The Constructive Model of Univalence in Cubical Sets

Literature review

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45 min. public seminar

Outline

Topology

Type Theory

Cubical model

Applications

Topology

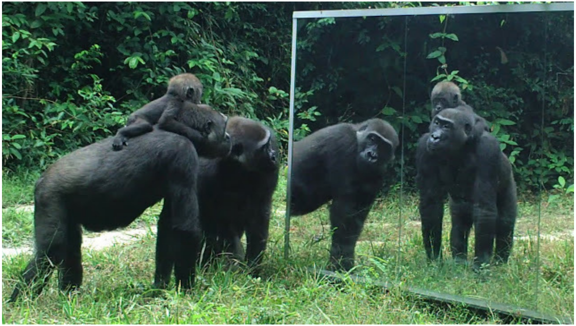

Figure: Hubert-Brierre, 2013

- is the mirror monkey really another monkey?
- which properties does he (not) satisfy?

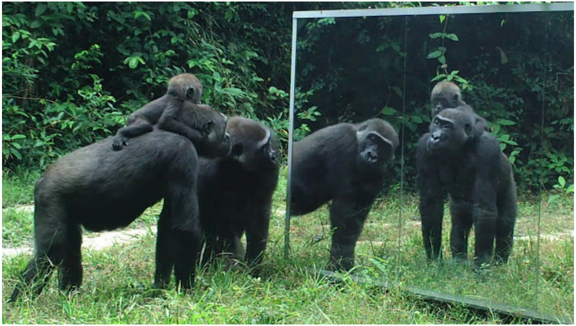

Figure: Hubert-Brierre, 2013

They are the same.

Mathematicians are monkeys

Take two similar objects A and B:

- Which properties does object B satisfy as well?
- Are objects A and B really different?

What is the sameness?

Definition

An isomorphism is a map that identifies *spaces and their structure*.

Notation	Space type	Isomorphisms
$\mathbb{E}^3$	Euclidean	rotation, translation, mirroring
Fin_n	Finite	permutations
G	Groups	homomorphisms
X	Topological	homotopy equivalences

Figure: Examples of isomorphisms

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Isomorphisms in mathematics

Used for classification of different but similar objects.

> *A mathematician is asked by a friend who is a devout Christian: "Do you believe in one God?"*

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Isomorphisms in mathematics

Used for classification of different but similar objects.

> *A mathematician is asked by a friend who is a devout Christian: "Do you believe in one God?"*
>
> *He answers: "Yes – up to isomorphism." (© Michael Benjamin Stepp)*

Homotopy equivalence

Figure: A mug

Figure: A donut

Definition (*homotopy equivalent*)

Can be smoothly deformed into eachother.

What does a donut think when he sees a mug?

What does a donut think when he sees a mug?

He has the same number of holes.

Definition (homotopy groups)

Number of n-dimensional holes

Computing holes

homotopy classes of smooth embeddings

$$S^n \to X, \quad \text{or } [0,1]^n \to X$$

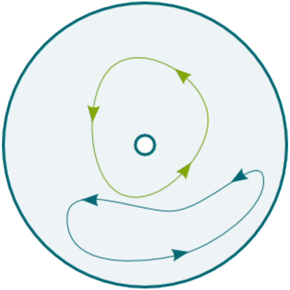

Figure: A donut has two 1-dimensional homotopy classes.

Figure: A ball without center has two 2-dimensional homotopy classes.

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Type Theory

Paradox in set theory

Zermelo, 1899

The set that contains all sets that do not contain itself:

$$R = \{x \mid x \notin x\}, R \in R \Leftrightarrow R \notin R$$

Weird set

Paradox in set theory

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Weird set
Paradox eliminated with:

- Universe hierarchy:

$$\mathscr{U}_0 \in \mathscr{U}_1 \in \ldots \mathscr{U}_\omega \in \ldots$$

- Rejection of “set comprehension principle”:

$$S = \{x \mid P(x)\}$$

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Constructivism

Brouwer, 1905

Mathematics is in essence the result of pure thought.

