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hw5

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IEOR4703 – Monte Carlo Simulation (Hirsa)

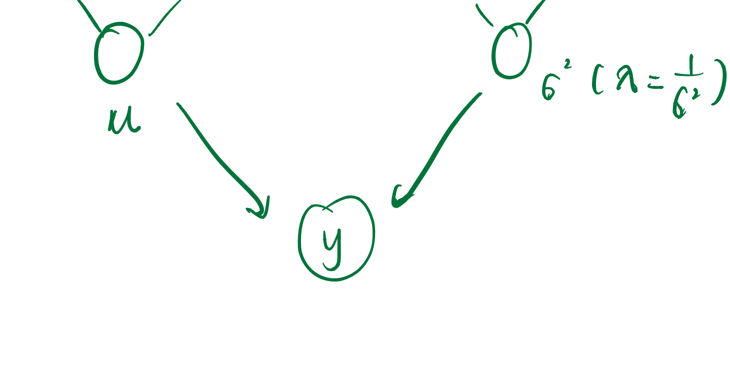
Assignment 5 – Due 17:40 on Tuesday April 10th, 2018

Problem 1 (Conjugate Prior for Normal Distribution): Extend the Matlab code *exampleConjugatePriorForNormalDistribution.m* for the case that both μ and σ^2 (or $\lambda = \frac{1}{\sigma^2}$) are unknown. First you have to derive the posterior in details and then modify the code to update the posterior as data being processed (like Cases 1 & 2).

Problem 2 (Conjugate Prior for Poisson Distribution): Assume the likelihood distribution is $Poisson(\theta)$ with θ unknown and prior is $\Gamma(\alpha, \beta)$ find the posterior $\pi(\theta|y)$. Write a Matlab code for it. In your code posterior should get updated as data being processed.

Problem 3 (Conjugate Prior for Gamma Distribution): Assume the likelihood distribution is $\Gamma(\nu, \theta)$ with θ unknown and prior is $\Gamma(\alpha, \beta)$ find the posterior $\pi(\theta|y)$. Write a Matlab code for it. In your code posterior should get updated as data being processed.

both u, $\lambda = \frac{1}{\sigma^2}$ unknown



P1. μ & σ^2 ($\lambda = \frac{1}{\sigma^2}$) unknown , $x_i \mid \mu, \lambda \sim N(\mu, \frac{1}{\lambda})$

let priors be : $\mu \mid \lambda \sim N(\mu_0, \frac{1}{k_0 \lambda})$

$\lambda \sim \text{Gamma}(\alpha_0, \beta_0)$

$$P(\lambda \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \lambda^{\alpha-1} e^{-\lambda\beta}$$

$$P(\mu, \lambda \mid x) = \frac{P(\mu \mid \lambda, x) \cdot P(\lambda \mid x) \cdot P(x)}{P(x)} = \underbrace{P(\mu \mid \lambda, x) \cdot P(\lambda \mid x)}_{\text{(posteriors)}}$$

$$P(\mu, \lambda \mid x) \propto P(x \mid \mu, \lambda) \cdot P(\mu \mid \lambda) \, P(\lambda)$$

$$\propto \lambda^{\frac{n}{2}} \cdot \exp\left(-\frac{\lambda}{2} \textstyle\sum_i (x_i - \mu)^2\right) \cdot (k_0 \lambda)^{\frac{1}{2}} \cdot \exp\left(-\frac{k_0 \lambda}{2} \cdot (\mu - \mu_0)^2\right)$$

$$\cdot \lambda^{\alpha_0 - 1} \cdot e^{-\lambda \beta_0}$$

check exp term : $\exp\left(-\frac{\lambda}{2}\left(\sum_i x_i^2 + n\mu^2 - 2\mu \underbrace{\sum_i x_i}_{2n\mu\bar{x}}\right) - \frac{k_0\lambda}{2}(\mu^2 - 2\mu\mu_0 + \mu_0^2) - \lambda\beta_0\right)$

check the μ term : $-\frac{\lambda}{2}(n\mu^2 - 2n\bar{x}\mu) - \frac{k_0\lambda}{2}\mu^2 + k_0\lambda\mu_0\mu$

