The Constructive Model of Univalence in Cubical Sets

Literature review

W. Vanhulle[1] A. Nuyts[2] D. Devriese[3]

[1]Student

[2]Supervisor

[3]Promoter

45 min. public seminar

Outline

A mathematician is asked by a friend who is a devout Christian:
"Do you believe in one God?"

What does he reply?

A mathematician is asked by a friend who is a devout Christian: “Do you believe in one God?”

What does he reply?

He answers: “Yes – up to isomorphism.” (© Michael Benjamin Stepp)

`figures/isomorphism.jpg`

Figure: Monkeys looking at isomorphic monkeys.

Algebraic Topology

Isomorphism in algebraic topology is "homotopy equivalence"

figures/mug.jpg

Figure: A mug

figures/donut.jpg

Figure: A donut

homotopy equivalent spaces have the same "number of n-dimensional holes"

Holes are homotopy groups

computed by looking at homotopy classes of *continous* embeddings

$$S^n \to X, \quad \text{or } [0,1]^n \to X$$

figures/loops.png

Figure: A donut has two 1-dimensional homotopy classes.

figures/basketball_hollow.png

Figure: A ball without center has two 2-dimensional homotopy classes.

Type theory's origin

Russel, 1907

Invented to prevent paradox:

$$R = \{x \mid x \notin x\}, R \in R \Leftrightarrow \notin R$$

Solution was:

- replace sets (and propositions) by types and elements by terms,

$$x \in R \Rightarrow x : R$$

- types belong to universe hierarchy

$$\exists i, R : \mathscr{U}_i, \quad \mathscr{U}_0 : \mathscr{U}_1 : \dots$$

- constructive logic and formation rules

$x \notin x$ is not a valid proposition anymore

Type theory

Two meanings/subfields:

- verifying computation in programming languages

```
filter :: (a -> Bool) -> [a] -> [a]
filter _pred []    = []
filter pred (x:xs)
  | pred x         = x : filter pred xs
  | otherwise      = filter pred xs
```

Figure: A typed recursive function in Haskell

- alternative constructive foundation of mathematics

```
_∘_ :   (∀ {x} (y : B x) → C y) → (g : (x : A) → B x) →
        ((x : A) → C (g x))
f ∘ g = λ x → f (g x)
```

Figure: Definition of the topological space S^1 in Agda

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Figure: Definition of the topological space S^1 in Agda

Type theory as foundation for mathematics

Deductive system of judgements with typing rules:

$$\frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash b : B}$$

- judgments express wether a type is inhabited
- all judgements have contexts
- typing rules tell how to form and combine types and terms

Equality in type theory

Martin-Löf, 1984

Definitional equality in type theory is for type checking, "denoted =" in code:

```
data Nat : Set where
  zero : Nat
  suc  : (n : Nat) → Nat


_+_ : Nat → Nat → Nat
zero  + m = m
suc n + m = suc (n + m)
```

Figure: An example of definitional equality.

Mathematics needs a "softer" equality as in:

Example (Leibniz's extensionality principle)

$$f = g \Leftrightarrow f(x) = g(x), \forall x$$

Identity eliminator

Martin-Löf, 1984

Definition (Introduction rule)

Given a $a : X$, $\texttt{refl}(a) : a = a$.

Elimination rule of equality type:

Definition (path induction)

Given the following terms:

- a predicate $C : \prod_{x,y:A}(x =_A y) \to \mathscr{U}$
- the base step $c : \prod_{x:A} C(x, x, \texttt{refl}_x)$

there is a function $f : \prod_{x,y:A} \prod_{p:x=_A y} C(x, y, p)$ such that $f(x, x, \texttt{refl}_x) \equiv c(x)$.

- weaker than equality "by definition".
- stronger than equivalence.

Intuition identity eliminator

Role of eliminator:

To prove a property C that depends on terms x, y and equalities $p : x = y$ it suffices to consider all the cases where

- *x is definitionally equal to y*
- *the term of the intensional equality type under consideration is $\texttt{refl}_x : x = x$.*

Implications:

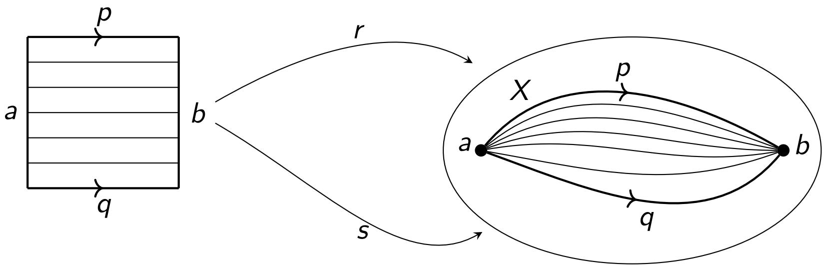
- proves transitivity, symmetry
- equality type can have multiple terms

