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For a 3-Dimensional geometry, the Navier-Stokes equations read

$$\begin{aligned}\rho\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}&=-\nabla p+\mu\nabla^2\mathbf{u}\\ \nabla\cdot\mathbf{u}&=0,\end{aligned}$$

with

$$\mathbf{u}=\begin{pmatrix}u & v & w\end{pmatrix}^T.$$

Assuming fully developed flow for a square cross section $(x,y)\in\Omega_z=(0,a)\times(0,b)$, $u=v=0$, $\frac{\partial w}{\partial z}=0$, and $\frac{\partial \mathbf{u}}{\partial t}=0$, so

$$\begin{aligned}\mu\nabla^2 w=\frac{\partial p}{\partial z}&=\mu\Big(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\Big)\\ w&=0\ \text{ on }\ \partial\Omega_z\end{aligned}$$

This equation can be solved for w by assuming that $w=\sum_{n,m\in\mathbb{Z}}a_{m,n}g_m(x)h_n(y)$. Eigenfunction decomposition gives

$$g_m(x)=\sin\left(\frac{(m+1)\pi x}{a}\right),\qquad h_n(x)=\sin\left(\frac{(n+1)\pi x}{b}\right)$$

and

$$\begin{aligned}a_{m,n}&=\frac{-4\nabla_z p}{\mu ab\pi^2\Big((m+1)^2/a^2+(n+1)^2/b^2\Big)}\iint_{\Omega_z}g_m h_n\,\mathrm{d}\Omega_z\\ &=\frac{-4\nabla_z p}{\mu\pi^2\Big((m+1)^2/a^2+(n+1)^2/b^2\Big)}\frac{\Big(1-(-1)^{m+1}\Big)\Big(1-(-1)^{n+1}\Big)}{\pi^2(m+1)(n+1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}w&=\sum_{n,m\in\mathbb{Z}}\frac{-4\nabla_z p}{\mu\pi^2\Big((m+1)^2/a^2+(n+1)^2/b^2\Big)}\frac{\Big(1-(-1)^{m+1}\Big)\Big(1-(-1)^{n+1}\Big)}{\pi^2(m+1)(n+1)}\sin\left(\frac{(m+1)\pi x}{a}\right)\sin\left(\frac{(n+1)\pi x}{b}\right)\\ &=\sum_{n,m\text{ even}}\frac{-16\nabla_z p}{\mu\pi^2\Big((m+1)^2/a^2+(n+1)^2/b^2\Big)\pi^2(m+1)(n+1)}\sin\left(\frac{(m+1)\pi x}{a}\right)\sin\left(\frac{(n+1)\pi x}{b}\right)\\ &=\sum_{n,m\text{ odd}}\frac{-16\nabla_z p\sin\big(m\pi x/a\big)\sin\big(n\pi y/b\big)}{\mu\pi^4\Big((m/a)^2+(n/b)^2\Big)mn}\end{aligned}$$

Now, note that the average velocity is given by

$$\begin{aligned}w_{\text{avg}}=\frac{1}{ab}\iint_{\Omega_z}w\,\mathrm{d}\Omega_z&=\frac{-16\nabla_z p}{\mu\pi^4 ab}\iint_{\Omega_z}\sum_{n,m\text{ odd}}\frac{\sin\big(m\pi x/a\big)\sin\big(n\pi y/b\big)}{\Big((m/a)^2+(n/b)^2\Big)mn}\,\mathrm{d}\Omega_z\\ &=\frac{-64\nabla_z p}{\mu\pi^6}\sum_{n,m\text{ odd}}\frac{1}{\Big((m/a)^2+(n/b)^2\Big)m^2n^2}=\frac{-64\nabla_z p}{\mu\pi^6}\mathcal{F}(a,b).\end{aligned}$$

Assuming that

$$\nabla_z p\approx\frac{-\Delta p}{L}$$

gives

$$\frac{L w_{\text{avg}} \mu \pi^6}{64 \mathcal{F}(a,b)} = \Delta p$$

Note that with $a = b = S$,

$$\mathcal{F}(S,S) = S^2 \sum_{n,m \text{ odd}} \frac{1}{\Big(m^2 + n^2\Big) m^2 n^2},$$

and with

$$\psi = \sum_{n,m \text{ odd}} \frac{1}{\Big(m^2 + n^2\Big) m^2 n^2} = 0.5279266\ldots,$$

it follows that

$$\Delta p = \frac{\pi^6}{64\psi} \frac{L w_{\text{avg}} \mu}{S^2} \approx 28.45415 \frac{\mu L w_{\text{avg}}}{S^2}.$$

Summary

For a rectangular cross-section with sides a and b,

$$\Delta p = \frac{L w_{\text{avg}} \mu \pi^6}{64 \mathcal{F}(a,b)} \quad \text{with} \quad \mathcal{F}(a,b) = \sum_{n,m \text{ odd}} \frac{1}{\Big((m/a)^2 + (n/b)^2\Big) m^2 n^2}.$$

For $a = b = S$,

$$\Delta p = \frac{\pi^6}{64\psi} \frac{L w_{\text{avg}} \mu}{S^2} \approx 28.45415 \frac{\mu L w_{\text{avg}}}{S^2} \quad \text{with} \quad \psi = 0.5279266\ldots$$