Evolved into constructive logic (Heyting, 1930)

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Constructivism

Brouwer, 1905

Mathematics is in essence the result of pure thought.

Evolved into constructive logic (Heyting, 1930)
Type theory (Russel, 1907 – ...):

- replace sets (and propositions) by types and elements by terms,

$$x \in R \Rightarrow x : R$$

- set universes become type universes
- constructive logic replaced by formation rules of terms

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Type theory as foundation for mathematics

Deductive system of judgements with typing rules:

$$\frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash b : B}$$

- judgments express that a type is inhabited
- all judgements have contexts
- typing rules tell how to form and combine types and terms

Definition (Type-checking)

Checking if the typing rules are respected.

Definitional equality

Definitional equality in type theory ("denoted =" in code):

```
data Nat : Set where
  zero : Nat
  suc  : (n : Nat) → Nat

_+_ : Nat → Nat → Nat
zero  + m = m
suc n + m = suc (n + m)
```

Figure: An example of definitional equality.

- used for stating terms, terms and typing rules
- defined such that type-checking decidable

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Problems definitional equality

Definitional equality distinguishes `0 + m` and `m + 0`: too *strong* for mathematics.

A weaker alternative?

Example (Leibniz's extensionality axiom)

Functions are identified with values:

$$f = g \Leftrightarrow f(x) = g(x), \forall x$$

Breaks decidability of type-checking $\Rightarrow$ not possible with definitional equality.

Identity type

Martin-Löf, 1984

A type of equality between terms $a, b : X$, denoted $a = b$. Terms $p : (a = b)$ are called "equalities".

Definition (Introduction rule)

Given a term $a : X$, there is an equality $\texttt{refl}(a) : a = a$.

Elimination rule of equality type:

Definition (Induction rule)

Given the following terms:

- a predicate $C : \prod_{x,y:A}(x =_A y) \to \mathscr{U}$
- the base step $c : \prod_{x:A} C(x, x, \texttt{refl}_x)$

there is a function $f : \prod_{x,y:A} \prod_{p:x=_A y} C(x, y, p)$ such that $f(x, x, \texttt{refl}_x) \equiv c(x)$.

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Role identity eliminator

To prove a property C that depends on terms x, y and equalities $p : x = y$ it suffices to consider all the cases where

- x is definitionally equal to y
- the term of the intensional equality type under consideration is $\mathtt{refl}_x : x = x$.

Implications:

- proves transitivity, symmetry
- less things equal $\Rightarrow$ weaker than equality "by definition".

Univalence axiom

Voevodsky, 2009

Definition (Type equivalence)

Given types $X, Y : \mathscr{U}$ for some universe $\mathscr{U}$, an equivalence $f : X \simeq Y$ of types is a map $f : X \to Y$ that is a bijection up to equality.

> *Equivalences are isomorphisms between topological spaces.*

Axiom (Univalence axiom)

Given types $X, Y : \mathscr{U}$ for some universe $\mathscr{U}$, the map $\Phi_{X,Y} : (X = Y) \to (X \simeq Y)$ is an equivalence of types.

> *Equivalences are (up to homotopy equivalence) the same as equalities.*

Type theory up to isomorphism (homotopy type equivalence)

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Consequences of univalence

Example (Natural numbers)

$\mathbb{N}$ is a type that behaves like a set.

