APPENDIX

***Proof of Proposition 1**:* Adding the constant term $(\lambda_i^{k-1})^2/2\rho$ to the right-hand side of (2a).

$$\begin{aligned}\min_{x_i\in\mathcal{X}_i} L_i^k &= \frac{1}{2}\boldsymbol{x}_{i,\mathrm{Re}}^T\boldsymbol{Q}_i\boldsymbol{x}_{i,\mathrm{Re}}+\boldsymbol{d}_i\boldsymbol{x}_{i,\mathrm{Re}}+\lambda_i^{k-1}(x_{i,\mathrm{Sh}}-y_{i,\mathrm{Sh}}^{k-1})\\&\quad+\frac{\rho}{2}(x_{i,\mathrm{Sh}}-y_{i,\mathrm{Sh}}^{k-1})^2+\frac{(\lambda_i^{k-1})^2}{2\rho}\\&=\frac{1}{2}\boldsymbol{x}_{i,\mathrm{Re}}^T\boldsymbol{Q}_i\boldsymbol{x}_{i,\mathrm{Re}}+\boldsymbol{d}_i\boldsymbol{x}_{i,\mathrm{Re}}+\frac{\rho}{2}\left(x_{i,\mathrm{Sh}}-y_{i,\mathrm{Sh}}^{k-1}+\frac{\lambda_i^{k-1}}{\rho}\right)^2\end{aligned}\tag{A1}$$

Set $x'^k_{i,\mathrm{Sh}}=x^k_{i,\mathrm{Sh}}-y^{k\text{-}1}_{i,\mathrm{Sh}}+\lambda_i^{k\text{-}1}/\rho$, $\boldsymbol{x}'_i=[\boldsymbol{x}^{\mathrm{T}}_{i,\mathrm{Re}},x'^k_{i,\mathrm{Sh}}]^{\mathrm{T}}$, $\boldsymbol{x}'_i\in\mathrm{R}^{R_i+1}$, $\boldsymbol{\theta}_{i,\mathrm{Re}}=[-\overline{x}_{i,1}\ \cdots\ -\overline{x}_{i,R_i}\ \ \underline{x}_{i,1}\ \cdots\ \underline{x}_{i,R_i}]^{\mathrm{T}}$, $\theta^k_{i,\mathrm{Sh}}=D_i-y^{k-1}_{i,\mathrm{Sh}}+\lambda_i^{k-1}/\rho$, then the (A1) can be expressed as:

$$\min_{\boldsymbol{x}'_i} L_i^k(\boldsymbol{x}'_i,\boldsymbol{\theta}_i)=\frac{1}{2}\boldsymbol{x}'^T_i\boldsymbol{Q}'_i\boldsymbol{x}'_i+\boldsymbol{d}'_i\boldsymbol{x}'_i,\tag{A2a}$$

$$\text{s.t. } \boldsymbol{A}_i\boldsymbol{x}'_i+\boldsymbol{\theta}_{i,\mathrm{Re}}\le\boldsymbol{0},:\boldsymbol{\mu}_i\tag{A2b}$$

$$\boldsymbol{1}^{1\times(R_i+1)}\boldsymbol{x}'_i-\theta^k_{i,\mathrm{Sh}}=0:\eta_i\tag{A2c}$$

$$\boldsymbol{Q}'_i=\begin{bmatrix}\boldsymbol{Q}_i & \boldsymbol{0}^{R_i}\\ \boldsymbol{0}^{1\times R_i} & \rho\end{bmatrix},\ \boldsymbol{d}'_i=\begin{bmatrix}\boldsymbol{d}_i & 0\end{bmatrix}\tag{A2d}$$

where, (A2b) corresponds to the resource constraint (1b), and (A2c) corresponds to prosumer's power balance constraint (1c). The multipliers for (A2b) and (A2c) are denoted as $\boldsymbol{\mu}_i$ and η_i.

Defining the optimal solution of (A2) as $\boldsymbol{x}'^*_i=[\boldsymbol{x}^{*\mathrm{T}}_{i,\mathrm{Re}},x'^{k*}_{i,\mathrm{Sh}}]^{\mathrm{T}}$, the KKT conditions can be written as:

$$\frac{\partial L_i(\boldsymbol{x}'^*_i,\boldsymbol{\theta}_i)}{\partial\boldsymbol{x}'^*_i}=\boldsymbol{Q}'_i\boldsymbol{x}'^*_i+\boldsymbol{d}'^{\mathrm{T}}_i+\boldsymbol{A}_i^{\mathrm{T}}\boldsymbol{\mu}_i+\boldsymbol{1}^{R+1}\eta_i=\boldsymbol{0},\tag{A3a}$$

$$(\boldsymbol{A}_i)_{\mathrm{A},s}\boldsymbol{x}'^*_i+(\boldsymbol{\theta}_{i,\mathrm{Re}})_{\mathrm{A},s}=\boldsymbol{0},\tag{A3b}$$

$$(\boldsymbol{\mu}_i)_{\mathrm{A}}\ge\boldsymbol{0},\tag{A3c}$$

$$(\boldsymbol{A}_i)_{\mathrm{I},s}\boldsymbol{x}'_i+(\boldsymbol{\theta}_{i,\mathrm{Re}})_{\mathrm{I},s}\le\boldsymbol{0},\tag{A3d}$$

$$(\boldsymbol{\mu}_i)_{\mathrm{I}}=\boldsymbol{0},\tag{A3e}$$

$$\boldsymbol{1}^{1\times(R_i+1)}\boldsymbol{x}'^*_i-\theta^k_{i,\mathrm{Sh}}=0\tag{A3f}$$

where, $(\cdot)_{\mathrm{A},s}$ and $(\cdot)_{\mathrm{I},s}$ are matrix reorganization operators, which that combine the rows corresponding to active and inactive constraints within the s segments into new matrices.

