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A Symmetric Function Generalization of the Chromatic Polynomial of a Graph

RICHARD P. STANLEY*

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

For a finite graph G with d vertices we define a homogeneous symmetric function X_G of degree d in the variables $x_1, x_2, \ldots$. If we set $x_1 = \cdots = x_n = 1$ and all other $x_i = 0$, then we obtain $\chi_G(n)$, the chromatic polynomial of G evaluated at n. We consider the expansion of X_G in terms of various symmetric function bases. The coefficients in these expansions are related to partitions of the vertices into stable subsets, the Möbius function of the lattice of contractions of G, and the structure of the acyclic orientations of G. The coefficients which arise when X_G is expanded in terms of elementary symmetric functions are particularly interesting, and for certain graphs are related to the theory of Hecke algebras and Kazhdan–Lusztig polynomials.

1. BACKGROUND

Let G be a (finite) graph. We will consider a symmetric function $X_G(x) = X_G(x_1, x_2, \ldots)$ associated with G. $X_G(x)$ will have the property that the specialization $X_G(1^n)$ (short for $X_G(x_1 = x_2 = \cdots = x_n = 1, x_{n+1} = x_{n+2} = \cdots = 0)$) is equal to $\chi_G(n)$, the chromatic polynomial of G evaluated at the positive integer n. We first review some properties of chromatic polynomials which will be generalized by $X_G(x)$.

Let $V = V(G)$ and $E = E(G)$ denote the vertex and edge sets of G, respectively. Set $d = \# V$ and $q = \# E$. (The more customary notation $p = \# V$ would conflict with our use of p for power sum symmetric functions.) We always assume $d \geqslant 1$. Let $\mathbb{P} = \{1, 2, \ldots\}$. A function $\kappa : V \to \mathbb{P}$ is called a *coloring* of G. The coloring is *proper* if $\kappa(u) \neq \kappa(v)$ whenever u and v are the vertices of an edge e of G. For a positive integer n, let $\chi_G(n)$ denote the number of proper colorings $\kappa : V \to \{1, 2, \ldots, n\}$. In particular, $\chi_G(n) = 0$ if G has a loop (an edge from a vertex to itself). Moreover, any multiple edge can be replaced by a single edge without affecting the set of proper colorings of G. Hence from now on we assume that G is *simple*, i.e., has no loops or multiple edges. Thus we think of an edge e as being a two-element

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set $\{u, v\}$ of vertices, abbreviated $e = uv$. It is a standard and easy result of graph theory that $\chi_G(n)$ is a polynomial function of n, called the *chromatic polynomial* of G. Moreover, $\chi_G(n)$ is monic of degree d, and the highest power of n which divides $\chi_G(n)$ is $n^{c(G)}$, where $c(G)$ denotes the number of connected components of G. In Theorem 1.1 we state two classical results (essentially equivalent) which interpret the coefficients of $\chi_G(n)$. Background information on set partitions, Möbius functions, etc., used here may be found in [30]. Let us call a partition $\pi = \{B_1, ..., B_k\}$ of V *connected* if the restriction of G to each block B_i of π is connected (as a graph). The *lattice of contractions* (or *bond lattice*) L_G of G is the set of all connected partitions of G, partially ordered by refinement. L_G is a geometric lattice, so it is ranked. The rank of $\pi \in L_G$ is given by $d - |\pi|$, where $|\pi|$ denotes the number of blocks of π. Moreover, the Möbius function μ of L_G strictly alternates in sign [24, Sect. 7, Thm. 4; 30, Prop. 3.10.1], which is equivalent to the inequality

$$(-1)^{d-|\pi|}\mu(\hat{0}, \pi) > 0, \tag{1}$$

for all $\pi \in L_G$. Here $\hat{0}$ denotes the unique minimal element of L_G (the partition into d one-element blocks).

1.1. THEOREM: (a) (Whitney [33]). *For a (loopless) graph G we have*

$$\chi_G(n) = \sum_{S \subseteq E} (-1)^{\# S} n^{c(S)},$$

where $c(S)$ is the number of components of the spanning subgraph G_S of G with edge set S.

(b) (Whitney [33]; equivalent to Birkhoff [2]). *We have*

$$\chi_G(n) = \sum_{\pi \in L_G} \mu(\hat{0}, \pi)\, n^{|\pi|},$$

where $|\pi|$ denotes the number of blocks of π.

An *acyclic orientation* $\mathfrak{o}$ of G is an orientation of each edge of G so that the resulting directed graph has no directed cycles. A *sink* of $\mathfrak{o}$ is a vertex v for which no edge points out of v. In particular, an isolated vertex of G is a sink in every acyclic orientation of G. Write $[n^k]\, f(n)$ for the coefficient of n^k in the polynomial $f(n)$.

1.2. THEOREM. (a) (Stanley [28]). *The number of acyclic orientations of G is equal to $(-1)^d \chi_G(-1)$. (Note that $\chi_G(-1)$ is defined using the fact that $\chi_G(n)$ is a polynomial in n.)*

(b) (Greene–Zaslavsky [16, Thm. 7.3]). *Let v be any vertex of G. Then the number of acyclic orientations of G whose unique sink is v is equal to $(-1)^{d-1}[n]\,\chi_G(n)$ (independent of v). (Note that if G is not connected then this number is 0, since every connected component of an acyclic orientation has at least one sink.)*

Suppose that the edges of G are labelled $1, 2, ..., q$ (each edge e receiving a different label $\alpha(e)$). A *circuit* is a minimal set of edges which is not a forest (i.e., which contains a cycle). A *broken circuit* is a circuit with its largest edge (with respect to the labeling α) removed. The *broken circuit complex* B_G of G (with respect to α) consists of all subsets of the edges which do not contain a broken circuit. To following result is known as the *Broken Circuit Theorem* and is due to Whitney [33, Sect. 7]. For further information on broken circuits see for instance [3; 5; 6; 24, Sect. 7].

1.3. THEOREM. *We have*

$$\chi_G(n) = \sum_{S \in B_G} (-1)^{|S|} n^{d-|S|}.$$

A fundamental property of chromatic polynomials, and the basis for inductive proofs of many of their properties, is the *deletion–contraction* property. Let e be an edge of G which is not a loop. (We are assuming anyway that G has no loops.) Let $G \backslash e$ denote G with the edge e deleted, and let G/e denote G with the edge e contracted to a point. Then

$$\chi_G(n) = \chi_{G \backslash e}(n) - \chi_{G/e}(n). \tag{2}$$

For instance, Theorems 1.1 and 1.2 can be readily proved using (2). We will be giving extensions of Theorems 1.1 and 1.2 which hold for $X_G(x)$, but which cannot be proved by deletion–contraction techniques.

2. BASIC PROPERTIES

Let $x_1, x_2, ...$ be (commuting) indeterminates, and suppose $V(G) = \{v_1, ..., v_d\}$.

2.1. DEFINITION. Define

$$X_G = X_G(x) = X_G(x_1, x_2, ...) = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)},$$

where the sum ranges over all proper colorings $\kappa : V \to \mathbb{P}$.

It is clear from the definition that $X_G(x)$ is a homogeneous symmetric function in $x = (x_1, x_2, \ldots)$ of degree $d = \# V$. We will use throughout this paper notation and terminology involving symmetric functions and partitions from Macdonald [18].

Note. An object closely related to X_G has been investigated by Ray, Wright, and Schmitt [21–22] (as well as some other papers less relevant to what we do here), called the *umbral chromatic polynomial* and denoted by $\chi^{\phi}(G; x)$. From $\chi^{\phi}(G; x)$ it is possible to compute X_G. Indeed, in the preceding references there is defined for each positive integer k a polynomial $\chi^{\phi}(G; k\phi)$ in the variables $\phi = (\phi_1, \ldots, \phi_d)$, which can be obtained from $\chi^{\phi}(G; x)$ by "umbral substitution." Knowing this polynomial is equivalent to knowing the symmetric function

$$\sum_{\substack{\lambda \vdash d \\ l(\lambda) = k}} a_{\lambda} \tilde{m}_{\lambda},$$

where the notation follows Eq. (3).

From the definition of X_G we see that knowing $X_G(x)$ is equivalent to knowing, for each partition λ of d, the number of proper colorings of G with λ_i vertices colored i. The following result is also immediate from the definition of X_G and shows its connection with the chromatic polynomial of G.

2.2. PROPOSITION. *Let $n \in \mathbb{P}$. Then $X_G(1^n) = \chi_G(n)$.*

Let us note that the specialization $f(1^n)$ of a symmetric function $f(x)$ is well-known in the theory of symmetric function; see, e.g., [18, Exam. 1, p. 18, and Exam. 4, pp. 28–29].

Many known graphical invariants related to graph coloring can be computed from X_G. For instance, the chromatic difference sequence of Albertson and Berman [1] can be computed from X_G. It is also immediate from Proposition 2.4 that X_G determines the *matchings polynomial* (as defined, e.g., in [15, p. 1]) of the complement $\bar{G}$ (and hence of G itself by [15, Thm. 2.3]). In particular, if $\bar{G}$ is bipartite then knowing X_G is equivalent to knowing the matchings polynomial of G or $\bar{G}$. For a special property of X_G when $\bar{G}$ is bipartite, see Corollary 3.6. The fact that X_G determines the matchings polynomial of G can also be seen from Theorem 2.6, since $\mu(\hat{0}, \pi) = (-1)^j$ when type $(\pi) = \langle 1^{d-2j} 2^j \rangle$.