- equivalences

$$\mathbb{N} \simeq \mathbb{N}_0$$

  are bijections

$$\mathbb{N} \leftrightarrow \mathbb{N}_0$$

- univalence implies

$$p : (\mathbb{N} \leftrightarrow \mathbb{N}_0) \Rightarrow p : (\mathbb{N} = \mathbb{N})$$

- forces multiple equalities $\mathbb{N} = \mathbb{N}_0$

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- forces multiple equalities $\mathbb{N} = \mathbb{N}_0$

  $\Rightarrow$ *terms of equality are paths*

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Consequences of path interpretation

A way to construct paths in topology:

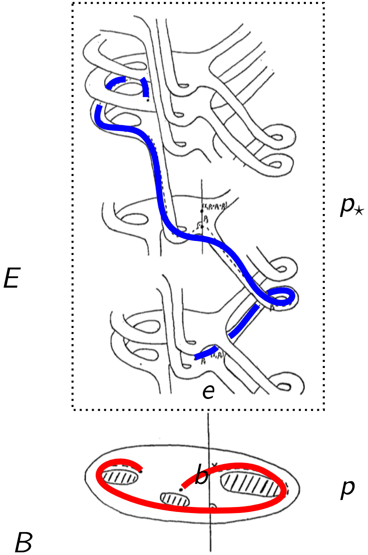
Definition (Covering)

A surjective smooth map $\pi : E \to B$ that is locally homeomorphic.

Defining transport:

- take path p in base space B and point b in $\pi^{-1}(p(1))$
- path p is lifted to path $p_\star$ ending in b
- *transport* gives start $p_\star(0)$

More paths means more equalities $\Rightarrow$ identity type is indeed *weak*.

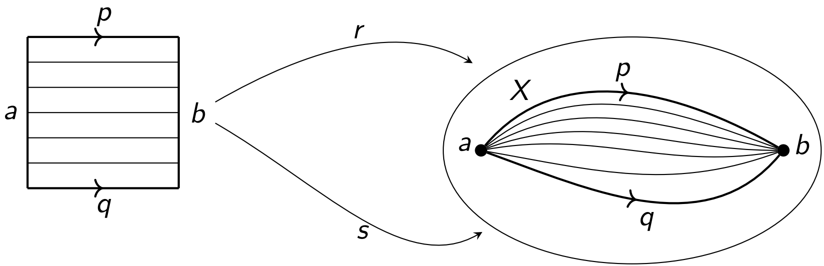


Homotopy type theory (HoTT)

Awodey, 2006

Gives *homotopy* interpretation to equality type:

$$p, q : a =_X b \quad r, s : p =_{\mathtt{Id}_X(a,b)} q$$



$\Rightarrow$ alternative foundation for mathematics based on type theory and topology

Origin univalence
Grayson, 2018

Figure: Voevodsky (1966 - 2017)

Definition (Univalent type theory)

Type theory + univalence axiom (also *homotopy type theory*)

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Figure: Voevodsky (1966 - 2017)

Definition (Univalent type theory)

Type theory + univalence axiom (also *homotopy type theory*)

> *... these foundations seem to be faithful to the way in which I think about mathematical objects in my head ...*

faithful = univalent in a Russian translation of Boardman (2006)

Practical limitations of univalence

The univalence axiom adds:

- intuitive explanation of equality
- a definition of equivalence and connection with equality
- field of mathematics: HoTT

But does not:

- make all proofs in mathematics easier or shorter
- eliminate need for proofs of equivalence

Computing with univalence

Huber, 2015

Question posed in 2013:

> *Can the univalence axiom be implemented in computers such that*
>
> - *equalities are really paths*
> - *calculations with very simple types as $\mathbb{N}$ still work?*

Canonicity of $\mathbb{N}$ in cubical type theory:

$$t \equiv ua(...) \rightsquigarrow u \equiv S(\dots(0)\dots) : \mathbb{N}$$

Cubical model

Cubical type theory

Cohen et al., 2015

A constructive extension of HoTT with dimension variables $i, j, k : \mathbb{I}$ (cubes) as primitives:

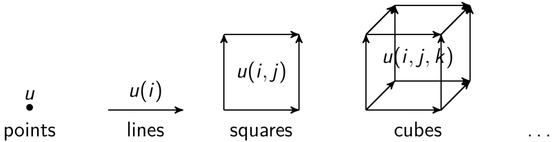

Figure: Discrete "*n*-cubes". Huber (2016)

- univalence becomes *constructable*
- computational interpretation for univalence

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Are cubes a good idea?

