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Chromatic Polynomials for Regular Graphs and Modified Wheels*

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Let $P(G; \lambda)$ denote the chromatic polynomial of a graph G, expressed in the variable λ. A graph G is chromatically unique if $P(G; \lambda) = P(H; \lambda)$ implies that H is isomorphic to G. We prove that complements of partial matching forests are chromatically unique. We construct an infinite family of counterexamples to the conjecture that all regular graphs are chromatically unique. We show that the coefficients of chromatic polynomials of certain connected graphs, relative to the tree basis, do not exhibit the strong logarithmic concavity property. We show that many of the coefficients have equal absolute value.

1. INTRODUCTION

The graphs which we consider here are finite, undirected, simple and loopless. Two graphs G and H are said to be *chromatically equivalent* if they have the same chromatic polynomial, i.e., $P(G; \lambda) = P(H; \lambda)$. A graph G is *chromatically unique* if $P(H; \lambda) = P(G; \lambda)$ implies that H is isomorphic to G. In [2, 3, 6], it was proven that cycle graphs, θ-graphs, generalized θ-graphs and two families of modified wheels are chromatically unique. Here, we prove that the complements of certain forests are chromatically unique. We give an infinite family of counterexamples to the conjecture that all regular graphs are chromatically unique.

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In [8], Read conjectured that the absolute value of the coefficients of the chromatic polynomial of any graph, relative to the null basis, are unimodal. *Unimodal* means that the coefficients first increase in magnitude, and then decrease; two successive coefficients may be equal, but there is never a coefficient surrounded by larger coefficients. This conjecture allows for any number of equal coefficients.

Read also made the (apparently unpublished) conjecture that the coefficients of the chromatic polynomial of any graph, relative to the falling factorial basis, are unimodal. The falling factorial basis is used in Section 3 of this paper. Chvátal [4] proved that the last few coefficients form a decreasing sequence.

Nijenhuis and Wilf [7] developed an algorithm which computes the coefficients of chromatic polynomials of connected graphs relative to the tree basis. Output from this algorithm led one of the authors (Whitehead) to make the following conjecture in a lecture at the University of Waterloo, during March of 1978.

Conjecture. The absolute value of the coefficients of the chromatic polynomial of any connected graph, relative to the tree basis, are unimodal.

In [5], Hoggar observed that the unimodal property is not preserved under multiplication of polynomials. Since the product of two chromatic polynomials is always a chromatic polynomial, Hoggar sought a property which not only was preserved under polynomial multiplication but which implied the unimodal conjecture as well. Hoggar found such a property—strong logarithmic concavity. A polynomial with coefficients a_k, $0 \leqslant k \leqslant n$, has *strong logarithmic concavity* if $a_i^2 > a_{i-1}a_{i+1}$ for $1 \leqslant i \leqslant n-1$. Hoggar conjectured that the chromatic polynomial of any graph, relative to the null graph basis, has strong logarithmic concavity. In Section 4 of this paper, we prove that the chromatic polynomials of modified wheels, relative to the tree basis, do not obey strong logarithmic concavity. Moreover, we prove that many of these coefficients have equal absolute value.

2. Chromatically Unique Forest Complements

Let F be a forest with n vertices, k disjoint edges and $n-2k$ isolated vertices. (A *forest* is a graph containing no cycles.) Let $\bar{F}$ denote the complement of F. $\bar{F}$ is sometimes called an almost-complete graph.

THEOREM 1. *$\bar{F}$ is chromatically unique for $0 \leqslant 2k \leqslant n$.*

Proof. The chromatic number $\chi(\bar{F}) = n - k$. A coloring of $\bar{F}$ with exactly $\chi(\bar{F})$ colors must assign a different color to each of the $n - 2k$ vertices that are isolated in F (completely connected in $\bar{F}$) and a different color to each of the k pairs of vertices that are joined by a disjoint edge in F (disconnected in $\bar{F}$). Hence, $(n - 2k) + k = n - k$ colors are used.

Suppose that there exists a graph H which is not isomorphic to F, where $P(\bar{H}; \lambda) = P(\bar{F}; \lambda)$. This implies that H has n vertices and k edges by Theorems 7 and 11 in [8]. We consider two cases.

