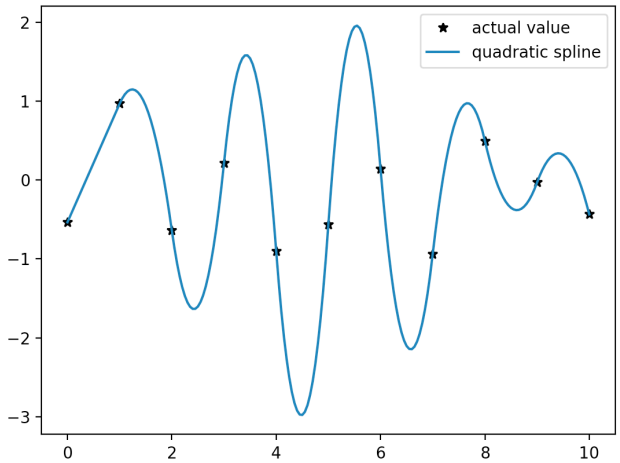
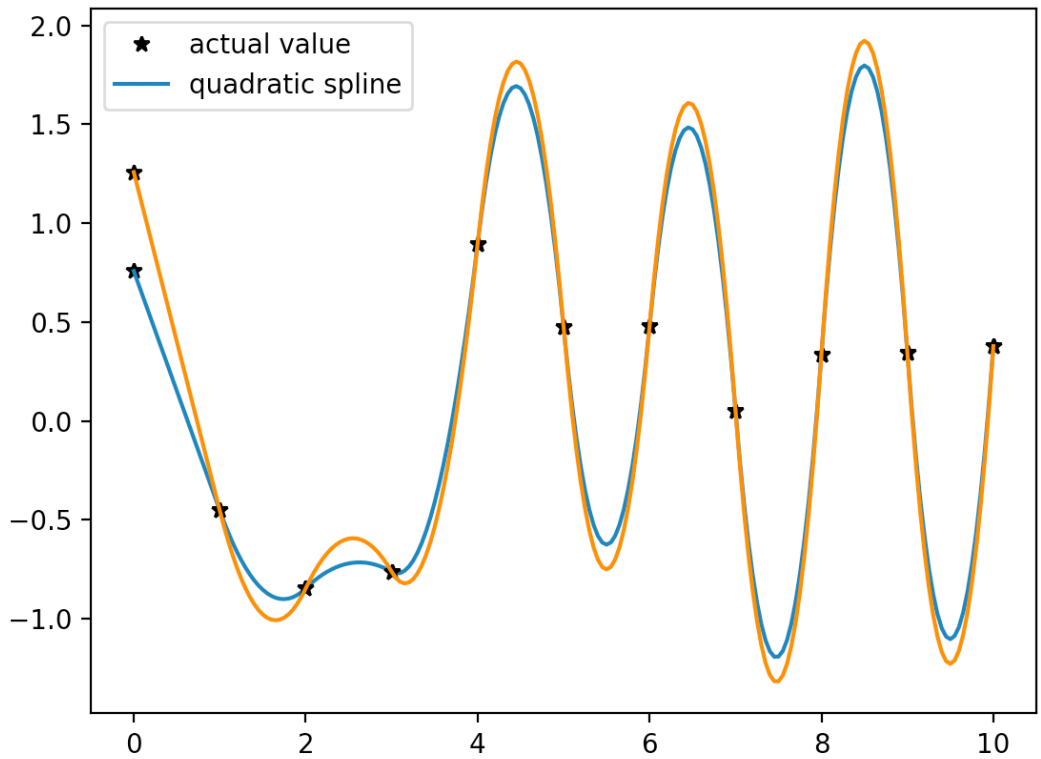
1. b)



c)



The spline for c) is like a vertically stretched version of the spline for b). Both of them pass all data points, and the differences between peaks for two splines seem to be the same, expect for the first few peaks.

d) I will use induction to prove this:

Assume $P(i) = \tilde{c}_i - c_i = (-1)^{(i+1)}\left(\frac{\tilde{y}_0 - y_0}{h^2}\right)$, $i = 1, \dots, n-1$