Homotopy type theory (HoTT)

Awodey, 2006

Gives *homotopy* interpretation to equality type:

$$p, q : a =_X b \quad r, s : p =_{\mathtt{Id}_X(a,b)} q$$



$\Rightarrow$ alternative foundations for mathematics based on type theory and topology

Role of univalence

Voevodsky, 2009

Axiom (Univalence axiom)

Given types $X, Y : \mathscr{U}$ for some universe $\mathscr{U}$, the map $\Phi_{X,Y} : (X = Y) \to (X \simeq Y)$ is an equivalence of types.

- equivalence of types is a bijection for set-like types

$$\mathbb{N} \simeq \mathbb{N}_0$$

- univalence implies

$$\mathbb{N} \simeq \mathbb{N}_0 \Rightarrow \mathbb{N} = \mathbb{N}$$

- forces multiple terms of equality

$\Rightarrow$ *terms of equality are like paths*

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Consequences of path interpretation

in general:

- equivalence behaves like homotopy equivalence
- mathematics "up to homotopy"
- lifting of path p as in algebraic topology: *transport* gives the other ending point of $p_{\star}$

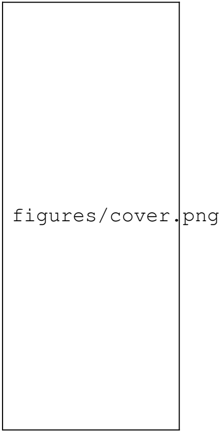

Figure: Jeff Erickson, 2009

Origin univalence

Grayson, 2018

figures/voevodsky.jpg

Figure: Vladimir Voevodsky (1966 - 2017)

Why is it called "univalent"?

> *... these foundations seem to be faithful to the way in which I think about mathematical objects in my head ...*

faithful = univalent in a Russion translation of Boardman (2006)

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Figure: Vladimir Voevodsky (1966 - 2017)

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Personal remark

The univalence axiom adds:

- intuitive explanation of equaity
- alternative foundations with types
- field of mathematics: HoTT

But does not:

- make all proofs easier or shorter
- eliminate proofs of equivalence

Computing with univalence

Huber, 2015

What about:

- implementing HoTT?
- calculations with very simple types as $\mathbb{N}$?

Can we, given a term $t : \mathbb{N}$ constructed using the univalence axiom, construct two terms $u : \mathbb{N}$ and $p : t =_{\mathbb{N}} u$ such that u does not involve the univalence axiom?

$\Rightarrow$ canonicity of $\mathbb{N}$ in cubical type theory (CTT)

$$t \equiv ua(...) \rightsquigarrow u \equiv S(\ldots(0)\ldots) : \mathbb{N}$$

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Cubical type theory

Cohen et al., 2015

A constructive extension of HoTT with dimension variables $i, j, k : \mathbb{I}$ (cubes) as primitives:

`figures/cubes.png`

Figure: Discrete "n-cubes". Huber (2016)

- univalence becomes *constructable*
- computational interpretation for univalence

Are cubes a good idea?

EnigmaChord, 2016

figures/cube-joke.jpg

Figure:

(yes, they model n-dimensional homotopies)

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Cubes model homotopy

Altenkirch, Brunerie, Licata, et. al 2013

Homotopy groups are defined as equivalence classes of *continous* embeddings:

$$[0,1]^n \to X$$

> *In HoTT, higher-dimensional eqalities behave like these embeddings*

Level	Types	Cubes	Topology
1	$p,q:(a=b)$	edge	line
2	$r,s:(p=q)$	face	path homotopy
...	...	...	\dots
n	...	n-hypercube	n-dimensional homotopy

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Operations on cubes

Bezem, Coquand, Huber et al., 2013

figures/kan.jpg

Figure: Daniel Kan (1927 — 2013)

Necessary for modelling HoTT:

- composition $\Rightarrow$ equality type
- glueing $\Rightarrow$ univalence

figures/extension.png

Presheaf model on $\mathscr{C}$

Dybjer, 1994

Give interpretation for stuff in type theory by modelling every context Γ as as presheaf.

