Overlap in observational studies with high-dimensional covariates

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December, 2024

D'Amour, A., Ding, P., Feller, A., Lei, L., & Sekhon, J.S. (2017). Overlap in observational studies with high-dimensional covariates. Journal of Econometrics.

Introduction

- Causal inference with observational studies relies on two key assumptions: Unconfoundedness and Overlap
- When more covariates are included in the analysis: Unconfoundedness is usually more plausible, while Overlap is more difficult to satisfy
- Main idea: the implications of overlap in observational studies with high-dimensional covariates
- Notations
  - $\{(Y_i(0), Y_i(1)), T_i, X_i\}_{i=1}^{n} \sim \{(Y(0), Y(1)), T, X\}$
  - Estimand: $\tau^{ATE} = E[Y(1) - Y(0)]$
  - $X_{1:p} \subset X$, the covariate sequence X is a stochastic process $(X_{(k)})_{k>0}$

Overlap Assumption

- Overlap $0 < e(X) < 1$ is important for the non-parametric identification of ATE, but many statistical analyses in fact require a strict version of overlap
- Strict Overlap: For some constant $\eta \in (0, 0.5)$,

$$\eta \le e(X) \le 1 - \eta$$

- Strict Overlap(Likelihood Ratio Form):

$$P_0(X \in A) := P(X \in A \mid T = 0), \; P_1(X \in A) := P(X \in A \mid T = 1)$$

$$\pi := P(T = 1)$$

By Bayes' Theorem, strict overlap is equivalent to the bound on the density ratio between P_1 and P_0

$$b_{min} =: \frac{1-\pi}{\pi}\frac{\eta}{1-\eta} \le \frac{dP_1(X)}{dP_0(X)} \le \frac{1-\pi}{\pi}\frac{1-\eta}{\eta} := b_{max}$$

Strict overlap bounds general discrepancies

$$b_{min} \leq \frac{dP_1(X_{1:p})}{dP_0(X_{1:p})} \leq b_{max}$$

- f-divergence('dissimilarity' of two probability distributions):

$$D_f(Q_1 || Q_0) := E_{Q_0}[f(\frac{dQ_1}{dQ_0})]$$

  - KL-divergence: $f(x) = xlog(x)$
  - χ^2-divergence: $f(x) = (x-1)^2$
- Theorem 1: Strict overlap implies

$$D_f(P_1(X_{1:p}) || P_0(X_{1:p})) \leq B_f(b_{min}, b_{max})$$

$$D_f(P_0(X_{1:p}) || P_1(X_{1:p})) \leq B_f(b_{max}^{-1}, b_{min}^{-1})$$

  - The upper bound B_f is actually free of p, while the left side may not
- KL-divergence
  - can be expanded into p terms, each corresponds to the discriminating information added by the new covariate
  - Strict overlap implies that the average unique discriminating information contained in each covariate converges to zero

Strict overlap bounds mean discrepancy

- Theorem 2: Strict overlap implies

$$p^{-1}\sum_{k=1}^{p}|E(X_k|T=0)-E(X_k|T=1)|$$
$$\leq p^{-\frac{1}{2}}\,min\{\lambda_0^{1/2}\cdot B_{\chi^2(1\|0)}^{1/2},\ \lambda_1^{1/2}\cdot B_{\chi^2(0\|1)}^{1/2}\}$$

- $B_{\chi^2(1\|0)}$ and $B_{\chi^2(0\|1)}$ are free of p, λ_1 and λ_0 are the largest eigenvalues of the covariance matrices $cov(X|T=1)$ and $cov(X|T=0)$
- The order of λ is usually smaller than $O(p)$ unless the components of X are highly correlated
- If the covariates are not highly correlated, the upper bound converges to zero, then strict overlap implies that the covariate means are, on average, arbitrarily close to balance

Summary

- Strict overlap restricts global discrepancies between the covariate distributions in the treated and control populations
- These restrictions become more restrictive as the dimension grows large
- Overlap assumptions should be carefully considered when adjusting for rich covariates