FOUR LECTURES ON SHEAVES OF DIVISORS

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Contents

Unlike [Tam94], we talk about stalks before sheaf of divisors.

0.1 Review

X a scheme. $X_{\text{ét}}$ for the étale site of X.
$\mathrm{PreSh}(X)$ the category of abelian presheaves on X.
$\widetilde{X_{\text{ét}}}$ the category of abelian sheaves on $X_{\text{ét}}$.
$f : X \to Y$ a morphism of sheaves.
$f_{\mathrm{PreSh}} : \mathrm{PreSh}(X) \to \mathrm{PreSh}(Y)$ the pushforward of presheaves.
$f^{\mathrm{PreSh}} : \mathrm{PreSh}(Y) \to \mathrm{PreSh}(X)$ the pullback of presheaves.
$f_{\mathrm{Sh}} : \widetilde{X_{\text{ét}}} \to \widetilde{Y_{\text{ét}}}$ the pushforward of sheaves, left exact.
$f^{\mathrm{Sh}} : \widetilde{Y_{\text{ét}}} \to \widetilde{X_{\text{ét}}}$ the pullback of sheaves, exact and commute with $\varinjlim$.

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1. March 2nd, Stalks of étale sheaves

1.1 Definition

Let X be a scheme. A **geometric point** of X is a scheme $\xi = \mathrm{Spec}(\Omega)$ for some separably closed field Ω with a structure morphism $u : \xi \to X$.

Let $x = u(\xi)$. Then Ω is a separably closed extension of its residue field $\kappa(x)$.

1.2 Example

When Ω is a separable closure of $\kappa(x)$, we write $\Omega = \overline{\kappa(x)}$ and $\xi = \overline{x}$. In other words, given a point $x \in X$, we have a geometric point $\overline{x} = \mathrm{Spec}(\overline{\kappa(x)})$ over X.

1.3 Definition

Let $u : \xi \to X$ be a geometric point of X. Let $\Gamma_\xi = \Gamma(\xi, \bullet) : \widetilde{\xi_{\text{ét}}} \to \mathbf{Ab}$ be the global section functor of ξ. We have $u^{\mathrm{Sh}} : \widetilde{X_{\text{ét}}} \to \widetilde{\xi_{\text{ét}}}$.

(1) The **functor fiber** relative to ξ is the composition $\Gamma_\xi \circ u^{\mathrm{Sh}} : \widetilde{X_{\text{ét}}} \to \mathbf{Ab}$.

(2) Let $F \in \widetilde{X_{\text{ét}}}$. We call $F_\xi = \Gamma_\xi \circ u^{\mathrm{Sh}}(F) = \Gamma(\xi, u^{\mathrm{Sh}} F)$ the **stalk** of F at ξ.

1.4 Example

Let G be an étale group scheme over X. Let $G_X = \mathrm{Hom}_X(\bullet, G)$. Let $u : \xi = \mathrm{Spec}(\Omega) \to X$ be a geometric point of X. Then

$$G_{X,\xi} = \Gamma(\xi, u^{\mathrm{Sh}}(G_X)) = \mathrm{Hom}_\xi(\xi, G \times_X \xi) \simeq \mathrm{Hom}_X(\xi, G) = G(\Omega)$$

1.5 Review

Let k be a field. Let $\overline{k}$ be a separable closure of k. Let $G_k = \mathrm{Gal}(\overline{k}/k)$ be the absolute Galois group of k. Let $C(G_k)$ be the category of continuous G_k-sets. Then there exists an equivalence of categories

$$\widetilde{\mathrm{Spec}(k)_{\text{ét}}} \simeq C(G_k),\ F \mapsto \varinjlim F(\mathrm{Spec}(k')).$$

where k' runs through all finite subextension of $\overline{k}/k$. In particular, $\Gamma_\xi : \xi_{\text{ét}} \to \mathbf{Ab}$ is an equivalence of categories for all geometric point ξ.

1.6 Lemma

Let $u : \xi = \mathrm{Spec}(\Omega) \to X$ and $u' : \xi' = \mathrm{Spec}(\Omega') \to X$ be geometric points of X with an X-morphism $v : \xi \to \xi'$. Then there exists an isomorphism $F_\xi \simeq F_{\xi'}$ which is functorial in $F \in \widetilde{X_{\text{ét}}}$.

Proof. By Review 1.5, $\Gamma_\xi : \widetilde{\xi_{\text{ét}}} \to \mathbf{Ab}$ and $\Gamma_{\xi'} : \widetilde{\xi'_{\text{ét}}} \to \mathbf{Ab}$ are equivalences of categories since Ω, Ω' are separably closed. Since $v : \xi' \to \xi$ is an isomorphism, $v^{\mathrm{Sh}} : \widetilde{\xi_{\text{ét}}} \to \widetilde{\xi'_{\text{ét}}}$ is an equivalence of categories. Since $u' = u \circ v$, we have ${u'}^{\mathrm{Sh}} = v^{\mathrm{Sh}} \circ u^{\mathrm{Sh}}$,

$$\Gamma_{\xi'} \circ {u'}^{\mathrm{Sh}} = \Gamma_{\xi'} \circ v^{\mathrm{Sh}} \circ u^{\mathrm{Sh}} = (\Gamma_{\xi'} \circ v^{\mathrm{Sh}} \circ \Gamma_{\xi'}^{-1}) \circ (\Gamma_{\xi'} \circ u^{\mathrm{Sh}})$$

The composition $\Gamma_{\xi'} \circ v^{\mathrm{Sh}} \circ \Gamma_{\xi'}^{-1}$ is an equivalence of categories. □

1.7 Lemma

Let $f : X' \to X$ be a morphism of schemes. Let $u' : \xi' \to X'$ be a geometric point of X'. Then $u = f \circ u' : \xi = \xi' \to X$ is a geometric point of X with an isomorphism $f^{\mathrm{Sh}}(F)_{\xi'} \simeq F_\xi$ which is functorial in $F \in \widetilde{X'_{\text{ét}}}$.

Proof. Since $\xi' = \xi$ and $u = f \circ u'$,

$$(\Gamma_{\xi'} \circ {u'}^{\mathrm{Sh}}) \circ f^{\mathrm{Sh}} = \Gamma_{\xi'} \circ ({u'}^{\mathrm{Sh}} \circ f^{\mathrm{Sh}}) = \Gamma_{\xi'} \circ (f \circ u')^{\mathrm{Sh}} = \Gamma_\xi \circ u^{\mathrm{Sh}}.$$

□

1.8 Proposition
For every geometric point $u : \xi \to X$ of X, the functor fiber $\Gamma_\xi \circ u^{\mathrm{Sh}}$, $F \mapsto F_\xi$ is exact and commutes with $\varinjlim$ and finite $\varprojlim$.

Proof. By Review 0.1, u^{Sh} is exact and commutes with $\varinjlim$. By Review 1.5, $\Gamma_\xi : \widetilde{\xi_{\text{ét}}} \simeq \mathbf{Ab}$ and hence Γ_ξ is exact and commutes with $\varinjlim$. So their composition $\Gamma_\xi \circ u^{\mathrm{Sh}}$ is exact and commutes with $\varinjlim$.
In particular, finite $\varprojlim$ is the corresponding $\varinjlim$ in the opposite category, the functor fiber commutes with finite $\varprojlim$. ∎

1.9 Corollary
For every geometric point ξ of X, the functor fiber $F \mapsto F_\xi$ commutes with kernel, cokernel and image.

Proof. (1) Since $\ker(u) = \varprojlim \left(F \underset{0}{\overset{u}{\rightrightarrows}} G \right)$, by Proposition 1.8, $\ker(u)_\xi = \ker(u_\xi)$.
(2) Since $\mathrm{Cok}(u) = \varinjlim \left(F \underset{0}{\overset{u}{\rightrightarrows}} G \right)$, by Proposition 1.8, $\mathrm{Cok}(u)_\xi = \mathrm{Cok}(u_\xi)$.
(3) By (1)(2), $\mathrm{im}(u)_\xi = \ker(\mathrm{Cok}(u))_\xi = \ker(\mathrm{Cok}(u)_\xi) = \ker(\mathrm{Cok}(u_\xi)) = \mathrm{im}(u_\xi)$. ∎

1.10 Definition
Let $u : \xi \to X$ be a geometric point of X. An **étale neighborhood** of ξ is a scheme X' with an étale structure morphism $v : X' \to X$ and an X-morphism $u' : \xi \to X'$, i.e. the following commutative diagram.

$$\begin{tikzcd} & X' \arrow[d, "v"] \\ \xi \arrow[ur, "u'"] \arrow[r, "u"] & X \end{tikzcd}$$

1.11 Review
Let P be a presheaf on X. Let $X' \mapsto \Gamma(X', P^+) = \check{H}^0(X', P)$ be the separated presheaf associated to P for all X' étale over X. Let $P^\#$ be the sheaf associated to P. Then $(P^+)^+ = P^\#$.

