The Constructive Model of Univalence in Cubical Sets

Literature review

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45 min. public seminar

Outline

Introduction

Cubical model

Applications

Introduction

A mathematician is asked by a friend who is a devout Christian: "Do you believe in one God?"

What does he reply?

Introduction

A mathematician is asked by a friend who is a devout Christian: "Do you believe in one God?"

What does he reply?

He answers: "Yes – up to isomorphism." (© Michael Benjamin Stepp)

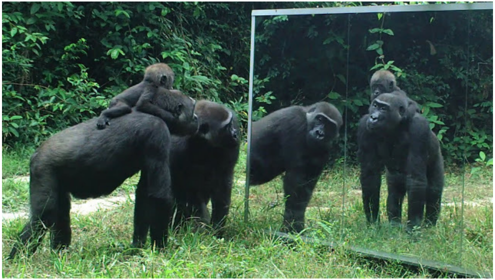

Figure: Isomorphisms in nature.

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Algebraic Topology

Isomorphism in algebraic topology is “homotopy equivalence”

Figure: A mug

Figure: A donut

homotopy equivalent spaces have the same “number of n-dimensional holes”

Holes are homotopy groups

computed by looking at homotopy classes of *continous* embeddings

$$S^n \to X, \quad \text{or } [0,1]^n \to X$$

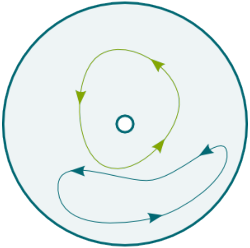

Figure: A donut has two 1-dimensional homotopy classes.

Figure: A ball without center has two 2-dimensional homotopy classes.

Type theory's origin
Russel, 1907

Invented to prevent paradox:

$$R = \{x \mid x \notin x\}, R \in R \Leftrightarrow \notin R$$

Solution was:

- replace sets (and propositions) by types and elements by terms,

$$x \in R \Rightarrow x : R$$

- types belong to universe hierarchy

$$\exists i, R : \mathscr{U}_i, \quad \mathscr{U}_0 : \mathscr{U}_1 : \dots$$

- constructive logic and formation rules

$x \notin x$ is not a valid proposition anymore

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Type theory

Two meanings/subfields:

- verifying computation in programming languages

```
filter :: (a -> Bool) -> [a] -> [a]
filter _pred []    = []
filter pred (x:xs)
  | pred x         = x : filter pred xs
  | otherwise      = filter pred xs
```

Figure: A typed recursive function in Haskell

- alternative constructive foundation of mathematics

```
_∘_ :   (∀ {x} (y : B x) → C y) → (g : (x : A) → B x) →
        ((x : A) → C (g x))
f ∘ g = λ x → f (g x)
```

Figure: Definition of the topological space S^1 in Agda

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Type theory as foundation for mathematics

Deductive system of judgements with typing rules:

$$\frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash b : B}$$

- judgments express wether a type is inhabited
- all judgements have contexts
- typing rules tell how to form and combine types and terms

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Equality in type theory

Martin-Löf, 1984

Definitional equality in type theory is for type checking, "denoted =" in code:

```
data Nat : Set where
  zero : Nat
  suc  : (n : Nat) → Nat

_+_ : Nat → Nat → Nat
zero  + m = m
suc n + m = suc (n + m)
```

Figure: An example of definitional equality.

Mathematics needs a "softer" equality as in:

Example (Leibniz's extensionality principle)

$$f = g \Leftrightarrow f(x) = g(x), \forall x$$

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Identity type

Martin-Löf, 1984

> ... *introduction of propositional equality in form of "identity type"*

Definition (Introduction rule)

Given a $a : X$, $\texttt{refl}(a) : a = a$.

Elimination rule of equality type:

Definition (path induction)

Given the following terms:

- a predicate $C : \prod_{x,y:A}(x =_A y) \to \mathscr{U}$
- the base step $c : \prod_{x:A} C(x, x, \texttt{refl}_x)$

there is a function $f : \prod_{x,y:A} \prod_{p:x=_A y} C(x, y, p)$ such that $f(x, x, \texttt{refl}_x) \equiv c(x)$.

- weaker than equality "by definition".
- stronger than equivalence.