Base Case: $i = 1$

$$\tilde{c}_1 - c_1 = \frac{1}{x_1 - x_2}[\tilde{c}_0(x_1 - x_0) + \tilde{b}_0 - b_1] - \frac{1}{x_1 - x_2}[c_0(x_1 - x_0) +$$

$$= \frac{\tilde{b}_0 - b_0}{x_1 - x_2} = \frac{\tilde{b}_0 - b_0}{-h}, \quad \text{where } b_0 = \frac{y_1 - a_0}{x_1 - x_0} \quad a$$

$$\tilde{b}_0 = \frac{y_1 - \tilde{a}_0}{x_1 - x_0},$$

$$a_0 = y_0, \ \tilde{a}_0 =$$

$$\text{Thus } \frac{\tilde{b}_0 - b_0}{-h} = \frac{\frac{y_1 - \tilde{a}_0}{x_1 - x_0} - \frac{y_1 - a_0}{x_1 - x_0}}{-h}$$

$$= \frac{\frac{y_1 - \tilde{a}_0}{h} - \frac{y_1 - a_0}{h}}{-h} = \frac{-\tilde{a}_0 + a_0}{-h^2} = \frac{-(\tilde{a}_0 - a_0)}{-h^2}$$

$$= (-1)^{1+1}\frac{\tilde{y}_0 - y_0}{h^2}$$

Thus, $P(1)$ holds.

Inductive Hypothesis: Assume $i > 1$ and $P(i-1)$ holds, we want to show that $P(i)$ holds.

$$\tilde{c}_i - c_i = \frac{1}{x_i - x_{i+1}}[\tilde{c}_{i-1}(x_i - x_{i-1}) + \tilde{b}_{i-1} - b_i] - \frac{1}{x_i - x_{i+1}}[c_{i-1}(x_i - x_{i-1}) + b_{i-1} - b_i]$$

$$\left(\text{since } i > 1, \ \tilde{b}_{i-1} = b_{i-1}\right)$$

$$= \frac{(\tilde{c}_{i-1} - c_{i-1})(x_i - x_{i-1})}{x_i - x_{i+1}} = \frac{(\tilde{c}_{i-1} - c_{i-1})h}{-h} = -(\tilde{c}_{i-1} - c_{i-1})$$

By I.H., we know $(\tilde{c}_{i-1} - c_{i-1}) = (-1)^{(i-1)+1} \frac{\tilde{y}_0 - y_0}{h^2}$,

Thus $\tilde{c}_i - c_i = (-1) \cdot (-1)^{(i-1+1)} \left(\frac{\tilde{y}_0 - y_0}{h^2}\right)$

$$= -1^{(i+1)} \left(\frac{\tilde{y}_0 - y_0}{h^2}\right)$$

Thus, $P(i)$ holds.

Therefore, $\tilde{c}_i - c_i = (-1)^{(i+1)} \left(\frac{\tilde{y}_0 - y_0}{h^2}\right)$ for $i = 1, \cdots, n-1$.

$$\tilde{Q}_i(x) - Q_i(x) = \tilde{a}_i + \tilde{b}_i(x - x_i) + \tilde{c}_i(x - x_i)(x - x_{i+1})$$