1.12 Lemma
Let P be a presheaf on a point $\xi = \mathrm{Spec}(\Omega)$. Then $\Gamma(\xi, P) = \Gamma(\xi, P^\#)$.

Proof. For all covering $\{X'_i \to \xi\}$, $\{\xi \xrightarrow{\mathrm{Id}} \xi\}$ is a refinement. Then

$$\Gamma(\xi, P^+) = \varprojlim H^0(\{X'_i \to \xi\}, P) = H^0(\{\xi \xrightarrow{\mathrm{Id}} \xi\}, P) = \Gamma(\xi, P).$$

Since $\# = + \circ +$, we have $\Gamma(\xi, P^\#) = \Gamma(\xi, P^+) = \Gamma(\xi, P)$. ∎

1.13 Proposition
Let $u : \xi \to X$ be a geometric point of X. There is an isomorphism

$$\varprojlim \Gamma(X', P) \simeq (P^\#)_\xi$$

for all presheaf P on X where X' runs through all étale neighborhoods of ξ.

In particular, when $P = i(F)$ for some $F \in \widetilde{X_{ét}}$ where i is the forgetful functor,

$$\varprojlim \Gamma(X', F) = \varprojlim F(X') \simeq F_\xi.$$

Proof.

$$\begin{array}{rll} & \varprojlim \Gamma(X', P) & \\ = & \Gamma(\xi, u^{\mathrm{PreSh}} P), & \text{by definition of } u^{\mathrm{PreSh}}, \\ = & \Gamma(\xi, (u^{\mathrm{PreSh}} P)^{\#}), & \text{by Lemma 1.12}, \\ = & \Gamma(\xi, u^{\mathrm{Sh}}(P^{\#})), & \text{by our discussion} \\ = & (P^{\#})_\xi, & \text{by Definition 1.3}. \end{array}$$

Therefore $\varprojlim \Gamma(X', P) \simeq (P^{\#})_\xi$.
In particular, when $P = i(F)$, by $\# \circ i = \mathrm{Id}$, we have $\varprojlim \Gamma(X', F) \simeq F_\xi$. □

1.14 Example
Let $\overline{k}$ be a separable closure of a field k. In other words, $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$ is a geometric point of $\mathrm{Spec}(k)$. The category of étale neighborhoods of $\mathrm{Spec}(\overline{k})$ is $\mathcal{I} = \{\mathrm{Spec}(k') \mid k \subset k' \subset \overline{k},\ k' \text{ field}\}$. Since every k' contains a finite subextension, $\mathcal{J} = \{\mathrm{Spec}(k') \in \mathcal{I} \mid k'/k \text{ finite}\}$ is a final full subcategory of $\mathcal{I}$, we have

$$\varprojlim_{\mathcal{I}} F(\mathrm{Spec}(k')) = \varprojlim_{\mathcal{J}} F(\mathrm{Spec}(k'))$$

By Proposition 1.13, $\varprojlim_{\mathcal{I}} \mathrm{Spec}(k') \simeq F_{\mathrm{Spec}(\overline{k})}$. Therefore $\varprojlim_{\mathcal{J}} \mathrm{Spec}(k') \simeq F_{\mathrm{Spec}(\overline{k})}$. By Review 1.5, $F \mapsto F_{\mathrm{Spec}(\overline{k})}$ defines the equivalence of categories $\widetilde{X_{ét}} \to C(\mathrm{Gal}(\overline{k}/k))$.

1.15 Theorem
Let $u : F \to G$ be a morphism of abelian sheaves on $\widetilde{X_{ét}}$. Then u is an isomorphism iff $u_{\overline{x}} : F_{\overline{x}} \to G_{\overline{x}}$ is an isomorphism for all $x \in X$.

Proof. Suppose u is an isomorphism. Let $\pi_x : \overline{x} \to X$ be the structure morphism. Since $\Gamma_{\overline{x}} \circ \pi^{\mathrm{Sh}}$ is a functor, $u_{\overline{x}} = (\Gamma_{\overline{x}} \circ \pi_x^{\mathrm{Sh}})(u)$ has inverse $u_{\overline{x}}^{-1} = (\Gamma_{\overline{x}} \circ \pi_x^{\mathrm{Sh}})(u^{-1})$.
Conversely, suppose $u_{\overline{x}}$ is an isomorphism for all $x \in X$. We need to show that $u(X') : F(X') \to G(X')$ is an isomorphism for all X' with an étale structure morphism $p : X' \to X$. We have a commutative diagrams

$$\begin{tikzcd} F(X') \arrow[r, "u(X')"] \arrow[d, "F(p)"'] & G(X') \arrow[d, "G(p)"] \\ F(X) \arrow[r, "u(X)"] & G(X) \end{tikzcd} \quad , \quad \begin{tikzcd} (p^*F)_{\overline{x'}} \arrow[r, "u_{\overline{x'}}"] \arrow[d, equal] & (p^*G)_{\overline{x'}} \arrow[d, equal] \\ F_{\overline{x}} \arrow[r, "u_{\overline{x}}"] & G_{\overline{x}} \end{tikzcd}$$

where in the second diagram $x = p(x')$ for all $x' \in X'$ and vertical identifications follow from Lemma 1.7. Then $u_{\overline{x'}}$ is an isomorphism for all $x' \in X'$ for all X'.

(1) $u(X')$ is a *monomorphism*. Suppose $s \in F(X')$ such that $u(X')(s) = 0$ in $G(X')$. Since $u_{\overline{x'}}(s_{\overline{x'}}) = u(s)_{\overline{x'}} = 0$, we have $s_{\overline{x'}} = 0$ for all $x' \in X'$. Since $u_{\overline{x'}}$ is injective, by Proposition 1.13, there exists an étale neighborhood $X'_{x'}$ of $\overline{x'}$ such that the image of s by $F(X') \to F(X'_{x'})$ is 0 for all $x' \in X'$. Since $F \in \widetilde{X_{ét}}$ and $\{X'_{x'} \to X'\}$ is an étale covering, we have $s = 0$.

(2) $u(X')$ is an *epimorphism*. Suppose $s \in G(X')$. Since $u_{\overline{x'}}$ is surjective, by Proposition 1.13, there exists an étale neighborhood $X'_{x'}$ of $\overline{x'}$ and $t_{x'} \in F(X'_{x'})$ such that $u(X'_{x'})(t_{x'})$ is the image of s by $G(X') \to G(X'_{x'})$ for all $x' \in X'$.

For two such neighborhoods $X'_{x'}$ and $X'_{y'}$, the image of $u(X'_{x'})(t_{x'})$ by $G(X'_{x'}) \to G(X'_{x'} \times_{X'} X'_{y'})$, the image of $u(X'_{y'})(t_{y'})$ by $G(X'_{y'}) \to G(X'_{x'} \times_{X'} X'_{y'})$ are both equal to the image of s by $G(X') \to G(X'_{x'} \times_{X'} X'_{y'})$.
By (1), $F(X'_{x'} \times_{X'} X'_{y'}) \to G(X'_{x'} \times_{X'} X'_{y'})$ is injective. Then the image of $t_{x'}$ by $F(X'_{x'}) \to F(X'_{x'} \times_{X'} X'_{y'})$ and the image of $t_{y'}$ by $F(X'_{y'}) \to F(X'_{x'} \times_{X'} X'_{y'})$ coincide. Since F is a sheaf, there exists $t \in F(X')$ whose image by $F(X') \to F(X'_{x'})$ is $t_{x'}$. Then the images of $u(X')(t)$ and s by $G(X') \to G(X'_{x'})$ coincide. Since G is a sheaf, we have $u(X')(t) = s$. $\square$

1.16 Corollary

Let $u : F \to G$ be a morphism of abelian sheaves on $\widetilde{X_{\text{ét}}}$.
(1) u is a monomorphism iff $u_{\overline{x}} : F_{\overline{x}} \to G_{\overline{x}}$ is injective for all $x \in X$.
(2) u is an epimorphism iff $u_{\overline{x}} : F_{\overline{x}} \to G_{\overline{x}}$ is surjective for all $x \in X$.