A natural question to ask about X_G is whether it determines G. Naturally we do not expect such a simply defined combinatorial invariant to determine G, and such is indeed the case. It turns out that X_G distinguishes all

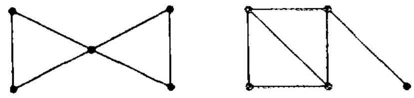

FIG. 1. Graphs G and H with $X_G = X_H$.

eleven nonisomorphic four-vertex graphs, but that there is a unique pair G, H of nonisomorphic five-vertex graphs for which $X_G = X_H$. These graphs are shown in Fig. 1. Using the notation of Proposition 2.4, we have

$$X_G = X_H = 2\tilde{m}_{221} + 4\tilde{m}_{2111} + \tilde{m}_{11111}.$$

We do not know whether X_G distinguishes trees. Of course all d-vertex trees have the same chromatic polynomial $n(n-1)^{d-1}$.

Let $G + H$ denote the disjoint union of the graphs G and H. The following result is an immediate consequence of Definition 2.1.

2.3. PROPOSITION. *We have* $X_{G+H} = X_G X_H$.

Given a symmetric function $f(x)$, it is natural to expand it in terms of the many known "natural" bases for the space of symmetric functions and ask for a formula or a combinatorial or algebraic interpretation of the coefficients. The first (and easiest) basis we consider is the monomial symmetric functions m_λ. It is more convenient to deal instead with the *augmented monomial symmetric functions* $\tilde{m}_\lambda$ [8], defined by

$$\tilde{m}_\lambda = r_1!\, r_2! \cdots m_\lambda,$$

where λ has r_i parts equal to i (denoted $\lambda = \langle 1^{r_1} 2^{r_2} \ldots \rangle$). Define a *stable partition* π of G to be a partition of the set $V(G)$ such that each block of π is totally disconnected (i.e., each block is a stable (or independent) set of vertices). The *type* of a partition π of a d-element set V, denoted type(π), is the partition λ of d whose parts λ_i are the sizes of the blocks of π. The following result is essentially just a restatement of the definition of X_G.

2.4. PROPOSITION. *Let* a_λ *be the number of stable partitions of* G *of type* λ. *Then*

$$X_G = \sum_{\lambda \vdash d} a_\lambda \tilde{m}_\lambda. \tag{3}$$

Proof. The coefficient of a monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ in $X_G(x)$ is equal to the number of ways to choose a stable partition π of G of type $\lambda = \langle 1^{r_1} 2^{r_2} \ldots \rangle$, and then color some block of size λ_i with the color i for each i. Once we choose π we have $r_1!\, r_2! \cdots$ ways to choose the coloring, and the proof follows. ∎

The next basis we consider consists of the power sum symmetric functions p_λ. Given $S \subseteq E$, let $\lambda(S)$ be the partition of d whose parts are equal to the vertex sizes of the connected components of the spanning subgraph of G with edge set S. For instance, if $S = \varnothing$ then $\lambda(S) = \langle 1^d \rangle$.

2.5. THEOREM. *We have*

$$X_G = \sum_{S \subseteq E} (-1)^{\# S} p_{\lambda(S)}. \tag{4}$$

Proof. Given $S \subseteq E$, we have

$$p_{\lambda(S)}(x) = \sum_{\kappa \in K_S} x^\kappa,$$

where K_S denotes the set of all colorings $\kappa : V \to \mathbb{P}$ which are monochromatic on the connected components of the spanning subgraph G_S of G with edge set S, and where $x^\kappa = x_{\kappa(v_1)} \cdots x_{\kappa(v_d)}$. Hence

$$\begin{aligned}\sum_{S \subseteq E} (-1)^{\# S} p_{\lambda(S)}(x) &= \sum_{S \subseteq E} (-1)^{\# S} \sum_{\kappa \in K_S} x^\kappa \\ &= \sum_\kappa x^\kappa \sum_{S \subseteq E_\kappa} (-1)^{\# S},\end{aligned}$$

where in the bottom line κ ranges over *all* colorings $\kappa : V \to \mathbb{P}$, and where E_κ is the set of all edges e of G such that the two vertices of e have the same color. Hence the sum on S is 0 unless E_κ is empty, in which case the sum is 1. Thus the bottom line becomes $\sum x^\kappa$, summed over proper colorings κ, and the proof follows. ∎

Theorem 2.5 is a direct generalization of Theorem 1.1(a) (Whitney's theorem), since $p_{\lambda(S)}(1^n) = n^{c(S)}$. We next give an analogous generalization of Theorem 1.1(b). An analogous result for the umbral chromatic polynomial of Ray *et al.* appears in [22, Theorem 4.3; 21, Theorem 3].

2.6. THEOREM. *We have*

$$X_G = \sum_{\pi \in L_G} \mu(\hat{0}, \pi)\, p_{\mathrm{type}(\pi)}.$$

Proof. We could deduce this result directly from Theorem 2.5, but a Möbius inversion argument analogous to the original proof of Theorem 1.1(b) [33] (see also [24, Sect. 9; 30, Exer. 3.44; 34, p. 128]) is instructive. Let $\sigma \in L_G$, and define

$$X_\sigma = X_\sigma(x) = \sum_\kappa x^\kappa, \tag{5}$$

summed over all colorings $\kappa : V \to \mathbb{P}$ such that (i) if u and v are in the same block of σ then $\kappa(u) = \kappa(v)$, and (ii) if u and v are in different blocks and there is an edge with vertices u and v, then $\kappa(u) \neq \kappa(v)$. Given *any* $\kappa : V \to \mathbb{P}$, there is a unique $\sigma \in L_G$ such that κ is one of the maps appearing in the sum (5). It follows that for any $\pi \in L_G$, we have

$$p_{\text{type}(\pi)} = \sum_{\sigma \geq \pi} X_\sigma.$$

By Möbius inversion,

$$X_\pi = \sum_{\sigma \geq \pi} p_{\text{type}(\sigma)}\, \mu(\pi, \sigma).$$

But $X_{\hat{0}} = X_G$, and the proof follows. ∎

Let ω denote the usual involution on symmetric functions [18, p. 14] satisfying in particular $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$ where $\varepsilon_\lambda = (-1)^{d-l(\lambda)}$. Here $\lambda \vdash d$ and $l(\lambda)$ denotes the length (number of parts) of λ. Given a $\mathbb{Q}$-basis u_λ for the space $\Lambda_{\mathbb{Q}}$ of symmetric functions with rational coefficients (any ordered field could be used in place of $\mathbb{Q}$), we say that a symmetric function f is *u-positive* if in the expansion $f = \sum d_\lambda u_\lambda$, the coefficients d_λ are all nonnegative. We also say that a graph G is *u-positive* if X_G is u-positive.

2.7. Corollary. *For any graph G, the symmetric function ωX_G is p-positive.*

Proof. Since for $\lambda \vdash d$ we have $\varepsilon_\lambda = (-1)^{d-l(\lambda)}$, there follows $\varepsilon_{\text{type}(\pi)} = (-1)^{d-|\pi|}$. Hence from (1) and Theorem 2.6 we have

$$\omega X_G = \sum_{\pi \in L_G} |\mu(\hat{0}, \pi)|\, p_{\text{type}(\pi)},$$

and the proof follows. ∎

The graphs with the simplest chromatic polynomials are the *forests* (graphs without cycles). If G is a forest with d vertices and q edges, then L_G is isomorphic to the boolean algebra B_q and $\chi_G(n) = n^{c(G)}(n-1)^q$. Moreover, $\mu(\hat{0}, \pi) = (-1)^{d-|\pi|}$ for all $\pi \in L_G$. However, the symmetric function X_G is not so trivial, and it seems difficult to give a simple formula for X_G even when G is a forest. (Even the case when G is a path is interesting and is treated in Proposition 5.3.) For forests we have an interesting relationship between counting connected partitions and stable partitions given by the next result. Note that for a forest, a connected partition is equivalent to a subset of the edges. The lattice L_G of contractions of G is just a boolean algebra of rank q.

2.8. COROLLARY. *Let G be a forest. Then knowing for all $\lambda \vdash d$ the number a_λ of stable partitions of G of type λ is equivalent to knowing for all $\lambda \vdash d$ the number b_λ of connected partitions of type λ.*

Proof. By Proposition 2.4 we have $X_G = \sum a_\lambda \tilde{m}_\lambda$. On the other hand, for a forest G we have $\mu(\hat{0}, \pi) = \varepsilon_{\text{type}(\pi)}$ for all $\pi \in L_G$. Hence Theorem 2.6 becomes

$$X_G = \sum_\lambda \varepsilon_\lambda b_\lambda p_\lambda. \tag{6}$$

Since both $\{\tilde{m}_\lambda\}$ and $\{p_\lambda\}$ are bases for $\Lambda_{\mathbb{Q}}$, the proof follows. ∎

Corollary 2.8 need not be true if G is not a forest. For instance, let G and H be the graphs of Fig. 1. G and H have the same number of stable partitions of each type λ since $X_G = X_H$. However, G has two connected partitions of type $(3, 2)$, while H has three such paritions.

Next we give an analogue of Theorem 1.3 (the Broken Circuit Theorem). We retain the notation B_G of Theorem 1.3 as well as the notation $\lambda(S)$ of Theorem 2.5.

