2. a)

$g_1(x) = \frac{x^2+2}{3}$:

- when $x=2$, $g_1(x) = \frac{2^2+2}{3} = 2$ (fixed point $x=2$)
- $g_1(x)$ has continuous derivative in an open interval containing $x=2$.
- $|g_1'(2)| = \left|\frac{2x}{3}\right| = \left|\frac{2\times 2}{3}\right| = \frac{4}{3} > 1$

Thus, $g_1(x)$ diverges.

$g_2(x) = \sqrt{3x-2}$:

- when $x=2$, $g_2(x) = \sqrt{3\times 2-2} = \sqrt{4} = 2$ (fixed point $x=2$)
- $g_2(x)$ has continuous derivative in an open interval containing $x=2$.
- $|g_2'(2)| = \left|\frac{3}{2\sqrt{3x-2}}\right| = \left|\frac{3}{2\times\sqrt{3\times 2-2}}\right| = \left|\frac{3}{2\times 2}\right| = \left|\frac{3}{4}\right| < 1$

Thus, $g_2(x)$ converges linearly with $C = \frac{3}{4}$.

$g_3(x) = 3 - \frac{2}{x}$:

- when $x=2$, $g_3(x) = 3 - \frac{2}{2} = 2$ (fixed point $x=2$)
- $g_3(x)$ has continuous derivative in an open interval containing $x=2$.
- $|g_3'(2)| = \left|\frac{2}{x^2}\right| = \left|\frac{2}{2^2}\right| = \left|\frac{1}{2}\right| < 1$

Thus, $g_3(x)$ converges linearly with $C = \frac{1}{2}$

$$g_4(x) = \frac{x^2-2}{2x-3}$$

- when $x=2$, $g_4(x) = \frac{2^2-2}{2\times2-3} = \frac{2}{1} = 2$ (fixed point $x=2$)

- $g_4(x)$ has continuous derivative in an open interval containing $x=2$.

- $|g_4'(x)| = \left|\frac{2x(2x-3)-2(x^2-2)}{(2x-3)^2}\right| = \left|\frac{2\times2(2\times2-3)-2(\cancel{2}-2)}{(2\times2-3)^2}\right| = |0|$

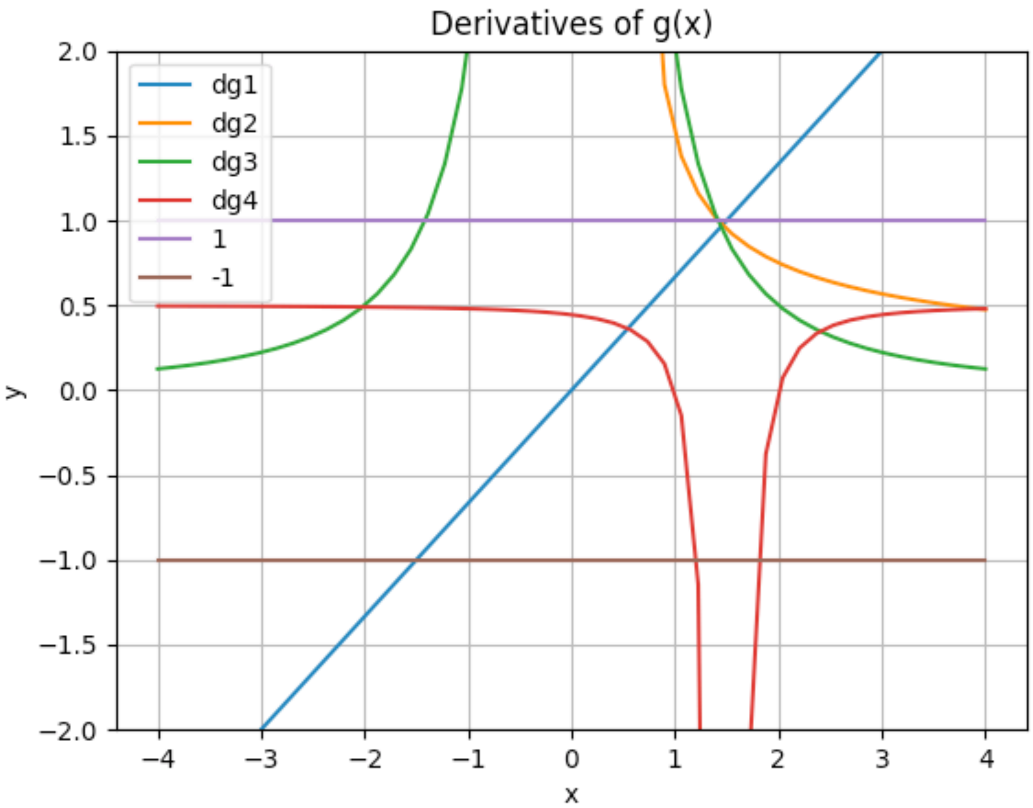
Thus, $g_4(x)$ converges quadratically.