Support p_{A} and p_{I} denote the number of active and inactive constraints. Combining (A3a), (A3b), and (A3f), we have

$$\boldsymbol{Q}'_i\boldsymbol{x}'^*_i+\boldsymbol{d}'^{\mathrm{T}}_i+\begin{bmatrix}(\boldsymbol{A}_i)^{\mathrm{T}}_{\mathrm{A},s} & \boldsymbol{1}^{R_i}\\ \boldsymbol{0}^{1\times p_{\mathrm{A}}} & 1\end{bmatrix}\begin{bmatrix}(\boldsymbol{\mu}_i)_{\mathrm{A}}\\ \eta_i\end{bmatrix}=\boldsymbol{0},\tag{A4a}$$

$$\begin{bmatrix}(\boldsymbol{A}_i)_{\mathrm{A},s} & \boldsymbol{0}^{p_{\mathrm{A}}}\\ \boldsymbol{1}^{1\times R_i} & 1\end{bmatrix}\boldsymbol{x}'^*_i+\begin{bmatrix}(\boldsymbol{\theta}_{i,\mathrm{Re}})_{\mathrm{A},s}\\ -\theta^k_{i,\mathrm{Sh}}\end{bmatrix}=\boldsymbol{0},\tag{A4b}$$

Simplifying (A3c)~(A3d) and (A4a)~(A4b), then:

$$\boldsymbol{H}_{1,i,s}\boldsymbol{\Gamma}_{i,s}\ge\boldsymbol{0},\boldsymbol{H}_{1,i,s}=\begin{bmatrix}(\boldsymbol{E}^{2R_i})_{\mathrm{A},s} & \boldsymbol{0}^{p_{\mathrm{A}}}\\ \boldsymbol{0}^{1\times 2R_i} & 0\end{bmatrix},\boldsymbol{\Gamma}_{i,s}=\begin{bmatrix}(\boldsymbol{\mu}_i)_{\mathrm{A},s}\\ \eta_i\end{bmatrix}\tag{A5a}$$

$$(\boldsymbol{A}_i)_{\mathrm{I},s}\boldsymbol{x}'^*_i+\boldsymbol{H}_{2,i,s}\boldsymbol{\theta}'^k_i\le\boldsymbol{0},\boldsymbol{H}_{2,i,s}=\begin{bmatrix}(\boldsymbol{E}^{2R_i})_{\mathrm{I},s} & \boldsymbol{0}^{p_{\mathrm{I}}}\\ \boldsymbol{0}^{1\times 2R_i} & 0\end{bmatrix}\tag{A5b}$$

$$\boldsymbol{Q}'_i\boldsymbol{x}'^*_i+\boldsymbol{d}'^{\mathrm{T}}_i+\boldsymbol{G}^{\mathrm{T}}_{i,s}\boldsymbol{\Gamma}_{i,s}=\boldsymbol{0},\boldsymbol{G}_{i,s}=\begin{bmatrix}(\boldsymbol{A}_i)_{\mathrm{A},s} & \boldsymbol{0}^{p_{\mathrm{A}}}\\ \boldsymbol{1}^{1\times R_i} & 1\end{bmatrix}\tag{A5c}$$

$$\boldsymbol{G}_i\boldsymbol{x}'^*_i+\boldsymbol{H}_{3,i,s}\boldsymbol{\theta}'^k_i=\boldsymbol{0},\boldsymbol{H}_{3,i,s}=\begin{bmatrix}(\boldsymbol{E}^{2R_i})_{\mathrm{A},s} & \boldsymbol{0}^{p_{\mathrm{A}}}\\ \boldsymbol{0}^{1\times 2R_i} & -1\end{bmatrix}\tag{A5d}$$

The $\boldsymbol{E}^{2R_i}$ denotes an identity diagonal matrix of size $2R_i\times 2R_i$. (A5a), (A5b), (A5c) and (A5d) correspond to (A3c), (A3d), (A5a) and (A5b), respectively.

Considering the mutual exclusivity of the upper and lower resource constraints, rank($\boldsymbol{G}_{i,s}$)= p_{A}+1 and $\boldsymbol{G}_{i,s}\boldsymbol{Q}'^{-1}_i\boldsymbol{G}^{\mathrm{T}}_{i,s}$ is invertible. Hence, combining (A5c) and (A5d), we have

$$\boldsymbol{\Gamma}_{i,s}=\boldsymbol{M}_{1,i,s}\boldsymbol{\theta}'^k_i+\boldsymbol{M}_{2,i,s}\tag{A6a}$$

$$\boldsymbol{M}_{1,i,s}=(\boldsymbol{G}_{i,s}\boldsymbol{Q}'^{-1}_i\boldsymbol{G}^{\mathrm{T}}_{i,s})^{-1}\boldsymbol{H}_{3,i,s}\tag{A6b}$$

$$\boldsymbol{M}_{2,i,s}=-(\boldsymbol{G}_{i,s}\boldsymbol{Q}'^{-1}_i\boldsymbol{G}^{\mathrm{T}}_{i,s})^{-1}\boldsymbol{G}_{i,s}\boldsymbol{Q}'^{-1}_i\boldsymbol{d}'^{\mathrm{T}}_i\tag{A6c}$$