2.9. THEOREM. *We have*

$$X_G = \sum_{S \in B_G} (-1)^{|S|} p_{\lambda(S)}.$$

Proof. Let $\pi \in L_G$, say of rank k. Let G_π be the spanning subgraph of G consisting of all edges uv of G for which the vertices u and v are in the same block of π. Thus L_{G_π} is isomorphic to the interval $[\hat{0}, \pi]$ of L_G. By Theorem 1.3, we have that $(-1)^k \mu(\hat{0}, \pi)$ is equal to the number of k-element subsets S of edges of G_π which contain no broken circuit of G (or equivalently of G_π, with the edge labeling obtained by restricting that of G). These subsets S are precisely the subsets of edges of G which contain no broken circuit and whose connected components have vertex sets equal to the blocks of π. The proof follows from Theorem 2.6. ∎

2.10. COROLLARY. *Let α be any linear ordering of the edges of G, and let $\nu \vdash d$. Then the number of subsets S of edges of G which contain no broken circuit (with respect to the ordering α) and for which $\lambda(S) = \nu$ is independent of α.*

Proof. By Theorem 2.9 this number is just $(-1)^{|S|}$ times the coefficient of p_ν in X_G. ∎

The differential operators $\partial/\partial p_j$ play a useful role in the theory of symmetric functions [18, Ex. 3, pp. 43–45]. Here $\partial/\partial p_j$ acts on a symmetric function expressed as a polynomial in the p_i's. The case $\partial/\partial p_1$ is especially interesting, since this is the same operator as "skewing by s_1," i.e., $(\partial/\partial p_1) s_\lambda = s_{\lambda/1}$, where s_λ denotes a Schur function and $s_{\lambda/1}$ a skew Schur function.

2.11. COROLLARY. *We have*

$$\frac{\partial}{\partial p_j} X_G = \sum_H \mu_H X_{G-H},$$

where H runs over all j-vertex induced connected subgraphs of G, $\mu_H = \mu_{L_H}(\hat{0}, \hat{1}) = [n]\, \chi_H(n)$, and $G-H$ denotes G with H (and all incident edges) deleted.

Proof. By Theorem 2.6 we have

$$\frac{\partial}{\partial p_j} X_G = \sum_{H,\sigma} \mu(\hat{0}, \pi)\, p_{\mathrm{type}(\sigma)},$$

where $\pi = \sigma \cup \{H\}$ is a connected partition of G with $\# H = j$. (Thus $\sigma \in L_{G-H}$.) Now $[\hat{0}, \pi] \cong L_H \times [\hat{0}, \sigma]$. By the product property of Möbius functions [24, Sect. 3, Prop. 5; 30, Prop. 3.8.2] there follows

$$\begin{aligned}\frac{\partial}{\partial p_j} X_G &= \sum_H \mu_H \sum_{\sigma \in L_{G-H}} \mu(\hat{0}, \sigma)\, p_{\mathrm{type}(\sigma)} \\ &= \sum_H \mu_H X_{G-H}. \quad \blacksquare\end{aligned}$$

When $\# H = 1$ we have $\mu_H = 1$, and when $\# H = 2$ we have $\mu_H = -1$. Hence we get the following corollary to Corollary 2.11.

2.12. COROLLARY. (a) *We have*

$$\frac{\partial}{\partial p_1} X_G = \sum_{v \in V} X_{G-v},$$

where X_{G-v} denotes G with vertex v (and all incident edges) removed.

(b) *We have*

$$\frac{\partial}{\partial p_2} X_G = -\sum_{e \in E} X_{G-V(e)},$$

where $G - V(e)$ denotes G with the two vertices of e (and all incident edges) removed.

3. Acyclic Orientations

In this section we generalize Theorem 1.2. Our results will involve the expansion of X_G in terms of the elementary symmetric functions e_λ. We will require some results from the theory of quasi-symmetric functions and P-partitions, which we now review.

Following Gessel [13], define a power series $F(x) = F(x_1, x_2, \ldots)$ (say with rational coefficients) to be *quasi-symmetric* if

$$[x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}]\, F(x) = [x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}]\, F(x)$$

whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$. (Here $[x^\alpha]\, F(x)$ denotes the coefficient of the monomial x^α in $F(x)$.) Clearly a symmetric function is quasi-symmetric. It is easily seen that the set $\mathscr{Q}_d$ of homogeneous quasi-symmetric functions of degree d is a vector space over $\mathbb{Q}$ of dimension 2^{d-1} (the number of sequences $a_1, a_2, \ldots, a_k$ of positive integers with sum d).

Given a subset S of $[d-1] := \{1, 2, \ldots, d-1\}$, define the *fundamental* quasi-symmetric function $Q_S(x) = Q_{S,d}(x)$ by

$$Q_S(x) = \sum_{\substack{i_1 \leqslant \cdots \leqslant i_d \\ i_j < i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_d}.$$

For instance, $Q_{[d-1]}$ is the elementary symmetric function e_d, and $Q_\varnothing$ is the complete symmetric function h_d. It is easy to see that the set $\{Q_S : S \subseteq [d-1]\}$ is a $\mathbb{Q}$-basis for $\mathscr{Q}_d$.

Let $P = \{v_1, \ldots, v_d\}$ be a d-element poset. Define

$$X_P(x) = \sum_\kappa x_{\kappa(v_1)} \cdots x_{\kappa(v_d)},$$

summed over all strict order-preserving maps $\kappa : P \to \mathbb{P}$, i.e., if $u < v$ in P, then $\kappa(u) < \kappa(v)$. Clearly X_P is a quasi-symmetric function, and the theory of P-partitions [27; 30, Sect. 4.5] allows us to expand X_P in terms of the basis $\{Q_S : S \subseteq [d-1]\}$. Namely, fix an order-reversing bijection $\omega : P \to [d]$. (Thus ω is a linear extension of the dual P^* of P.) Given a linear extension $\alpha : P \to [d]$, we can identify α with the permutation $(\omega(\alpha^{-1}(1)), \omega(\alpha^{-1}(2)), \ldots, \omega(\alpha^{-1}(d))) := (a_1, \ldots, a_d)$ of $[d]$. Define the *descent set* $D(\alpha)$ of α by

$$D(\alpha) = \{j : a_j > a_{j+1}\}.$$

Let $\mathscr{L}(P, \omega)$ denote the set of all linear extensions of P (regarded as permutations of $[d]$ *via* ω). The following result is a consequence of [27, Thm. 6.2; 30, Lemma 4.5.1; 13, Eq. (1)].

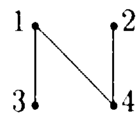

FIG. 2. A labelled poset (P, ω).

3.1. THEOREM. *With notation as above we have*

$$X_P = \sum_{\pi \in \mathscr{L}(P, \omega)} Q_{D(\pi)}.$$

3.2. EXAMPLE. Let (P, ω) be given by Fig. 2. Then $\mathscr{L}(P, \omega) = \{3412, 3421, 4312, 4321, 4231\}$. The descent sets of these permutations are $\{2\}$, $\{2, 3\}$, $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 3\}$. Hence

$$X_P = Q_2 + Q_{2,3} + Q_{1,2} + Q_{1,2,3} + Q_{1,3}.$$

We can now give our main result on acyclic orientations. Following [18], we let $l(\lambda)$ denote the length (number of parts) of the partition λ.

3.3. THEOREM. *Suppose*

$$X_G = \sum_{\lambda \vdash d} c_\lambda e_\lambda \tag{7}$$

is the expansion of X_G in terms of elementary symmetric functions e_λ. Let $\operatorname{sink}(G, j)$ *be the number of acyclic orientations of G with j sinks. Then*

$$\operatorname{sink}(G, j) = \sum_{\substack{\lambda \vdash d \\ l(\lambda) = j}} c_\lambda.$$

Proof. Let $\mathfrak{o}$ be an acyclic orientation of G and κ a proper coloring. We say that κ is $\mathfrak{o}$-*compatible* if $\kappa(u) < \kappa(v)$ whenever (v, u) is an edge of $\mathfrak{o}$ (i.e., the edge uv of G is directed from v to u). Every proper coloring is compatible with exactly one acyclic orientation $\mathfrak{o}$, viz., if uv is an edge of G with $\kappa(u) < \kappa(v)$, then let (v, u) be an edge of $\mathfrak{o}$. (This observation was the basis for one of the two proofs of Theorem 1.2(a) given in [28].) Thus if $K_\mathfrak{o}$ denotes the set of $\mathfrak{o}$-compatible proper colorings of G, and if K_G denotes the set of all proper colorings of G, then we have a disjoint union $K_G = \bigcup_\mathfrak{o} K_\mathfrak{o}$. Hence $X_G = \sum_\mathfrak{o} X_\mathfrak{o}$, where $X_\mathfrak{o} = \sum_{\kappa \in K_\mathfrak{o}} x^\kappa$. Let $\bar{\mathfrak{o}}$ denote the transitive closure of $\mathfrak{o}$. Since $\mathfrak{o}$ is acyclic, $\bar{\mathfrak{o}}$ is a poset. By the definition of X_P for a poset P and of $X_\mathfrak{o}$, we have $X_{\bar{\mathfrak{o}}} = X_\mathfrak{o}$, so

$$X_G = \sum_\mathfrak{o} X_{\bar{\mathfrak{o}}}. \tag{8}$$

We now come to the crucial step in the proof. Let t be an indeterminate, and define a linear transformation $\varphi : \mathcal{Q}_d \to \mathbb{Q}[t]$ as follows:

$$\varphi(Q_S) = \begin{cases} t(t-1)^i, & \text{if } S = \{i+1, i+2, ..., d-1\} \\ 0, & \text{otherwise.} \end{cases}$$

Claim. For any d-element poset P, we have $\varphi(X_P) = t^m$, where m is the number of minimal elements of P.