Proof. Suppose $u : F \to G$ is the composition of $u' : F \to \mathrm{im}(u)$ and the inclusion $i : \mathrm{im}(u) \to G$, i.e. $u = i \circ u'$.
(1)

$$
\begin{array}{lll}
 & u : F \to G \text{ is a monomorphism} & \\
\iff & u' : F \to \mathrm{im}(u) \text{ is an isomorphism} & \\
\iff & u'_{\overline{x}} : F_{\overline{x}} \to \mathrm{im}(u)_{\overline{x}} \text{ is an isomorphism for all } x \in X, & \text{by Theorem 1.15,} \\
\iff & u'_{\overline{x}} : F_{\overline{x}} \to \mathrm{im}(u_{\overline{x}}) \text{ is an isomorphism for all } x \in X, & \text{by Corollary 1.9,} \\
\iff & u_{\overline{x}} : F_{\overline{x}} \to G_{\overline{x}} \text{ is injective for all } x \in X. &
\end{array}
$$

(2)

$$
\begin{array}{lll}
 & u : F \to G \text{ is an epimorphism} & \\
\iff & i : \mathrm{im}(u) \to G \text{ is an isomorphism} & \\
\iff & i_{\overline{x}} : \mathrm{im}(u)_{\overline{x}} \to G_{\overline{x}} \text{ is an isomorphism for all } x \in X, & \text{by Theorem 1.15,} \\
\iff & i_{\overline{x}} : \mathrm{im}(u_{\overline{x}}) \to G_{\overline{x}} \text{ is an isomorphism for all } x \in X, & \text{by Corollary 1.9,} \\
\iff & u_{\overline{x}} : F_{\overline{x}} \to G_{\overline{x}} \text{ is surjective for all } x \in X. &
\end{array}
$$

$\square$

1.17 Corollary

(1) Let $u, v : F \to G$ be a morphisms of abelian sheaves on $\widetilde{X_{\text{ét}}}$. Then $u = v$ iff $u_{\overline{x}} = v_{\overline{x}} : F_{\overline{x}} \to G_{\overline{x}}$ for all $x \in X$.
(2) Let u, v be a sections of $F \in \widetilde{X_{\text{ét}}}$. Then $u = v$ iff $u_{\overline{x}} = v_{\overline{x}} \in F_{\overline{x}}$ for all $x \in X$.

Proof. (1)

$$
\begin{array}{lll}
 & u = v & \\
\iff & 0 \to \mathrm{im}(u - v) \text{ is an isomorphism} & \\
\iff & 0 \to \mathrm{im}(u - v)_{\overline{x}} \text{ is an isomorphism for all } x \in X, & \text{by Theorem 1.15,} \\
\iff & 0 \to \mathrm{im}(u_{\overline{x}} - v_{\overline{x}}) \text{ is an isomorphism for all } x \in X, & \text{by Corollary 1.9,} \\
\iff & u_{\overline{x}} = v_{\overline{x}} \text{ for all } x \in X. &
\end{array}
$$

(2) Define $\varphi : \Gamma(X', F) \to \mathrm{Hom}(\underline{\mathbb{Z}}_{X'}, F)$ for X' étale over X. Let $s \in \Gamma(X', F)$. Its image $\varphi(s) : \underline{\mathbb{Z}}_{X'} \to F$ is defined stalk-wise by $\underline{\mathbb{Z}}_{X',\overline{x}} = \mathbb{Z} \to F_{\overline{x}}$, $1 \mapsto s_{\overline{x}}$. Then φ is an isomorphism of abelian groups. The result follows from (1). $\square$

1.18 Corollary

Let $F \xrightarrow{v} G \xrightarrow{u} H$ be a sequence of morphisms of abelian sheaves on $\widetilde{X_{\text{ét}}}$. Then it is exact iff $F_{\overline{x}} \xrightarrow{v_{\overline{x}}} G_{\overline{x}} \xrightarrow{u_{\overline{x}}} H_{\overline{x}}$ is exact for all $x \in X$.

Proof. $\ker(u) = \operatorname{im}(v)$ iff $u \circ v = 0$ and the inclusion $i : \operatorname{im}(v) \to \ker(u)$ is an isomorphism.
First, by Corollary 1.17, $0 = u \circ v$ iff $0 = (u \circ v)_{\overline{x}} = u_{\overline{x}} \circ v_{\overline{x}}$ for all $x \in X$.
Also, by Theorem 1.15, $i : \operatorname{im}(v) \to \ker(u)$ is an isomorphism iff $i_{\overline{x}} : \operatorname{im}(v)_{\overline{x}} \to \ker(u)_{\overline{x}}$ is an isomorphism for all $x \in X$. By Corollary 1.9, iff the inclusion $\operatorname{im}(v_{\overline{x}}) \to \ker(u_{\overline{x}})$ is an isomorphism for all $x \in X$.
Together, we have $\ker(u_{\overline{x}}) = \operatorname{im}(v_{\overline{x}})$ for all $x \in X$. □

2. March 9th, Zariski sheaves of meromorphic and rational functions

Today, we introduce the Zariski sheaf of Cartier divisors $\mathcal{D}iv_{\mathrm{Zar}}$.

2.1 Definition

Let A be a ring. Let S be the set of non zero-divisors of A. Then S is a multiplicative set. We call $S^{-1}A$ the **total quotient ring of** A. We write $\mathrm{Frac}(A) = S^{-1}A$.

For all open subset U of X, let $S(U)$ be the set of elements of $\Gamma(U, \mathcal{O}_X)$ whose images in $\mathcal{O}_{X,x}$ are not zero-divisors for all $x \in U$. Let $\mathscr{M}_X$ be the sheaf associated to the presheaf $U \mapsto (S(U))^{-1}\Gamma(U, \mathcal{O}_X)$. We call $\mathscr{M}_X$ the **sheaf of total quotient rings** (or the **sheaf of meromorphic functions**) of $\mathcal{O}_X$. Elements of $\Gamma(U, \mathscr{M}_X)$ are called **meromorphic functions** on U.

We call $\mathcal{D}iv_{\mathrm{Zar}} = \mathscr{M}_X^*/\mathcal{O}_X^*$ the Zariski **sheaf of Cartier divisors** on X_{Zar}. Elements of $\Gamma(U, \mathcal{D}iv_{\mathrm{Zar}})$ are called **Cartier divisors** on U.

2.2 Remark

Then $U \mapsto \mathrm{Frac}(\Gamma(U, \mathcal{O}_X))$ does not define a presheaf. [Gro67, 20.1] was wrong, see its correction [Kle79].

2.3 Proposition

Let A be a noetherian ring. Let U be an open subset of $X = \mathrm{Spec}(A)$. Then $U \supset D(t) = \{x \in X \mid t \notin x\}$ for some non zero-divisor t iff the restriction $\Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$ is injective for all open subset V of X.

Proof. Suppose $U \supset D(t)$. Take $s \in \Gamma(V, \mathcal{O}_X)$ such that $s|_{U\cap V} = 0$. Then $s|_{D(t)\cap V} = 0$. For any affine open subset $W = \mathrm{Spec}(B)$ of V, if $t \in B$, then the image of $s|_W$ in $B[t^{-1}]$ is 0. Then there exist some integer $n > 0$ such that $t^n s|_W = 0$. Since t is not a zero divisor, $s|_W = 0$ for all W. Therefore $s = 0$.

Conversely, suppose $\Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$ are all injective. Since U is open, there exists an ideal $\mathfrak{a}$ of A such that $U = \{x \in X \mid \mathfrak{a} \not\subset x\}$.

Next, we show that $\mathrm{Ann}(\mathfrak{a}) = 0$. Suppose $s \in \mathrm{Ann}(\mathfrak{a})$. For each $x \in U$, there exists $t \in \mathfrak{a} \not\subset x$. Then t_x is invertible in $\mathcal{O}_{X,x}$. Since $st = 0$, $s_x t_x = 0$ in $\mathcal{O}_{X,x}$. Then $s_x = 0$ for all $x \in U$ and hence $s|_U = 0$. By injectivity of $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$, we have $s = 0$.

Thus U contains every prime ideal of the form $x = \mathrm{Ann}(s)$ for $0 \neq s \in A$. Otherwise $\mathfrak{a} \subset x = \mathrm{Ann}(s)$ for some $0 \neq s \in A$ and hence $s \in \mathrm{Ann}(\mathfrak{a})$, a contradiction to $\mathrm{Ann}(\mathfrak{a}) = 0$. Since A is noetherian, maximal annihilators are prime ideals. In fact, if $bc \in \mathrm{Ann}(x)$ maximal and $c \notin \mathrm{Ann}(x)$, then $b \in \mathrm{Ann}(cx) = \mathrm{Ann}(x)$. Since the union of all maximal annihilators is the set of zero divisors, U contains all zero divisors of A. Hence all zero divisors of A are in some element of U (see [Mat80, 1.B]). Therefore $\mathfrak{a}$ contains some non zero-divisor t of A, $U \supset D(t)$. $\square$

2.4 Proposition

Let X be a reduced scheme. Let U be an open subset of X. Then U is dense in X iff the restriction $\Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$ is injective for all V open in X.