By substituting (A6a) into (A5c), we obtain

$$\boldsymbol{x}'^*_i=\boldsymbol{W}_{1,i,s}\boldsymbol{\theta}'^k_i+\boldsymbol{W}_{2,i,s}\tag{A7a}$$

$$\boldsymbol{W}_{1,i,s}=-\boldsymbol{Q}'^{-1}_i\boldsymbol{G}^{\mathrm{T}}_{i,s}\boldsymbol{M}_{1,i,s}\tag{A7b}$$

$$\boldsymbol{W}_{2,i,s}=-\boldsymbol{Q}'^{-1}_i\boldsymbol{d}'^{\mathrm{T}}_i-\boldsymbol{Q}'^{-1}_i\boldsymbol{G}^{\mathrm{T}}_{i,s}\boldsymbol{M}_{2,i,s}\tag{A7c}$$

Combining (A7a) with the definitions of $\boldsymbol{x}'^*_i$, we have

$$x^{k*}_{i,\mathrm{Sh}}=\boldsymbol{w}'_{1,i,s}\boldsymbol{\theta}'^k_i+\boldsymbol{w}'_{2,i,s}+y^{k-1}_{i,\mathrm{Sh}}-\frac{\lambda_i^{k-1}}{\rho}\tag{A8c}$$

where, $\boldsymbol{w}'_{1,i,s}$=$\boldsymbol{W}_{1,i,s,R_i+1}$ and $\boldsymbol{w}'_{2,i,s}$=$\boldsymbol{W}_{2,i,s,R_i+1}$ represent the vectors corresponding to the (R_i+1)-th row of $\boldsymbol{W}_{1,i,s}$ and $\boldsymbol{W}_{2,i,s}$, respectively. To ensure the complementary slackness condition holds, (A8c) is further substituted into (A5a)~(A5b) (A5b), yielding the following domain:

$$\mathit{\Theta}'_{i,s}:=\left\{\boldsymbol{\theta}'^k_i\ \middle|\ \begin{aligned}&\boldsymbol{H}_{1,i,s}(\boldsymbol{M}_{1,i,s}\boldsymbol{\theta}'^k_i+\boldsymbol{M}_{2,i,s})\ge\boldsymbol{0},\\&(\boldsymbol{A}_i)_{\mathrm{I},s}(\boldsymbol{W}_{1,i,s}\boldsymbol{\theta}'^k_i+\boldsymbol{W}_{2,i,s})+\boldsymbol{H}_{2,i,s}\boldsymbol{\theta}'^k_i\le\boldsymbol{0}\end{aligned}\right\}\tag{A9}$$

According to (A8c)~(A9), we can derive Proposition 1. The prosumer's best response can be characterized by piecewise linear function. The domain of each segment depends on the cost coefficients and the active constraints.

■End of proof of ***Proposition 1***.

***Proof of Proposition 3**:*

Since prosumers are not coupled in (4), it is equivalent to show that the $\hat{\boldsymbol{x}}^k_i$ of each prosumer satisfies the (15).

1) For prosumers in $\mathcal{I}^k_{\mathrm{C}}$. According to the KKT conditions, (15) is satisfied with $\hat{\boldsymbol{x}}^k_i$=$\boldsymbol{x}^{k*}_i$ and $\hat{\boldsymbol{\alpha}}^k_i$=$\boldsymbol{\alpha}^{k*}_i$.

2) For prosumers in $\mathcal{I}/\mathcal{I}^k_{\mathrm{C}}$. Combining (1c) and (13) and ***A1***

$$\sum_{r\in\mathcal{R}_i}(\underline{x}_{i,r}-x^*_{i,r})\le x^{k*}_{i,\mathrm{Sh}}-\tilde{x}^k_{i,\mathrm{Sh}}\le\sum_{r\in\mathcal{R}_i}(\overline{x}_{i,r}-x^*_{i,r})\tag{A10a}$$

$$-\xi\le x^{k*}_{i,\mathrm{Sh}}-\tilde{x}^k_{i,\mathrm{Sh}}\le\xi\tag{A10b}$$

According to (13) and (1c) existing $\boldsymbol{\delta}_i$ have

$$\max\{(\underline{x}_{i,r}-x^{k*}_{i,r}),-\xi\}\le\delta_{i,r}\le\min\{(\overline{x}_{i,r}-x^{k*}_{i,r}),\xi\},\tag{A11a}$$

$$\sum_{r\in\mathcal{R}_i}\delta_{i,r} = x_{i,\mathrm{Sh}}^{k*} - \tilde{x}_{i,\mathrm{Sh}}^{k} \tag{A11b}$$

Assume the $\hat{x}_{i,r}^k = x_{i,r}^{k*} + \delta_{i,r}$, then we have

$$\tilde{x}_{i,\mathrm{Sh}}^{k} + \sum_{r\in\mathcal{R}_i}\hat{x}_{i,r}^k = x_{i,\mathrm{Sh}}^{k*} + \sum_{r\in\mathcal{R}_i} x_{i,r}^{k*} = D_i \tag{A12a}$$

$$\underline{x}_{i,r} \le \hat{x}_{i,r}^k \le \overline{x}_{i,r}, -\boldsymbol{\xi} \le \hat{\boldsymbol{x}}_i^k - \boldsymbol{x}_i^{k*} \le \boldsymbol{\xi} \tag{A12b}$$

Under $\hat{\boldsymbol{x}}_i^k$, the multipliers of the inactive constraints in (4) are set to 0, and that of active constraints are consistent with $\alpha_{i,r}^{k*}$. The complementary slackness conditions are satisfied.