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Intuition identity eliminator

Role of eliminator:

To prove a property C that depends on terms x, y and equalities $p : x = y$ it suffices to consider all the cases where

- *x is definitionally equal to y*
- *the term of the intensional equality type under consideration is $\texttt{refl}_x : x = x$.*

Implications:

- proves transitivity, symmetry
- equality type can have multiple terms

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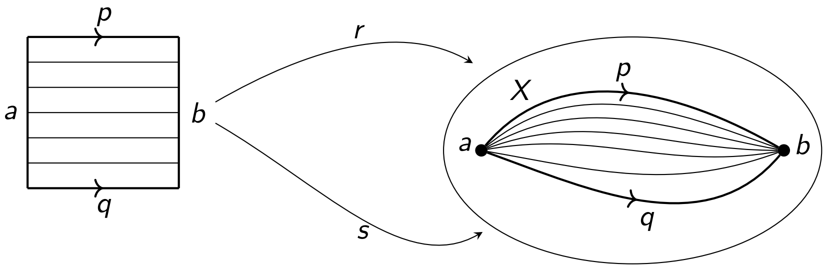
Homotopy type theory (HoTT)

Awodey, 2006

Gives *homotopy* interpretation to equality type:

$$p, q : a =_X b \quad r, s : p =_{\mathtt{Id}_X(a,b)} q$$



$\Rightarrow$ alternative foundations for mathematics based on type theory and topology

Consequences of univalence axiom

Voevodsky, 2009

Axiom (Univalence axiom)

Given types $X, Y : \mathscr{U}$ for some universe $\mathscr{U}$, the map $\Phi_{X,Y} : (X = Y) \to (X \simeq Y)$ is an equivalence of types.

- equivalence of types is a bijection for set-like types

$$\mathbb{N} \simeq \mathbb{N}_0$$

- univalence implies

$$\mathbb{N} \simeq \mathbb{N}_0 \Rightarrow \mathbb{N} = \mathbb{N}$$

- forces multiple terms of equality

$\Rightarrow$ *terms of equality are paths*

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Consequences of path interpretation

in general:

- equivalence behaves like homotopy equivalence
- mathematics "up to homotopy"
- lifting of path p as in algebraic topology: *transport* gives the other ending point of $p_\star$

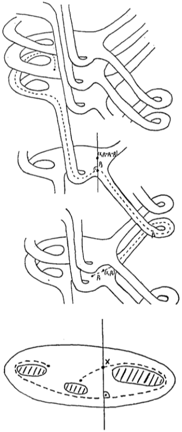

Figure: Jeff Erickson, 2009

Origin univalence

Grayson, 2018

Figure: Vladimir Voevodsky (1966 - 2017)

Why is it called “univalent”?

... these foundations seem to be faithful to the way in which I think about mathematical objects in my head ...

faithful = univalent in a Russion translation of Boardman (2006)

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Grayson, 2018

Figure: Vladimir Voevodsky (1966 - 2017)

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Personal remark

The univalence axiom adds:

- intuitive explanation of equaity
- alternative foundations with types
- field of mathematics: HoTT

But does not:

- make all proofs easier or shorter
- eliminate proofs of equivalence

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Computing with univalence

Huber, 2015

What about:

- implementing HoTT?
- calculations with very simple types as $\mathbb{N}$?

Can we, given a term $t : \mathbb{N}$ constructed using the univalence axiom, construct two terms $u : \mathbb{N}$ and $p : t =_{\mathbb{N}} u$ such that u does not involve the univalence axiom?

$\Rightarrow$ canonicity of $\mathbb{N}$ in cubical type theory (CTT)

$$t \equiv ua(...) \rightsquigarrow u \equiv S(\ldots(0)\ldots) : \mathbb{N}$$

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Cubical type theory

Cohen et al., 2015

A constructive extension of HoTT with dimension variables $i, j, k : \mathbb{I}$ (cubes) as primitives:

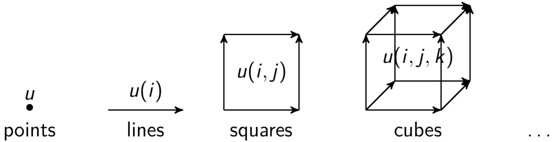

Figure: Discrete "*n*-cubes". Huber (2016)

- univalence becomes *constructable*
- computational interpretation for univalence

Are cubes a good idea?