Proof of Claim. Let $\omega : P \to [d]$ be an order-reversing bijection as above. Since ω is order-reversing, the only way to obtain a linear extension $\alpha = (a_1, ..., a_d)$ with descent set $\{i+1, i+2, ..., d-1\}$ is as follows. Let v be the minimal element of P with the largest label $\omega(v)$. Choose any i minimal elements $u_1, ..., u_i$ of P other than v, list them in increasing order of their labels, then list v, and finally list the remaining elements of P in decreasing order of their labels. Since there are $\binom{m-1}{i}$ choices for $u_1, ..., u_i$, we obtain

$$\begin{aligned} \varphi(X_P) &= \sum_{i=0}^{m-1} \binom{m-1}{i} t(t-1)^i \\ &= t^m. \end{aligned}$$

This proves the claim.

Note that if $\mathfrak{o}$ is an acyclic orientation of G, then the number of sinks of $\mathfrak{o}$ is the number of minimal elements of $\bar{\mathfrak{o}}$. Hence applying φ to Eq. (8) yields

$$\varphi(X_G) = \sum_j \text{sink}(G, j)\, t^j. \tag{9}$$

We now want to compute $\varphi(e_\lambda)$ for $\lambda \vdash d$. The easiest way to proceed is to let P_λ be the poset which is a disjoint union of chains of cardinalities $\lambda_1, \lambda_2, ...$. Clearly $X_{P_\lambda} = e_\lambda$, so by the claim we have $\varphi(e_\lambda) = t^{l(\lambda)}$. Applying φ to Eq. (7) and comparing with (9) yields

$$\sum_\lambda c_\lambda t^{l(\lambda)} = \sum_j \text{sink}(G, j)\, t^j.$$

Taking the coefficient of t^j on both sides completes the proof. ∎

Note that Theorem 1.2(a) is an easy consequence of Theorem 3.3. Namely, since $e_\lambda(1^n) = \binom{n}{\lambda_1}\binom{n}{\lambda_2}\cdots$, we have from (7) that

$$\chi_G(n) = \sum_{\lambda \vdash d} c_\lambda \binom{n}{\lambda_1}\binom{n}{\lambda_2}\cdots. \tag{10}$$

Set $n=-1$ and use the formula $\binom{-1}{j}=(-1)^j$ for all $j\geqslant 0$ to get $\chi_G(-1)=(-1)^d\sum c_\lambda$. By Theorem 3.3, $\sum c_\lambda$ is just the number of acyclic orientations of G, which yields Theorem 1.2(a).

We don't get Theorem 1.2(b) directly from Theorem 3.3, but rather the somewhat weaker result

$$\operatorname{sink}(G,1)=(-1)^{d-1}\,d[n]\,\chi_G(n).$$

For if $l(\lambda)>1$ then $\binom{n}{\lambda_1}\binom{n}{\lambda_2}\cdots$ is divisible by n^2. Hence from (10) and Theorem 3.3 we get

$$\begin{aligned}[n]\,\chi_G(n)&=c_{(d)}[n]\binom{n}{d}\\&=(-1)^{d-1}\frac{c_{(d)}}{d}\\&=(-1)^{d-1}\frac{\operatorname{sink}(G,1)}{d},\end{aligned}$$

as desired.

Although $\sum_j \operatorname{sink}(G,j)$ and $\operatorname{sink}(G,1)$ are computable from $\chi_G(n)$, one cannot in general obtain $\operatorname{sink}(G,j)$ from $\chi_G(n)$, or even from the more informative Tutte polynomial [4]. For instance, the two trees with four vertices (the path P_4 and claw K_{13}) have the same Tutte polynomial (as do any two trees with the same number of vertices), but $\operatorname{sink}(P_4,3)=0$ and $\operatorname{sink}(K_{13},3)=1$.

In view of Theorem 3.3 it is natural to ask whether the coefficients c_λ themselves in the expansion $X_G=\sum c_\lambda e_\lambda$ have any combinatorial significance. Unfortunately, in general we need not have $c_\lambda\geqslant 0$ (in other words, G need not be e-positive), so a combinatorial interpretation cannot be too simple. Of the eleven nonisomorphic (simple) graphs G with four vertices, exactly one fails to be e-positive. This is the claw K_{13}, for which

$$X_{K_{13}}=e_4+5e_{31}-2e_{22}+e_{211}.$$

On the other hand, there are some special classes of graphs for which more can be said about the coefficients c_λ. See Corollary 3.6 and Section 5 for further details.

Although we can have $c_\lambda<0$, there is a refinement of Theorem 3.3 which shows that some additional sums of certain c_λ are nonnegative. Call a partition $\mu=(\mu_1,\mu_2,...)$ of some integer $r\leqslant d$ *allowable* if there exists a stable partition of type μ of some subset W of V. (Hence $\#W=r$.) We partially order all allowable partitions of r by *dominance* (called the *natural* order in [18]), i.e., $\mu\leqslant\nu$ if $\mu_1+\mu_2+\cdots+\mu_i\leqslant\nu_1+\nu_2+\cdots+\nu_i$ for all i. An

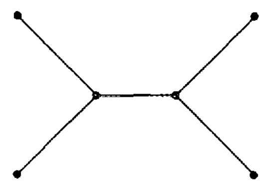

Fig. 3. A graph with two maximal allowable partitions.

allowable partition $\mu \vdash r$ is *maximal* if it is maximal with respect to dominance among all allowable partitions of r. For instance, let G be the graph of Fig. 3. Then both $(4, 1, 1)$ and $(3, 3)$ are maximal. G is the smallest graph which possesses more than one maximal partition of some r (here $r = 6$). As in [18] we write $\mu' = (\mu'_1, \mu'_2, \ldots)$ to denote the conjugate partition to μ. In particular, $\mu'_1 = l(\mu)$.

Given an acyclic orientation $\mathfrak{o}$ of G, define its *sink sequence* $\mathrm{ss}(\mathfrak{o}) = (s_1, s_2, \ldots, s_j)$ as follows: s_1 is the number of sinks of $\mathfrak{o}$. When these s_1 sinks are removed (together with all incident edges), then s_2 is the number of sinks of the resulting acyclic digraph. When these s_2 sinks are also removed, then s_3 is the number of new sinks, etc., until at the jth step we reach a totally disconnected graph with s_j vertices. Note that when the sequence $\mathrm{ss}(\mathfrak{o})$ is sorted into decreasing order, we get an allowable partition of d. If P is a poset (e.g., the transitive closure $\bar{\mathfrak{o}}$ of $\mathfrak{o}$), then P may be regarded as an acyclic digraph (with an edge $v \to u$ if $u < v$ in P). Thus $\mathrm{ss}(P)$ is defined, and in particular we have $\mathrm{ss}(\mathfrak{o}) = \mathrm{ss}(\bar{\mathfrak{o}})$. We can now state our generalization of Theorem 3.3.

3.4. THEOREM. *Let $\mu = (\mu_1, \ldots, \mu_l)$ be a maximal partition of $r \leqslant d$, with $\mu_l > 0$. Let $X_G = \sum c_\lambda e_\lambda$ as in (7). Given $0 \leqslant j \leqslant d - r$, let* $\mathrm{sink}(G, \mu, j)$ *be the number of acyclic orientations $\mathfrak{o}$ of G whose sink sequence has the form* $\mathrm{ss}(\mathfrak{o}) = (\mu_1, \ldots, \mu_l, j, \ldots)$. *(We can have $j = 0$ only when $r = d$.) Then*

$$\mathrm{sink}(G, \mu, j) = \sum_{\lambda} c_\lambda, \tag{11}$$

summed over all partitions $\lambda \vdash d$ such that

$$\lambda'_1 = \mu_1, \lambda'_2 = \mu_2, \ldots, \lambda'_l = \mu_l, \lambda'_{l+1} = j.$$

Proof. The proof is analogous to that of Theorem 3.3. Define a homomorphism $\varphi_\mu : \mathcal{Q}_d \to \mathbb{Q}[t]$ as follows:

$$\varphi_\mu(Q_S) = \begin{cases} t(t-1)^i, & \text{if } S = \{\mu_1, \mu_2, \ldots, \mu_l, \mu_l + i + 1, \mu_l + i + 2, \ldots, d - 1\} \\ 0, & \text{otherwise.} \end{cases}$$

CLAIM. *Let P be a d-element poset. If* $\mathrm{ss}(P)=(\mu_1,\mu_2,...,\mu_l,m,...)$, *then* $\varphi_\mu(X_P)=t^m$. *If* $\mathrm{ss}(P)=(\nu_1,\nu_2,...)$ *where* $\nu_1+\cdots+\nu_i<\mu_1+\cdots+\mu_i$ *for some* $1\leqslant i\leqslant l$, *then* $\varphi_\mu(X_P)=0$. (*We do not care about the case* $\mathrm{ss}(P)=(\nu_1,\nu_2,...)$ *where* $\nu_1+\cdots+\nu_i>\mu_1+\cdots+\mu_i$ *for some i and* $\nu_1+\cdots+\nu_i\geqslant\mu_1+\cdots+\mu_i$ *for all i.*)

Proof of Claim. Let $\omega: P\to[d]$ be an order-reversing bijection as in the proof of Theorem 3.3. Assume $\mathrm{ss}(P)=(\mu_1,...,\mu_l,m,...)$. Since ω is order-reversing, the only way to obtain a linear extension $\alpha=(\alpha_1,...,\alpha_d)$ with descent set $\{\mu_1,...,\mu_l,\mu_l+i+1,...,d-1\}$ is as follows. First list in increasing order of their labels the μ_1 minimal elements of P. Then list in increasing order of their labels the μ_2 next smallest elements of P, etc., until we have listed the bottom $\mu_1+\cdots+\mu_l$ elements of P. Let v be the minimal element of the remaining poset Q with the largest label $\omega(v)$. Choose any i minimal elements $u_1,...,u_i$ of Q other than v, list them in increasing order of their labels, then list v, and finally list the remaining elements of Q in decreasing order of their labels. Since there are at most $\binom{m-1}{i}$ choices for $u_1,...,u_i$, we obtain $\varphi_\mu(X_P)=t^m$ exactly as in the proof of Theorem 3.3. This proves the claim for $\mathrm{ss}(P)=(\mu_1,\mu_2,...,\mu_l,m,...)$.