EnigmaChord, 2016

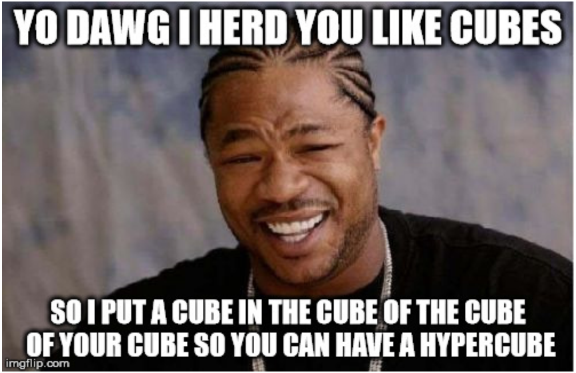

Figure:

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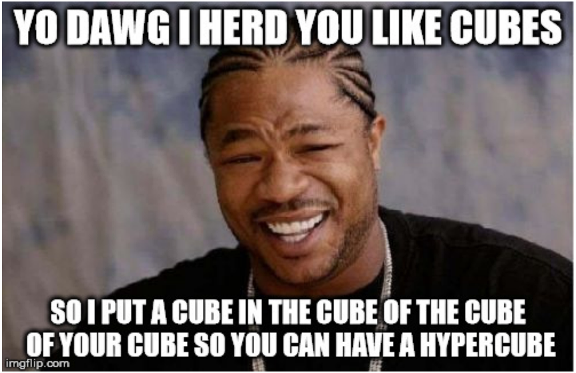

Figure:

Cubes model homotopy

Altenkirch, Brunerie, Licata, et. al 2013

Homotopy groups are defined as equivalence classes of smooth embeddings:

$$[0,1]^n \to X$$

In HoTT, higher-dimensional eqalities behave like these embeddings

Cubes model homotopy

Altenkirch, Brunerie, Licata, et. al 2013

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$$[0,1]^n \to X$$

In HoTT, higher-dimensional eqalities behave like these embeddings

Level	Types	Cubes	Topology
1	$p, q : (a = b)$	edge	line
2	$r, s : (p = q)$	face	path homotopy
...	...	...	...
n	...	n-hypercube	n-dimensional homotopy

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Operations on cubes

Bezem, Coquand, Huber et al., 2013

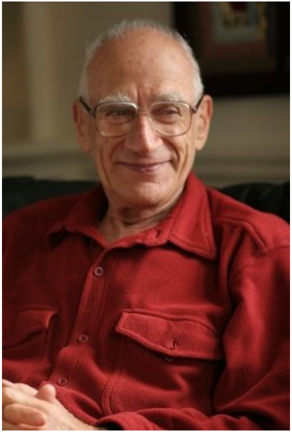
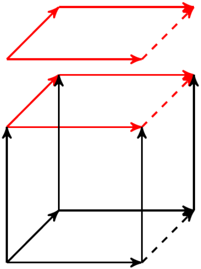


Figure: Daniel Kan (1927 — 2013)

Necessary for modelling HoTT:

- composition ⇒ equality type
- glueing ⇒ univalence



Presheaf model on $\mathscr{C}$

Dybjer, 1994

Give interpretation for stuff in type theory:

- base category $\mathscr{C}$ contains "extra tools" for model
- every context Γ is modelled as presheaf on $\mathscr{C}$, denoted $\widehat{\mathscr{C}}$.
- types and terms also interpreted in $\widehat{\mathscr{C}}$

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Goals:

- verify consistency of type theory in sets
- justify primitives for implementations

Denoted as "presheaf model $\widehat{\mathscr{C}}$".