Case 1. H has no cycles. This implies that H is a forest with p components containing edges and $n - (k + p)$ isolated vertices. Since H and F are nonisomorphic, H must contain a component (which must be a tree) with at least three vertices. The vertices of this component require at least two colors in a coloring of $\bar{H}$. Thus, in $\bar{H}$, at least $p + 1$ colors are used in coloring the p components, and $n - (k + p)$ colors are used for vertices that are isolated in H. Therefore,

$$\chi(\bar{H}) \geqslant (p + 1) - (n - (k + p)) = n - k + 1 > \chi(\bar{F}).$$

Hence, $\bar{H}$ and $\bar{F}$ are not chromatically equivalent.

Case 2. H contains at least one cycle. This implies that H has p components containing edges and at least $n - (k + p) + 1$ isolated vertices. Therefore,

$$\chi(\bar{H}) \geqslant p + (n - (k + p) + 1) = n - k + 1 > \chi(\bar{F}).$$

Again, $\bar{H}$ and $\bar{F}$ are not chromatically equivalent.

Combining these two cases, we conclude that $\bar{F}$ is chromatically unique.

It should be observed that when $n = 2k$, F is a complete matching. Also, F and $\bar{F}$ are regular graphs when $n = 2k$. Thus we have the following corollary.

COROLLARY 2. *$\bar{F}$ is a chromatically unique regular graph when $n = 2k$.*

3. AN INFINITE FAMILY OF COUNTEREXAMPLES

In April of 1978, R. C. Read privately communicated the following conjecture:

Conjecture. All regular graphs are chromatically unique.

In addition to the family in Corollary 2, the null graphs, cycles and complete graphs form three families of chromatically unique regular graphs. In this section, we construct an infinite family of regular graphs which are not chromatically unique.

Given two disjoint graphs G and H, we construct a new graph by forming the *Zykov* [8, 11] *product* of G and H, denoted $G \odot H$. The Zykov product is also called the join. In Zykov's theorem [8, 11], the *umbral product* [9] of two polynomials expressed in the falling factorial basis is used.

THEOREM 3 (Zykov's theorem). *If G and H are two disjoint graphs with chromatic polynomials $P(G; \lambda)$ and $P(H; \lambda)$ expressed in the falling factorial basis, then*

$$P(G \odot H; \lambda) = P(G; \lambda) \odot P(H; \lambda).$$

Let M_{2k} denote the complete matching forest with k disjoint edges and $2k$ vertices. Let N_{2k-1} denote the null graph with no edges and $2k-1$ vertices.

THEOREM 4. *For each $k \geqslant 2$, the graph $M_{2k} \odot N_{2k-1}$ is a regular graph which is not chromatically unique.*

Proof. We claim that $M_{2k} \odot N_{2k-1}$ is regular. Each vertex of M_{2k} has one edge joining it to another vertex of M_{2k} and $2k-1$ edges joining it to the vertices of N_{2k-1}. Thus each vertex of M_{2k} has degree $2k$. Each vertex of N_{2k-1} has $2k$ edges joining it to the vertices of M_{2k}. Thus $M_{2k} \odot N_{2k-1}$ is a regular graph of degree $2k$.

We claim that $M_{2k} \odot N_{2k-1}$ for $k \geqslant 2$ is not chromatically unique. To prove this claim, we replace M_{2k} by a chromatically equivalent nonisomorphic forest F_{2k} with $2k$ vertices and k edges. Actually any forest (with $2k$ vertices and k edges) nonisomorphic to M_{2k} will do. To be precise, we choose F_{2k} to be the forest with one three-vertex tree, $k-2$ disjoint edges and one isolated vertex. See Fig. 1. From Theorem 13 in [7], it follows that

$$P(F_{2k}; \lambda) = P(M_{2k}; \lambda).$$

By Zykov's theorem, it follows that

$$P(F_{2k} \odot N_{2k-1}; \lambda) = P(M_{2k} \odot N_{2k-1}; \lambda).$$

Therefore, $M_{2k} \odot N_{2k-1}$ is not chromatically unique.