■End of proof of ***Proposition 3.***

Proof of Proposition 4:

According to (4), (5c) and the definition of $\boldsymbol{d}_{\mathrm{P}}^k$, we have

$$\boldsymbol{d}_{\mathrm{P}}^k = \frac{\partial f(\boldsymbol{x}^k)}{\partial \boldsymbol{x}} + \boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^k + \rho\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}^{k-1}+\boldsymbol{y}^k) + \boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\alpha}^k \tag{A13}$$

Combining (5c) and the KKT conditions of (5a)~(5b), then

$$\boldsymbol{P}_{\mathrm{G}} - \boldsymbol{B}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^k + \mathbf{1}^{I+1}\boldsymbol{\gamma}^k = \mathbf{0} \tag{A14}$$

Support the auxiliary variable $\boldsymbol{u}^k$, where

$$\boldsymbol{u}^k := \boldsymbol{d}_{\mathrm{P}}^k - \boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^k - \rho\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}^{k-1}+\boldsymbol{y}^k) - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\boldsymbol{\alpha}^k = \frac{\partial f(\boldsymbol{x}^k)}{\partial \boldsymbol{x}} \tag{A15}$$

Since $f(\boldsymbol{x})$ is a convex function, we have

$$\left\langle \boldsymbol{u}^k + \boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^* + \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\boldsymbol{\alpha}^*, \boldsymbol{x}^k - \boldsymbol{x}^* \right\rangle \ge 0 \tag{A16a}$$

$$\left\langle \boldsymbol{B}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^k - \mathbf{1}^{I+1}\boldsymbol{\gamma}^k - \boldsymbol{B}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^* + \mathbf{1}^{I+1}\boldsymbol{\gamma}^*, \boldsymbol{y}^k - \boldsymbol{y}^* \right\rangle = 0 \tag{A16b}$$

Bring (5c) and (A15) to (A16), then we obtain

$$\begin{aligned} 0 \le &\left\langle \boldsymbol{d}_{\mathrm{P}}^k, \boldsymbol{x}^k - \boldsymbol{x}^* \right\rangle - \left\langle \boldsymbol{\lambda}^k - \boldsymbol{\lambda}^*, \boldsymbol{A}_{\mathrm{C}}(\boldsymbol{x}^k - \boldsymbol{x}^*) \right\rangle \\ &- \left\langle \rho\boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}^{k-1}+\boldsymbol{y}^k), \boldsymbol{A}_{\mathrm{C}}(\boldsymbol{x}^k - \boldsymbol{x}^*) \right\rangle \\ &- \left\langle (\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^*), -\boldsymbol{B}_{\mathrm{C}}(\boldsymbol{y}^k - \boldsymbol{y}^*) \right\rangle \\ &+ \left\langle -(\boldsymbol{\alpha}^k - \boldsymbol{\alpha}^*), \boldsymbol{A}_{\mathrm{P}}(\boldsymbol{x}^k - \boldsymbol{x}^*) \right\rangle \\ &+ \left\langle -(\boldsymbol{\gamma}^k - \boldsymbol{\gamma}^*), \mathbf{1}^{I+1}(\boldsymbol{y}^k - \boldsymbol{y}^*) \right\rangle \end{aligned} \tag{A17}$$

According to ***Proposition 3***, we have that

$$\left\langle -(\boldsymbol{\alpha}^k - \boldsymbol{\alpha}^*), \boldsymbol{A}_{\mathrm{P}}(\boldsymbol{x}^k - \boldsymbol{x}^*) \right\rangle \le 0 \tag{A18a}$$

$$\left\langle -(\boldsymbol{\gamma}^k - \boldsymbol{\gamma}^*), \mathbf{1}^{I+1}(\boldsymbol{y}^k - \boldsymbol{y}^*) \right\rangle = 0 \tag{A18b}$$

Combining (A17) and (A18), we can get

$$\begin{aligned} &\left\langle \boldsymbol{d}_{\mathrm{P}}^k, \boldsymbol{x}^k - \boldsymbol{x}^* \right\rangle - \left\langle \rho\boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}_{\mathrm{Sh}}^{k-1}+\boldsymbol{y}_{\mathrm{Sh}}^k), \boldsymbol{A}_{\mathrm{C}}(\boldsymbol{x}^k - \boldsymbol{x}^*) \right\rangle \\ &- \left\langle \boldsymbol{\lambda}^k - \boldsymbol{\lambda}^*, -\boldsymbol{B}_{\mathrm{C}}(\boldsymbol{y}^k - \boldsymbol{y}^*) + \boldsymbol{A}_{\mathrm{C}}(\boldsymbol{x}^k - \boldsymbol{x}^*) \right\rangle \ge 0 \end{aligned} \tag{A19}$$

Bringing (5c) into the third term on (A19), we have

$$\begin{aligned} &\left\langle \boldsymbol{d}_{\mathrm{P}}^k, \boldsymbol{x}^k - \boldsymbol{x}^* \right\rangle - \left\langle \boldsymbol{\lambda}^k - \boldsymbol{\lambda}^*, \frac{1}{\rho}(\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^{k-1}) \right\rangle \\ &\ge \left\langle \rho\boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}^{k-1}+\boldsymbol{y}^k), \boldsymbol{A}_{\mathrm{C}}(\boldsymbol{x}^k - \boldsymbol{x}^*) \right\rangle \end{aligned} \tag{A20}$$

With multiplying both sides of (A20) by 2ρ, (19) is obtained

■End of proof of ***Proposition 4***.