$$= -\frac{(n+k_0)}{2}\lambda \cdot \mu^2 + (\lambda n \bar{x} + k_0 \lambda \mu_0)\mu$$

$$\Rightarrow -\frac{(n+k_0)\lambda}{2}\left(\mu - \frac{n\bar{x} + k_0 \mu_0}{n+k_0}\right)^2$$

remaining terms $\exp\left(-\frac{\lambda}{2}\sum_i x_i^2 - \frac{k_0\lambda}{2}\mu_0^2 - \lambda\beta_0 + \frac{\lambda}{2}\frac{(n\bar{x}+k_0\mu_0)^2}{n+k_0}\right)$

$$\Rightarrow -\frac{\lambda}{2}\sum_i (x_i - \bar{x})^2 - \frac{\lambda}{2}n\bar{x}^2 - \frac{k_0\lambda}{2}\mu_0^2 - \lambda\beta_0 + \frac{\lambda}{2}n\bar{x}^2 + \frac{\lambda}{2}k\mu_0^2 - \frac{\lambda}{2}\frac{nk_0(\bar{x}-\mu_0)^2}{n+k_0}$$

$$\Rightarrow -\lambda\left(\frac{1}{2}\sum_i (x_i - \bar{x})^2 + \beta_0 + \frac{nk_0(\bar{x}-\mu_0)^2}{2(n+k_0)}\right)$$

so $P(\mu, \lambda \mid x) \propto \lambda^{\alpha_0 + \frac{n}{2} - 1} \cdot \exp\left(-\lambda \left(\frac{1}{2} \sum_i (x_i - \bar{x})^2 + \beta_0 + \frac{n k_0 (\bar{x} - \mu_0)^2}{2(n+k_0)} \right) \right)$

$$\cdot \lambda^{\frac{1}{2}} \exp\left(-\frac{(n+k_0)\lambda}{2} \left(\mu - \frac{n\bar{x} + k_0 \mu_0}{n+k_0} \right)^2 \right)$$

which $\propto \text{Gamma}\left(\alpha_0 + \frac{n}{2}, \ \frac{1}{2} \sum_i (x_i - \bar{x})^2 + \beta_0 + \frac{n k_0 (\bar{x} - \mu_0)^2}{2(n+k_0)} \right)$

$$\propto N\left(\frac{n\bar{x} + k_0 \mu_0}{n+k_0}, \ \frac{1}{(n+k_0)\lambda} \right)$$

$\Rightarrow$ Posteriors :

$$\mu \mid \lambda, x \sim N\left(\frac{n\bar{x} + k_0 \mu_0}{n+k_0}, \ \frac{1}{(n+k_0)\lambda} \right)$$

$$\lambda \mid x \sim \text{Gamma}\left(\alpha_0 + \frac{n}{2}, \ \frac{1}{2} \sum_i (x_i - \bar{x})^2 + \beta_0 + \frac{n k_0 (\bar{x} - \mu_0)^2}{2(n+k_0)} \right)$$

$\Rightarrow$ conjugate priors

Moreover since $\mu \mid \lambda \sim N\left(\mu_0, \frac{1}{k_0 \lambda} \right)$

$$P(\mu) \propto \int_0^\infty p(\mu, \lambda) \, d\lambda = \int_0^\infty P(\mu \mid \lambda) P(\lambda) \, d\lambda$$

$$\propto \int_0^\infty \underbrace{\lambda^{\alpha_0 + \frac{1}{2} - 1} \cdot e^{-\lambda \left(\beta_0 + \frac{k_0}{2} (\mu - \mu_0)^2 \right)}}_{\propto \text{Gamma}\left(\alpha_0 + \frac{1}{2}, \ \beta_0 + \frac{k_0}{2} (\mu - \mu_0)^2 \right), \ \int \text{pdf} = 1} d\lambda$$

$$\Rightarrow P(\mu) \propto \frac{\Gamma(\alpha_0 + \frac{1}{2})}{\left(\beta_0 + \frac{k_0}{2} (\mu - \mu_0)^2 \right)^{(\alpha_0 + \frac{1}{2})}}$$

$$\propto \left(\beta_0 + \frac{k_0}{2} (\mu - \mu_0)^2 \right)^{-(\alpha_0 + \frac{1}{2})} \propto \left(1 + \frac{1}{2\alpha_0} \frac{(\mu - \mu_0)^2}{\left(\sqrt{\frac{\beta_0}{\alpha_0 k_0}} \right)^2} \right)^{-\frac{2\alpha_0 + 1}{2}}$$