Now assume $\mathrm{ss}(P)=(\nu_1,\nu_2,...)$ where $\nu_1+\cdots+\nu_i<\mu_1+\cdots+\mu_i$ for some i. If $\alpha=(a_1,a_2,...)$ is a linear extension of P with descent set $S=\{\mu_1,...,\mu_l,\mu_l+i+1,...,d-1\}$, then the elements of P labelled $a_1,a_2,...,a_{\mu_1+\cdots+\mu_i}$ form an order ideal of P which is a union of i antichains and hence contains no $(i+1)$-element chain. But if $\mathrm{ss}(P)=(\nu_1,\nu_2,...)$, then any order ideal of P which contains no $(i+1)$-element chain has at most $\nu_1+\cdots+\nu_i<\mu_1+\cdots+\mu_i$ elements. Hence α cannot exist, so $\varphi_\mu(X_P)=0$. This completes the proof of the claim.

Now note that since μ is maximal, no acyclic orientation $\mathfrak{o}$ of G can satisfy $\mathrm{ss}(\mathfrak{o})=(\nu_1,\nu_2,...)$ where (a) $\nu_1+\cdots+\nu_i>\mu_1+\cdots+\mu_i$ for some $i\in[l]$, and (b) $\nu_1+\cdots+\nu_i\geqslant\mu_1+\cdots+\mu_i$ for all $i\in[l]$. Thus either $\mathrm{ss}(\mathfrak{o})=(\mu_1,...,\mu_l,m,...)$, in which case $\varphi_\mu(X_\mathfrak{o})=t^m$; or else $\mathrm{ss}(\mathfrak{o})=(\nu_1,\nu_2,...)$ where $\nu_1+\cdots+\nu_i<\mu_1+\cdots+\mu_i$ for some $i\in[l]$, in which case $\varphi_\mu(X_\mathfrak{o})=0$. Hence applying φ_μ to (8) yields

$$\varphi_\mu(X_G)=\sum_j \mathrm{sink}(G,\mu,j)\,t^j. \tag{12}$$

We now want to compute $\varphi_\mu(e_\lambda)$ for $\lambda\vdash d$ and apply φ_μ to (7). By the triangularity property of the expansion of e_λ in terms of the m_μ's [18, (2.3)], if some λ satisfied (a) $c_\lambda\neq 0$, (b) $\lambda'_1+\cdots+\lambda'_i>\mu_1+\cdots+\mu_i$ for some $i\in[l]$, and (c) $\lambda'_1+\cdots+\lambda'_i\geqslant\mu_1+\cdots+\mu_i$ for all $i\in[l]$, then we would also have $a_\lambda\neq 0$, where a_λ is given by (3). This contradicts maximality of μ. Hence we may assume that either $\lambda'=(\mu_1,...,\mu_l,...)$ or $\lambda'=(\nu_1,\nu_2,...)$ with some $\nu_1+\cdots+\nu_i<\mu_1+\cdots+\mu_i$.

Let P_λ be as in the proof of Theorem 3.3. By the assumption on λ and what we have just proved about $\varphi_\mu(P)$, we have

$$\varphi_\mu(P_\lambda)=\begin{cases} t^m, & \text{if } \lambda'=(\mu_1, ..., \mu_l, m, ...) \\ 0, & \text{otherwise.} \end{cases}$$

Applying φ_μ to Eq. (7) and comparing with (12) yields

$$\sum c_\lambda t^m = \sum_j \operatorname{sink}(G, \mu, j) t^j,$$

where the first sum ranges over all $\lambda \vdash d$ and m such that $\lambda'=(\mu_1, ..., \mu_l, m, ...)$. Taking the coefficient of t^j on both sides completes the proof. ▮

It should be remarked that the maximal partitions $\mu \vdash r$ which appear in Theorem 3.4 can easily be read off from the expansions (3) or (7) of X_G. Namely, $\mu \vdash r$ is allowable if and only if some $\lambda \vdash d$ has $\mu_1, \mu_2, ...$ among its parts and $a_\lambda \neq 0$. Of course from the allowable μ we can compute which are maximal. By triangularity, the set of maximal elements (in dominance order) among all partitions $\nu=(\nu_1, ..., \nu_k) \vdash r$ for which there is a $\lambda'=(\nu_1, \nu_2, ..., \nu_k, ...)$ with $c_\lambda \neq 0$ coincides with the set of maximal allowable partitions of r. Hence we can also read off the maximal partitions directly from (7).

In certain cases Thoerem 3.4 shows that $c_\lambda \geqslant 0$ for suitable λ. In particular we have the following result.

3.5. Corollary. *Given the graph G, let μ be a maximal (with respect to G) partition of d. Then $c_{\mu'}>0$. In fact, $c_{\mu'}=r_1!\, r_2! \cdots a_\mu$, where $\mu=\langle 1^{r_1} 2^{r_2} \ldots \rangle$ and a_μ is given by* (3).

Proof. In Theorem 3.4 let $j=0$. Then Eq. (11) reduces to $\operatorname{sink}(G, \mu, 0)= c_{\mu'}$, so $c_{\mu'}>0$. We don't really need Theorem 3.4 here, since the triangularity of the basis $\{e_{\lambda'}\}$ with respect to $\{m_\mu\}$, together with (3), immediately gives $c_{\mu'}=r_1!\, r_2! \cdots a_\mu$. ▮

For certain graphs we can use Theorem 3.4 to show that all $c_\lambda \geqslant 0$.

3.6. Corollary. *Suppose that the vertices of G can be partitioned into two disjoint cliques. (Equivalently, the complement of G is bipartite.) Then $c_\lambda \geqslant 0$ for all λ.*

Proof. The allowable partitions of G all have the form $\langle 1^a 2^b \rangle$, where $a+2b=d$. Thus if $\langle 2^b \rangle$ is allowable, then it is maximal. If $a>0$ then choose $\mu=\langle 2^b \rangle$ and $j=1$ in Theorem 3.4; if $a=0$ then choose $\mu=\langle 2^b \rangle$

and $j=0$. The only partition $\lambda \vdash d$ appearing in (11) is then $\lambda = \langle 1^a 2^b \rangle$, so $c_\lambda \geqslant 0$ as desired. ∎

With notation and hypotheses as above, we in fact have that if $\lambda = \langle 1^a 2^b \rangle$ is allowable, then

$$c_\lambda = \begin{cases} \operatorname{sink}(G, \langle 2^b \rangle, 1), & a>0 \\ \operatorname{sink}(G, \langle 2^b \rangle, 0), & a=0. \end{cases}$$

A closely related result appears in [32, Remark 4.4]. This result is equivalent to a completely different interpretation of c_λ in terms of nonattacking rooks which shows that $c_\lambda \geqslant 0$. We will have more to say about the paper [32] in Section 5.

4. Reciprocity and Superfication

In [28, Thm. 1.2] Theorem 1.2(a) of this paper is extended to the following interpretation of $\chi_G(-n)$ for any $n \in \mathbb{P}$.

4.1. Proposition. *Let $n \in \mathbb{P}$. Let $\bar{\chi}_G(n)$ be the number of pairs $(\mathfrak{o}, \kappa)$, where $\mathfrak{o}$ is an acyclic orientation of G and $\kappa : V \to [n]$ is a coloring satisfying $\kappa(u) \leqslant \kappa(v)$ if (v, u) is an edge of $\mathfrak{o}$. Then $\bar{\chi}_G(n) = (-1)^d \chi_G(-n)$.*

We wish to give an analogue of Proposition 4.1 for the symmetric function X_G. The result will come as no surprise to experts on symmetric functions and combinatorial reciprocity theorems.

4.2. Theorem. *Define*

$$\bar{X}_G(x) = \sum_{(\mathfrak{o}, \kappa)} x^\kappa,$$

summed over all pairs $(\mathfrak{o}, \kappa)$ where $\mathfrak{o}$ is an acyclic orientation of G and $\kappa : V \to \mathbb{P}$ is a coloring satisfying $\kappa(u) \leqslant \kappa(v)$ if (v, u) is an edge of $\mathfrak{o}$. Then $\bar{X}_G = \omega X_G$, where ω is the standard involution on symmetric functions (as in Corollary 2.6).