Proof. Suppose all restrictions $\Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$ are injective. Let V be a nonempty open subset of X. If $U \cap V = \emptyset$, then $\Gamma(U \cap V, \mathcal{O}_X) = \{0\}$. By injectivity, $\Gamma(V, \mathcal{O}_X) = \{0\}$, a contradiction to $1 \in \Gamma(V, \mathcal{O}_X)$. Therefore $U \cap V \neq \emptyset$ for all nonempty open subset V of X, i.e. U is dense in X.

Conversely, suppose U is dense in X. Since $\Gamma(V, \mathcal{O}_X) \simeq \mathrm{Hom}(V, \mathbb{A}^1_{\mathbb{Z}})$ and $\Gamma(U \cap V, \mathcal{O}_X) \simeq \mathrm{Hom}(U \cap V, \mathbb{A}^1_{\mathbb{Z}})$, it suffices to show that the restriction $\mathrm{Hom}(V, \mathbb{A}^1_{\mathbb{Z}}) \to \mathrm{Hom}(U \cap V, \mathbb{A}^1_{\mathbb{Z}})$ is injective for all open subset V of X.
Suppose $f, g \in \mathrm{Hom}(V, \mathbb{A}^1_{\mathbb{Z}})$ such that $f|_{U\cap V} = g|_{U\cap V}$. Then $U \cap V \subset K = \{x \in V \mid f(x) = g(x)\}$. Since $\mathbb{A}^1_{\mathbb{Z}}$ is separated, the diagonal Δ of $\mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}}$ is closed. Since

$$(f, g) : V \to \mathbb{A}^1_{\mathbb{Z}} \times \mathbb{A}^1_{\mathbb{Z}},\ x \mapsto (f(x), g(x))$$

is a morphism, $K = (f, g)^{-1}(\Delta)$ is closed. Since V the closure of $U \cap V$, the underlying topological space of K is V. Finally, since X is reduced, V is reduced and hence $K = V$. Therefore $f = g$, the restriction is injective. □

2.5 Lemma

Let X, Y be schemes. Let U, V be dense open subsets of X. Let $f : U \to Y$, $g : V \to Y$ be morphisms. We say that f and g are equivalent if there exists a dense open subset W of X such that $W \subset U \cap V$ and $f|_W = g|_W$. It is an equivalence relation and commutes with restrictions of morphisms.

Proof. Omit. □

2.6 Definition

A **rational map from X to Y** is an equivalence class of Lemma 2.5. A **rational function** on a scheme X is a rational map from X to $\mathbb{A}^1_{\mathbb{Z}}$.
For all open subscheme U of X, we write $R(U)$ for the ring of rational functions on X. Let $\mathscr{R}_X$ be the sheaf associated to the presheaf $U \mapsto R(U)$. We call $\mathscr{R}_X$ the **sheaf of rational functions**.

2.7 Lemma

Let X be a locally noetherian, reduced scheme. For all meromorphic function $f \in \Gamma(X, \mathscr{M}_X)$. Let $\mathrm{dom}(f) = \{x \in X \mid f_x \in \mathcal{O}_{X,x}\}$. Then the equivalence class of $f|_{\mathrm{dom}(f)}$ defines a rational function on X.

Proof. (1) $\underline{\mathrm{dom}(f) \text{ is open.}}$ In fact, for all $x \in \mathrm{dom}(f)$, $f_x \in \mathcal{O}_{X,x}$, there exists an open neighborhood W of x and $g \in \Gamma(W, \mathcal{O}_X)$ such that $g_x = f_x$. Hence there exist an open neighborhood $W' \subset W$ such that $g|_{W'} = f|_{W'}$. Therefore $f_y \in \mathcal{O}_{X,y}$ for all $y \in W'$, $W' \subset \mathrm{dom}(f)$.

(2) $\underline{\mathrm{dom}(f) \text{ is dense in } X.}$ In fact, for all $x \in X$, since $f_x \in \mathrm{Frac}(\mathcal{O}_{X,x})$, there exists a non zero-divisor $s \in \mathcal{O}_{X,x}$ such that $s \cdot (f_x) \in \mathcal{O}_{X,x}$. Since X is locally noetherian, there exists an affine open neighborhood $U_x = \mathrm{Spec}(B)$ of x (where B is noetherian) and a non zero-divisor $t \in \Gamma(U_x, \mathcal{O}_X) = B$ such that $t_x = s$ and $t \cdot (f|_{U_x}) \in \Gamma(U_x, \mathcal{O}_X)$. Then $D(t) \subset \mathrm{dom}(f)$. In fact, for all $y \in D(t)$, $t \notin y$ and hence t_y is a unit of $\mathcal{O}_{X,y}$. We have $f_y = (t_y)^{-1}(t \cdot (f|_{U_x}))_y \in \mathcal{O}_{X,y}$.

Since $D(t) \subset U_x$ and B is noetherian, by Proposition 2.3, the restriction $\Gamma(V, \mathcal{O}_X) \to \Gamma(D(t) \cap V, \mathcal{O}_X)$ is injective for all open subset V of U_x. Since $\{U_x \mid x \in X\}$ is a covering of X, the restriction $\Gamma(V, \mathcal{O}_X) \to \Gamma(\bigcup_{x\in X} D(t) \cap V, \mathcal{O}_X)$ is injective for all open subset V of X. Since X is reduced, by Proposition 2.4, $\bigcup_{x\in X} D(t)$ is dense in X. Therefore, $\mathrm{dom}(f)$ is dense in X since $\bigcup_{x\in X} D(t) \subset \mathrm{dom}(f)$. □

2.8 Definition

Lemma 2.7 defines a morphism of sheaves $\mathscr{M}_X \to \mathscr{R}_X$.

Proof. Since $f|_{\mathrm{dom}(f)} \in \Gamma(\mathrm{dom}(f), \mathcal{O}_X) \simeq \mathrm{Hom}(\mathrm{dom}(f), \mathbb{A}^1_{\mathbb{Z}})$ corresponds a morphism from an open dense subset $\mathrm{dom}(f)$ of X to $\mathbb{A}^1_{\mathbb{Z}}$. By Definition 2.6, its equivalence class $[f|_{\mathrm{dom}(f)}]$ is a rational function on X.
For all $x \in X$ and $s \in \mathscr{M}_{X,x}$, there exists an open neighborhood U of x and $f \in \Gamma(U, \mathscr{M}_X)$ such that $f_x = s$. Then $[f|_{U\cap\mathrm{dom}(f)}] \in R(U)$ and $[f|_{\mathrm{dom}(f)}]_x \in \mathscr{R}_{X,x}$ is the image of s. We can verify that $\mathscr{M}_{X,x} \to \mathscr{R}_{X,x}$ is well-defined.

If $g \in \Gamma(V, \mathscr{M}_X)$ and $g_x = s$, then there exists an open neighborhood W of x such that $W \subset U \cap V$ and $f|_W = g|_W$. Then $f|_{W\cap\mathrm{dom}(f)\cap\mathrm{dom}(g)} = g|_{W\cap\mathrm{dom}(f)\cap\mathrm{dom}(g)}$. Since $W\cap\mathrm{dom}(f)\cap\mathrm{dom}(g)$ is open dense in W, we have $[f|_{W\cap\mathrm{dom}(f)}] = [g|_{W\cap\mathrm{dom}(g)}]$ in $R(W)$. Therefore $[f|_{U\cap\mathrm{dom}(f)}]_x = [g|_{V\cap\mathrm{dom}(g)}]_x$ in $\mathscr{R}_{X,x}$. □

2.9 Theorem
Let X be a locally noetherian, reduced scheme. Then $\mathscr{M}_X \simeq \mathscr{R}_X$.

Proof. For any affine open subset U of X, assume that $U = \mathrm{Spec}(A)$ where A is a noetherian ring without nonzero nilpotent elements. We show that

$$r : \Gamma(U, \mathscr{M}_X) = \bigcup A_t = \varinjlim \Gamma(D(t), \mathcal{O}_X) \to \Gamma(U, \mathscr{R}_X) = \varinjlim \mathrm{Hom}(V, \mathbb{A}^1_{\mathbb{Z}}) = \varinjlim \Gamma(V, \mathcal{O}_X)$$

is bijective. Here t runs through all non zero-divisors of A and V runs through open dense subsets of U.

By Proposition 2.3, $V \supset D(t_V)$ for some t_V for all V. the restriction $\Gamma(D(t_V), \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X)$, if exists, is surjective for all t and V such that $V \subset D(t_V)$. Taking direct limit, r is surjective.

By Proposition 2.4, the restriction $\Gamma(D(t_V), \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X)$, if exists, is injective for all t and V such that $V \subset D(t_V)$. Taking direct limit, r is injective since U is noetherian.

