You Wu
yw3027
MCS

Columbia University
IEOR4703 – Monte Carlo Simulation (Hirsa)
Assignment 4 – Due 17:40 on Thursday March 22nd, 2018

Problem 1 (Pricing Heston Utilizing Various Discretization Schemes): Extend the matlab code *exampleSimulationOfSDE.m* for Heton stochastic volatility model

$$\begin{aligned} dS_t &= (r-q)S_t dt + \sqrt{v_t} S_t dW_t^1, \\ dv_t &= \kappa(\theta - v_t)dt + \lambda\sqrt{v_t} dW_t^2 \end{aligned}$$

with the following parameter set: spot price of $S_0 = \$100$, strike price of $K = \$110$, risk-free rate $r = 0.025$, continuous divided rate of $q = 0.0125$, volatility of $\sigma = 0.30$, maturity of $T = 1$ year, $\Theta = \{\theta = 0.0625, \kappa = 2.75, \lambda = 0.0125, v_0 = 0.05, \rho = -0.65\}$.

Problem 2 (Calculating Greeks via Simulation): Extend the matlab code *exampleGreek_BMS.m* for (a) Digital Call, (b) Up-and-Out Call, (c) Average Option Call with the following parameter set: spot price of $S_0 = 100$, strike price of $K = \$115$, risk-free rate $r = 0.04$, continuous divided rate of $q = 0.015$, volatility of $\sigma = 0.28$ for all options, maturity of $T = 1$ year for all options , barrier of $H = 130$ for up-and-out call, $m = 12$ monitoring times for average option.

P1. $Z_1, Z_3 \sim N(0,1)$, $Z_2 = \rho Z_1 + \sqrt{1-\rho^2} \cdot Z_3$

Euler
$$\begin{cases} S_{t_{j+1}} = S_{t_j} + (r-q) S_{t_j} \Delta t + \sqrt{V_{t_j}}\, S_{t_j} \cdot \sqrt{\Delta t} \cdot Z_1 \\ V_{t_{j+1}} = V_{t_j} + k(\theta - V_{t_j}) \Delta t + \lambda \sqrt{V_{t_j}}\, \sqrt{\Delta t} \cdot Z_2 \end{cases}$$

Milsten
$$\begin{cases} S_{t_{j+1}} = S_{t_j} + (r-q) S_{t_j} \Delta t + \sqrt{V_{t_j}}\, S_{t_j} \cdot \sqrt{\Delta t} \cdot Z_1 \\ \qquad + \frac{1}{2} \sqrt{V_{t_j}}\, S_{t_j} \cdot \sqrt{V_{t_j}} \cdot \Delta t \, (Z_1^2 - 1) \\ V_{t_{j+1}} = V_{t_j} + k(\theta - V_{t_j}) \Delta t + \lambda \sqrt{V_{t_j}}\, \sqrt{\Delta t} \cdot Z_2 \\ \qquad + \frac{1}{2} \lambda \sqrt{V_{t_j}} \cdot \frac{1}{2} \lambda \cdot \frac{1}{\sqrt{V_{t_j}}} \cdot \Delta t \, (Z_2^2 - 1) \end{cases}$$

Runge-Kutta
$$\begin{cases} \tilde{S}_{t_j} = S_{t_j} + (r-q) S_{t_j} \Delta t + \sqrt{V_{t_j}}\, S_{t_j} \cdot \sqrt{\Delta t} \\ S_{t_{j+1}} = S_{t_j} + (r-q) S_{t_j} \Delta t + \sqrt{V_{t_j}}\, S_{t_j} \cdot \sqrt{\Delta t} \cdot Z_1 \\ \qquad + \frac{1}{2\sqrt{\Delta t}} \left(\sqrt{V_{t_j}} (\tilde{S}_{t_j} - S_{t_j}) \right) \cdot \Delta t \, (Z_1^2 - 1) \\ \tilde{V}_{t_j} = V_{t_j} + k(\theta - V_{t_j}) \Delta t + \lambda \sqrt{V_{t_j}}\, \sqrt{\Delta t} \\ V_{t_{j+1}} = V_{t_j} + k(\theta - V_{t_j}) \Delta t + \lambda \sqrt{V_{t_j}}\, \sqrt{\Delta t} \cdot Z_2 \\ \qquad + \frac{1}{2\sqrt{\Delta t}} \left(\lambda \left(\sqrt{\tilde{V}_{t_j}} - \sqrt{V_{t_j}} \right) \right) \cdot \Delta t \, (Z_2^2 - 1) \end{cases}$$