Proof. Define a linear transformation $\omega' : \mathcal{Q}_d \to \mathcal{Q}_d$ (where $\mathcal{Q}_d$ denotes the space of quasi-symmetric functions of degree d) by $\omega'(Q_S) = Q_{\bar{S}}$, where $\bar{S} = [d-1] - S$. Thus ω' is an involution. Given a d-element poset P, let X_P be as in Theorem 3.1, and define $\bar{X}_P = \sum_\kappa x^\kappa$, summed over all order-preserving functions $\kappa : P \to \mathbb{P}$, i.e., if $s \leqslant t$ in P then $\kappa(s) \leqslant \kappa(t)$. The reciprocity theorem for P-partitions [27, Thm. 10.1; 30, Thm. 4.5.7] implies that $\omega' X_P = \bar{X}_P$. In particular, if $P = P_\lambda$ is a disjoint union of chains as in the proof of Theorem 3.3, then $X_P = e_\lambda$ and $\bar{X}_P = h_\lambda$, so $\omega' e_\lambda = h_\lambda$. Thus ω' is an extension of ω, so we henceforth denote it also as ω.

Now apply ω to Eq. (8). We get

$$\omega X_{\mathfrak{o}} = \bar{X}_{\mathfrak{o}} = \sum_{\kappa \in \bar{K}_{\mathfrak{o}}} x^\kappa,$$

where

$$\bar{K}_{\mathfrak{o}} = \{\kappa : V \to \mathbb{P} \mid \kappa(u) \leqslant \kappa(v) \text{ if } (v, u) \text{ is an edge of } \mathfrak{o}\}.$$

Hence $\omega X_G = \bar{X}_G$, as desired. ▮

A *supersymmetric function* is a formal power series $f = f(x, y)$ in the two sets of variables $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ which is symmetric in the x_i's and in the y_i's, and which satisfies the "cancellation property"

$$f(x, y)|_{y_1 = -x_1} = f(x, y)|_{y_1 = x_1 = 0}.$$

Let ω_y denote the involution ω acting only on the y variables, so $x_1, x_2, \ldots$ are regarded as constants. If $f(x)$ is a symmetric function, then define the *superfication* $f(x/y)$ of f by

$$f(x/y) = \omega_y f(x, y),$$

i.e., replace the variables $x_1, x_2, \ldots$ by $x_1, x_2, \ldots, y_1, y_2, \ldots$ and then apply ω_y. It is well-known (see, e.g., [19], where their bisymmetric functions $f(x \mid y)$ are our $f(x/-y)$) that superfication is a (degree-preserving) isomorphism from the ring of symmetric functions (over $\mathbb{Q}$, say) to the ring of bisymmetric functions (over $\mathbb{Q}$). Note that $f(x/0) = f(x)$ and $f(0/x) = \omega f(x)$. In particular (by Theorem 4.2), $X_G(x/0) = X_G(x)$ and $X_G(0/x) = \bar{X}_G(x)$. Hence we can ask whether there is some interpretation of $X_G(x/y)$ which generalizes Theorem 4.2. We state our result using the colors $\mathbb{P} = \{1, 2, \ldots\}$ and $\bar{\mathbb{P}} = \{\bar{1}, \bar{2}, \ldots\}$. Let $<*$ be a fixed linear ordering of the set $\mathbb{P} \cup \bar{\mathbb{P}}$. (Since the expression of $X_G(x/y)$ given below is invariant under permuting colors in $\mathbb{P}$ and colors in $\bar{\mathbb{P}}$, we may as well assume that $1 <* 2 <* 3 <* \cdots$ and $\bar{1} <* \bar{2} <* \bar{3} <* \cdots$.)

4.3. Theorem. *We have*

$$X_G(x/y) = \sum_{\mathfrak{o}, \kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots y_1^{\#\kappa^{-1}(\bar{1})} y_2^{\#\kappa^{-1}(\bar{2})} \cdots, \tag{13}$$

summed over all pairs $(\mathfrak{o}, \kappa)$ *where* $\mathfrak{o}$ *is an acyclic orientation of* G *and* $\kappa : V \to \mathbb{P} \cup \bar{\mathbb{P}}$ *is a coloring of* G *satisfying*: (a) *If* (v, u) *is an edge of* $\mathfrak{o}$ *then*

$\kappa(u) \leqslant^* \kappa(v)$, *and* (b) *if* (v, u) *is an edge of* $\mathfrak{o}$, $\kappa(u) \in \mathbb{P}$, *and* $\kappa(v) \in \mathbb{P}$, *then* $\kappa(u) <^* \kappa(v)$. (*Let us call such a pair* $(\mathfrak{o}, \kappa)$ *an* okay pair.)

Proof. Let $S \subseteq [d-1]$ as in the previous section. Define the "superfication" (with respect to the ordering $<*$) $\mathbb{S}Q_S$ of the quasisymmetric function Q_S by

$$\mathbb{S}Q_S = \sum_{\substack{i_1, \ldots, i_d \in \mathbb{P} \cup \bar{\mathbb{P}} \\ i_1 <^* \cdots <^* i_d \\ j \in S \text{ and } i_j, i_{j+1} \in \mathbb{P} \Rightarrow i_j < i_{j+1} \\ j \notin S \text{ and } i_j, i_{j+1} \in \bar{\mathbb{P}} \Rightarrow i_j < i_{j+1}}} z_{i_1} z_{i_2} \cdots z_{i_d},$$

where

$$z_i = \begin{cases} x_i, & i \in \mathbb{P} \\ y_j, & i = \bar{j} \in \bar{\mathbb{P}}. \end{cases}$$

Extend this map to the space $\mathcal{Q}_d$ by linearity. We claim that if $f \in \mathcal{Q}_d$ is a symmetric function, then

$$\mathbb{S}f = f(x/y). \tag{14}$$

To prove the claim, we may assume that f is a *Schur function* s_λ, since the Schur functions $\{s_\lambda : \lambda \vdash d\}$ form a basis for the space $\Lambda_{\mathbb{Q}}^d$ of all homogeneous symmetric functions of degree d. (See e.g. [18, 25, 26] for the definition of this important basis.) If τ is a standard Young tableau (SYT) of shape λ [18, p. 5; 25, Def. 2.5.1] then the *descent set* $D(\tau)$ of τ is defined by

$$D(\tau) = \{i : i+1 \text{ is in a lower row of } \tau \text{ than } i\}.$$

Here we are using "English notation" for SYT's, e.g.,

$$\begin{array}{ccccc} 1 & 2 & 3 & 5 & 9 \\ 4 & 7 & & & \\ 6 & 8 & & & \end{array}$$

is an SYT of shape $(5, 2, 2)$ and descent set $\{3, 5, 7\}$. It is known (implicit in [27, Sect. 21] and more explicit in [13]) that

$$s_\lambda = \sum_\tau Q_{D(\tau)}, \tag{15}$$

where τ ranges over all SYT of shape λ.

It is also known [29, Theorem 5.1] that

$$s_\lambda(x/y) = \sum_\sigma x_1^{m_1} x_2^{m_2} \cdots y_1^{\bar{m}_1} y_2^{\bar{m}_2} \cdots, \tag{16}$$

where σ ranges over all ways of filling the shape λ with entries from $\mathbb{P} \cup \bar{\mathbb{P}}$ such that (a) every row and column is weakly increasing with respect to $<*$, (b) if $i \in \mathbb{P}$ then i appears at most once in every column, and (c) if $\bar{i} \in \bar{\mathbb{P}}$ then $\bar{i}$ appears at most once in every row; and where m_i denotes the number of $i \in \mathbb{P}$ and $\bar{m}_i$ the number of $\bar{i} \in \bar{\mathbb{P}}$ appearing in σ. Let us call σ a *supertableau* (with respect to $<*$). Write σ_{ij} for the (i, j)-entry of σ, i.e., the entry in the ith row from the top and the jth column from the left, and similarly τ_{ij}. We say that a supertableau σ is *compatible* with an SYT τ if the following three conditions are satisfied: (i) if $\tau_{ij} < \tau_{kl}$ then $\sigma_{ij} <* \sigma_{kl}$, (ii) if $\sigma_{ij} = \sigma_{kl} \in \mathbb{P}$ and $i < k$, then $\tau_{ij} > \tau_{kl}$, and (iii) if $\sigma_{ij} = \sigma_{kl} \in \bar{\mathbb{P}}$ and $i < k$, then $\tau_{ij} < \tau_{kl}$. The following facts are straightforward to verify, analogous to [27, Lemma 6.1] or [30, Lemma 4.5.3].

- Every supertableaux σ is compatible with a unique SYT τ.
- Let K_τ be the set of all supertableaux compatible with the SYT τ.

Let

$$R_\tau(x/y) = \sum_{\sigma \in K_\tau} x_1^{m_1} x_2^{m_2} \cdots y_1^{\bar{m}_1} y_2^{\bar{m}_2} \cdots,$$

where m_i and $\bar{m}_i$ have the same meaning as in (16). Then $R_\tau(x/y) = \mathbb{S} Q_{D(\tau)}$.

It now follows from applying $\mathbb{S}$ to (15) that

$$\begin{aligned} \mathbb{S} s_\lambda &= \sum_\tau \mathbb{S} Q_{D(\tau)} \\ &= \sum_\tau R_\tau(x/y) \\ &= s_\lambda(x/y), \end{aligned}$$

as claimed.