Therefore $\mathscr{M}_{X,x} = \varinjlim \Gamma(U, \mathscr{M}_X) \to \varinjlim \Gamma(U, \mathscr{R}_X) = \mathscr{R}_{X,x}$ is an isomorphism for all $x \in X$. Then $\mathscr{M}_X \simeq \mathscr{R}_X$. □

2.10 Remark
By [Kle79], it is possible that $\mathrm{Frac}(A) \subsetneq \mathscr{M}_X(\mathrm{Spec}(A)) = \Gamma(\mathrm{Spec}(A), \mathscr{M}_X)$

2.11 Lemma
Let $\mathfrak{I}(X)$ be the set of irreducible closed subsets of a scheme X. There exists a bijection $X \to \mathfrak{I}(X)$, $x \mapsto \overline{\{x\}}$, i.e. every irreducible closed subset of a scheme has a unique generic point.

Proof. (1) Suppose $X = \mathrm{Spec}(A)$ for a commutative ring A. Every irreducible closed subset of A has the form $V(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}(A) \mid \mathfrak{q} \supset \mathfrak{p}\} = \overline{\{\mathfrak{p}\}}$ where $\mathfrak{p}$ is a prime ideal of A. By definition, $V(\mathfrak{p}) = V(\mathfrak{p}')$ iff $\mathfrak{p} = \mathfrak{p}'$.
(2) Suppose X is not affine. Let Z be an irreducible closed subset of X. There exists an open affine subset U of X such that $U \cap Z \neq \emptyset$.
Existence. Since $U \cap Z$ is an irreducible closed subset of U, by the existence of (1), there exists $x \in U \cap Z$ such that $\{x\}$ is dense in $U \cap Z$. Since $U \cap Z$ is a nonempty open subset of Z and Z is irreducible, $U \cap Z$ is dense in Z. By transitivity of density, $\{x\}$ is dense in Z.
Uniqueness. Let $x, x' \in Z$ such that $\{x\}$ and $\{x'\}$ are dense in Z. Since $U \cap Z$ is a nonempty open subset of Z, x, x' are dense in $U \cap Z$. Since $U \cap Z$ is an irreducible closed subset of the affine scheme U, by the uniqueness of (1), $x = x'$. □

3. March 16th, Zariski and étale sheaves of Cartier and Weil divisors

A point $x \in X$ is called the **generic point** of the irreducible closed set $\overline{\{x\}}$.

3.1 Definition
For $x, y \in X$, we say that $\overline{\{x\}} \preceq \overline{\{y\}}$ if $\dim \overline{\{x\}} \leq \dim \overline{\{y\}}$. An **embedded component** is a non-maximal $\overline{\{x\}}$ with respect to $\preceq$.

3.2 Definition
For $x, y \in X$, we say that $x \leq y$ (i.e. x is a **specialization** of y and y is a **generalization** of x) if $\overline{\{x\}} \subset \overline{\{y\}}$. A point maximal relative to $\leq$ of X is the generic point of an irreducible component of X. A point minimal relative to $\leq$ is a **closed point** of X.

3.3 Lemma
Let X be irreducible noetherian without embedded components. Let x be the maximal (generic) point of X. Let $K = \mathcal{O}_{X,x}$, the function field of the irreducible component $\overline{\{x\}}$. Let $i : \operatorname{Spec}(K) \to X$ be the morphism with image $\{x\}$. Let $\mathbb{G}_m$ be the multiplicative group scheme over X. Then

$$(\mathbb{G}_m)_X \to (i)_*(\mathbb{G}_m)_K$$

is an injection of Zariski (resp. étale) sheaves.

Proof. Let X' be a scheme over X with an étale structure morphism $\pi : X' \to X$.

$$\operatorname{Hom}_X(X', \mathbb{G}_m) \to \operatorname{Hom}_{\operatorname{Spec}(K)}(X' \times_X \operatorname{Spec}(K), \mathbb{G}_m)$$

is injective for all X'.

In fact, $X' \times_X \operatorname{Spec}(K)$ is identified with $\pi^{-1}(x)$ and the above morphism is identified with restriction to $\pi^{-1}(x)$. Suppose $g_1, g_2 : X' \to \mathbb{G}_m$ such that $g_1|_{\pi^{-1}(x)} = g_2|_{\pi^{-1}(x)}$. Let $K = \{y \in X' \mid g_1(y) = g_2(y)\}$. Then $\pi^{-1}(x) \subset K$. Since $\mathbb{G}_m$ is separated, $K = (g_1, g_2)^{-1}(\Delta_{\mathbb{G}_m})$ is closed. Since π is étale, $\pi^{-1}(x)$ is the set of all maximal points of X', $\overline{\pi^{-1}(x)} = \pi^{-1}(\overline{\{x\}}) = \pi^{-1}(X) = X'$. (*The first equality does not necessarily hold in general.*) Then $X' = K$, i.e. $g_1 = g_2$. Then

$$(\mathbb{G}_m)_X(X') \to i_*(\mathbb{G}_m)_{\operatorname{Spec}(K)}(X')$$

is injective for all X'. Therefore $(\mathbb{G}_m)_X \to i_*(\mathbb{G}_m)_{\operatorname{Spec}(K)}$ is injective. $\square$

3.4 Lemma
Let X be noetherian without embedded components. Let $\mathbb{G}_m$ be the multiplicative group scheme over X. Let x_k be maximal points of X. Let $K_k = \mathcal{O}_{X,x_k}$. Let $i_k : \operatorname{Spec}(K_k) \to X$ be inclusion morphisms.
(1) There is an isomorphism $j_*(\mathbb{G}_m)_{R(X)} = \bigoplus_k (i_k)_*(\mathbb{G}_m)_{K_k}$.
(2) There is an injection $\iota : (\mathbb{G}_m)_X \to j_*(\mathbb{G}_m)_{R(X)}$ associated to the canonical injection $j : \operatorname{Spec}(R(X)) \to X$ in both Zariski and étale topology.

Proof. (1) By Definition 2.6, $R(X) = \bigoplus_k \mathcal{O}_{X,x_k}$. Then $\operatorname{Spec}(R(X)) = \coprod_k \operatorname{Spec}(K_k)$. Then we have a morphism $j : \operatorname{Spec}(R(X)) \to X$ whose restriction to $\operatorname{Spec}(K_k)$ is i_k. Then for all X' which is an affine open subset of X or a scheme étale over X,

$$j_*(\mathbb{G}_m)_{R(X)}(X') = (\mathbb{G}_m)_{R(X)}(X' \times_X \operatorname{Spec}(R(X))) = (\mathbb{G}_m)_{R(X)}(X' \times_X \coprod_k \operatorname{Spec}(K_k))$$

$$= \bigoplus_k (\mathbb{G}_m)_{R(X)}(X' \times_X \operatorname{Spec}(K_k)) = \bigoplus_k (i_k)_*(\mathbb{G}_m)_{K_k}(X')$$

Therefore $j_*(\mathbb{G}_m)_{R(X)} = \bigoplus_k (i_k)_*(\mathbb{G}_m)_{K_k}$.

(2) By Lemma 3.3 and (1), we obtain an injection $\iota : (\mathbb{G}_m)_X \to j_*(\mathbb{G}_m)_{R(X)}$. □

We did not use Corollary 1.16(1) in the proofs of Lemma 3.3 and Lemma 3.4 because any presheaf kernel is already a sheaf (exercise). Hint: snake lemma.

3.5 Definition

In $X_{\text{ét}}$, we call $\mathcal{D}iv_{\text{ét}} = \operatorname{Cok}(\iota_{\text{ét}})$ the **étale sheaf of Cartier divisors** of X where ι comes from Lemma 3.4.

Recall that *all open immersions are étale.*

3.6 Lemma

$\mathcal{D}iv_{\text{Zar}} = \operatorname{Cok}(\iota_{\text{Zar}})$ where ι comes from Lemma 3.4.

Proof. For all affine subset U of X,

$$\Gamma(U, j_*(\mathbb{G}_m)_{R(X)}) = \Gamma(j^{-1}(U), (\mathbb{G}_m)_{R(X)}) = \Gamma(U, \mathbb{G}_m)\otimes R(X) = \Gamma(U, \mathcal{O}_X)^* \otimes R(X) = \Gamma(U, \mathscr{R}_X^*).$$

Then $j_*(\mathbb{G}_m)_{R(X)} = \mathscr{R}_X^*$. By Theorem 2.9 $\mathscr{R}_X^* = \mathscr{M}_X^*$. Recall that $(\mathbb{G}_m)_X = \mathcal{O}_X^*$. We have $\operatorname{Cok}(\iota_{\text{Zar}}) = \mathscr{M}_X^*/\mathscr{O}_X^* = \mathcal{D}iv_{\text{Zar}}$. □

3.7 Review

Let $\varepsilon : X_{\text{Zar}} \to X_{\text{ét}}$ be the inclusion morphism of sites. The functor $\varepsilon^{\text{Sh}} : \widetilde{X_{\text{ét}}} \to \widetilde{X_{\text{Zar}}}$ is left exact. Hilbert 90: $R^1\varepsilon^{\text{Sh}}(\mathbb{G}_m)_X = 0$.