b) For g_1, we can't find any interval such that $|g'(x)|<1$ for root $x=2$.

For g_2, we can find on the Derivatives of $g(x)$ graph that interval $(1.45, \infty)$ has $|g'(x)|<1$ and contains root $x=2$.

For g_3, we can find on the Derivatives of $g(x)$ graph that interval $(1.45, \infty)$ has $|g'(x)|<1$ and contains root $x=2$.

For g_4, we can find on the Derivatives of $g(x)$ graph that interval $(1.85, \infty)$ has $|g'(x)|<1$ and contains root $x=2$.



c)

```
Method g1
i x
0 1.5
1 1.4166666666666667
2 1.3356481481481481
3 1.2613186585505258
4 1.1969749194692325
5 1.1442496526127919
6 1.1031024225014983
7 1.0722783181762248
8 1.049926930543611
9 1.0341155198269094
10 1.0231316361156264
11 1.0155994482735462
12 1.0104807464445105
13 1.0070237796450183
14 1.0046989642568462
15 1.00314000292626
16 1.0020966218236322
17 1.0013992131567786
18 1.0009334613703385
19 1.000622598030269
20 1.0004151945629485
21 1.000276853837474
22 1.0001845947743317
23 1.0001230745412981
24 1.000082054743313
25 1.0000547054065356
26 1.000036471268584
27 1.0000243146224406
28 1.0000162099453607
29 1.000010806717828
30 1.0000072045174804
31 1.0000048030289552
32 1.000003202026993
33 1.0000021346880796
34 1.0000014231269054
```

```
Method g2
i x
0 1.45
1 1.532970971675589
2 1.612114423676796
3 1.684144670457496
4 1.74712163611252
5 1.8003791012832713
6 1.8442172604793108
7 1.8795349907458314
8 1.9075127711859479
9 1.9293880671233157
10 1.9463206830761335
11 1.9593269378101248
12 1.969258950323795
13 1.9768097660046566
14 1.9825310333041373
15 1.98685507773275
16 1.9901168893304357
17 1.9925738801839463
18 1.9944226333833657
19 1.995812591439912
20 1.9968569739267097
21 1.9976413396253416
22 1.9982302216901897
23 1.9986722255213758
24 1.999003921097737
25 1.9992528012468087
26 1.999439522401322
27 1.9995795976164505
28 1.9996846733546148
29 1.999763491031838
30 1.9998226104071115
31 1.9998669533799829
32 1.9999002125456031
33 1.9999251580088715
34 1.9999438677189454
35 1.9999579003461139
36 1.9999684250103402
37 1.999976318617553
38 1.9999822388843005
39 1.999986679118864
40 1.9999900093141947
41 1.9999925069716098
42 1.9999943802208118
43 1.9999957851611676
44 1.9999968388683775
45 1.9999976291498778
```

```
Method g3
i x
0 1.45
1 1.6206896551724137
2 1.7659574468085106
3 1.8674698795180722
4 1.929032258064516
5 1.9632107023411371
6 1.981260647359455
7 1.9905417024935512
8 1.9952483801295897
9 1.997618532149816
10 1.9988078465373362
11 1.9994035677492816
12 1.9997016949152542
13 1.9998508252078275
14 1.999925407040219
15 1.9999627021290303
16 1.999981350716726
17 1.9999906752714132
18 1.9999953376139687
19 1.99999766880155
20 1.9999988343994164
21 1.9999994171993685
```

```
Method g4
i x
0 1.85
1 2.032142857142857
2 2.0009707574304887
3 2.000000940543909
4 2.0000000000008846
```