Let $(\mathfrak{o}, \kappa)$ be an okay pair (as defined in Theorem 4.3). We say that κ is *compatible* with a linear extension $\alpha = (a_1, ..., a_d) \in \mathscr{L}(\bar{\mathfrak{o}}, \omega)$ if (a) $\kappa(\omega^{-1}(i)) <* \kappa(\omega^{-1}(i+1))$ for $1 \leqslant i \leqslant d-1$, (b) if $\kappa(\omega^{-1}(i)) \in \mathbb{P}$, $\kappa(\omega^{-1}(i+1)) \in \mathbb{P}$, and $a_i > a_{i+1}$, then $\kappa(\omega^{-1}(i)) <* \kappa(\omega^{-1}(i+1))$, and (c) if $\kappa(\omega^{-1}(i)) \in \bar{\mathbb{P}}$, $\kappa(\omega^{-1}(i+1)) \in \bar{\mathbb{P}}$, and $a_i < a_{i+1}$, then $\kappa(\omega^{-1}(i)) <* \kappa(\omega^{-1}(i+1))$. Once again it is straightforward to check that for each okay pair $(\mathfrak{o}, \kappa)$ there is a unique linear extension $\alpha \in \mathscr{L}(\bar{\mathfrak{o}}, \omega)$ with which ω is compatible. Moreover,

$$\sum_{\kappa} x_1^{\#\kappa^{-1}(1)} x_2^{\#\kappa^{-1}(2)} \cdots y_1^{\#\kappa^{-1}(1)} y_2^{\#\kappa^{-1}(2)} \cdots = \mathbb{S} Q_{D(\alpha)},$$

where κ ranges over all colorings compatible with α. Hence summing over all acyclic orientations $\mathfrak{o}$ and all $\alpha \in \mathscr{L}(\bar{\mathfrak{o}}, \omega)$ yields by Eq. (8) that $\mathbb{S}X_G$ agrees with the right-hand side of (13). But by (14) we have $\mathbb{S}X_G = X_G(x/y)$, and the proof follows. ∎

5. Special Graphs

The main question to concern us in this section is the following: Which graphs G are e-positive? If P is a finite poset then let $\operatorname{inc}(P)$ denote its incomparability graph; i.e., uv is an edge of $\operatorname{inc}(P)$ if u and v are incomparable in P. Let $\mathbf{a}+\mathbf{b}$ denote the poset which is a disjoint union of an a-element chain and a b-element chain. We say that P is $(\mathbf{a}+\mathbf{b})$*-free* if P contains no induced subposet isomorphic to $\mathbf{a}+\mathbf{b}$. The following conjecture is equivalent to [32, Conj. 5.5].

5.1. Conjecture. If P is $(\mathbf{3}+\mathbf{1})$-free, then $\operatorname{inc}(P)$ is e-positive.

As mentioned in [32, pp. 277–278], this conjecture has been verified for all posets P with $\#P \leqslant 7$. For instance, 639 seven-element posets are $(\mathbf{3}+\mathbf{1})$-free, and for all of them $\operatorname{inc}(P)$ is e-positive. Recently Stembridge has verfied (private communication) that Conjecture 5.1 holds for eight-element posets. There are 2469 $(\mathbf{3}+\mathbf{1})$-free eight-element posets P (out of a total of 16999 eight-element posets), and for all 2469 $\operatorname{inc}(P)$ is e-positive.

Although Conjecture 5.1 remains mysterious, a weaker result was recently proved by Gasharov [11] and is closely related to [14, Sect. 7]. This weaker result concerns the expansion of X_G in terms of Schur functions. It is well-known that each e_λ is s-positive (e.g., by [18, (5.17)] and the fact that $e_i = s_{\langle 1^i \rangle}$), so an e-positive symmetric function is also s-positive. If G is the claw K_{13} then $X_G = s_{31} - s_{22} + 5s_{211} + 8s_{1111}$, so X_G is not always s-positive. The result of Gasharov is the following.

5.2. Theorem. *Let G be the incomparability graph of a $(\mathbf{3}+\mathbf{1})$-free poset. Then X_G is s-positive.*

A special class of $(\mathbf{3}+\mathbf{1})$-free posets are of particular interest. These are posets which are both $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free. Such posets are called *semiorders*, and their incomparability graphs are known as *indifference graphs* or *unit interval graphs*. They have the following characterization:

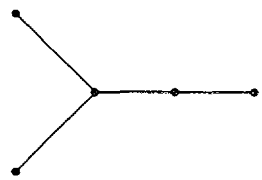

FIG. 4. An e-positive tree.

Choose a collection $\mathscr{C}$ of intervals $[i, j] = \{i, i+1, ..., j\}$ of $[d]$. (Without loss of generality we may assume that no interval in $\mathscr{C}$ contains another.) Let G have vertex set $V = [d]$ and edge set

$$E = \{uv : u, v \text{ belong to some } I \in \mathscr{C}\}.$$

In other words, G is an edge-union of cliques whose vertex sets are intervals. Then G is an indifference graph, and every indifference graph is isomorphic to such a G. A good reference for this subject is [10].

One reason for our interest in indifference graphs stems from their connection with immanants of Jacobi–Trudi matrices, as explained in [32, Sect. 4]. One can deduce easily from a result of Haiman [17, Thm. 1.4] that indifference graphs are s-positive (a special case of Theorem 5.2). Haiman's proof uses the Kazhdan–Lusztig conjectures on composition series of Verma modules (proved by Beilinson–Bernstein and Brylinski–Kashiwara), but the proof of Theorem 5.2 by Gasharov is much more elementary. Let us mention that Haiman also has a conjecture [17, Conj. 2.1] on the Hecke algebra $H_n(q)$ of the symmetric group $\mathscr{S}_n$ which is easily seen to imply Conjecture 2.1 for indifference graphs.

Are there "nice" graphs G more general that those of Conjecture 5.1 for which X_G is e-positive? A complete characterization of such graphs appears hopeless; X_G can be e-positive only "by accident." For instance, of all trees with at most six vertices which are not paths (for the situation with paths, see Proposition 5.3), exactly one is e-positive, viz., the five-vertex tree of Fig. 4. It seems just an "accident" that this tree is e-positive. On the other hand, one class of graphs not covered by Conjecture 5.1 which seem to be e-positive are the circular analogues of indifference graphs, i.e., graphs with vertex set $V = \mathbb{Z}/d\mathbb{Z}$ and edge set $E = E_1 \cup \cdots \cup E_j$, where each E_j is a clique on a subset of V of the form $\{a+1, a+2, ..., a+b\}$. Such graphs need not be indifference graphs, e.g., odd cycles. One might ask whether the hypothesis that G is an incomparability graph in Conjecture 5.1 is really necessary. In other words, since $\mathrm{inc}(\mathbf{3}+\mathbf{1}) = K_{13}$, we can ask whether any graph not containing an induced K_{13} is e-positive. Unfortunately the answer is negative; the graph G of Fig. 5 satisfies

$$X_G = 12e_6 + 18e_{51} + 12e_{42} - 6e_{33} + 6e_{411} + 6e_{321},$$

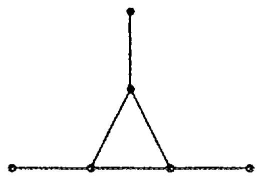

FIG. 5. A non-e-positive clawfree graph.

yet G contains no induced K_{13}. On the other hand, G is *contractible* to K_{13}. We don't know of a graph which is not contractible to K_{13} (even regarding multiple edges of a contraction as a single edge) which is not e-positive. However, there do exist indifference graphs which can be contracted to K_{13} (treating multiple edges as single edges). One such example, due to J. Kahn, is given in Fig. 6. Thus the hypothesis of non-contractibility to K_{13} is too weak to be relevant to Conjecture 5.1.

The simplest (connected) indifference graph is the path P_d, and the simplest circular indifference graph which is not an ordinary indifference graph is the cycle C_d. For these graphs G we can compute the expansion of X_G in e_λ explicitly and thereby verify that they are e-positive. The result for paths seems first to have been proved by Carlitz *et al.* [7, p. 242]. A combinatorial proof was given by Dollhopf *et al.* [9].

5.3. PROPOSITION. *We have*

$$\sum_{d \geqslant 0} X_{P_d} \cdot t^d = \frac{\sum_{i \geqslant 0} e_i t^i}{1 - \sum_{i \geqslant 1} (i-1)\, e_i t^i}. \tag{17}$$

Hence P_d is e-positive.

First Proof (sketch). We give two different proofs. The first proof is the most complicated but uses the most general techniques. Let P_d have vertex set $V = \{v_1, ..., v_d\}$ with edges $v_i v_{i+1}$, $1 \leqslant i \leqslant d-1$. Fix $n \in \mathbb{P}$, and let $X_d^i = \sum_\kappa x^\kappa$, summed over all proper colorings $\kappa : V \to [n]$ of P_d such that $\kappa(v_d) = i$. Let

$$F^i(t) = \sum_{d \geqslant 1} X_d^i \cdot t^d.$$

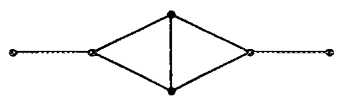

FIG. 6 An incomparability graph of a $(3+1)$-free poset contractible to K_{13}.

