An Inverted Problem of Boros and Füredi

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Abstract

Boros and Füredi proved that for any finite set of points in the plane in general position there is a point lying in $\frac{2}{9}$ of all the triangles determined by these points. We consider an inverted version of this problem, and prove that for any finite set of points in the plane in general position, any another point can lies on at most asymptotically $\frac{2}{9}$ of all the triangles determined by these points. Moreover, we will discuss a modified case of adapting lines to circles.

1 Introduction

Let P be a set of n points in $\mathbb{R}^2$ in general position, and p be another point in $\mathbb{R}^2$. Let $T(P,p)$ be the number of simplices containing p. Boros and Füredi constructed a set P of n points in $\mathbb{R}^2$ for which $T(P,p) \leq \frac{2}{9}\binom{n}{3} + O(n^2)$ for every point p. Besides, they also proved that $\frac{2}{9}$ is the best constant. Here we consider an inverted question. We want to caculate the possible maximum of $T(P,p)$, and we will answer this question in this paper. We will prove that, for any set P and point p, $T(P,p) \leq \frac{1}{4}\binom{n}{3} + O(n^2)$, and $\frac{1}{4}$ is the best constant.

Moreover, we will consider the case of extending the problems on triangles to circles. We define $C(P,p)$ as the number of circles containing p that have a diameter generated by connecting two points in P. Notice that we don't need P being in general position now. We can ask two questions if we change the $T(P,p)$ to $C(P,p)$ on the problems above:

P1. Find the minimum constant C such that for any set P and point p, $C(P,p) \leq C\binom{n}{2} + O(n)$.

P2. Find the maximum constant C such that for any set P there is a point p outside satisfying $C(P,p) \geq C\binom{n}{2} + O(n)$.

Notice that change the $C(P,p)$ to $T(P,p)$(and $\binom{n}{2}$ to $\binom{n}{3}$) in the two problems we can get the problems in triangle situation. In this paper, we will answer the problem 1, and give a simple bound about problem 2. We will prove that, for any set P and point p, $C(P,p) \leq \frac{2}{3}\binom{n}{2} + O(n)$, and $\frac{2}{3}$ is the best constant. We will also prove that for any set P there is an another point p such that $C(P,p) \geq \frac{3}{8}\binom{n}{2} + O(n)$, and we construct a set P such that $C(P,p) \leq \frac{1}{2}\binom{n}{2} + O(n)$ for every point p.

2 Triangle Case

Theorem 1. For any set P and point p, $T(P,p) \leq \frac{1}{4}\binom{n}{3} + O(n^2)$, and $\frac{1}{4}$ is the best constant.

Proof. Define a_n as the maximum possible value of $T(P,p)$ for any set P of n points and some other point p outside P. Obviously, a_n is nondecreasing. We will prove that

$$a_n = \frac{n^3 - n}{24}$$

when n is odd, and hence

$$\frac{(n-1)^3-n+1}{24} \le a_n \le \frac{(n+1)^3-n-1}{24}$$

when n is even. The result follows from the above.

We prove that $a_n \ge \frac{n^3-n}{24}$ first. Construct a regular n-gon $X = x_1x_2....x_n$, and let P be $\{x_1, x_2, ..., x_n\}$ and p be the center of the circumcircle of X. We calculate $T(P,p)$ now. Assume $\triangle x_1x_ix_j$ contains p, and $i < j$. Notice that the condition is equivalent to $i \le \frac{n+1}{2} \le j \le \frac{n-1}{2} + j$. Thus, for each $i \le \frac{n+1}{2}$, there are $i-1$ possible j's. Therefore, there are $\frac{n^2-1}{8}$ triangles $x_1x_ix_j$ containing p. By symmetry, there are $\frac{n^3-n}{24}$ triangles $x_ix_jx_k$ containing p in total.

On the other hand, we prove that for any set P of n points and an another point p, $T(P,p) \le \frac{n^3-n}{24}$. The proof is by induction on n. Obviously $a_3 = 1$. Assume that the statement holds for $n-2$. Consider the case n. Assume $P = \{x_1, x_2, ..., x_n\}$. Note that point p lies in $\triangle x_ix_jx_k$ if and only if line segments px_i, px_j, and px_k do not lie in the same side for every line passing through p, which only depends on directions $\overrightarrow{px_i}$, $\overrightarrow{px_j}$, and $\overrightarrow{px_k}$. Thus, we may lengthen px_i for every i, and assume that all x_i's lie on a big circle with center p.

Without loss of generality, suppose $\angle x_1px_2$ is the biggest among all angles $\angle x_ipx_j$ where $1 \le i < j \le n$. Define a and b be the numbers of points in P and lying on minor arc x_1x_2 and major arc x_1x_2 separately. Obviously $a + b = n - 2$. We consider all triangles $x_ix_jx_k$ $(i < j < k)$ in the following three groups, and calculate the number of these triangles containing p:

- $|\{i,j,k\} \cap \{1,2\}| = 0$. By induction hypothesis, the number of these triangles containing p is at most $\frac{(n-2)^3-(n-2)}{24}$.
- $|\{i,j,k\} \cap \{1,2\}| = 1$. Then $i = 1$ or 2 and $j \ge 3$.
  - If x_j, x_k occupies one of major arc x_1x_2 and minor arc x_1x_2, then x_j, x_k have to be on different sides of line px_1, and hence on major arc x_1x_2. Suppose without loss of generality that x_j and x_2 lie on the same side of line px_1, then $\angle x_1px_j > \angle x_1px_2$, which is a contradiction.
  - If x_j, x_k occupies both major arc x_1x_2 and minor arc x_1x_2, then the points are in order x_1, x_i, x_2, x_j on the circle, and p can only be in one of $\triangle A_1A_iA_j$ and $\triangle A_2A_iA_j$. We have no more than ab such triangles.
- $|\{i,j,k\} \cap \{1,2\}| = 2$. Obviously x_k lies on the major arc, so the number of such triangles containing p is no more than b.