$$- a_i + b_i(x - x_i) + c_i(x - x_i)(x - x_{i+1})$$

since $i \geq 1 \Rightarrow \tilde{a}_i = a_i$, $\tilde{b}_i = b_i$

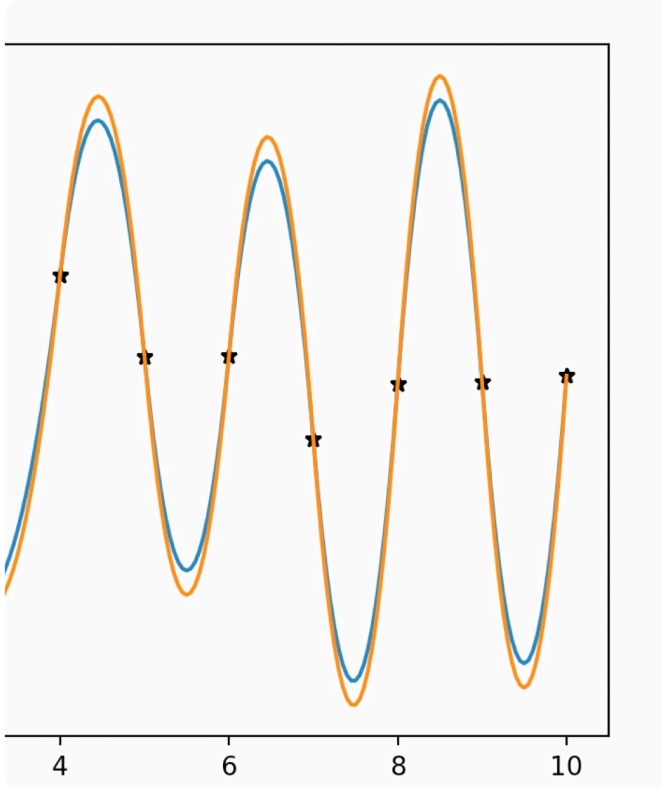
Thus,

$$\tilde{Q}_i(x) - Q_i(x) = \tilde{c}_i(x - x_i)(x - x_{i+1}) - c_i(x - x_i)(x - x_{i+1})$$

$$= (\tilde{c}_i - c_i)(x - x_i)(x - x_{i+1})$$

From above derivation, we know $\tilde{c}_i - c_i = (-1)^{(i+1)} \left(\frac{\tilde{y}_0 - y_0}{h^2}\right)$ for $i = 1, \cdots, n-1$,

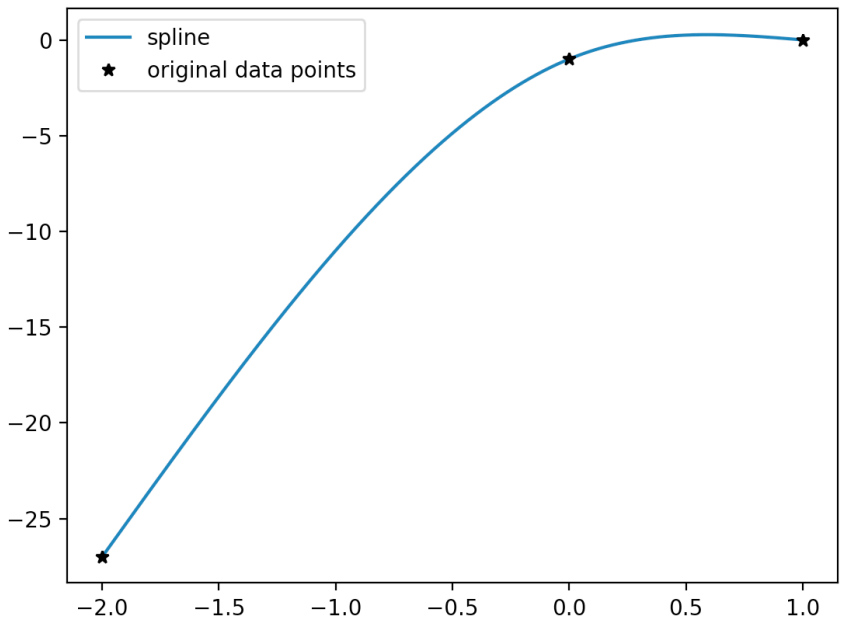
thus $\tilde{Q}_i(x) - Q_i(x) = (-1)^{(i+1)} \left(\frac{\tilde{y}_0 - y_0}{h^2}\right)(x - x_i)(x - x_{i+1})$ for $i = 1, \cdots, n-1$