It is clear that

$$F^i(t) = x_i + \sum_{\substack{1 \leqslant j \leqslant n \\ j \neq i}} F^j(t)\, x_i, \qquad 1 \leqslant i \leqslant n. \tag{18}$$

We can solve this system of n linear equations in the n unknowns $F^j(t)$ by Cramer's rule. We omit the derivation of the solution, but at any rate it is easy to check that the unique solution is given by

$$F^j(t) = \frac{x_j \sum_{i \geqslant 0} e_i(x_1, ..., \hat{x}_j, ..., x_n)\, t^{i+1}}{1 - \sum_{i \geqslant 1} (i-1)\, e_i(x_1, ..., x_n)\, t^i},$$

where $\hat{x}_j$ denotes that the variable x_j is missing. Hence

$$\begin{aligned}\sum_{d \geqslant 0} X_{P_d}(x_1, ..., x_n)\, t^d &= 1 + \sum_{j=1}^{n} F^j(t) \\ &= \frac{\sum_{i \geqslant 0} e_i(x_1, ..., x_n)\, t^i}{1 - \sum_{i \geqslant 1} (i-1)\, e_i(x_1, ..., x_n)\, t^i}\end{aligned}$$

Letting $n \to \infty$ completes the proof. ∎

Second Proof. Let $\lambda \vdash d$. The number b_λ of connected partitions of P_d of type λ is just the number of distinct permutations of the parts of λ. Hence if $\lambda = \langle 1^{r_1} 2^{r_2} \ldots \rangle$, then $b_\lambda = \binom{l(\lambda)}{r_1, r_2, \ldots}$. Since P_d is a forest (even a tree), we get from (6) that

$$X_{P_d} = \sum_{\lambda \vdash d} \varepsilon_\lambda \binom{l(\lambda)}{r_1, r_2, \ldots} p_\lambda.$$

Hence

$$\sum_{d \geqslant 0} X_{P_d} \cdot t^d = \frac{1}{1 - p_1 + p_2 - p_3 + \cdots}.$$

By a known identity in the theory of symmetric functions (e.g., apply the involution ω to [32, Prop. 2.2]), we get (17). ∎

The power series appearing in (17) (or the essentially equivalent image of it under ω) has appeared before in several contexts [12; 31, Prop. 12, when $q = 1$; 32, Prop. 2.2]. We do not know whether all these appearances of the same series are just a coincidence.

The "circular analogue" of Proposition 5.3 is given by the next result.

5.4. PROPOSITION. *We have*

$$\sum_{d\geq 2} X_{C_d}\cdot t^d = \frac{\sum_{i\geq 0} i(i-1)\,e_i t^i}{1-\sum_{i\geq 1}(i-1)\,e_i t^i}.$$

Hence C_d *is e-positive.*

Proof. In the first proof of Proposition 5.3, the denominator

$$D(t) = 1 - \sum_{i\geq 1}(i-1)\,e_i t^i$$

of each $F^i(t)$ (in the limit $n\to\infty$) is the determinant of the coefficient matrix of the homogeneous part of (18). By a well-known result concerning periodic initial conditions [30, Cor. 4.7.3], we get

$$\sum_{d\geq 1} X_{C_d}\cdot t^d = -\frac{tD'(t)}{D(t)}.$$

Since $X_{C_1}=0$ (because C_1 is a loop), the proof follows. ∎

The proof technique of the first proof of Proposition 5.3 can in principle be carried over to some other graphs. For instance, let P_{d2} be the graph with vertex set $V_d=\{v_1,\ldots,v_d\}$ and edges v_iv_j if $|j-i|=1$ or 2. The graph P_{d2} is an indifference graph and so by Conjecture 5.1 should be e-positive. If for $i\neq j$ we let

$$F^{ij}(t)=\sum_{d\geq 2}\left(\sum_{\kappa} x^{\kappa}\right)t^d,$$

where κ ranges over proper colorings $\kappa: V_d\to[n]$ with $\kappa(v_{d-1})=i$ and $\kappa(v_d)=j$, then we get analogously to (18) that

$$F^{ij}(t)=x_ix_j+\sum_{\substack{k\neq i\\ k\neq j}} F^{ki}(t)x_j.$$

Solving by Cramer's rule and letting $n\to\infty$ yields, up to degree eight, that

$$\sum_{i\neq j} F^{ij}(t)=\sum_{d\geq 2} X_{P_{d2}}t^d=\frac{N}{D},$$

where

$$\begin{aligned} D = {} & 1-2e_3t^3-6e_4t^4-24e_5t^5-(64e_6+6e_{51}-e_{33})t^6\\ & -(174e_7+30e_{61}+6e_{52}-6e_{43})t^7\\ & -(426e_8+120e_{71}+30e_{62}-24e_{53}+3e_{44})t^8+\cdots. \end{aligned}$$

Unfortunately the coefficients of $1-D$ are not all nonnegative, so there is no obvious reason why N/D should be e-positive. (We have not given the expansion of N since it is sensitive to how we treat the cases $d \leqslant 2$, and therefore may not be too meaningful.) It remains open whether P_{d2} is e-positive. For reference we record (computed with the aid of John Stembridge's SF package for Maple):

$$X_{P_{12}} = e_1$$
$$X_{P_{22}} = 2e_2$$
$$X_{P_{32}} = 6e_3$$
$$X_{P_{42}} = 16e_4 + 2e_{31}$$
$$X_{P_{52}} = 40e_5 + 12e_{41} + 2e_{32}$$
$$X_{P_{62}} = 96e_6 + 44e_{51} + 16e_{42} + 6e_{33}$$
$$X_{P_{72}} = 224e_7 + 136e_{61} + 66e_{52} + 52e_{43} + 4e_{511} + 2e_{421} + 2e_{331}$$
$$X_{P_{82}} = 512e_8 + 384e_{71} + 208e_{62} + 178e_{53} + 96e_{44} + 30e_{611} + 18e_{521} + 30e_{431} + 2e_{332}.$$

Note that although the symmetric function $X_{P_{d2}}$ seems quite mysterious, the chromatic polynomial $\chi_{P_{d2}}$ is easy to compute, viz.,

$$\chi_{P_{d2}}(n) = n(n-1)(n-2)^{d-2}.$$

In general, chromatic polynomials of indifference graphs are easy to compute and have only integer zeros, since indifference graphs are chordal (= supersolvable).

We mentioned after Corollary 3.6 that for complements of bipartite graphs, the coefficients c_λ have an interpretation in terms of nonattacking rooks. We now show that $c_{(d)}$ has a similar interpretation for any incomparability graph G. This result is really about the chromatic polynomial $\chi_G(n)$ and not the symmetric function $X_G(x)$ since $(-1)^{d-1} d[n]\,\chi_G(n) = c_{(d)}$, but it seems worthwhile to include it here. Note that $[n]\,\chi_G(n) = \mu_{L_G}(\hat{0}, \hat{1})$, so our result may be regarded as a formula for the "Möbius number" $\mu_{L_G}(\hat{0}, \hat{1})$ of an incomparability graph. We use the notation per(A) for the permanent of the $m \times m$ matrix $A = (a_{ij})$, defined by

$$\operatorname{per}(A) = \sum_{w \in \mathfrak{S}_m} a_{1,w(1)} \cdots a_{m,w(m)}.$$

If A is a $0-1$ matrix, then per(A) is the number of ways to place m nonattacking rooks on the board $B = \{(i, j) : a_{ij} \neq 0\}$.

5.5. PROPOSITION. *Let $P=\{v_1,\dots,v_d\}$ be a poset. Let $A=(a_{ij})_{i,j=1}^{d}$ be the $d\times d$ matrix defined by*

$$a_{ij}=\begin{cases}0, & \text{if } v_i<v_j\\ 1, & \text{otherwise.}\end{cases}$$

Let v_r be a minimal element and v_s a maximal element of P. Let A_0 be A with row s and column r removed. (Note that the entries in row s and column r are all 1's.) Suppose that $X_{\mathrm{inc}(P)}=\sum c_\lambda e_\lambda$. Then

$$c_{(d)}=d\cdot \mathrm{per}(A_0).$$

Proof. Write $G=\mathrm{inc}(P)$. Let g_i be the number of stable partitions of G with i blocks, so

$$\chi_G(n)=\sum_i g_i(n)_i,$$

where $(n)_i=n(n-1)\cdots(n-i+1)$. Hence

$$\begin{aligned}c_{(d)}&=(-1)^{d-1}\,d[n]\,\chi_G(n)\\ &=(-1)^{d-1}d\sum_i(-1)^{i-1}(i-1)!\,g_i.\end{aligned}$$

A stable partition of G with i blocks is the same as a partition π of P into chains $C_1,\dots,C_i$. For each such chain $u_1<u_2<\cdots<u_k$, place rooks on the squares (u_r,u_{r+1}) of the board $[d]\times[d]$. We get a total of $d-i$ nonattacking rooks on the board $\bar{B}=\{(j,k):a_{jk}=0\}$, and conversely any placement of $d-i$ nonattacking rooks on $\bar{B}$ corresponds to a partition of P into i chains. Let B be the complement of $\bar{B}$ in the $(d-1)\times(d-1)$ board K obtained by removing row s and column r from $[d]\times[d]$, so $\bar{B}\subset K$. The number of ways to extend the $d-i$ nonattacking rooks on $\bar{B}$ to $d-1$ nonattacking rooks on K is $(i-1)!$. Hence by a standard Inclusion–Exclusion argument (e.g., [23, Sect. 7.2; 30, Thm. 2.3.1]), the number of ways to place $d-1$ nonattacking rooks on B is just $\sum_i(-1)^{d-i}(i-1)!\,g_i$, and the proof follows. ∎

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