Proof of Proposition 5:

According to the definition of inner product, we have

$$\frac{2}{\rho}\left|\left\langle \boldsymbol{\omega}^{k-1} - \boldsymbol{x}^k, \boldsymbol{d}_{\mathrm{P}}^k \right\rangle\right| \le \frac{2}{\rho}\left\langle \left|\boldsymbol{\omega}^{k-1}\right| + \left|\boldsymbol{x}^k\right|, \left|\boldsymbol{d}_{\mathrm{P}}^k\right| \right\rangle \tag{A21}$$

According to (16) and the definition of $\boldsymbol{d}_{\mathrm{P}}^k$, there are

$$\left|\frac{\partial L'_{\mathrm{P}}(\boldsymbol{x}^k, \boldsymbol{\alpha}^k)}{\partial \boldsymbol{x}} - \frac{\partial L'_{\mathrm{P}}(\boldsymbol{x}^{k*}, \boldsymbol{\alpha}^{k*})}{\partial \boldsymbol{x}}\right| = \left|d_{\mathrm{P}}^k\right| \le \boldsymbol{\xi}' \tag{A22}$$

Combining (17b)~(17c) and (A22), we can get

$$\frac{2}{\rho}\left\langle \left|\boldsymbol{\omega}^{k-1}\right| + \left|\boldsymbol{x}^k\right|, \left|\boldsymbol{d}_{\mathrm{P}}^k\right| \right\rangle \le \frac{2}{\rho}\left\langle \left|\boldsymbol{\omega}^{k-1}\right| + \overline{\boldsymbol{x}}_{\max}^k, \boldsymbol{\xi}' \right\rangle \tag{A23a}$$

$$\left|\boldsymbol{\omega}^{k-1}\right| \le \overline{\boldsymbol{\omega}}^{k-1} \tag{A23b}$$

Then, we can obtain

$$\left\langle \left|\boldsymbol{\omega}^{k-1}\right| + \left|\boldsymbol{x}^k\right|, \left|\boldsymbol{d}_{\mathrm{P}}^k\right| \right\rangle \le \left\langle \overline{\boldsymbol{\omega}}^{k-1} + \overline{\boldsymbol{x}}_{\max}^k, \boldsymbol{\xi}' \right\rangle \tag{A24}$$

According to (17a), (A22) and (A24), the (20) is satisfied.

■End of proof of ***Proposition 5***.

Proof of Proposition 6:

According to (17c) and ***Proposition 5***, we can get

$$\begin{aligned} &\left\|\boldsymbol{\omega}^{k-1} - x^*\right\|_2^2 - \left\|\boldsymbol{\omega}^k - x^*\right\|_2^2 \\ &= -\left\|\boldsymbol{\omega}^k - \boldsymbol{\omega}^{k-1}\right\|_2^2 - 2\left\langle \boldsymbol{\omega}^{k-1} - x^*, \boldsymbol{\omega}^k - \boldsymbol{\omega}^{k-1} \right\rangle \\ &= -\rho^2\left\|\boldsymbol{d}_{\mathrm{P}}^k\right\|_2^2 + 2\rho\left\langle \boldsymbol{\omega}^{k-1} - x^k, \boldsymbol{d}_{\mathrm{P}}^k \right\rangle + 2\rho\left\langle x^k - x^*, \boldsymbol{d}_{\mathrm{P}}^k \right\rangle \\ &\ge -\rho^2\left\|\boldsymbol{d}_{\mathrm{P}}^k\right\|_2^2 + 2\rho\left\langle x^k - x^*, \boldsymbol{d}_{\mathrm{P}}^k \right\rangle - 2\rho\left|\left\langle \boldsymbol{\omega}^k - x^*, \boldsymbol{d}_{\mathrm{P}}^k \right\rangle\right| \\ &\ge 2\rho\left\langle x^k - x^*, \boldsymbol{d}_{\mathrm{P}}^k \right\rangle - \rho^2\sigma\left\|\boldsymbol{A}_{\mathrm{C}}\boldsymbol{x}^k - \boldsymbol{B}_{\mathrm{C}}\boldsymbol{y}^{k-1}\right\|_2^2 \end{aligned} \tag{A25}$$

Similarly, we have

$$\begin{aligned} &\left\|\boldsymbol{\lambda}^{k-1} - \boldsymbol{\lambda}^*\right\|_2^2 - \left\|\boldsymbol{\lambda}^k - \boldsymbol{\lambda}^*\right\|_2^2 \\ &= \left\|\boldsymbol{\lambda}^{k-1} - \boldsymbol{\lambda}^k\right\|_2^2 + 2\left\langle \boldsymbol{\lambda}^{k-1} - \boldsymbol{\lambda}^k, \boldsymbol{\lambda}^k - \boldsymbol{\lambda}^* \right\rangle \end{aligned} \tag{A26}$$