For t-distribution : $P(x \mid \nu, \mu, \sigma) = c \cdot \left(1 + \frac{1}{\nu} \left(\frac{x - \mu}{\sigma} \right)^2 \right)^{-\frac{\nu + 1}{2}}$

$$\Rightarrow \quad v = 2\alpha_0 , \quad \mu = \mu_0 , \quad \sigma^2 = \frac{\beta_0}{\alpha_0 k_0} \qquad \Rightarrow \quad \mu = t_{2\alpha_0}\left(\mu_0 , \frac{\beta_0}{\alpha_0 k_0} \right)$$

$\Rightarrow$ posterior $\quad P(\mu \mid x) \sim t_{2\alpha}\left(\mu' , \frac{\beta'}{\alpha' k'} \right)$

$$\mu' = \frac{n\bar{x} + k_0 \mu_0}{n + k_0} , \quad \beta' = \frac{1}{2} \sum_i (x_i - \bar{x})^2 + \beta_0 + \frac{n k_0 (\bar{x} - \mu_0)^2}{2(n + k_0)}$$

$$\alpha' = \alpha_0 + \frac{1}{2} , \qquad k' = k_0 + n$$

P2. $\quad \mathrm{Pois}(\theta) : P(X = k) = \frac{\theta^k e^{-\theta}}{k!}$

Prior : $\Gamma(\alpha, \beta) \quad \Rightarrow \quad \pi(\theta \mid \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha - 1} e^{-\beta\theta}$

(use def given in class)

$$\pi(\theta \mid x) \propto \pi(x \mid \theta) \cdot \pi(\theta \mid \alpha, \beta)$$

$$= \frac{\theta^{\sum_i x_i} e^{-\theta n}}{\prod x_i !} \cdot \theta^{\alpha - 1} e^{-\beta\theta}$$

$$\propto \theta^{\sum_i x_i + \alpha - 1} \cdot e^{-\theta(n + \beta)}$$

$\Rightarrow$ Gamma $\left(\sum_i x_i + \alpha , \; n + \beta \right)$

posterior $\quad \pi(\theta \mid x) = \frac{\beta'^{\alpha'}}{\Gamma(\alpha')} \theta^{\alpha' - 1} e^{-\beta'\theta}$

where $\alpha' = \sum x_i + \alpha$, $\beta' = n + \beta$

P3. $\Gamma(\nu,\theta)$: $P(x \mid \nu,\theta) = \frac{\theta^{\nu}}{\Gamma(\nu)} x^{\nu-1} e^{-\theta x}$

Prior : $\Gamma(\alpha,\beta)$ $\Rightarrow$ $\pi(\theta \mid \alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$

(use def given in class)

$$\pi(\theta \mid x) = \pi(x \mid \theta) \cdot \pi(\theta \mid \alpha,\beta)$$

$$\propto \frac{\theta^{n\nu}}{\Gamma(\nu)^n} (\Pi x_i)^{\nu-1} e^{-\theta \Sigma x_i} \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$

$$\propto \theta^{n\nu+\alpha-1} \cdot e^{-\theta(\Sigma x_i + \beta)}$$

$\Rightarrow$ Gamma $(n\nu + \alpha, \ \Sigma x_i + \beta)$

posterior $\pi(\theta \mid x) = \frac{\beta'^{\alpha'}}{\Gamma(\alpha')} \theta^{\alpha'-1} e^{-\beta'\theta}$

where $\alpha' = n\nu + \alpha$, $\beta' = \Sigma x_i + \beta$