EnigmaChord, 2016

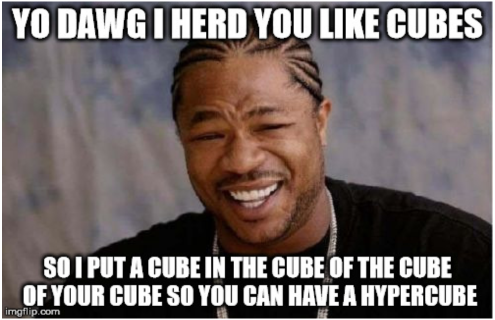

Figure:

(yes, they model n-dimensional homotopies)

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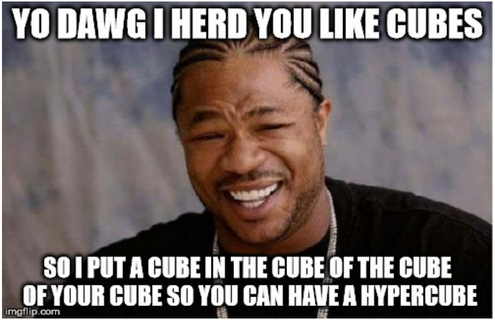

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Cubes model homotopy

Altenkirch, Brunerie, Licata, et. al 2013

Homotopy groups are defined as equivalence classes of *continous* embeddings:

$$[0,1]^n \to X$$

In HoTT, higher-dimensional eqalities behave like these embeddings

Level	Types	Cubes	Topology
1	$p, q : (a = b)$	edge	line
2	$r, s : (p = q)$	face	path homotopy
...	...	...	...
n	...	n-hypercube	n-dimensional homotopy

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Operations on cubes

Bezem, Coquand, Huber et al., 2013

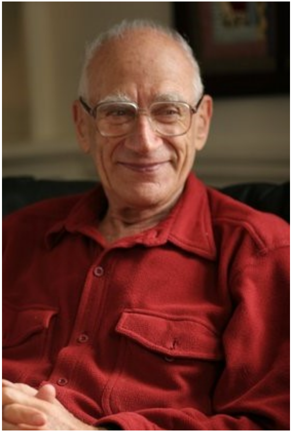
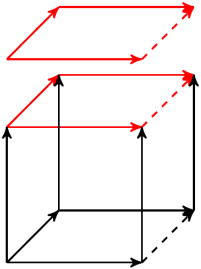


Figure: Daniel Kan (1927 — 2013)

Necessary for modelling HoTT:

- composition $\Rightarrow$ equality type
- glueing $\Rightarrow$ univalence



Presheaf model on $\mathscr{C}$

Dybjer, 1994

Give interpretation for stuff in type theory:

- base category $\mathscr{C}$ contains "extra tools" for model
- every context Γ is modelled as presheaf on $\mathscr{C}$, denoted $\widehat{\mathscr{C}}$.
- types and terms also interpreted in $\widehat{\mathscr{C}}$

Goals:

- verify consistency of type theory in sets
- justify primitives for implementations

Denoted as "presheaf model $\widehat{\mathscr{C}}$".

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Contexts in presheaf model $\widehat{\mathscr{C}}$

Definition (Presheaves $\widehat{\mathscr{C}}$)

Contravariant functors $\mathscr{C} \to \mathbf{Set}$

- generalize sheaves (see sheafication)
- model contexts

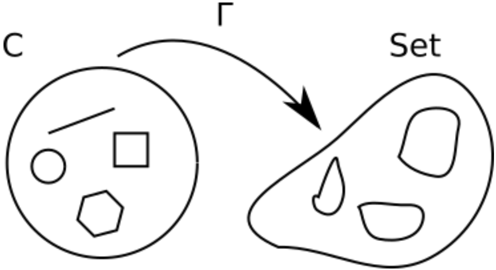

Figure: A representation of a preseheaf

Contexts in presheaf model $\widehat{\{0,1\}}$

Hofstra, 2014

Example (Reflexive directed graph)

Take $\mathscr{C} = \{0,1\}$ and $\mathrm{Hom}_{\mathscr{C}} = \{B, E, R\}$, $\Gamma \in \widehat{\{0,1\}}$, then

Applying functorial identities:

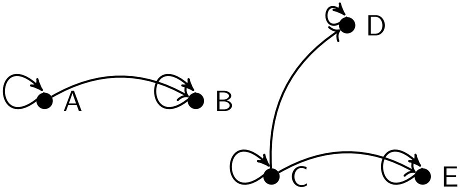

Figure: Reflexive graph

Types in presheaf model $\widehat{\mathscr{C}}$

Lemma (Types in a presheaf model)

If $\Gamma \in \widehat{\mathscr{C}}$ *a context, then the types are* $\left\{(\Delta, \sigma) \mid \Delta \in \widehat{\mathscr{C}}, \sigma \in Hom_{Ctx}(\Delta, \Gamma)\right\}$.