Contexts in presheaf model $\widehat{\mathscr{C}}$

Definition (Presheaves $\widehat{\mathscr{C}}$)

Contravariant functors $\mathscr{C} \to \mathbf{Set}$

- generalize sheaves (see sheafication)
- model contexts

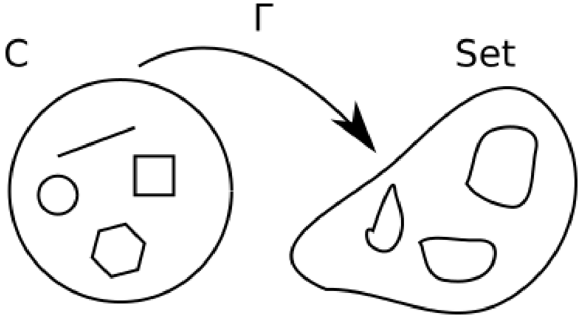

Figure: A representation of a preseheaf

Contexts in presheaf model $\widehat{\{0,1\}}$

Hofstra, 2014

Example (Reflexive directed graph)

Take $\mathscr{C} = \{0,1\}$ and $\mathrm{Hom}_{\mathscr{C}} = \{B, E, R\}$, $\Gamma \in \widehat{\{0,1\}}$, then
Applying functorial identities:

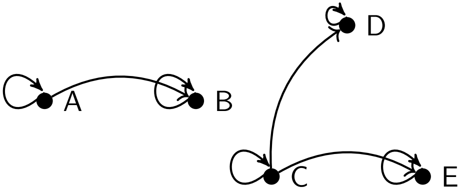

Figure: Reflexive graph

Types in presheaf model $\widehat{\mathscr{C}}$

Lemma (Types in a presheaf model)

If $\Gamma \in \widehat{\mathscr{C}}$ a context, then the types are
$\left\{(\Delta, \sigma) \mid \Delta \in \widehat{\mathscr{C}}, \sigma \in Hom_{Ctx}(\Delta, \Gamma)\right\}$.

Helps to characterize types without using presheaves explicitly.

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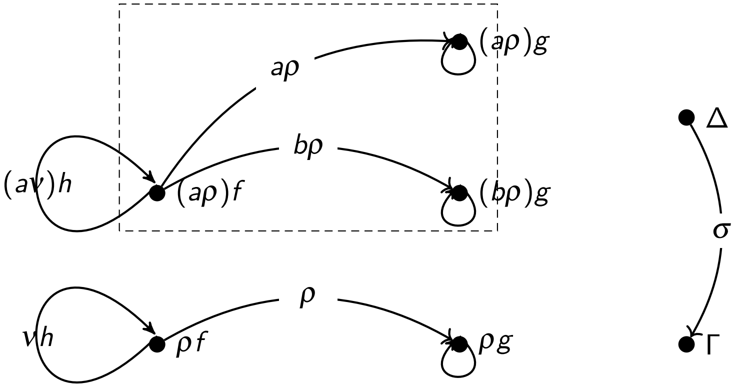
Types in a presheaf model $\widehat{\{0,1\}}$

Example (Dependent directed reflexive graph)

Applying previous lemma to the type A in $\widehat{\{0,1\}}$:



CTT as a presheaf model

Dimension variables and cubes have an abstract representation as "hypercubes" in a base category:

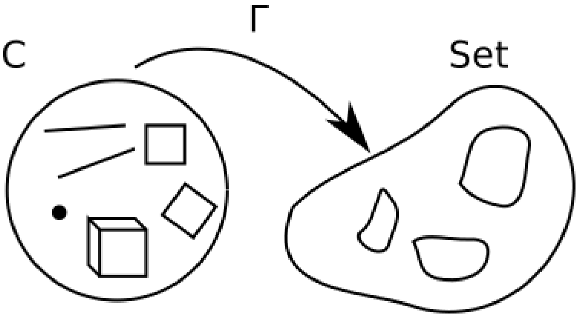


Figure: Presheaf acting on cubes

- proves consistency of CTT and HoTT
- justifies primitives used in implementations