Furthermore, according to Proposition 4, (A25)~(A26) and (5c), we have

$$\begin{aligned} &\Phi^{k-1} - \Phi^k \\ &\ge \rho^2\left\|\boldsymbol{A}_{\mathrm{C}}\boldsymbol{x}^k - \boldsymbol{B}_{\mathrm{C}}\boldsymbol{y}^k\right\|_2^2 - \rho^2\sigma\left\|\boldsymbol{A}_{\mathrm{C}}\boldsymbol{x}^k - \boldsymbol{B}_{\mathrm{C}}\boldsymbol{y}^{k-1}\right\|_2^2 \\ &-\rho^2\left\|\boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}^k + \boldsymbol{y}^*)\right\|_2^2 + \rho^2\left\|\boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}^{k-1} + \boldsymbol{y}^*)\right\|_2^2 \\ &-2\rho^2\left\langle \boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}^k + \boldsymbol{y}^{k-1}), \boldsymbol{A}_{\mathrm{C}}(\boldsymbol{x}^k - \boldsymbol{x}^*) \right\rangle \end{aligned} \tag{A27}$$

Due to $\boldsymbol{A}_{\mathrm{C}}\boldsymbol{x}^* - \boldsymbol{B}_{\mathrm{C}}\boldsymbol{y}^* = 0$ and Cauchy-Schwarz Inequality, the fifth term on the right-hand side of (A27) can be written as

$$\begin{aligned} &2\rho^2\left\langle \boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}^k + \boldsymbol{y}^{k-1}), \boldsymbol{A}_{\mathrm{C}}(\boldsymbol{x}^k - \boldsymbol{x}^*) \right\rangle \\ &= \rho^2\left(\left\|\boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}_{=}^{k-1} + \boldsymbol{y}_{=}^k)\right\|_2^2 + \left\|\boldsymbol{A}_{\mathrm{C}}\boldsymbol{x}^k - \boldsymbol{B}_{\mathrm{C}}\boldsymbol{y}^k\right\|_2^2 - \left\|\boldsymbol{A}_{\mathrm{C}}\boldsymbol{x}^k - \boldsymbol{B}_{\mathrm{C}}\boldsymbol{y}^{k-1}\right\|_2^2\right) \\ &-\rho^2\left(\left\|\boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}^{k-1} + \boldsymbol{y}^k)\right\|_2^2 + \left\|\boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}^k + \boldsymbol{y}^*)\right\|_2^2 - \left\|\boldsymbol{B}_{\mathrm{C}}(-\boldsymbol{y}^{k-1} + \boldsymbol{y}^*)\right\|_2^2\right) \end{aligned} \tag{A28}$$

Bring (A28) to (A27), we obtain

$$\Phi^{k-1} - \Phi^k \ge \rho^2(1-\sigma)\left\|\boldsymbol{A}_{\mathrm{C}}\boldsymbol{x}^k - \boldsymbol{B}_{\mathrm{C}}\boldsymbol{y}^{k-1}\right\|_2^2 \tag{A29}$$

Obviously, is a non-increasing and positive sequence. Then, $\{\boldsymbol{y}^k\}$, $\{\boldsymbol{\lambda}^k\}$, and $\{\boldsymbol{\omega}^k\}$ are bounded sequences. And we can get

$$\lim_{k\to\infty} \boldsymbol{A}_{\mathrm{C}}\boldsymbol{x}^k - \boldsymbol{B}_{\mathrm{C}}\boldsymbol{y}^{k-1} = \boldsymbol{0} \tag{A30}$$

Besides, based on ***Proposition 3***, $\{\boldsymbol{x}^k\}$ is bounded sequence. Since $\{\boldsymbol{x}^k\}$, $\{\boldsymbol{y}^k\}$, $\{\boldsymbol{\lambda}^k\}$ is bounded, $\{\boldsymbol{\alpha}^k\}$ and $\{\boldsymbol{\gamma}^k\}$ are bounded, according to KKT conditions of (4) and (5).

■End of proof of ***Proposition 6***.

Proof of Proposition 7:

With Bolzano-Weierstrass Theorem, there are $\mathcal{K}$ satisfying

$$\lim_{k\to\infty,k\in\mathcal{K}} \boldsymbol{\lambda}^k = \overline{\boldsymbol{\lambda}},\ \lim_{k\to\infty,k\in\mathcal{K}} \boldsymbol{\alpha}^k = \overline{\boldsymbol{\alpha}},\ \lim_{k\to\infty,k\in\mathcal{K}} \boldsymbol{\gamma}^k = \overline{\boldsymbol{\gamma}} \tag{A31}$$

According to ***Proposition 5***, (A30) and the definition of $\boldsymbol{\omega}^k$, we have

$$\left\langle \boldsymbol{\omega}^{k-1} - \boldsymbol{x}^k, \boldsymbol{d}_{\mathrm{P}}^k \right\rangle = \left\| \boldsymbol{d}_{\mathrm{P}}^k \right\|^2 = 0 \tag{A32}$$

Since $\{\boldsymbol{\omega}^k\}$ and $\{\boldsymbol{x}^k\}$ are bounded, we obtain

$$\left\langle \boldsymbol{x}^k, \boldsymbol{d}_{\mathrm{P}}^k \right\rangle = \left\langle \boldsymbol{\omega}^k, \boldsymbol{d}_{\mathrm{P}}^k \right\rangle = \left\langle \boldsymbol{A}_{\mathrm{C}}\boldsymbol{x}^k - \boldsymbol{B}_{\mathrm{C}}\boldsymbol{y}^{k-1}, \boldsymbol{A}\boldsymbol{x}^k \right\rangle = 0 \tag{A33}$$

Support the Fenchel conjugate function of f is f^*. According to Legendre Transform Definition, there are

$$\boldsymbol{u}^k = \partial f(\boldsymbol{x}^k) \Leftrightarrow \boldsymbol{x}^k = \partial f^*(\boldsymbol{u}^k) \tag{A34}$$