(Please find output in the end)

P2. Likelihood Score function for Digital Option and Up & Out :

$$g(x) = \frac{1}{x\sigma\sqrt{T}} \cdot \phi(\zeta(x)) \quad , \quad \zeta(x) = \frac{\log(x/S_0) - (r-q-\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$\frac{dg(x)/dS_0}{g(x)} = \frac{\frac{1}{x\sigma\sqrt{T}}\phi(\zeta(x)) \cdot (-\zeta(x)) \cdot \frac{d\zeta(x)}{dS_0}}{g(x)}$$

$$= \frac{\log(x/S_0) - (r-q-\frac{\sigma^2}{2})T}{S_0\sigma^2 T} = T(x)$$

$$\frac{d^2g(x)/dS_0^2}{g(x)} = \frac{\frac{1}{x\sigma\sqrt{T}}\phi(\zeta(x)) \cdot (-\zeta(x)) \cdot \frac{d\zeta(x)}{dS_0} \cdot T(x)}{g(x)}$$

$$+ \frac{g(x)}{g(x)} \cdot - \left(\frac{1 + \log(x/S_0) - (r-q-\frac{\sigma^2}{2})T}{S_0^2\sigma^2 T}\right)$$

$$= T(x)^2 - \frac{1}{S_0^2\sigma^2 T} - \frac{T(x)}{S_0}$$

$$x = S_T \quad , \quad \zeta(x) = z$$

$$\Rightarrow \frac{dg(x)/dS_0}{g(x)} = \frac{z}{S_0\sigma\sqrt{T}}$$

$$\frac{d^2g(x)/dS_0^2}{g(x)} = \frac{z^2 - 1 - \sigma\sqrt{T}\cdot z}{S_0^2\sigma^2 T}$$

$$\frac{dg(x)/d\sigma}{g(x)} = \left(-\frac{1}{x\sigma^2\sqrt{T}} \cdot \phi(\zeta(x)) + \frac{1}{x\sigma\sqrt{T}} \cdot \phi(\zeta(x))(-\zeta(x)) \cdot \frac{d\zeta(x)}{d\sigma}\right) / g(x)$$

$$= \frac{1}{\sigma}\left(z^2 - \sigma\sqrt{T}z - 1\right)$$

① Digital Option : $e^{-rT} \cdot 1_{\{S_T > K\}}$

② Up & Out : $e^{-rT} (S_T - K)^+ 1_{\{S_T < H\}}$

③ Average Option : $e^{-rT} (\bar{S} - K)^+$, $\bar{S} = \frac{1}{m} \sum_{i=1}^{m} S_{t_i}$

Pathwise for Average Option :

$$\frac{\partial h}{\partial S_0} = \frac{\partial h}{\partial \bar{S}} \cdot \frac{\partial \bar{S}}{\partial S_0} = e^{-rT} \cdot 1_{\{S_T > K\}} \cdot \frac{\partial \bar{S}}{\partial S_0}$$

$$* \quad \frac{\partial E[h]}{\partial S_0} = E\left[\frac{\partial h}{\partial S_0}\right] = E\left[e^{-rT} \cdot 1_{\{\bar{S} > K\}} \cdot \frac{\bar{S}}{S_0}\right]$$

$$\frac{\partial h}{\partial \sigma} = \frac{\partial h}{\partial \bar{S}} \cdot \frac{\partial \bar{S}}{\partial \sigma} = e^{-rT} \cdot 1_{\{\bar{S} > K\}} \cdot \frac{\partial \bar{S}}{\partial \sigma}$$

$$S_{t_i} = S_0 \cdot e^{(r - q - \frac{1}{2}\sigma^2) t_i} \cdot e^{\sigma \cdot \sqrt{\Delta t} \cdot \sum_{j=1}^{i} Z_j}$$

$$\frac{\partial S_{t_i}}{\partial \sigma} = S_{t_i} \cdot \left(-\sigma t_i + \sqrt{\Delta t} \cdot \sum_{j=1}^{i} Z_j\right)$$

$$* \quad \frac{\partial E[h]}{\partial \sigma} = E\left[\frac{\partial h}{\partial \sigma}\right] = E\left[e^{-rT} \cdot 1_{\{\bar{S} > K\}} \cdot \frac{1}{m\sigma} \sum_{i=1}^{m} \left(S_{t_i}\left(\log \frac{S_{t_i}}{S_0} - (r - q + \frac{\sigma^2}{2})(t_i)\right)\right)\right]$$