Table 1

	r (convergency rate)	C
g_2	(1.0000016032783208,	0.7500175869738762)
g_3	(1.0000025226992548,	0.500018107336575)
g_4	(1.9994415977414204,	0.9961312021156257)

For g_1: The table 1 proves that it does not converges at all, which matches the analysis from (a)

The table 1 shows g_2, g_3, g_4 converges, which matches the analysis from (a).

For g_2: $r \approx 1$ and $C \approx 0.75\ (\frac{3}{4}) < 1 \Rightarrow$ linear convergence
matches the analysis from (a)

For g_3: $r \approx 1$ and $C \approx 0.5\ (\frac{1}{2}) < 1 \Rightarrow$ linear convergence
matches the analysis from (a)

For g_4: $r \approx 2$ and $C \approx 1 > 0 \Rightarrow$ quadric convergence (superlinear convergence)
matches the analysis from (a)

The convergence rate for $g_{2,3}$ is nearly 1, which means it converges linearly, and for g_4 is nearly 2, which means it converges much faster than $g_{2,3}$, since it converges quadritically.

d) From Newton's method,

$$x^{(k+1)} = x^{(k)} - \frac{f(x^{(k)})}{f'(x^{(k)})}$$ from W8 slides

Thus, if we consider $g(x) = x - \frac{f(x)}{f'(x)}$,

Then, $x^{(k+1)} = g(x^{(k)})$

$\Rightarrow$ We can have Newton's method in the format of fixed point method.

In the current problem,

$f(x) = x^2 - 3x + 2$,

$f'(x) = 2x - 3$,

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 3x + 2}{2x - 3}$$

$$= \frac{2x^2 - 3x}{2x - 3} - \frac{x^2 - 3x + 2}{2x - 3}$$

$$= \frac{x^2 - 2}{2x - 3}$$

Thus, we have $g_4(x)$ corresponds to Newton's method.

3. a) $x_1 = x_0 - a_0 \frac{f(x_0)}{f'(x_0)}$

$$= x_0 - a_0 \cdot \frac{(x_0 - x^*)^m}{m(x_0 - x^*)^{m-1}}$$

$$= x_0 - a_0 \cdot \frac{x_0 - x^*}{m}$$

α_0 should be equal to m since

if $\alpha_0 = m$:

$$x_1 = x_0 - m \cdot \frac{x_0 - x^*}{m}$$

$$= x_0 - x_0 + x^*$$

$$= x^*$$

Thus, when $\alpha_0 = m$, it will make x_1 closest to x^*, which means it will get the good approximation most quickly.

b)

Since $C = 1 - \frac{1}{m}$

$$Cm = m - 1$$

$$1 = m - Cm$$

$$1 = m(1-C)$$

$$\frac{1}{1-C} = m$$

$$m = \frac{1}{1 - \frac{|X_2 - X_1|}{|X_1 - X_0|}}$$

Pseudo code:

- pick an initial x value x_0
- run two iterations of Newton's method (assume $\alpha=1$), thus we get x_1, x_2
- Use the equation calculated above,

  $$m = \frac{1}{1 - \frac{|x_2 - x_1|}{|x_1 - x_0|}}$$, plug x_2, x_1 and x_0,

  we can get m value.
- set $\alpha = m$
- repeat above process (expect set $\alpha_k = \alpha = m$) until we get the α_k we want

Since we know $C = 1 - \frac{1}{m}$, we can compute that $m = \frac{1}{1-C}$. Also we can approximate C by $\frac{|x_{k+1} - x_k|}{|x_k - x_{k-1}|}$, then we can get the relationship between m and x.

Then, we can approximate m value after we compute x values using Newton's method.

From a), we conclude when $\alpha = m$ we can find good approximation for x^* quickly, thus we return $\alpha = m$ in our algorithm.