Point Stablity Problem

1 Introduction

Given that the independence number of simple graph G keeps the same no matter how we delete a small subset of vertices, what can we say about the behavior of the independence number $\alpha(G)$? There can be a lot of related questions to ask under such a setting. Specifically, Prof. Boris Bukh conjectured that $\alpha(G) \leqslant \frac{1}{2}|V(G)|$ when $\alpha(G)$ does not drop up to every possible 1-vertex deletion. I will give a proof of this conjecture, and then generalize slightly to solve this problem under the condition that $\alpha(G)$ does not drop up to every possible 2-vertex deletion.

2 1-Point Stability

Theorem 1. Suppose $G = (V, E)$ is a simple graph on n vertices with $\alpha(G) = \alpha(G \backslash v)$ for every $v \in V$. Then $\alpha(G) \leqslant \lfloor \frac{n}{2} \rfloor$. The bound is tight by considering $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$ or $K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor, 1}$.

Proof. Take a maximal independent set Y of V, and hence $|Y| = \alpha(G)$. It suffices to show that $|N(Y)| \geqslant |Y|$.

We prove by contradiction. Assume that $|N(Y)| < |Y|$, then we can find a minimal subset Z of Y such that $|N(Z)| < |Z|$. Choose an arbitrary $z_0 \in Z$ since $Z \neq \varnothing$.

Note that $\alpha(G) = \alpha(G \backslash z_0)$, we can find another maximal independent set $X \neq Y$ with $z_0 \notin X$. Define $Z_1 = X \cap Z$ and $Z_2 = Z \backslash Z_1$. We construct a set

$$U = (X \backslash (N(Z) \backslash N(Z_1))) \cup Z_2$$

and claim that U is independent with $|U| > \alpha(G)$, which is a contradiction.

First, we show that U is independent. Note that both X and Z are independent, hence both $X \backslash (N(Z_1) \backslash N(Z_2))$ and Z_2 are independent. It suffices to argue that there is no edge between $X \backslash (N(Z_1) \backslash N(Z_2))$ and Z_2.

Suppose $x \in X \backslash (N(Z) \backslash N(Z_1))$ and $z \in Z_2$ are connected. Then $x \in N(Z_2)$ and hence $x \in N(Z)$. Since $Z_1 \subseteq X$ and X is independent, there is no edge between Z_1 and X, hence $x \notin N(Z_1)$. Thus, $x \in X$ while $x \in N(Z) \backslash N(Z_1)$, which is a contradiction. We conclude that U is independent.

Next, we show that $|U| > |X| = \alpha(G)$. Note that $Z_2 \cap X = \varnothing$ by definition, we have

$$|U| = |X \backslash (N(Z) \backslash N(Z_1))| + |X \backslash (N(Z) \backslash N(Z_1))| \geqslant |X| - |N(Z) \backslash N(Z_1)| + |Z_2|.$$

Then it suffices to show that $|N(Z) \backslash N(Z_1)| < |Z_2|$. Note that $z_0 \in Z \backslash X$, then $Z_1 \subsetneq Z$, and hence $|N(Z_1)| \geqslant |Z_1|$ by the minimality assumption of Z. Thus,

$$|N(Z) \backslash N(Z_1)| = |N(Z)| - |N(Z_1)| \leqslant |N(Z)| - |Z_1| < |Z| - |Z_1| = |Z_2|,$$

which concludes the proof. $\square$

3 2-Point Stability

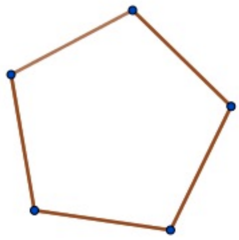
Theorem. Suppose $G=(V,E)$ is a simple graph on n vertices with $\alpha(G)=\alpha(G\backslash\{u,v\})$ for every $u,v\in V$. Then $\alpha(G)\leqslant\lfloor\frac{n-1}{2}\rfloor$. The bound is sharp.

Proof. By removing one point and then apply the theorem of 1-point removal, we see that

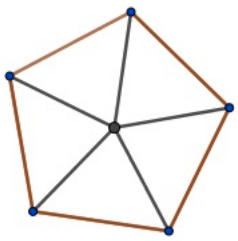
$$\alpha(G)\leqslant\lfloor\frac{n-1}{2}\rfloor.$$

The sharpness is seen by the following constructions:

- When n is odd, take $G=C_n$ which is the n-cycle.



- When n is even, Take $H=C_{n-1}$ and let v be an extra point. Then take G be the union of H and $\{v\}$ and connect v to every point of H (which seems like a *wheel*).



Now we prove that the constructions satisfy the condition. Obviously, we only need to prove the odd case. According to clockwise direction, let all the points be $1,2,...,n$. By symmetry, Assume we delete the points 1 and k. Consider two independence sets $\{2,4,...,n-1\}$ and $\{3,5,...,n\}$. Obviously, One of them don't contain k, so it is a $\lfloor\frac{n-1}{2}\rfloor$ points independence set in Graph $G\backslash\{1,k\}$. Thus, we have finished the proof. □