Helps to characterize types without using presheaves explicitly.

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Types in a presheaf model $\widehat{\{0,1\}}$

Example (Dependent directed reflexive graph)

Applying previous lemma to the type A in $\widehat{\{0,1\}}$:

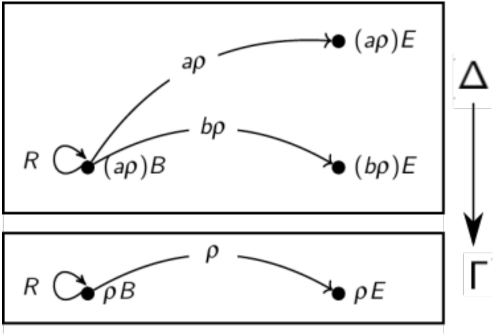

Figure: Modelled by two contexts and a surjective morphism

CTT as a presheaf model

Dimension variables and cubes have an abstract representation as "hypercubes" in a base category:

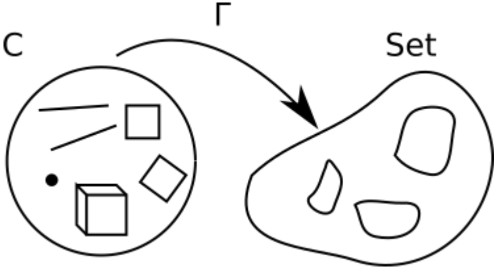

Figure: Presheaf acting on cubes

- proves consistency of CTT and HoTT
- justifies primitives used in implementations

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Cube category

Definition ("Cube" $\square$)

Category with:

- objects: $\{I \mid |I| < \infty, I \subset \mathbb{A}\}$
- morphisms $J \to I$: maps $I \mapsto dM(J)$
  - distributive lattice
  - $x \wedge 0 = 0, x \vee 1 = 1$
  - $\neg 0 = 1$ and $\neg 1 = 0$

$\mathbb{A}$: countable set of "dimension variables"

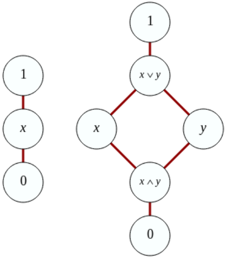


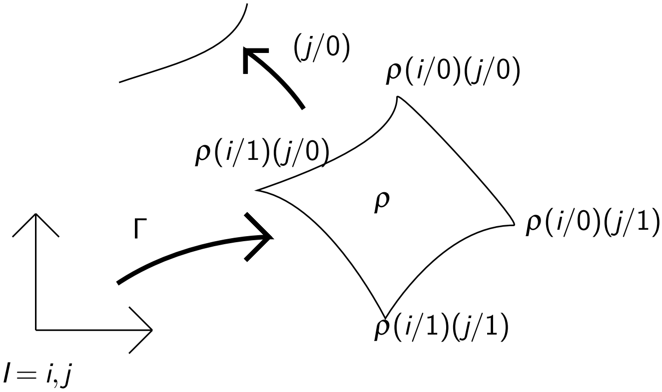
Figure: A simple lattice

Contexts in presheaf model $\widehat{\square}$

Example (Cubical contexts ("cubical sets"))

A presheaf $\Gamma \in \widehat{\square}$ is a functor $\square \to \mathbf{Set}$

- $\Gamma \in \widehat{\square}$ applied to $\{i,j\}$ gives square $\rho \in \Gamma(i,j)$
- morphisms in lattice $dM(i,j)$ give corners of ρ



Types in presheaf model $\widehat{\square}$

Type A can be represented by a context and a morphism on top of Γ:

- the $\rho \in \Gamma(i)$ is an edge of a square
- endpoints $\rho(0), \rho(1)$ can be lifted to points in the type $u(0), u(1) \in A(i, \rho)$

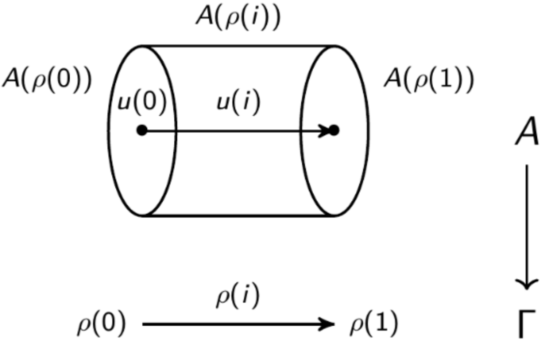


Figure: A type A within context Γ. Huber (2016)