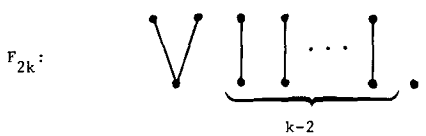

FIGURE 1

4. Modified Wheels and Strong Logarithmic Concavity

The *wheel*, W_n, is the graph formed by taking the Zykov product of the $(n-1)$-vertex cycle, C_{n-1}, with another vertex called the *hub*.

$$W_n = C_{n-1} \odot N_1.$$

A *modified wheel*, $W_{n,m}$, is obtained from the wheel W_n by deleting all but m consecutive *spokes*, where a spoke is an edge joining the cycle to the hub. In [3], it was shown that $W_{n,3}$ and $W_{n,4}$ are chromatically unique.

The purpose of this section is to show that the chromatic polynomials of modified wheels, relative to the tree basis, fail to obey strong logarithmic concavity. In Theorem 5, we actually prove a stronger result, namely, that the middle coefficients have absolute value equal to 2^{m-1}. In [7], Nijenhuis and Wilf call the expansion of a chromatic polynomial $P(G; \lambda)$ in the tree basis "the Tutte polynomial form" of $P(G; \lambda)$. In [1], Biggs interprets the coefficients of the Tutte polynomial form of $P(G; \lambda)$ in terms of the number of spanning trees of G with internal activity i and external activity zero, where $1 \leqslant i \leqslant n-1$.

Theorem 5. *Let* $P(W_{n,m}; \lambda) = \sum_{k=2}^{n} (-1)^{n-k} c_k T_k$, *where* $T_k = \lambda(\lambda-1)^{k-1}$, $n \geqslant 4$ *and* $3 \leqslant m \leqslant n-2$. *Then*

$$\begin{aligned}
c_k &= \sum_{i=0}^{n-k} \binom{m-1}{i} && \text{for } n-m+2 \leqslant k \leqslant n,\\
&= 2^{m-1} && \text{for } m \leqslant k \leqslant n-m+1,\\
&= \sum_{i=0}^{k-1} \binom{m-1}{i} && \text{for } 3 \leqslant k \leqslant m-1,\\
&= m-1 && \text{for } k = 2.
\end{aligned}$$

Remark. If $m > (n+1)/2$, then there is no integer k which satisfies $m \leqslant k \leqslant n-m+1$. If $m = 3$, then there is no integer k which satisfies $3 \leqslant k \leqslant m-1$.

Before proceeding with the proof of Theorem 5, we need a lemma. In this lemma, $G_{n,m}$ will denote the graph obtained by deleting the edge $(1, n-1)$ from the modified wheel $W_{n,m}$ as shown in Fig. 2.

Lemma 6. $P(G_{n,m}; \lambda) = \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} T_{n-i}$, *where* $n \geqslant 4$ *and* $1 \leqslant m \leqslant n-1$.

Proof. First, we add the new edge $(1, m)$ and apply Theorem 1 in [8]. See Fig. 3. Since a path of length $n-m-1$ attached to a graph adds a

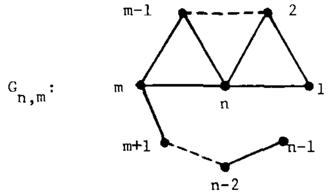

FIGURE 2

multiplicative factor of $(\lambda-1)^{n-m-1}$ to the chromatic polynomial of the graph, we have the following equation:

$$P(G_{n,m};\lambda)=(\lambda-1)^{n-m-1}\{P(W_{m+1};\lambda)+P(W_m;\lambda)\}.$$

In [10], Tutte gave a formula for $P(W_n;\lambda)$ valid for $n\geqslant 4$. Expanding Tutte's formula in terms of the tree basis $\{T_i\}_{i=1}^{\infty}$ where $T_i=\lambda(\lambda-1)^{i-1}$ and substituting for $P(W_{m+1};\lambda)$ and $P(W_m;\lambda)$ in the preceding equation, we obtain

$$P(G_{n,m};\lambda)=(\lambda-1)^{n-m-1}\left\{\sum_{j=3}^{m+1}(-1)^{m+1-j}\binom{m}{j-1}T_j+(-1)^{m-1}(m-1)T_2\right.$$
$$\left.+\sum_{k=3}^{m}(-1)^{m-k}\binom{m-1}{k-1}T_k+(-1)^{m-2}(m-2)T_2\right\}.$$

After combining the two summations, applying Pascal's identity and replacing the summation variable, we have

$$P(G_{n,m};\lambda)=\sum_{i=0}^{m-1}(-1)^i\binom{m-1}{i}T_{n-i}\qquad\text{for } 3\leqslant m\leqslant n-1.$$

For $m=2$, we have the special case illustrated in Fig. 4. For $m=1$, the graph is simply a tree. Thus,

$$P(G_{n,1};\lambda)=T_n.$$

Thus the lemma is true for $1\leqslant m\leqslant n-1$.

