$$X \to Y \to Z \to W$$

chain rule $\quad P(X,Y,Z,W) = P(X)\,P(Y|X)\,P(Z|X,Y)\,P(W|X,Y,Z)$

Bayes net $\quad P(X,Y,Z,W) = P(X)\,P(Y|X)\,P(Z|Y)\,P(W|Z)$

$$Z \perp\!\!\!\perp X \mid Y \qquad W \perp\!\!\!\perp X,Y \mid Z$$

Additional implied conditional independence assumptions

e.g. $W \perp\!\!\!\perp X \mid Y$

$$P(W|X,Y) \overset{?}{=} P(W|Y)$$

$$P(W|X,Y) = \frac{P(W,X,Y)}{P(X,Y)}$$

$$= \frac{\sum_Z P(X,Y,Z,W)}{P(X,Y)} = \frac{\sum_Z \cancel{P(X)}\,\cancel{P(Y|X)}\,P(Z|\cancel{X},Y)\,P(W|Z)}{\cancel{P(X)}\,\cancel{P(Y|X)}}$$

$$= \sum_Z P(Z|Y)\,P(W|Z)$$

because $W \perp\!\!\!\perp X,Y \mid Z$

$$= \sum_Z P(Z|Y)\,P(W|Z,Y)$$

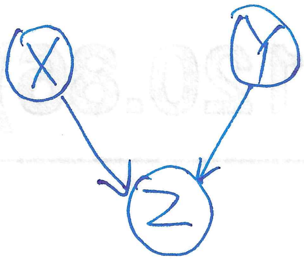
conditional probability definition

$$= \sum_Z P(Z,W|Y)$$

$$= P(W|Y)$$

Common Effect

$X \perp\!\!\!\perp Y$



Proof #1 :

$$P(X,Y,Z) = P(X)P(Y)P(Z|X,Y)$$

$$\sum_Z P(X,Y,Z) = \sum_Z P(X)P(Y)P(Z|X,Y)$$

$$\begin{aligned} P(X,Y) &= P(X)P(Y)\underbrace{\sum_Z P(Z|X,Y)}_{=1} \\ &= P(X)P(Y) \end{aligned}$$

Proof #2 :

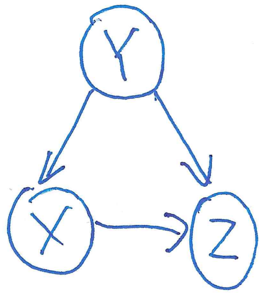
$X_i \perp\!\!\!\perp X_1, \cdots, X_{i-1} - Pa(X_i) \mid Pa(X_i)$.

X, Y, Z. $Pa(Y) = \phi$

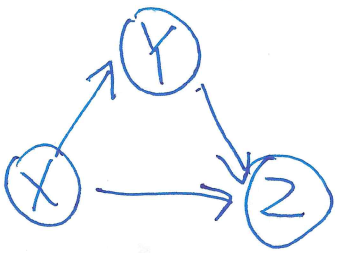
$Y \perp\!\!\!\perp X \mid \phi$

Chain Rule: joint probability of N variables can be factored in $N!$ ways.

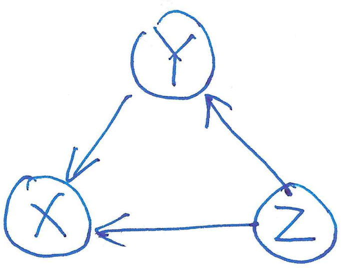
Example: $N=3$. $3 \times 2 \times 1 = 6$ ways to factorize



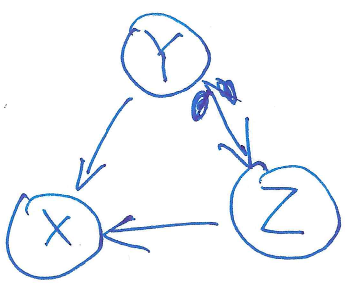
$P(\underline{Y})\,P(\underline{X}|Y)\,P(\underline{Z}|X,Y)$



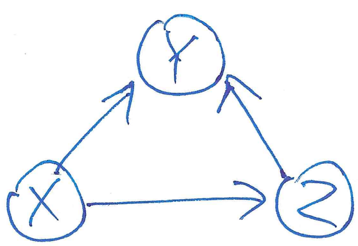
$P(\underline{X})\,P(\underline{Y}|X)\,P(\underline{Z}|X,Y)$



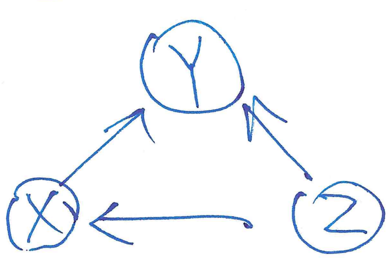
$P(\underline{Z})\,P(\underline{Y}|Z)\,P(\underline{X}|Y,Z)$



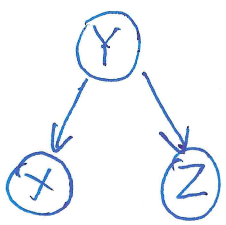
$P(\underline{Y})\,P(\underline{Z}|Y)\,P(\underline{X}|Y,Z)$



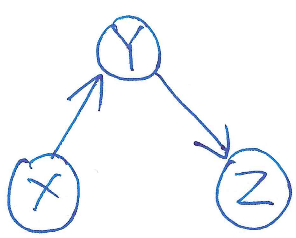
$P(\underline{X})\,P(\underline{Z}|X)\,P(\underline{Y}|X,Z)$



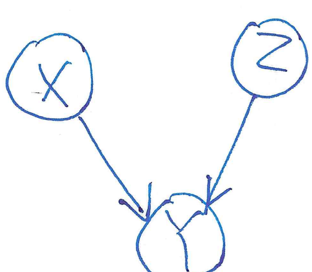
$P(\underline{Z})\,P(\underline{X}|Z)\,P(\underline{Y}|X,Z)$



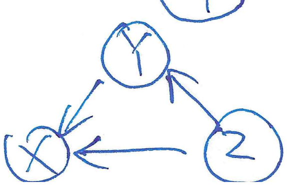
$\{ X \perp\!\!\!\perp Z \mid Y \}$



$\{ X \perp\!\!\!\perp Z \mid Y \}$



$\{ X \perp\!\!\!\perp Z \}$



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