Due to convexity, we have

$$\begin{aligned} & f^*(-\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^* - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\boldsymbol{\alpha}^*) \\ & \quad \ge f^*(\boldsymbol{u}^k) + \left\langle \partial f^*(\boldsymbol{u}^k),\ -\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^* - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\boldsymbol{\alpha}^* - \boldsymbol{u}^k \right\rangle \end{aligned} \tag{A35}$$

Bring (A15) into (A35), we can get

$$\begin{aligned} & f^*(-\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^* - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\boldsymbol{\alpha}^*) \\ & \quad \ge f^*(\boldsymbol{u}^k) + \left\langle \boldsymbol{x}^k,\ \boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}(\boldsymbol{\lambda}^{k-1} - \boldsymbol{\lambda}^*) \right\rangle \\ & \qquad + \left\langle \boldsymbol{A}_{\mathrm{C}}\boldsymbol{x}^k,\ \rho(\boldsymbol{A}_{\mathrm{C}}\boldsymbol{x}^k - \boldsymbol{B}_{\mathrm{C}}\boldsymbol{y}^{k-1}) \right\rangle \\ & \qquad - \left\langle \boldsymbol{x}^k, \boldsymbol{d}_{\mathrm{P}}^k \right\rangle + \left\langle \boldsymbol{x}^k,\ \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}(\boldsymbol{\alpha}^k - \boldsymbol{\alpha}^*) \right\rangle \end{aligned} \tag{A36}$$

According to (A15) and (A30)~(A33), we have

$$\lim_{k\to\infty,k\in\mathcal{K}} \boldsymbol{u}^k = -\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\overline{\boldsymbol{\lambda}} - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\overline{\boldsymbol{\alpha}} \tag{A37}$$

Besides, $\boldsymbol{\gamma}^k$ and $\boldsymbol{\gamma}^*$ satisfy the KKT conditions of problem (1) and (5a)~(5b). Hence, there are

$$\boldsymbol{1}^{1\times(I+1)}\boldsymbol{y}^k = \boldsymbol{1}^{1\times(I+1)}\boldsymbol{y}^* = \boldsymbol{0} \tag{A38a}$$

$$\boldsymbol{P}_{\mathrm{G}} - \boldsymbol{B}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^k + \boldsymbol{1}^{I+1}\boldsymbol{\gamma}^k = \boldsymbol{0}, \boldsymbol{P}_{\mathrm{G}} - \boldsymbol{B}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^* + \boldsymbol{1}^{I+1}\boldsymbol{\gamma}^* = \boldsymbol{0} \tag{A38b}$$

Based on (A31)~(A33) and (A38b), the (A36) can be written as following when $k \to \infty, k \in \mathcal{K}$.

$$\begin{aligned} & f^*(-\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^* - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\boldsymbol{\alpha}^*) \\ & \quad \ge f^*(-\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\overline{\boldsymbol{\lambda}} - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\overline{\boldsymbol{\alpha}}) + \left\langle \boldsymbol{A}_{\mathrm{C}}\boldsymbol{x}^k, \overline{\boldsymbol{\lambda}} - \boldsymbol{\lambda}^* \right\rangle \\ & \qquad + \left\langle \boldsymbol{A}_{\mathrm{P}}\boldsymbol{x}^k, \boldsymbol{\alpha}^k - \boldsymbol{\alpha}^* \right\rangle - \left\langle \boldsymbol{B}_{\mathrm{C}}\boldsymbol{y}^{k-1}, \overline{\boldsymbol{\lambda}} - \boldsymbol{\lambda}^* \right\rangle \\ & \qquad + \left\langle \boldsymbol{y}^{k-1}, \boldsymbol{1}^{I+1}(\boldsymbol{\gamma}^{k-1} - \boldsymbol{\gamma}^*) \right\rangle \end{aligned} \tag{A39}$$

Bringing (A30), (A38b) and (15a) into (A39) yields

$$\begin{aligned} & f^*(-\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^* - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\boldsymbol{\alpha}^*) \\ & \quad \ge f^*(-\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\overline{\boldsymbol{\lambda}} - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\overline{\boldsymbol{\alpha}}) - \left\langle \boldsymbol{b}_{\mathrm{P}}, \overline{\boldsymbol{\alpha}} \right\rangle \\ & \qquad - \left\langle \boldsymbol{A}_{\mathrm{P}}\boldsymbol{x}^k + \boldsymbol{b}_{\mathrm{P}}, \boldsymbol{\alpha}^* \right\rangle + \left\langle \boldsymbol{b}_{\mathrm{P}}, \boldsymbol{\alpha}^* \right\rangle \\ & \quad \ge f^*(-\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\overline{\boldsymbol{\lambda}} - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\overline{\boldsymbol{\alpha}}) - \left\langle \boldsymbol{b}_{\mathrm{P}}, \overline{\boldsymbol{\alpha}} \right\rangle + \left\langle \boldsymbol{b}_{\mathrm{P}}, \boldsymbol{\alpha}^* \right\rangle \end{aligned} \tag{A40}$$

That is

$$f^*(-\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda}^* - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\boldsymbol{\alpha}^*) - \left\langle \boldsymbol{b}_{\mathrm{P}}, \boldsymbol{\alpha}^* \right\rangle \ge f^*(-\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\overline{\boldsymbol{\lambda}} - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\overline{\boldsymbol{\alpha}}) - \left\langle \boldsymbol{b}_{\mathrm{P}}, \overline{\boldsymbol{\alpha}} \right\rangle \tag{A41}$$