3.8 Theorem

(1) $\varepsilon^{\text{Sh}}(\mathcal{D}iv_{\text{ét}}) = \mathcal{D}iv_{\text{Zar}}$.
(2) $H^0(X_{\text{Zar}}, \mathcal{D}iv_{\text{Zar}}) \simeq H^0(X_{\text{ét}}, \mathcal{D}iv_{\text{ét}})$.

Proof. (1) From the exact sequence in $\widetilde{X_{\text{ét}}}$

$$0 \to (G_m)_X \xrightarrow{\iota_{\text{ét}}} j_*(\mathbb{G}_m)_{R(X)} \to \mathcal{D}iv_{\text{ét}} \to 0,$$

we obtain a long exact sequence in $\widetilde{X_{\text{Zar}}}$

$$0 \to (G_m)_X \xrightarrow{\iota_{\text{Zar}}} j_*(\mathbb{G}_m)_{R(X)} \to \varepsilon^{\text{Sh}}\,\mathcal{D}iv_{\text{ét}} \to R^1\varepsilon^{\text{Sh}}(\mathbb{G}_m)_X \to \cdots$$

Since $R^1\varepsilon^{\text{Sh}}(\mathbb{G}_m)_X = 0$, $\operatorname{Cok}(\iota_{\text{Zar}}) = \varepsilon^{\text{Sh}}(\mathcal{D}iv_{\text{ét}})$. By Lemma 3.6, $\varepsilon^{\text{Sh}}(\mathcal{D}iv_{\text{ét}}) = \mathcal{D}iv_{\text{Zar}}$.

(2) By (1) $H^0(X_{\text{Zar}}, \mathcal{D}iv_{\text{Zar}}) = H^0(X_{\text{Zar}}, \varepsilon^{\text{Sh}}\,\mathcal{D}iv_{\text{ét}})$. By the Leray spectral sequence

$$E_2^{p,q} = H^p(X_{\text{Zar}}, R^q\varepsilon^{\text{Sh}}\,\mathcal{D}iv_{\text{ét}}) \Rightarrow L^{p+q} = H^{p+q}(X_{\text{ét}}, \mathcal{D}iv_{\text{ét}}).$$

Therefore $H^0(X_{\text{Zar}}, \varepsilon^{\text{Sh}}\,\mathcal{D}iv_{\text{ét}}) = E_2^{0,0} = E_\infty^{0,0} \simeq F^0L^0 = L^0 = H^0(X_{\text{ét}}, \mathcal{D}iv_{\text{ét}})$. Together, we have $H^0(X_{\text{Zar}}, \mathcal{D}iv_{\text{Zar}}) \simeq H^0(X_{\text{ét}}, \mathcal{D}iv_{\text{ét}})$. □

3.9 Definition

(1) Let X be a noetherian scheme. By Lemma 2.11, $\mathfrak{I}(X) = \{\overline{\{x\}} \mid x \in X\}$ is the set of irreducible closed subsets of X. We call elements of $\mathfrak{I}(X)$ **prime cycles**.

(2) Let $\mathscr{Z}(X)$ be the free abelian group generated by $\mathfrak{I}(X)$. We write

$$\mathscr{Z}(X) = \{Z = \sum_{x\in X} n_x\overline{\{x\}} \mid \text{The set } \{x \in X \mid n_x \neq 0\} \text{ is finite}\}$$

We call elements of $\mathscr{Z}(X)$ **cycles**.

(3) Suppose $Z = \sum\limits_{x\in X} n_x\overline{\{x\}}$ and $Z' = \sum\limits_{x\in X} n'_x\overline{\{x\}}$.

We defined an **order**: $Z \leq Z'$ if $n_x \leq n'_x$ for all $x \in X$.

(4) Let $X^{(1)} = \{x \in X \mid \dim(\mathcal{O}_{X,x}) = 1\}$. Let $\mathscr{Z}^1(X)$ be the free abelian group generated by $\mathfrak{I}(X^{(1)})$. We write

$$\mathscr{Z}^1(X) = \{ \sum_{x \in X^{(1)}} n_x \overline{\{x\}} \}.$$

We call elements of $\mathscr{Z}^1(X)$ **Weil divisors**.

3.10 Lemma

$\mathscr{Z}^1$ is a sheaf of ordered abelian groups (called the Zariski **sheaf of Weil divisors**) and

$$\mathscr{Z}^1 = \bigoplus_{x \in X^{(1)}} (i_x)_*(\underline{\mathbb{Z}}_x)$$

where $i_x : \{x\} \to X$ is the inclusion and $\underline{\mathbb{Z}}_x$ is the constant sheave of $\mathbb{Z}$ on $\{x\}$.

Proof. (1) For all open subsets U, V of X such that $U \supset V$, the restriction $\mathscr{Z}^1(U) \to \mathscr{Z}^1(V)$ is defined by $Z = \sum_{x \in U} n_x \overline{\{x\}} \mapsto Z|V = \sum_{x \in V} n_x (V \cap \overline{\{x\}})$. By definition, $Z \mapsto Z|V$ is a homomorphism of abelian groups. Since the restriction does not change the value of n_x, the restriction preserves orders.

(2) For all open subsets U, V, W of X such that $U \supset V \supset W$, we have $W \cap (V \cap \overline{\{x\}}) = W \cap \overline{\{x\}}$ and hence $(Z|V)|W = Z|W$. Therefore $\mathscr{Z}^1$ is a presheaf of ordered abelian groups.

(3) By definition of restriction, we have $\mathscr{Z}^1_x = \{n_x x\} \simeq \mathbb{Z}$. Since

$$(i_x)_*(\underline{\mathbb{Z}}_x)_y = \begin{cases} \mathbb{Z}, & \text{if } y = x. \\ 0, & \text{if } y \in X \setminus \{x\}, \end{cases}$$

we have $\mathscr{Z}^1_y = \left(\bigoplus_{x \in X} (i_x)_*(\underline{\mathbb{Z}}_x) \right)_y = \mathbb{Z}$ for all $y \in X$. Therefore $\mathscr{Z}^1 \simeq \bigoplus_{x \in X} (i_x)_*(\underline{\mathbb{Z}}_x)$ and it is a sheaf. □

3.11 Lemma

If a ring A is noetherian and $\dim(A) = 0$, then it has finite length.

Proof. By [AM69, Th. 8.5], A is artinian. By [AM69, Prop. 6.8], A has finite length. □

3.12 Lemma

Let A be a noetherian local ring of Krull dimension one. Then $\operatorname{ord}_A : \operatorname{Frac}(A)^* \to \mathbb{Z}$, $f = \frac{a}{b} \mapsto \operatorname{ord}_A(f) = \operatorname{length}(A/(a)) - \operatorname{length}(A/(b))$ is a group homomorphism such that $A^* \subset \ker(\operatorname{ord})$. Here a, b are non zero-divisors.

Proof. If $a \in A^*$, then $A/(a) = A/(1) = A/A = 0$ and $\operatorname{length}(A/(a)) = 0$. Therefore $A^* \subset \ker(\operatorname{ord}_A)$.

If a is a non unit, non zero-divisor of A, then $\dim(A/(a)) = \dim(A) - 1 = 1 - 1 = 0$. By Lemma 3.11, $A/(a)$ is a noetherian ring of dimension 0, it is an artinian ring of finite length. Hence ord_A is well-defined.

By the exact sequence $0 \to A/(a) \xrightarrow{b} A/(ab) \to A/(b) \to 0$, we have $\operatorname{length}(A/(ab)) = \operatorname{length}(A/(a)) + \operatorname{length}(A/(b))$. Since $\operatorname{length}(A/(\bullet))$ is a homomorphism, ord_A is also a homomorphism. □

3.13 Definition

We define $\mathrm{cyc}_X : \Gamma(X, \mathcal{D}iv_{\mathrm{Zar}}) \to \Gamma(X, \mathscr{Z}^1) = \mathscr{Z}^1(X)$. Suppose $D = (U_i, f_i) \in \Gamma(X, \mathscr{M}_X^*/\mathcal{O}_X^*) = \Gamma(X, \mathcal{D}iv_{\mathrm{Zar}})$ where (U_i) is a covering of X, $f_i \in \Gamma(U_i, \mathscr{M}_X^*)$ such that $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$. For $x \in U_i \cap X^{(1)}$ for some i, let $n_x = \mathrm{ord}_{\mathcal{O}_{X,x}}((f_i)_x)$, otherwise let $n_x = 0$. We define $\mathrm{cyc}_X(D) = \sum_{x \in X^{(1)}} n_x \overline{\{x\}}$.