Cube category

Definition ("Cube" $\square$)

Category with:

- objects: $\{I \mid |I| < \infty, I \subset \mathbb{A}\}$
- morphisms $J \to I$: maps $I \mapsto dM(J)$
  - distributive lattice
  - $x \wedge 0 = 0, x \vee 1 = 1$
  - $\neg 0 = 1$ and $\neg 1 = 0$

$\mathbb{A}$: countable set of "dimension variables"

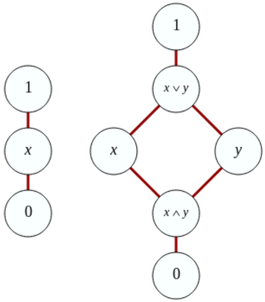


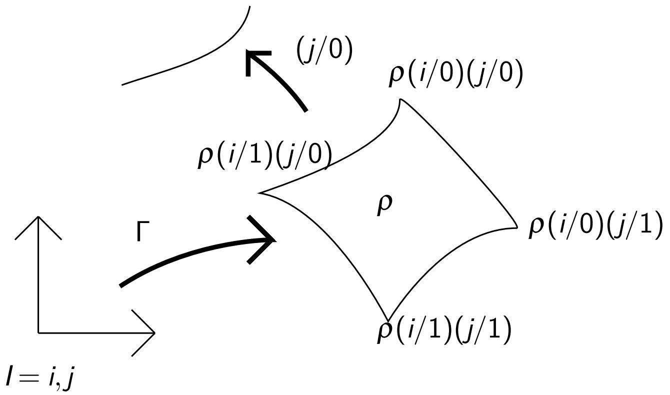
Figure: A simple lattice

Contexts in presheaf model $\widehat{\square}$

Example (Cubical contexts ("cubical sets"))

A presheaf $\Gamma \in \widehat{\square}$ is a functor $\square \to \mathbf{Set}$

- $\Gamma \in \widehat{\square}$ applied to $\{i, j\}$ gives square $\rho \in \Gamma(i, j)$
- morphisms in lattice $dM(i, j)$ give corners of ρ



Types in presheaf model $\widehat{\square}$

Type A can be represented by a context and a morphism on top of Γ:

- the $\rho \in \Gamma(i)$ is an edge of a square
- endpoints $\rho(0), \rho(1)$ can be lifted to points in the type $u(0), u(1) \in A(i, \rho)$

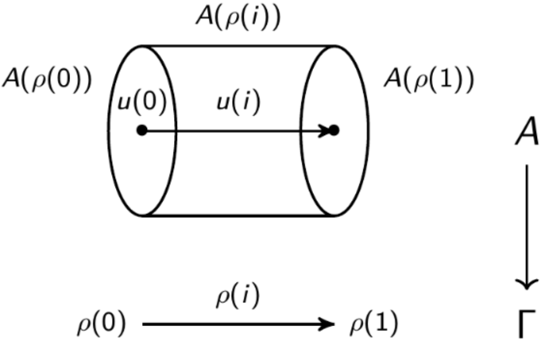


Figure: A type A within context Γ. Huber (2016)

Partial types in $\widehat{\square}$

Bezem, Coquand, Huber 2013

Partial type $A(i/0)$ is a subtype of $A\ [(i=0) \mapsto A(i/0)]$ on top of sub-subpolyhedron of cubes $(i=0) \vee (i=1)$:

- A behaves like a new context Δ by lemma
- partial type $A(i/0)$ is the left side of A:

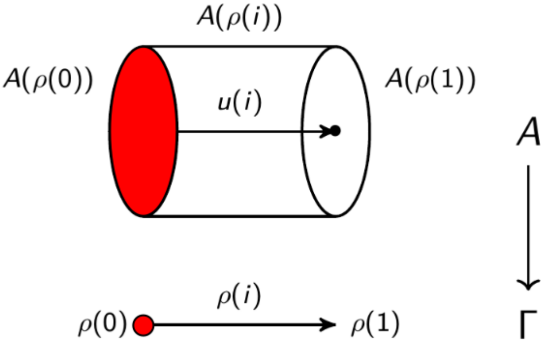


Figure: Huber, 2015

Other types in presheaf model $\widehat{\square}$

In the presheaf model on $\square$:

- types more complicated
- types no longer simply nested graphs

Interpreting types in presheaf model $\square$ hard but possible.