Partial types in $\widehat{\square}$

Bezem, Coquand, Huber 2013

Partial type $A\ [(i = 0) \mapsto A(i/0)]$ is a subtype of A on top of sub-subpolyhedron of cubes:

- u 1-dimensional cube, line and type A:
- partial type $A(i/0)$ in type A:

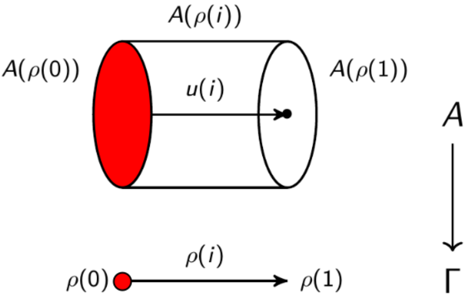


Figure: Huber, 2015

Other types in presheaf model $\widehat{\square}$

In the presheaf model on $\square$:

- types more complicated
- types no longer simply nested graphs

Interpreting types in presheaf model $\square$ hard but possible.

> *... transition from interpretation in model to syntax of types*

Path type

Bezem, Coquand, 2013

Syntactical definition of `Path` type with typing rules:

$$\frac{i : \mathbb{I} \vdash t : A \qquad i : \mathbb{I} \vdash t(i/0) = a : A \qquad i : \mathbb{I} \vdash t(i/1) = b : A}{() \vdash \langle i \rangle\, t : \texttt{Path}\ a\ b}$$

- almost models equality type
- not necessarily transitive $\Rightarrow$ composition operation

$$\begin{tikzcd}
a \arrow[r, dashed] & c \\
a \arrow[u, "refl"] \arrow[r, "p\ i"] & b \arrow[u, "q\ j"']
\end{tikzcd}$$

Figure: Transitivity can be proven with composition operation.

CTT as extension for HoTT

Other HoTT types interpreted in CTT:

- product, sum types
- natural numbers

Univalence proven with concepts from (Streicher, Voevodsky, Kapulkin et al. 2006 – 2012):

- simplicial sets replaced by cubical sets
- partial types and glueing construction conserved

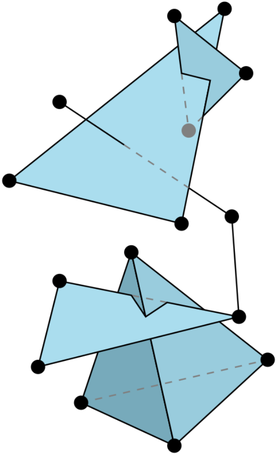


Figure: Every simplicial complex is a simplicial set

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Proving univalence

Cohen, Coquand, Huber, Moertberg (2015)

Axiom (Univalence axiom)

Given types $X, Y : \mathcal{U}$ for some universe $\mathcal{U}$, there is a map $\Phi_{X,Y} : (X = Y) \to (X \simeq Y)$ that is an equivalence of types.

Proof.

1. existence $\texttt{ua} : (X \simeq Y) \to (X = Y)$ with `Glue` construction.
2. for any partial $f : X \to Y$, $\texttt{Glue}\ [\varphi \mapsto (X, f)]\ Y \simeq Y$
3. for any type Y, $\sum_X X \simeq Y$ contractible
4. existence of an equivalence "eliminator".
5. the trivial map $p : X = Y \to X \simeq Y$ is an inverse "up-to-path" of `ua`

$\Rightarrow$ the map $\texttt{ua} \equiv \Phi_{X,Y}$ is equivalence $\square$

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Constructing `ua`

Definition

$\mathtt{ua} : \forall\ X\ Y : \mathscr{U}, X \simeq Y \to X = Y$

$$\begin{tikzcd} X \arrow[r, dashed, dash, "\mathtt{ua}\ f"] \arrow[d, "f"] & Y \arrow[d, "\mathtt{idEquiv}\ Y"] \\ Y \arrow[r, dash] & Y \end{tikzcd}$$

Input:

- partial equivalence f, type X
- a dimension variable or parameter i

Output:

- $\mathtt{ua}\ f \equiv \mathtt{Glue}\ [(i = 0) \mapsto (X, f), (i = 1) \mapsto (X, \mathtt{idEquiv}\ Y)]\ Y$
- $\mathtt{ua}\ f$ is path with endpoints X and Y.