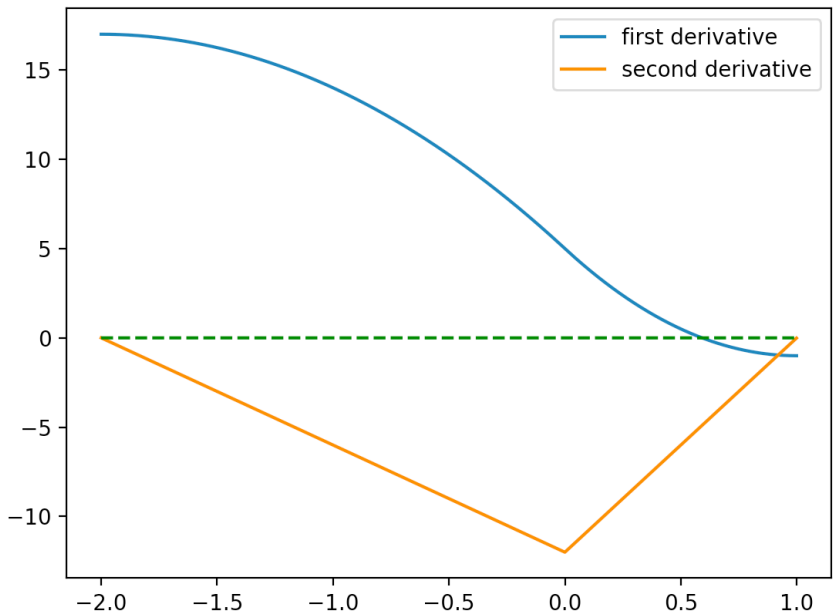


I think the analysis supports what we observed in (c).

For any subinterval in above graph, the different between peaks for two splines are the same, which means $|\tilde{C}_i - C_i|$ are the same for all intervals (C is the coefficient for the dominant term, and a_i, b_i are the same for two splines), also both C values are alternating between positive and negative, and when C is negative, the peak for spline for (b) is lower then the peak for spline for (c), and vice versa, which means the value of $\tilde{C}_i - C_i$ is fluctuating around 0. This also proves the correctness of our derivation.

2.





From graph 1 we can know $C(x_1) = y_1$, $C(x_2) = y_2$ and $C(x_3) = y_3$

and from graph 2 we can observe that $C''(-2) = 0$ and

$C''(1) = 0$.

All conditions are satisfied.

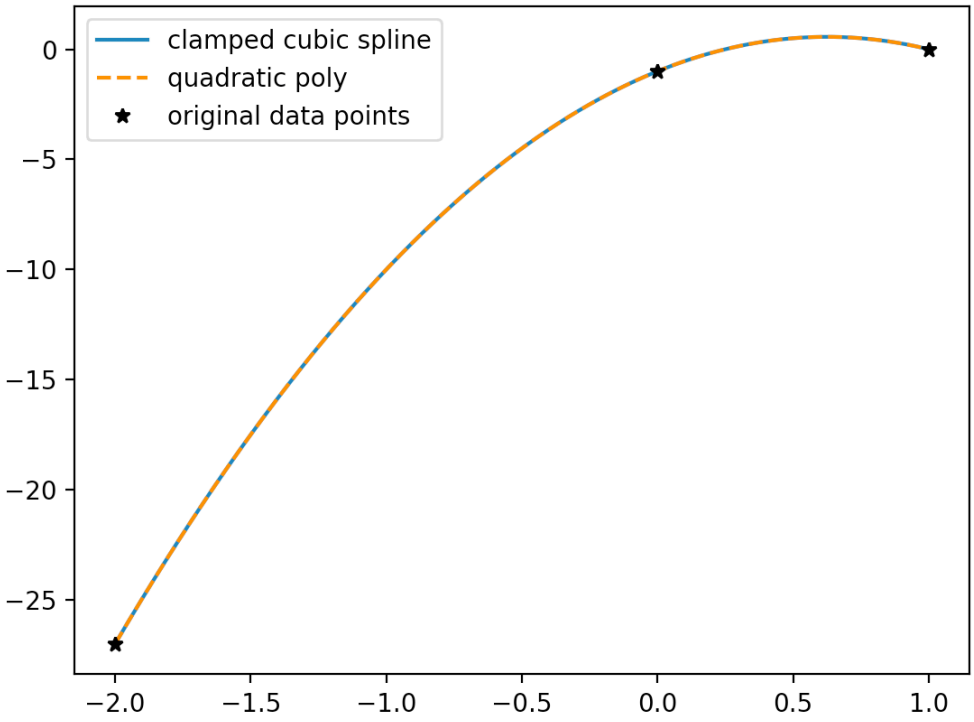
3. Conditions:

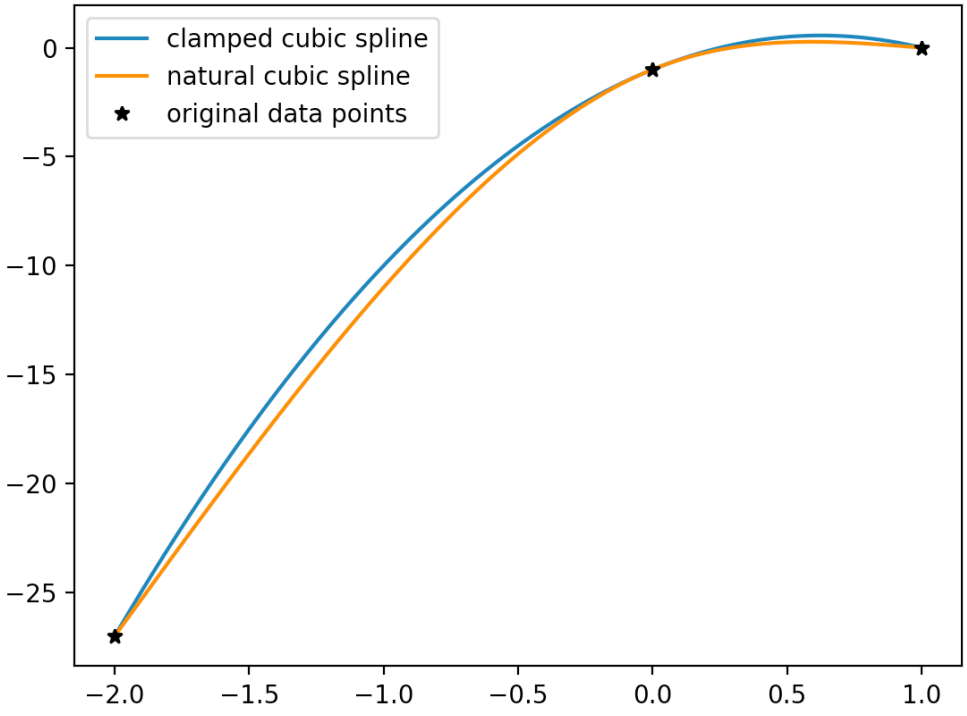
$$C(x_i) = y_i \quad , \quad i = 0, \cdots, n$$

