IEOR 4703 - Monte Carlo Simulation
Solutions for Assignment 1

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Spring 2018

Problem 1. Using the identities $F(x) = \Phi\left(\dfrac{x-\mu}{\sigma}\right)$ and $F^{-1}(u) = \mu + \sigma\Phi^{-1}(u)$, the algorithm is as follows:

1. Simulate $u \sim \mathrm{U}(0,1)$.
2. Set $v = F(-\infty) + (F(a) - F(-\infty))u = F(a)u$.
3. Set $x = F^{-1}(v)$.
4. Return x.

To see why this algorithm works observe that

$$\begin{aligned} P(X \leq x) &= P(F^{-1}(V) \leq x) \\ &= P(V \leq F(x)) \\ &= P\left(u \leq \frac{F(x)}{F(a)}\right) \\ &= \frac{F(x)}{F(a)}, \end{aligned}$$

which is the c.d.f. of X given $X < a$.

Problem 2. (a) $\kappa = \frac{1}{100}$.

(b) The c.d.f. is given by

$$F(x) = \begin{cases} 0, & x \leq -10 \\ \dfrac{x^2}{200} + \dfrac{x}{10} + \dfrac{1}{2}, & x \in (-10, 0] \\ -\dfrac{x^2}{200} + \dfrac{x}{10} + \dfrac{1}{2}, & x \in (0, 10] \\ 1, & x > 10. \end{cases}$$

By solving the corresponding quadratic equations we obtain the inverse function

$$F^{-1}(u) = \begin{cases} -10(1 - \sqrt{2u}), & u \in [0, \frac{1}{2}] \\ 10(1 - \sqrt{2(1-u)}), & u \in (\frac{1}{2}, 1]. \end{cases}$$

The inverse transform algorithm is the following:

1. Simulate $u \sim \mathrm{U}(0,1)$.
2. If $u \leq \frac{1}{2}$, set $x = -10(1 - \sqrt{2u})$.
   Else, set $x = 10(1 - \sqrt{2(1-u)})$.

3. Return x.

(c) We use $Y \sim \mathrm{U}(-10, 10)$ as a proposal distribution. It is easy to see that $\dfrac{f(x)}{g(x)} \leq 2$ for all $x \in [-10, 10]$ and we thus obtain $c^* = 2$. The acceptance-rejection algorithm would be:

1. Simulate $u_1, u_2 \sim \mathrm{U}(0, 1)$.
2. Set $y = -10 + 20u_1$
3. If $u_2 \leq \dfrac{10 - |y|}{10}$, set $x = y$.
   Else, go to (1).
4. Return x.

The expected number of trials is equal to $c^* = 2$ and since each iteration requires the simulation of 2 uniform r.v.s, we would require to generate 4 uniform r.v.s on average to obtain one sample of X.

Problem 3. (a) $\kappa = \frac{1}{2}$.

(b) The c.d.f. is given by

$$F(x) = \begin{cases} \frac{1}{2}\exp(x), & x \leq 0 \\ 1 - \frac{1}{2}\exp(-x) & x > 0, \end{cases}$$

and the inverse function is then

$$F^{-1}(u) = \begin{cases} \log(2u), & u \in [0, \frac{1}{2}] \\ -\log(2(1-u)), & u \in (\frac{1}{2}, 1]. \end{cases}$$

The simulation algorithm would be:

1. Simulate $u \sim \mathrm{U}(0, 1)$.
2. If $u \leq \frac{1}{2}$, set $x = \log(2u)$.
   Else, set $x = -\log(2(1-u))$.
3. Return x.

Problem 4. We do the change of variable $y = \exp(x)$ to obtain

$$\begin{aligned} \theta &= \int_{-\infty}^{1} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \\ &= \int_{0}^{e} \frac{1}{\sqrt{2\pi}y} \exp\left(-\frac{\log(y)^2}{2}\right) dx \\ &= \frac{e}{\sqrt{2\pi}} E\left(\frac{\exp\left(-\frac{\log(Y)^2}{2}\right)}{Y}\right), \end{aligned}$$

where $Y \sim \mathrm{U}(0, e)$. Therefore, we can estimate θ via simulation, that is

$$\theta \approx \frac{1}{n}\frac{e}{\sqrt{2\pi}} \sum_{i=1}^{n} \frac{\exp\left(-\frac{\log(Y_i)^2}{2}\right)}{Y_i}, \tag{1}$$

where the $Y_1, Y_2, \ldots Y_n$ are i.i.d. $\mathrm{U}(0, e)$. The real value is $\theta = \Phi(1) \approx 0.8413$.

Problem 5. Suppose that we wish to simulate a non-homogeneous Poisson process $N(t)$ with intensity function $\lambda(t)$, $t \geq 0$. One method to do so is via the *thinning* algorithm. Fix a time horizon $T > 0$ and suppose that there exists λ^* such that $\lambda(t) \leq \lambda^*$ for all $t \leq T$. The idea of the algorithm is to simulate a homogeneous Poisson process with intensity λ^* and to accept its arrivals with probability $p(t) = \frac{\lambda(t)}{\lambda^*}$. The number of accepted arrivals is precisely the desired non-homogeneous Poisson process. The pseudo-code is as follows:

1. Set $t = 0$ and $N = 0$.
2. Simulate $u_1 \sim \mathrm{U}(0,1)$.
3. Set $t = t - \frac{\log(u_1)}{\lambda^*}$.
4. While $t < T$
   (a) Simulate $u_2 \sim \mathrm{U}(0,1)$.
   (b) If $u_2 \leq \frac{\lambda(t)}{\lambda^*}$, then set $N = N + 1$.
   (c) Simulate $u_1 \sim \mathrm{U}(0,1)$.
   (d) Set $t = t - \frac{\log(u_1)}{\lambda^*}$.
5. Return N.

Problem 6. Denote V_{t_i} the cumulative cost of the delta hedging strategy at time t_i with $i = 0, 1, \ldots, n$ ($t_0 = 0$ and $t_n = T$). To simplify notation, assume that the times are equidistant with $dt = \frac{T}{n}$. Then we can establish the following recursion:

$$
\begin{aligned}
V_{t_0} &= \Delta_{t_0} S_{t_0}, \\
V_{t_m} &= V_{t_{m-1}}(1 + rdt) + (\Delta_{t_m} - \Delta_{t_{m-1}}) S_{t_m} - qdt\Delta_{t_{m-1}} S_{t_{m-1}}, \quad m = 1, 2, \ldots, n-1, \\
V_{t_n} &= V_{t_{n-1}}(1 + rdt) + (S_{t_n} - K)^+ - \Delta_{t_{n-1}} S_{t_n} - qdt\Delta_{t_{n-1}} S_{t_{n-1}}.
\end{aligned}
$$

Observe that at each new time the cost of the strategy considers the interest generated by the cost up to the previous time and also takes into account the dividends obtained by the stock held during the last period. For the last period we take into account the payoff of the option (if any) and the selling of the stock, just as seen in class. It follows that the price of the call should be close to the discounted value of V_{t_n} (assuming n large), that is

$$C \approx (1 + rdt)^{-n} V_{t_n} \approx \exp(-rT) V_{t_n}.$$