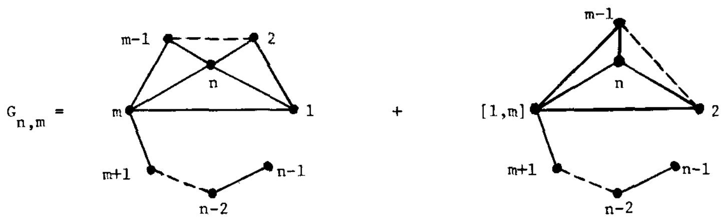

FIGURE 3

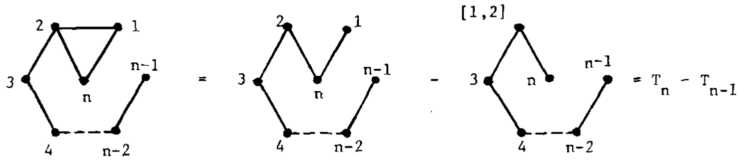

FIGURE 4

Proof of Theorem 5. By deleting the edge $(1, n-1)$ from $W_{n,m}$, we obtain the following equation:

$$P(W_{n,m}; \lambda) = P(G_{n,m}; \lambda) - P(W_{n-1,m}; \lambda).$$

Next, we apply Lemma 6, obtaining

$$P(W_{n,m}; \lambda) = \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} T_{n-i} - P(W_{n-1,m}; \lambda).$$

We proceed by recursion, stopping with

$$P(W_{m+2,m}; \lambda) = P(G_{m+2,m}; \lambda) - P(W_{m+1,m}; \lambda)$$

because $W_{m+1,m}$ is the wheel W_{m+1}. Therefore,

$$\begin{aligned} P(W_{n,m}; \lambda) = & \sum_{l=m+2}^{n} (-1)^{n-l} \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} T_{l-i} \\ & + (-1)^{n-(m+1)} \left[\sum_{j=3}^{m+1} (-1)^{m+1-j} \binom{m}{j-1} T_j \right. \\ & \left. + (-1)^{m-1} (m-1) T_2 \right]. \end{aligned}$$

After a considerable amount of algebraic manipulation including an application of Pascal's identity, we arrive at the following equation:

$$\begin{aligned} P(W_{n,m}; \lambda) = & \sum_{k=m}^{n} (-1)^{n-k} \sum_{i=0}^{n-k} \binom{m-1}{i} T_k \\ & + \sum_{k=3}^{m-1} (-1)^{n-k} \sum_{i=0}^{k-1} \binom{m-1}{i} T_k + (-1)^{n-2} (m-1) T_2. \end{aligned}$$

For $m = 3$, the second double sum vanishes. For $m \leqslant (n+1)/2$, the first double sum has terms which equal zero, of the form $(-1)^p \binom{m-1}{p} T_p$, where $p > m-1$. Indeed, in this case, for $m \leqslant k \leqslant n-m+1$, $c_k = 2^{m-1}$ because of the well-known identity $\sum_{i=0}^{r} \binom{r}{i} = 2^r$. This completes the proof of Theorem 5.

THEOREM 7. *Let* $P(W_{n,m}; \lambda) = \sum_{k=2}^{n} (-1)^{n-k} c_k T_k$, *where* $T_k = \lambda(\lambda-1)^{k-1}$ *and* $n \geqslant 4$. *The coefficients* c_k *do not obey strong logarithmic concavity (as discussed in Section* 1) *provided* $3 \leqslant m \leqslant (n-1)/2$.

Proof. By Theorem 5, $c_k = 2^{m-1}$ for $m \leqslant k \leqslant n-m+1$. The assumption $m \leqslant (n-1)/2$ implies that $2m \leqslant n-1$ which implies that $m \leqslant n-m-1$. Therefore, $m+2 \leqslant n-m+1$ which implies that $c_m = c_{m+1} = c_{m+2} = 2^{m-1}$. Hence, $(c_{m+1})^2 = c_m c_{m+2}$ which violates strong logarithmic concavity.

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