Applying `ua` from univalence

Example (Monoids)

$$M_1 \equiv (\mathbb{N}, (m,n) \mapsto m+n, 0)$$

and

$$M_2 \equiv (\mathbb{N}_0, (m,n) \mapsto m+n-1, 1)$$

- are isomorphic by
$$\lambda n \to n+1$$
- (path-) equal in CTT

Definition of a ~~monoid~~ magma

setoid encoding uses operator "•" and equivalence "≈":

```
notZero n = Σ ℕ (λ m → (n ≡ (suc m)))
ℕ₀ = Σ ℕ (λ n → notZero n)

op₂ : Op₂ ℕ₀
op₂ (x , p)  (y , q) =
    (predℕ (x + y) , (predℕ (predℕ (x + y)) , sumLem x y

M₂ : Algebra.Magma _ _
M₂ = record {
  Carrier = ℕ₀ ;
  _≈_ = (_≡_) ;
  _•_ = op₂ ;
  isMagma = ... ,
  }
```

Equality of carrier sets

$\mathbb{N} \to \mathbb{N}_0 : n \mapsto n+1$ is bijection

- is equivalence of (set-like) types
- univalence/`ua` returns equality `ℕ ≡ ℕ₀`

```
f : ℕ → ℕ₀
f n = (suc n , ( n , refl ) )
...
fEquiv : ℕ ≃ ℕ₀
fEquiv = (f ,  isoToIsEquiv (iso f g l' r'))

fEq : ℕ ≡ ℕ₀
fEq = ua fEquiv
```

Equality of ~~monoids~~ magmas

Defined for every component of record type:

```
mPath : s₁ ≡ s₂
mPath = λ i → record {
    Carrier = (fEq i) ;
    _≈_ =  _≡_;
    _•_ = transOp' i ;
    isMagma = ...
      }
    }
```

`transOp'` defined by transporting along $\mathbb{N} \equiv \mathbb{N}_0$, proofs can be transported over $s_1 \equiv s_2$.

Higher homotopies of spheres

Types in HoTT and CTT are topological spaces. Higher homotopy groups compute number of higher-dimensional holes in S^n:

	S^0	S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^8
π_1	0	$\mathbb{Z}$	0	0	0	0	0	0	0
π_2	0	0	$\mathbb{Z}$	0	0	0	0	0	0
π_3	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0
π_4	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
π_5	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
π_6	0	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
π_7	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
π_8	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
π_9	0	0	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
π_{10}	0	0	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$

Figure: HoTT Book, 2013

Homotopy groups can be defined in CTT as datatypes

Implementation in CTT

Brunerie, 2016

Theorem
$\pi_4(S^3) \cong \mathbb{Z}_n$ *for* $n = 2$

- proven in HoTT with univalence
- n implemented in CTT as a function
- canonicity predicts termination

(bug in Agda or CTT prevents evaluation)

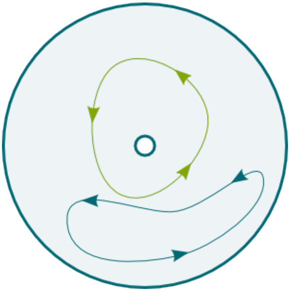

Figure: The case S^1 is simply $\mathbb{Z}$ (drawing from science4all)

Recent papers

Licata, Harper, Cavallo, Orton et al.

- computational type theory is an alternative implementation [AHH18]
- composition operation CTT may not be too strong [CM19]
- modelling $\widehat{\square}$ and `Glue` with language of topoi to simplify CTT and composition operations [Ort19]

Summary

- HoTT redefines equality
- CTT implements HoTT
- HoTT can be verified in computers

For Further Reading I

Thanks for watching!

Carlo Angiuli, Robert Harper, and Kuen-Bang Hou, *Cartesian cubical computational type theory: Constructive reasoning with paths and equalities*, 27$^{\text{th}}$ EACSL Annual Conference on Computer Science Logic (CSL 2018), Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018, Available on `https://www.cs.cmu.edu/~rwh/papers/cartesian/paper.pdf`.

Evan Cavallo and Anders Mörtberg, *A unifying cartesian cubical type theory*, Available on `http://www.cs.cmu.edu/~ecavallo/works/unifying-cartesian.pdf`.

For Further Reading II

Richard Ian Orton, *Cubical models of homotopy type theory-an internal approach*, Ph.D. thesis, University of Cambridge, 2019, Text available at https://www.repository.cam.ac.uk/handle/1810/289441 and code on https://doi.org/10.17863/CAM.35681.

KU LEUVEN