000 \Rightarrow \hat{\theta} = 0.8433$ while $\theta \approx 0.8413$, estimate is close to exact value

5. $N(t)$: non-homogeneous Poisson process with intensity $\lambda(t)$

Thinning Algorithm

Intuition : find a λ such that $\lambda(t) \le \lambda \quad \forall t \le T$

generate from the homogeneous Poisson process with intensity λ

then reject generations based on the $\lambda(t)$

$\Rightarrow$ ① set $t = 0$, $N = 0$, $t = t - \log(U_1)/\lambda$

② generate U_1 from $U(0,1)$

③ while $t < T$

generate U_2 from $U(0,1)$

if $U_2 \le \lambda(t)/\lambda$ $\Rightarrow$ reject to achieve $\lambda(t)$

set $N = N+1$, $T(N) = t$

generate U_1 from $U(0,1)$

set $t = t - \log(U_1)/\lambda$

N : # of arrivals

$T(N)$: time of the N^{th} arrival

6.

	Cash	Stock Value	Option Value
t_0,	$C - \Delta_{t_0} S_0$	$\Delta_{t_0} S_0$	
t_1,	$(C - \Delta_{t_0} S_0)\, e^{r(t_1 - t_0)}$	$\Delta_{t_1} S_1$	
	$+ (\Delta_{t_0} S_0 (e^{q(t_1 - t_0)} - 1))$		
	$- (\Delta_{t_1} - \Delta_{t_0}) S_1$		

t_i $\quad (C - \Delta t_0 S_0)\, e^{r(t_i - t_0)}$ $\qquad \Delta t_i S_i$

$$+ \sum_{k=1}^{i} \left(\Delta t_{k-1} S_{k-1} \left(e^{q(t_k - t_{k-1})} - 1 \right) \right) e^{r(t_i - t_k)}$$

$$- \sum_{k=1}^{i} (\Delta t_k - \Delta t_{k-1})\, S_k\, e^{r(t_i - t_k)}$$

t_n $\quad (C - \Delta t_0 S_0)\, e^{r(t_n - t_0)}$ $\qquad - (S_T - K)^+$

$$+ \sum_{k=1}^{n} \left(\Delta t_{k-1} S_{k-1} \left(e^{q(t_k - t_{k-1})} - 1 \right) \right) e^{r(t_n - t_k)}$$

$$- \sum_{k=1}^{n-1} (\Delta t_k - \Delta t_{k-1})\, S_k\, e^{r(t_n - t_k)}$$

$$+ \Delta t_{n-1} S_T$$

$V_{t_n} = 0 \Rightarrow$

$$C = \Delta t_0 S_0 - \sum_{i=1}^{n} \left(\Delta t_{i-1} S_{t_{i-1}} \left(e^{q(t_i - t_{i-1})} - 1 \right) \right) \cdot e^{-r t_i}$$

$$+ \sum_{i=1}^{n-1} (\Delta t_i - \Delta t_{i-1})\, S_{t_i}\, e^{-r t_i}$$

$$- \Delta t_{n-1} \cdot S_T \cdot e^{-rT} + (S_T - K)^+ \cdot e^{-rT}$$

Code for Monte Carlo simulation

Editor – /Users/Johnson/Desktop/Courses/Monte Carlo Simulation/HW/hw1/hw1.m

hw1.m

```
u = rand(1000,1);

theta = mean(1/sqrt(2*pi)*exp(-(u.^2)/2)) + 0.5;

disp(theta)
```