- verify consistency of type theory in sets
- justify primitives for implementations

Presheaves on $\mathscr{C}$

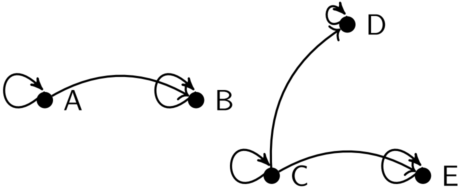
Hofstra, 2014

Maps (contravariant functors) $\mathscr{C} \to \mathbf{Set}$ denoted by $\hat{\mathscr{C}}$

- generalize sheaves (see sheafication)
- model type theories, Dybjer (1994)

Example (Reflexive directed graph)

Take $\mathscr{C} = \{0, 1\}$ and $\mathrm{Hom}_{\mathscr{C}} = \{B, E, R\}$



Types in presheaves

Lemma (Types in a presheaf model)

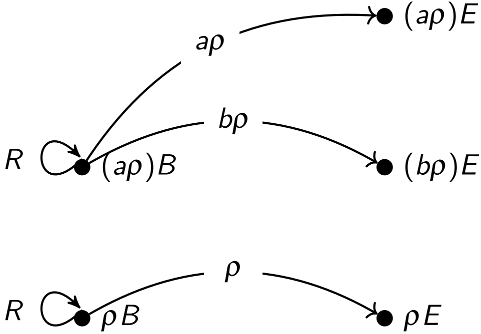
If $\Gamma \in \widehat{\mathscr{C}}$ a context, then the types are
$\left\{(\Delta,\sigma) \mid \Delta \in \widehat{\mathscr{C}}, \sigma \in Hom_{Ctx}(\Delta,\Gamma)\right\}$.

Helps to characterize types without using presheaves explicitly.

Types in a simple presheaf model

Example (Dependent directed reflexive graph)

A type A in $\widehat{\{0,1\}}$:



CTT as presheaf model

dimension variables and cubes can also be modelled with a presheaf model:

- proves consistency of CTT and HoTT
- justifies primitives used in implementations

Distributive lattice

$\mathbb{A}$: countable set of "dimension variables"
CTT built with presheaves on "cube" category $\square$:

- objects: $\{I \mid |I| < \infty, I \subset \mathbb{A}\}$
- morphisms $J \to I$: maps $I \mapsto dM(J)$
  - distributive lattice
  - $x \wedge 0 = 0, x \vee 1 = 1$
  - $\neg 0 = 1$ and $\neg 1 = 0$

figures/lattice.png

Figure: A simple lattice

Cubical contexts

Example (presheaf model on $\square$)

A presheaf $\Gamma \in \widehat{\square}$ is a functor $\square \to \mathbf{Set}$

- $u \in \Gamma(i, j)$ is a square
- morphisms in lattice $dM(i, j)$ give corners of a square

figures/context.png

Figure: a context $\Gamma \in \widehat{\square}$ applied to $\{i, j\}$

Types in $\widehat{\square}$

Type A is presheaf $\widehat{\int_{\mathscr{C}} \Gamma}$, a functor $\int_{\mathscr{C}} \Gamma \to \mathbf{Set}$:

- $\rho \in \Gamma(i)$ is a line
- endpoints $\rho(0), \rho(1)$ lifted to $u(0), u(1) \in A(i, \rho)$

`figures/types.png`

Figure: A type A within context Γ. Huber (2016)

Types in presheaf models

In the presheaf model on $\square$:

- types more complicated
- types no longer simply nested graphs

Interpreting types in presheaf model $\square$ hard but possible!

Path type

Bezem, Coquand, 2013

Syntactical definition of `Path` type with typing rules:

$$\frac{i : \mathbb{I} \vdash t : A \qquad i : \mathbb{I} \vdash t(i/0) = a : A \qquad i : \mathbb{I} \vdash t(i/1) = b : A}{() \vdash \langle i \rangle\, t : \texttt{Path}\ a\ b}$$

- almost models equality type
- not necessarily transitive $\Rightarrow$ composition operation

$$\begin{tikzcd}
a \arrow[r, dashed] & c \\
a \arrow[u, "refl"] \arrow[r, "p\ i"] & b \arrow[u, "q\ j"']
\end{tikzcd}$$

Figure: Transitivity can be proven with composition operation.

Constructive model of type theory

Other types can be interpreted in the presheaf model $\widehat{\square}$ for constructive model for type theory:

- product, sum types
- natural numbers

Univalence proven with:

- concepts from simplicial set model (Streicher, Voevodsky, Kapulkin et al. 2006 – 2012)
- partial types and glueing construction

figures/simplex.png

Figure: Related concept of a simplicial complex

Partial types

Bezem, Coquand, Huber 2013

Partial types only defined on subpolyhedra of cubes.