Moreover, solving the primary problem (1) is equivalent to solving its dual problem:

$$\min_{\lambda,\alpha,\gamma} f^*(-\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\boldsymbol{\lambda} - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\boldsymbol{\alpha}) - \boldsymbol{P}_{\mathrm{G}} - \left\langle \boldsymbol{b}_{\mathrm{P}}, \boldsymbol{\alpha} \right\rangle - \left\langle \boldsymbol{0}, \boldsymbol{\lambda} \right\rangle - \left\langle \boldsymbol{0}, \boldsymbol{\gamma} \right\rangle \tag{A42}$$

Since (1) is a convex programming, $\overline{\boldsymbol{\lambda}}$, $\overline{\boldsymbol{\alpha}}$, $\overline{\boldsymbol{\gamma}}$ is the global optimal solution $\boldsymbol{\lambda}^*$, $\boldsymbol{\alpha}^*$, $\boldsymbol{\gamma}^*$. Similarly, there are $\mathcal{K}' \subseteq \mathcal{K}$ satisfying

$$\lim_{k\to\infty,k\in\mathcal{K}'} \boldsymbol{x}^k = \overline{\boldsymbol{x}},\ \lim_{k\to\infty,k\in\mathcal{K}'} \boldsymbol{y}^k = \overline{\boldsymbol{y}} \tag{A43}$$

According to (A15) and (A31)~(A33), we have

$$\lim_{k\to\infty,k\in\mathcal{K}'} \boldsymbol{u}^k = \overline{\boldsymbol{u}} = -\boldsymbol{A}_{\mathrm{C}}^{\mathrm{T}}\overline{\boldsymbol{\lambda}} - \boldsymbol{A}_{\mathrm{P}}^{\mathrm{T}}\overline{\boldsymbol{\alpha}} = \frac{\partial f(\overline{\boldsymbol{x}})}{\partial \boldsymbol{x}} \tag{A43}$$

According to $\overline{\boldsymbol{\lambda}} = \boldsymbol{\lambda}^*$, $\overline{\boldsymbol{\alpha}} = \boldsymbol{\alpha}^*$, $\overline{\boldsymbol{\gamma}} = \boldsymbol{\gamma}^*$, (15), (A30), (A38) and (A43), the KKT conditions of (1) are satisfied by $\left\{\overline{\boldsymbol{\lambda}}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\gamma}}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}\right\}$. Therefore, it is the optimal solution.

Based on (A32), $\{\boldsymbol{\omega}^k\}$ is a global convergence sequence that converges to $\boldsymbol{\omega}^\infty$. Similarly, there is $\mathcal{K}'' \subseteq \mathcal{K}'$ satisfying

$$\lim_{k\to\infty} \boldsymbol{\omega}^k = \boldsymbol{\omega}^\infty \tag{A44a}$$

$$\lim_{k\to\infty,k\in\mathcal{K}''} \left\| -\boldsymbol{y}^k + \boldsymbol{y}^* \right\|_2^2 = \liminf_{k\to\infty} \left\| -\boldsymbol{y}^k + \boldsymbol{y}^* \right\|_2^2 \tag{A44b}$$

Introduce an auxiliary variable which satisfies

$$\phi^k = \left\| \boldsymbol{\omega}^k - x^* \right\|_2^2 + \rho^2 \left\| -\boldsymbol{y}^k + \boldsymbol{y}^* \right\|_2^2 \tag{A45}$$

According to (A44), we have

$$\lim_{k\to\infty,k\in\mathcal{K}''} \phi^k = \liminf_{k\to\infty} \phi^k = \left\| \boldsymbol{\omega}^\infty - x^* \right\|^2 + \rho^2 \liminf_{k\to\infty} \left\| -\boldsymbol{y}^k + \boldsymbol{y}^* \right\|^2 \tag{A46}$$

From (A21a), Φ^k is global convergence. Then, we can get

$$\limsup_{k\to\infty} \Phi^k = \lim_{k\to\infty} \Phi^k = \lim_{k\to\infty,k\in\mathcal{K}''} \Phi^k \tag{A47}$$

According to $\mathcal{K}'' \subseteq \mathcal{K}'$, (21b), (A31), and $\overline{\boldsymbol{\lambda}} = \boldsymbol{\lambda}^*$, we have

$$\lim_{k\to\infty,k\in\mathcal{K}''} \Phi^k = \lim_{k\to\infty,k\in\mathcal{K}''} \phi^k = \liminf_{k\to\infty} \phi^k \tag{A48}$$

Combining (21b) and (A45), we obtain

$$\begin{aligned} \limsup_{k\to\infty} \left\| \boldsymbol{\lambda}^k - \boldsymbol{\lambda}^* \right\|^2 &= \limsup_{k\to\infty} (\Phi^k - \phi^k) \\ &\ge \limsup_{k\to\infty} \Phi^k - \liminf_{k\to\infty} \phi^k = 0 \end{aligned} \tag{A49}$$

That is $\lim_{k\to\infty} \boldsymbol{\lambda}^k = \boldsymbol{\lambda}^*$. Since Φ^k, $\boldsymbol{\lambda}^k$, $\boldsymbol{\omega}^k$ is global convergence, $\boldsymbol{y}^k$ is also global convergence. So

$$\lim_{k\to\infty} \boldsymbol{y}^k = \boldsymbol{y}^* \tag{A50}$$

■End of proof of ***Proposition 7***.