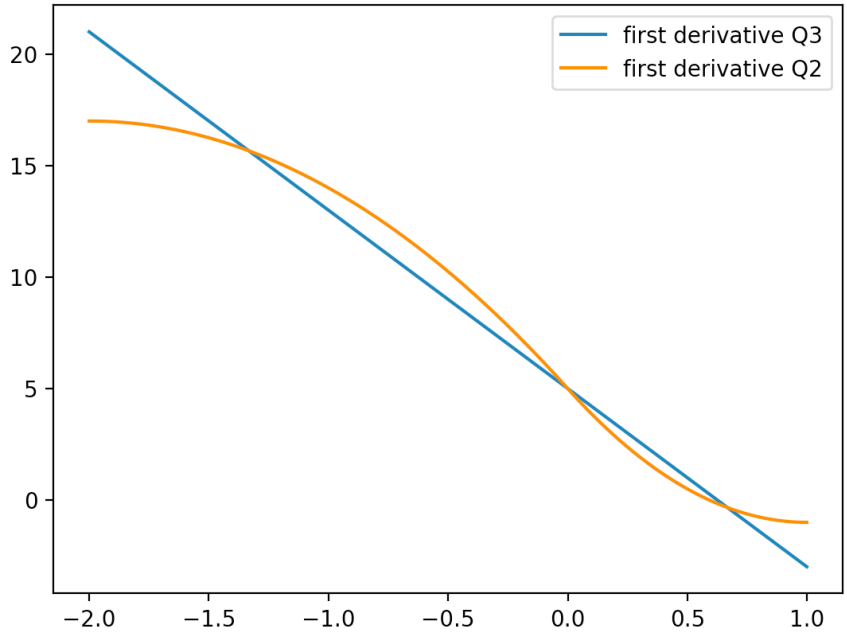
$$C'(x_0) = \tilde{y}_0'$$

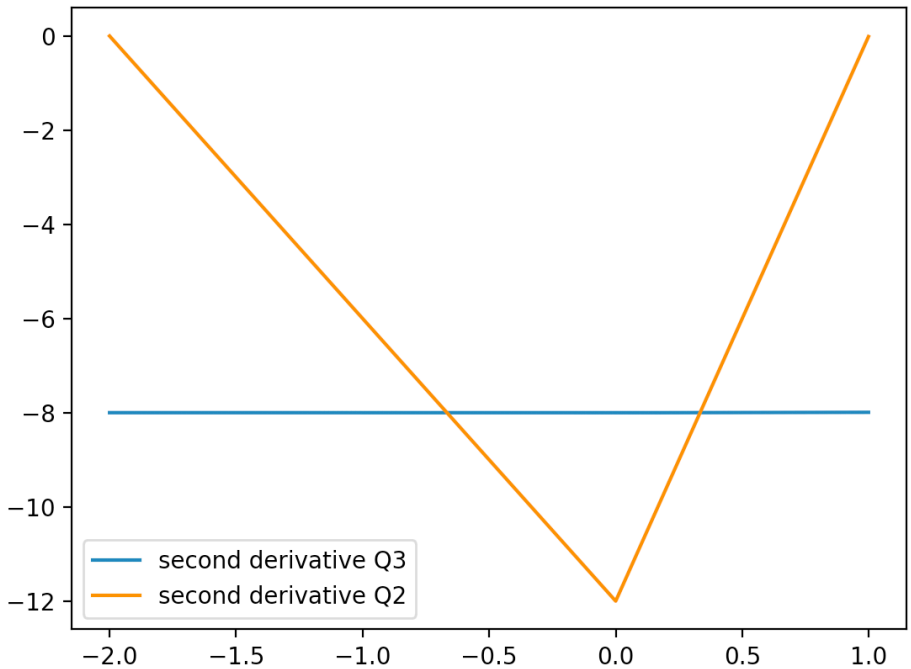
$$C'(x_n) = \tilde{y}_n'$$

$$\begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 & 0 & 0 & 0 & 0 \\ 1 & t_2 & t_2^2 & t_2^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t_2 & t_2^2 & t_2^3 \\ 0 & 0 & 0 & 0 & 1 & t_3 & t_3^2 & t_3^3 \\ 0 & 1 & 2t_2 & 3t_2^2 & 0 & 1 & -2t_2 & -3t_2^3 \\ 0 & 0 & 2 & 6t_2 & 0 & 0 & -2 & -6t_2 \\ 0 & 1 & 2t_1 & 3t_1^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2t_3 & 3t_3^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_2 \\ y_3 \\ 0 \\ 0 \\ \tilde{y}_0' \\ \tilde{y}_n' \end{bmatrix}$$


clamped cubic spline
quadratic poly
original data points
0
−5
−10
−15
−20
−25
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0


clamped cubic spline
natural cubic spline
original data points
0
−5
−10
−15
−20
−25
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0





By comparing the second graph, we can observe that the shape of clamped spline and natural spline are extremely similar. The only thing can say is that the clamped cubic spline has a greater arc. However, the first and second derivatives of two splines looked very different. The first derivative of clamped spline is a linear line, and thus its second derivative is just a horizontal line at -8. Therefore within this interval, the value

approximated by clamped spline is increasing at the beginning, but the speed of increase is decreasing or a constant speed, and finally the speed became less than 0 and the value starts to decrease at that rate. As for the natural spline, the second derivative started from 0, and decreases at the beginning, until it reaches the point near (0, -11.8), it began to increases, and ended at 0. Thus, the first derivative is like a "snake shape" fluctuated around the first derivative of clamped spline, but overall, the differences are very small. This results that two splines look very similar to each other.