Likelihood Score function for Average option :

$$g(S_{t_1}, \cdots, S_{t_m}) = g(S_{t_1} | S_{t_0})\, g(S_{t_2} | S_{t_1}) \cdots g(S_{t_m} | S_{t_{m-1}})$$

$$g(S_{t_i} | S_{t_{i-1}}) = \frac{1}{S_{t_i} \sigma \cdot \sqrt{\Delta t}} \cdot \phi(\zeta(S_{t_i} | S_{t_{i-1}}))$$

$$\zeta(S_{t_i} | S_{t_{i-1}}) = \frac{\log(S_{t_i} / S_{t_{i-1}}) - (r - q - \frac{1}{2}\sigma^2) \cdot \Delta t}{\sigma \sqrt{\Delta t}}$$

$$\frac{dg(S_{t_1},\cdots,S_{t_m})/dS_0}{g} = \frac{dg(S_{t_1}|S_0)/dS_0}{g(S_{t_1}|S_{t_0})} = \frac{Z_1}{S_0\cdot\sigma\cdot\sqrt{\Delta t}}$$

$$\frac{d^2g/dS_0^2}{g} = \frac{Z_1^2 - 1 - \sigma\sqrt{\Delta t}\cdot Z_1}{S_0^2\sigma^2\Delta t}$$

$$\frac{dg/d\sigma}{g} = \frac{\sum_1^m \frac{dg(S_{t_i}|S_{t_{i-1}})}{d\sigma}\cdot\prod_{j\neq i} g(S_{t_j}|S_{t_{j-1}})}{\prod g(S_{t_j}|S_{t_{j-1}})}$$

$$= \sum_1^m \frac{dg(S_{t_i}|S_{t_{i-1}})}{d\sigma}\Big/ g(S_{t_i}|S_{t_{i-1}})$$

$$= \sum_1^m \frac{1}{\sigma}\left(Z_i^2 - \sigma\sqrt{\Delta t}\,Z_i - 1\right)$$

(Please find output in the end)

Output (Please find attached code for Q1 & Q2)

```
>> hw4_q1

ans =

    " Euler: 6.179451
     Milstein: 6.182216
     Runge-Kutta: 6.182250
     "


ans =

    " Exact solution: 6.188100
     "
```

The results generated by different schemes are close to each other, and are close to the exact solution.

```
>> hw4_q2
===============
   Delta
---------------
      Digital Call
 W/ CRN:0.011578 W/O CRN:0.011606 Pathwise Estimator:NaN Likelihood Ratio:0.011744

      Up and Out Call
 W/ CRN:0.020365 W/O CRN:-0.0013269 Pathwise Estimator:NaN Likelihood Ratio:0.025625

      Average Option Call
 W/ CRN:0.25114 W/O CRN:0.32545 Pathwise Estimator:0.25117 Likelihood Ratio:0.25402

===============
   Gamma
---------------
      Digital Call
 W/ CRN:-0.012106 W/O CRN:0.14988 Pathwise Estimator:NaN Likelihood Ratio:0.00011442

      Up and Out Call
 W/ CRN:0.013658 W/O CRN:0.9037 Pathwise Estimator:NaN Likelihood Ratio:-0.000577

      Average Option Call
 W/ CRN:0.018926 W/O CRN:-0.97191 Pathwise Estimator:NaN Likelihood Ratio:0.018797

===============
   Vega
---------------
---------------
      Digital Call
 W/ CRN:0.33243 W/O CRN:-0.033628 Pathwise Estimator:NaN Likelihood Ratio:0.32038

      Up and Out Call
 W/ CRN:-2.2634 W/O CRN:-2.5855 Pathwise Estimator:NaN Likelihood Ratio:-1.6156

      Average Option Call
 W/ CRN:19.6766 W/O CRN:27.0959 Pathwise Estimator:19.6771 Likelihood Ratio:19.7701
```