- u 1-dimensional cube, line and type A:
- partial type $A(i/0)$ in type A:

figures/types_side.png

Figure: Huber, 2015

- red $= A\ [(i = 0) \mapsto A(i/0)]$ (syntax)

Glueing equivalences

Kapulkin, 2012

Equivalences $f\colon T \to A$ are crucial ingredient for the univalence axiom:

- correspond to homotopy equivalences
- have inverse up-to some paths

`Glue` type was introduced to prove univalence:

- let T be a partial type, defined on $i \in \{0,1\}$
- glues partial equivalence f and partial type T together:

figures/glue.png

Proving univalence

Cohen, Coquand, Huber, Moertberg (2015)

Axiom (Univalence axiom)

Given types $X, Y : \mathscr{U}$ for some universe $\mathscr{U}$, the map $\Phi_{X,Y} : (X = Y) \to (X \simeq Y)$ is an equivalence of types.

Proof.

- The existence of a map $\texttt{ua} : (X \simeq Y) \to (X = Y)$ proven with $\texttt{Glue}$ construction:

  $$i : \mathbb{I} \vdash E = \texttt{Glue}\,[(i = 0) \mapsto (X, f), (i = 1) \mapsto (Y, \texttt{id}_Y)]\,Y$$

  E is a path (equality) from X to Y.
- Remainder proven with "contractibility of singletons".

$\square$

Applying `ua` from univalence

Example (Monoids)

$$M_1 \equiv (\mathbb{N}, (m,n) \mapsto m+n, 0)$$

and

$$M_2 \equiv (\mathbb{N}_0, (m,n) \mapsto m+n-1, 1)$$

- are isomorphic by
$$\lambda\, n \to n+1$$
- (path-) equal in CTT

Definition of a ~~monoid~~ magma

setoid encoding uses operator "•" and equivalence "≈":

```
notZero n = Σ ℕ (λ m → (n ≡ (suc m)))
ℕ₀ = Σ ℕ (λ n → notZero n)

op₂ : Op₂ ℕ₀
op₂ (x , p)  (y , q) =
    (predℕ (x + y) , (predℕ (predℕ (x + y)) , sumLem x y p q) )

M₂ : Algebra.Magma _ _
M₂ = record {
  Carrier = ℕ₀ ;
  _≈_ = (_≡_) ;
  _•_ = op₂ ;
  isMagma = ... ,
  }
```

Equality of carrier sets

$\mathbb{N} \to \mathbb{N}_0 : n \mapsto n+1$ is bijection

- is equivalence of (set-like) types
- univalence/`ua` returns equality `ℕ ≡ ℕ₀`

```
f : ℕ → ℕ₀
f n = (suc n , ( n , refl ) )
...
fEquiv : ℕ ≃ ℕ₀
fEquiv = (f ,  isoToIsEquiv (iso f g l' r'))

fEq : ℕ ≡ ℕ₀
fEq i = ua fEquiv i
```

Equality of ~~monoids~~ magmas

Defined for every component of record type:

```
mPath : s₁ ≡ s₂
mPath = λ i → record {
   Carrier = (fEq i) ;
   _≈_ =  _≡_;
   _•_ = transOp' i ;
   isMagma = record {
    isEquivalence = ≡equiv  ;
     •-cong   = ?
     }
   }
```

`transOp'` defined by transporting along `ℕ ≡ ℕ₀`, proofs can be transported over `s₁ ≡ s₂`.

Higher homotopies of spheres

Types in HoTT and CTT are topological spaces. Higher homotopy groups compute number of higher-dimensional holes in S^n:

figures/groups.png

Figure: HoTT Book, 2013

Homotopy groups can be defined in CTT as datatypes

Implementation in CTT

Brunerie, 2016

Theorem

$\pi_4(S^3) \cong \mathbb{Z}_n$ for $n = 2$

- proven in HoTT with univalence
- n implemented in CTT as a function
- canonicity predicts termination

(bug in Agda or CTT prevents evaluation)

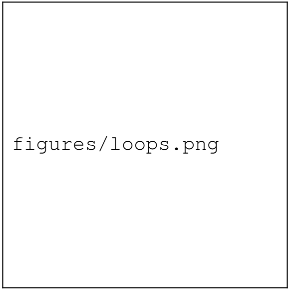


Figure: The case S^1 is simply $\mathbb{Z}$ (drawing from science4all)

Other research

Licata, Harper, Cavallo, Orton et al., 2018

- computational type theory is an alternative implementation
- composition operation may not be necessary
- alternatives to complicated glue types: fundamental axioms and language of topoi

Summary

- HoTT redefines equality
- CTT implements HoTT
- HoTT can be verified in computers

Thanks for watching!

For Further Reading I