A Complicated Inequality Problem

One day, My friend Zichao ask me a true or false inequality problem. I heard that this problem was derived from some research on Ramsey theory conducted by Debsoumya Chakraborti and Po-shen Loh. Finally, I worked the problem out, and here is my solution.

Problem. Prove or disprove that there is a function $C(t)$ satisfying the following conditions:

- $\lim_{t \to +\infty} C(t) = +\infty$, and
- For any non-negative real numbers $a_1, a_2, ..., a_t$ with $a_1 + ... + a_t = 1$,

$$P = \sum_{i=1}^{t} \frac{j a_j^2}{\sum_{i=j}^{t} a_i} \geq \frac{C(t)}{\ln t}.$$

Solution. The answer is negative.

Let $\sum_{i=j}^{t} a_i = x_j$, Then

$$P = \sum_{j=1}^{t} \frac{j(x_j - x_{j+1})^2}{x_j},$$

and $1 = x_1 \geq x_2 \geq ... \geq x_t \geq x_{t+1} = 0$. Making P homogeneous, we're to find the minimum of

$$P = \sum_{j=1}^{t} \frac{j(x_j - x_{j+1})^2}{x_j x_1}$$

in which $x_1 \geq x_2 \geq ... \geq x_t \geq x_{t+1} = 0$.

Suppose

$$P_l = \sum_{j=t+1-l}^{t} \frac{j(x_j - x_{j+1})^2}{x_j x_{t+1-l}}$$

has a minimum b_l (which exists obviously). We then use the recurrence method to find b_l. Obviously $b_1 = t$. Notice that

$$P_{l+1} = \frac{(t-l)(x_{t-l} - x_{t+1-l})^2}{x_{t-l}^2} + \frac{x_{t-l+1}}{x_{t-l}} P_l,$$

as long as the ratio x_{t-l}/x_{t-l+1} is a constant, P_{l+1} reaches its minimum if and only if P_l reaches its minimum. Thus, suppose $x_{t-l} = s_l x_{t-l+1}$, then b_{l+1} is the minimum of

$$\frac{(t-l)(s_l - 1)^2}{s_l^2} + \frac{1}{s_l} b_l$$

in which $s_l \geq 1$. Taking the derivative (considering as a function of s_l), we see that it reaches the minimum when $s_l = \frac{2(t-l)}{2(t-l)-b_l} > 1$, hence

$$b_{l+1} = b_l - \frac{b_l^2}{4(t-l)}.$$

Suppose $c_l = \frac{b_l}{4}$, then $c_1 = \frac{t}{4}$, and

$$\frac{c_l^2}{(t-l)} = c_l - c_{l+1}.$$

Evidently c_l is decreasing.

Consider the situation when $t = 2^M$, and suppose $d_l = c_{2^M - 2^{M-l}}$. Notice that

$$c_u - c_v = \sum_{i=u}^{v-1}(c_i - c_{i+1}) = \sum_{i=u}^{v-1} \frac{c_i^2}{t-i} \geq c_v^2 \left(\sum_{i=u}^{v-1} \frac{1}{t-i}\right) \geq c_v^2 \frac{v-u}{t-u},$$

we have $d_l - d_{l+1} \geq \frac{d_{l+1}^2}{2}$. If $d_k \geq 8$, then for each integer $l < k$, $d_l \geq 5d_{l+1}$, hence $d_1 \geq 5^k$, $k \leq \frac{M}{2}$.

Define $g_k = d_{\frac{M}{2}+k}$. We show $g_l \leq \frac{8}{l}$ for $l \leq \frac{M}{2}$ by induction on l. This is definitely true when $l = 1$. Suppose it is true for l. For $l+1$, if $g_{l+1} \geq \frac{8}{l+1}$, Then $\frac{8}{l+1} + \frac{64}{2(l+1)^2} < \frac{8}{l}$, which is a contradiction. So it is also true for $l+1$, and thus true for all $l \leq \frac{M}{2}$. Especially, $c_t \leq d_M = g_{\frac{M}{2}} \leq \frac{16}{M}$. Thus, $c_t \leq \frac{20}{\ln t}$ when $t = 2^M$, which disproves the statement.