3.14 Lemma

The map cyc_X is a well-defined homomorphism of abelian groups.

Proof. We need to show that $\mathrm{ord}_{\mathcal{O}_{X,x}}((f_i)_x) = \mathrm{ord}_{\mathcal{O}_{X,x}}((f_j)_x)$ for all $x \in U_i \cap U_j \cap X^{(1)}$. By Lemma 3.12, $(f_i)_x/(f_j)_x = (f_i/f_j)_x \in \mathcal{O}_{X,x}^* \subset \ker(\mathrm{ord}_{\mathcal{O}_{X,x}})$. In other words, $\mathrm{ord}_{\mathcal{O}_{X,x}}((f_i)_x/(f_j)_x) = 0$. Again, by Lemma 3.12, $\mathrm{ord}_{\mathcal{O}_{X,x}}$ is a homomorphism. Therefore $\mathrm{ord}_{\mathcal{O}_{X,x}}((f_i)_x) - \mathrm{ord}_{\mathcal{O}_{X,x}}((f_j)_x) = 0$ and also cyc_X is a homomorphism. □

Similarly, we have homomorphism of abelian groups

$$\mathrm{cyc}_U : \Gamma(U, \mathcal{D}iv_{\mathrm{Zar}}) \to \Gamma(U, \mathscr{Z}^1)$$

for all open subset U of X and they are compatible with restrictions. This defines a morphism of sheaves

$$\mathrm{cyc} : \mathcal{D}iv_{\mathrm{Zar}} \to \mathscr{Z}^1$$

Next time, we show that for some X, cyc is an isomorphism.

4. March 30th, H^2 of regular schemes

4.1 Lemma
Let A be a noetherian commutative ring. If A is a UFD, then every height one prime ideal $\mathfrak{p}$ of A is principal.

Proof. Suppose $x \in \mathfrak{p}$. Since A is noetherian, $x = a_1 a_2 \cdots a_n$ where a_i are irreducible. Then there exists some $a_i \in \mathfrak{p}$ since $\mathfrak{p}$ is a prime ideal. We have (a_i) is a prime ideal since A is a UFD. Since $(a_i) \subset \mathfrak{p}$ and $\mathrm{ht}(\mathfrak{p}) = 1$, we have $(a_i) = \mathfrak{p}$. $\square$

The converse is also true, see [Kap74, Th. 5].

4.2 Theorem
If $\mathcal{O}_{X,x}$ is a UFD for all $x \in X$, then cyc is an isomorphism of sheaves.

Proof. Injectivity. Suppose $D = (U_i, f_i)_{i\in I} \in \Gamma(U, \mathscr{M}_X^*/\mathscr{O}_X^*)$ for some open neighborhood U of x. Suppose $\mathrm{cyc}(D)_x = 0$.

Suppose $x \in U \cap X^{(1)}$. Since $\mathcal{O}_{X,x}$ is integrally closed, by [AM69, Prop. 9.2], it is a discrete valuation ring. Every element of $\mathrm{Frac}(\mathcal{O}_{X,x})^*$ has the form $u\pi^{n_x}$ for some $u \in \mathcal{O}_{X,x}^*$. Then $\mathrm{cyc}(D)_x = 0$ iff $n_x = 0$ iff $(f_i)_x \in \mathcal{O}_{X,x}^*$ for all U_i containing x. Then there exists an open neighborhood V_i of x such that $V_i \subset U_i$ and $(f_i)|_{V_i} \in \Gamma(V_i, \mathcal{O}_X^*)$. Let $I' = \{i \in I \mid x \in U_i\}$, $V = \bigcup_{i\in I'} V_i$. Let $D' = (V_i, (f_i)|_{V_i})_{i\in I'} \in \Gamma(V, \mathscr{M}_X^*/\mathscr{O}_X^*)$. We have $D_x = D'_x$ and $D' = 0$. Hence $D_x = 0$.

Suppose $x \in U \setminus X^{(1)}$. Similar to the first paragraph, we can show that $D_y = 0$ for all $y \in U_i \cap X^{(1)}$ with $x \in U_i$. Then $\mathrm{dom}(f_i) \supset U_i \cap X^{(1)}$ for all i. We may assume that U_i is affine. Let $U_i = \mathrm{Spec}(A)$ for some integrally closed domain A. We obtain the restriction $A = \Gamma(U_i, \mathcal{O}_X) \to \Gamma(\mathrm{dom}(f_i), \mathcal{O}_X) \subset \bigcap A_\mathfrak{p} = A$ where p runs through all height one prime ideals of A. By Proposition 2.4, it is injective and hence bijective. Then $\mathrm{dom}(f_i) = U_i$ and hence $D_x = 0$.

Surjectivity. Since cyc is a morphism, it suffices to find the inverse image of every prime Weil divisor $Z = \overline{\{x\}}$, $x \in X^{(1)}$. Let I be the sheaf of ideals of Z in $\mathcal{O}_X$. Then I_y is a height one prime ideal of $\mathcal{O}_{X,y}$ for all $y \in X$. Since $\mathcal{O}_{X,y}$ is a noetherian UFD, by Lemma 4.1, its height one prime ideals are all principal. Suppose $I_y = (\pi_y)$. There exists an open neighborhood U_y of y and $f \in \Gamma(U_y, \mathcal{O}_X)$ such that $f_y = \pi_y$ and $I|_{U_y} = (f)$.

Then there exists (U_i, f_i) such that (U_i) is a covering of X, $f_i \in \Gamma(U_i, \mathcal{O}_X)$, $I|_{U_i} = (f_i)$ and $(f_i|_{U_i\cap U_j}) = (f_j|_{U_i\cap U_j})$ iff $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X)^*$. Therefore $D = (U_i, f_i)$ is a Cartier divisor. Since $\mathcal{O}_{X,x}$ is a discrete valuation ring, $\mathrm{ord}_{\mathcal{O}_{X,x}}((f_i)_x) = \mathrm{ord}_{\mathcal{O}_{X,x}}(\pi_x) = 1$. Hence $\mathrm{cyc}(D)_x = Z = \overline{\{x\}}$ is the generator of $\mathscr{Z}_x^1$. $\square$

4.3 Review
Let k be a field. Let $\overline{k}$ be a separable closure of k. Let $G_k = \mathrm{Gal}(\overline{k}/k)$ be the absolute galois group of k. Let $C(G_k)$ be the category of continuous G-sets. Then there exists an equivalence of categories $\widetilde{\mathrm{Spec}(k)}_{\text{ét}} \simeq C(G_k)$. Furthermore, $H^q(\mathrm{Spec}(k)_{\text{ét}}, \mathscr{F}) = \varinjlim H^q(G_k, \mathscr{F}(k'))$ where $\mathscr{F}$ is an abelian sheaf on $\mathrm{Spec}(k)_{\text{ét}}$ and k' runs through all finite extensions of k in $\overline{k}$, $q \geq 0$.

4.4 Lemma
Let $i : \{x\} \to X$ be the inclusion of a point $\{x\} = \mathrm{Spec}(\kappa(x))$. Let $\underline{A}_x$ be the

constant sheaf on $\{x\}_{\text{ét}}$ for a torsion-free abelian group A. Then $H^1(X_{\text{ét}}, i_*\underline{A}_x) = 0$. In particular, $H^1(X_{\text{ét}}, i_*\underline{\mathbb{Z}}_x) = 0$.

Proof. Leray spectral sequence gives

$$E_2^{p,q} = H^p(X_{\text{ét}}, R^q i_*\underline{A}_x) \Rightarrow L^{p+q} = H^{p+q}(\{x\}_{\text{ét}}, \underline{A}_x).$$

Since $0 = E_2^{-1,1} \to E_2^{1,0} \to E_2^{3,-1} = 0$, we have $E_2^{1,0} = E_\infty^{1,0} = \dfrac{F^1L^1}{F^2L^1} = F^1L^1 \subset L^1$. i.e. $H^1(X_{\text{ét}}, i_*\underline{A}_x) \subset H^1(\{x\}_{\text{ét}}, \underline{A}_x)$. We have

$$\begin{array}{rcll} H^1(\{x\}_{\text{ét}}, \underline{A}_x) & = & H^1(\mathrm{Spec}(\kappa(x))_{\text{ét}}, \underline{A}_x) & \\ & = & H^1(G_{\kappa(x)}, A), & \text{by } \mathrm{Spec}\widetilde{(\kappa(x))}_{\text{ét}} \simeq C(G_{\kappa(x)}) \\ & = & \mathrm{Hom}_{\text{cont}}(G_{\kappa(x)}, A), & G_{\kappa(x)} \text{ acts on } A \text{ trivially.} \\ & = & 0, & G_{\kappa(x)} \text{ is torsion and } A \text{ is torsion free.} \end{array}$$

where $\kappa(x)$ is the residue field of $\mathcal{O}_{X,x}$ and cont means continuous homomorphisms. Therefore $H^1(X_{\text{ét}}, i_*\underline{A}_x) = 0$. □

4.5 Definition
A noetherian local ring $(A, \mathfrak{m}_A)$ satisfies $\dim_{A/\mathfrak{m}_A}(\mathfrak{m}_A/\mathfrak{m}_A^2) \geq \dim(A)$, see [AM69, cor. 11.15].
We call A **regular** if the equality holds. A scheme X is **regular** if $\mathcal{O}_{X,x}$ is regular for all $x \in X$.