Thus, the numbers of all the triangles containing p is at most

$$\frac{(n-2)^3-(n-2)}{24} + b + ab \le \frac{(n-2)^3-(n-2)}{24} + \left(\frac{a+b+1}{2}\right)^2 = \frac{n^3-n}{24},$$

which concludes the statement. $\square$

Remark. We have shown that $a_n = \frac{n^3-n}{24}$ for odd n. Using the same method, it is not difficult to prove that $a_n = \frac{n^3-4n}{24}$ when n is even. We omit the proof here.

3 Circle Case

Theorem 2. For any set P and any another point p, $C(P,p) \leq \frac{2}{3}\binom{n}{2} + O(n)$, and $\frac{2}{3}$ is the best constant.

Proof. Assume $P = \{x_1, x_2, ..., x_n\}$. Notice that p is contained in the circle with diameter $x_i x_j$ is equivalent to $\angle x_i p x_j > 90°$. Let G be a graph on $\{1, 2, ..., n\}$ where we join $\{i, j\}$ if and only if $\angle x_i p x_j > 90°$, then $C(P,p)$ is the number of edges in G.

We claim that there is no K_4 in graph G. Otherwise, by symmetry, assume x_1, x_2, x_3, x_4 are connected to each other. Starting from p, draw a line perpendicular to px_1 that divide the plane into two half-planes. Then x_2, x_3, x_4 has to be in the half without x_1, and hence $\angle x_2 p x_3$, $\angle x_2 p x_4$, $\angle x_3 p x_4$ cannot be all obtuse, which is a contradiction. Thus, by Turán's theorem, $C(P,p) \leq \frac{2}{3}\binom{n}{2} + O(n)$.

On the other hand, We will give a construction here. The idea is let p be the center of a regular triangle and make all points in P near the vertex of the triangle. So, we consider $P = \{(\mathrm{Re}(w^m) + \frac{1}{100m}, \mathrm{Im}(w^m)) : m = 1, 2, ..., n\}$($w$ is the cubic unit root) and $p = (0, 0)$. It is easily seen that when $3 \nmid i - j$, p lies on the circle determined by $(\mathrm{Re}(w^i) + \frac{1}{100i}, \mathrm{Im}(w^i))$ and $(\mathrm{Re}(w^j) + \frac{1}{100j}, \mathrm{Im}(w^j))$. So there are at least $3(\frac{n}{3} - 1)^2 = \frac{2}{3}\binom{n}{2} + O(n)$ circles containing p, which finishes the proof. $\square$

Theorem 3. For any set P there is an another point p such that $C(P,p) \geq \frac{3}{8}\binom{n}{2} + O(n)$, and there is a set P such that $C(P,p) \leq \frac{1}{2}\binom{n}{2} + O(n)$ for every point p not belongs to P.

To prove theorem 3, we need the following lemma:

Lemma. Let $P \subset \mathbb{R}^2$ be a set of n points. Then there exist two orthogonal lines which separate P into four groups with at least $\frac{n}{4} - 1$ points each.

The conclusion about this lemma can be found in [1].

Proof. According to the lemma, for any set P of n points, we can choose two orthogonal lines which separate P into four groups with at least $\frac{n}{4} - 1$ points each, and separate the plane into four areas. Then, for any two point x, y in nonadjacent areas, $\angle xpy > 90°$, and hence p is contained in the circle with diameter xy. There are at least

$$2\left(\frac{n}{4} - 1\right)^2 = \frac{1}{4}\binom{n}{2} + O(n)$$

such circles.

Besides, we consider the 4-tuple (x, y, z, w) (counter-clockwise) from the four areas, there are at least $(\frac{n}{4} - 1)^4$ of them. For any such 4-tuple (x, y, z, w), at least one of $\angle xpy, \angle ypz, \angle zpw, \angle wpx$ is greater than or equal to $90°$. On the other hand, note that if we apply a sufficiently small translation for p along any direction, the property from the lemma still holds, hence we may assume without loss of generality that no such angle is exactly $90°$. Thus, p lies in the interior of at least one of the four circles. Notice that there are at most $\frac{n}{4}+3$ points lie in each area, so such a circle is counted by at most $(\frac{n}{4}+3)^2$ times, hence the number of circles with a diameter consisting of two points from adjacent areas that contains p is at least

$$\frac{(\frac{n}{4}-1)^4}{(\frac{n}{4}+3)^2} = \frac{1}{16}\binom{n}{2} + O(n).$$

Thus,

$$D(P,p) \geq \frac{3}{8}\binom{n}{2} + O(n).$$

On the other hand, consider $P = \{(0,m) : m = 1, 2, ..., n\}$. If point (u,v) is contained in the circle with diameter $(0,i)$ and $(0,j)$, we have $i < v < j$. Thus, at most

$$u(n-u) \leq \left(\frac{n}{2}\right)^2 = \frac{1}{2}\binom{n}{2} + O(n)$$

circles contains p. $\square$