4.6 Theorem
Auslander-Buchsbaum-Nagata: Every Regular local ring is a UFD.

Proof. [AB59, Nag58]. □

4.7 Lemma
Let X' be any scheme with an étale structure morphism $p : X' \to X$. If X is regular, then X' is regular.

Proof. For all $x' \in X'$ and $x = p(x') \in X$, $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{X',x'})$ since $p : X' \to X$ is étale. Since X is regular, we have $\dim(\mathfrak{m}_x/\mathfrak{m}_x^2) = \dim(\mathcal{O}_{X,x})$. Then $\dim(\mathfrak{m}_x/\mathfrak{m}_x^2) = \dim(\mathcal{O}_{X',x'})$. Since $\mathfrak{m}_{x'} = \mathfrak{m}_x\mathcal{O}_{X',x'}$,

$$\dim(\mathfrak{m}_{x'}/\mathfrak{m}_{x'}^2) = \dim((\mathfrak{m}_x/\mathfrak{m}_x^2) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}) \leq \dim(\mathfrak{m}_x/\mathfrak{m}_x^2) = \dim(\mathcal{O}_{X',x'}).$$

Together with Definition 4.5, $\dim(\mathfrak{m}_{x'}/\mathfrak{m}_{x'}^2) = \dim(\mathcal{O}_{X',x'})$. Hence X' is regular. □

4.8 Corollary
If X is a regular noetherian scheme, then $H^1(X_{\text{ét}}, \mathcal{D}iv_{\text{ét}}) = 0$.

Proof. Let X' be any scheme étale over X. Since X is regular, by Lemma 4.7, X' is regular. Since X' is regular, $\mathcal{O}_{X',x}$ is a regular local ring for all $x \in X'$. By Theorem 4.6, $\mathcal{O}_{X',x}$ is a UFD. By Theorem 4.2, $\mathcal{D}iv_{X',\text{Zar}} \simeq \mathscr{Z}^1_{X'}$. By Lemma 3.10, $\mathscr{Z}^1_{X'} \simeq \bigoplus_{x \in X'^{(1)}} (i_x)_*(\underline{\mathbb{Z}}_x)$. Then $\Gamma(X', \mathcal{D}iv_{\text{Zar}}) \simeq \Gamma(X', \bigoplus_{x \in X'^{(1)}} (i_x)_*(\underline{\mathbb{Z}}_x))$.
Since affine open sets are étale, by Proposition 1.13, $(\mathcal{D}iv_{\text{ét}})_{\overline{x}} = \varinjlim(\mathcal{D}iv_{\text{Zar}})_{x'}$ for all geometric point $\overline{x}$ of X and for all $x' \in \mathrm{Spec}(\kappa(x)) \times_X X'$. Hence $\mathcal{D}iv_{\text{ét}} \simeq$

$\bigoplus_{x\in X^{(1)}}(i_x)_*(\underline{\mathbb{Z}}_x)$ on $X_{\text{ét}}$. By Lemma 4.4, $H^1(X_{\text{ét}}, i_*\underline{\mathbb{Z}}_x) = 0$. Therefore

$$H^1(X_{\text{ét}}, \mathcal{D}iv_{\text{ét}}) = H^1(X_{\text{ét}}, \bigoplus_{x\in X^{(1)}}(i_x)_*(\underline{\mathbb{Z}}_x)) = \bigoplus_{x\in X^{(1)}} H^1(X_{\text{ét}}, (i_x)_*(\underline{\mathbb{Z}}_x)) = 0$$

□

4.9 Lemma

Let $i : \{x\} \to X$ be the inclusion of a point $\{x\} = \operatorname{Spec}(\kappa(x))$. Then
(1) $R^1 i_*(\mathbb{G}_m)_{\kappa(x)} = 0$.
(2) $H^2(X_{\text{ét}}, i_*(\mathbb{G}_m)_{\kappa(x)}) \to H^2(\{x\}_{\text{ét}}, (\mathbb{G}_m)_{\kappa(x)})$ is injective.

Proof. (1) Any étale scheme over $\{x\}$ has form $\operatorname{Spec}(\bigoplus K_i)$ with each K_i a finite separable extension of $\kappa(x)$. By Hilbert 90,

$$H^1(\operatorname{Spec}(\bigoplus_i K_i)_{\text{ét}}, (\mathbb{G}_m)_{\kappa(x)}) = \bigoplus_i H^1(\operatorname{Spec}(K_i)_{\text{ét}}, (\mathbb{G}_m)_{\kappa(x)}) = 0.$$

The sheaf associated to $X' \mapsto H^1(X' \times_X \operatorname{Spec}(\kappa(x)), (\mathbb{G}_m)_{\kappa(x)}) = 0$ is $R^1 i_*(\mathbb{G}_m)_{\kappa(x)} = 0$.
(2) Leray spectral sequence gives

$$E_2^{p,q} = H^p(X_{\text{ét}}, R^q i_*(\mathbb{G}_m)_{\kappa(x)}) \Rightarrow L^{p+q} = H^{p+q}(\{x\}_{\text{ét}}, (\mathbb{G}_m)_{\kappa(x)}).$$

By (1), $E_2^{0,1} = 0$. Then $0 = E_2^{0,1} \to E_2^{2,0} \to E_2^{4,-1} = 0$.
Hence $H^2(X_{\text{ét}}, i_*(\mathbb{G}_m)_{\kappa(x)}) = E_2^{2,0} = E_\infty^{2,0} \simeq \dfrac{F^2L^2}{F^3L^2} = F^2L^2 \subset L^2 = H^2(\{x\}, \mathbb{G}_m)_{\kappa(x)})$. □

4.10 Proposition

Let X be a regular noetherian scheme. There exists an injection

$$H^2(X_{\text{ét}}, (\mathbb{G}_m)_X) \to \bigoplus_k H^2(\operatorname{Spec}(K_k)_{\text{ét}}, (\mathbb{G}_m)_{K_k})$$

where K_k runs through function fields of irreducible components of X.

Proof. From the exact sequence in $\widetilde{X_{\text{ét}}}$

$$0 \to (G_m)_X \xrightarrow{\iota_{\text{ét}}} j_*(\mathbb{G}_m)_{R(X)} \to \mathcal{D}iv_{\text{ét}} \to 0.$$

We obtain a long exact sequence

$$\cdots \to H^1(X_{\text{ét}}, \mathcal{D}iv_{\text{ét}}) \to H^2(X_{\text{ét}}, (G_m)_X) \to H^2(X_{\text{ét}}, j_*(\mathbb{G}_m)_{R(X)}) \to \cdots$$

By Corollary 4.8, $H^1(X_{\text{ét}}, \mathcal{D}iv_{\text{ét}}) = 0$, then $H^2(X_{\text{ét}}, (G_m)_X) \to H^2(X_{\text{ét}}, j_*(\mathbb{G}_m)_{R(X)})$ is injective. By Lemma 4.9, $H^2(X_{\text{ét}}, i_*(\mathbb{G}_m)_{\kappa(x_k)}) \to H^2(\{x_k\}, (\mathbb{G}_m)_{\kappa(x_k)})$ is injective for all maximal points x_k of X. By Lemma 3.4,

$$H^2(X_{\text{ét}}, i_*(\mathbb{G}_m)_{R(X)}) = \bigoplus_k H^2(X_{\text{ét}}, i_*(\mathbb{G}_m)_{K_k}) \to \bigoplus_k H^2(\operatorname{Spec}(K_k), (\mathbb{G}_m)_{K_k})$$

is injective. where $K_k = \mathcal{O}_{X,x_k}$. Therefore the composition $H^2(X, (\mathbb{G}_m)_X) \to \bigoplus_k H^2(\operatorname{Spec}(K_k)_{\text{ét}}, (\mathbb{G}_m)_{K_k})$ is injective. □

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