... transition from interpretation in model to syntax of types

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Path type

Bezem, Coquand, 2013

Syntactical definition of `Path` type with typing rules:

$$\frac{i : \mathbb{I} \vdash t : A \qquad i : \mathbb{I} \vdash t(i/0) = a : A \qquad i : \mathbb{I} \vdash t(i/1) = b : A}{() \vdash \langle i \rangle\, t : \texttt{Path}\ a\ b}$$

- almost models equality type
- not necessarily transitive $\Rightarrow$ composition operation

$$\begin{tikzcd}
a \arrow[r, dashed] & c \\
a \arrow[u, "refl"] \arrow[r, "p\ i"] & b \arrow[u, "q\ j"]
\end{tikzcd}$$

Figure: Transitivity can be proven with composition operation.

CTT as extension for HoTT

Other HoTT types interpreted in CTT:

- product, sum types
- natural numbers

Univalence proven with concepts from (Streicher, Voevodsky, Kapulkin et al. 2006 – 2012):

- simplicial sets replaced by cubical sets
- partial types and glueing construction conserved

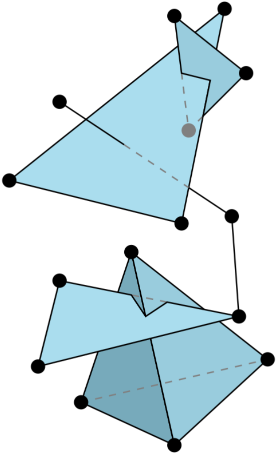

Figure: Every simplicial complex is a simplicial set

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Proving univalence

Cohen, Coquand, Huber, Moertberg (2015)

Axiom (Univalence axiom)

Given types $X, Y : \mathscr{U}$ for some universe $\mathscr{U}$, there is a map $\Phi_{X,Y} : (X = Y) \to (X \simeq Y)$ that is an equivalence of types.

Proof.

1. construction inverse map `ua` $: (X \simeq Y) \to (X = Y)$ with `Glue` construction.
2. for any partial $f : X \to Y$, `Glue` $[\varphi \mapsto (X, f)]\ Y \simeq Y$
3. for any type Y, $\sum_X X \simeq Y$ contractible
4. existence of induction principle for $X \simeq Y$.
5. the trivial map $p : X = Y \to X \simeq Y$ is an inverse "up-to-path" of `ua`

$\Rightarrow$ the map `ua` $\equiv \Phi^{-1}_{X,Y}$ is proper inverse and $\Phi_{X,Y}$ equivalence $\square$

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Constructing `ua`

Definition

$\mathtt{ua} : \forall\ X\ Y : \mathscr{U}, X \simeq Y \to X = Y$

$$
\begin{tikzcd}
X \arrow[r, dash, dashed, "\mathtt{ua}\ f"] \arrow[d, "f"] & Y \arrow[d, "\mathtt{idEquiv}\ Y"] \\
Y \arrow[r, dash] & Y
\end{tikzcd}
$$

Input:

- partial equivalence f, type X
- a dimension variable or parameter i

Output:

- $\mathtt{ua}\ f \equiv \mathtt{Glue}\ [(i=0) \mapsto (X, f), (i=1) \mapsto (X, \mathtt{idEquiv}\ Y)]\ Y$
- $\mathtt{ua}\ f$ is path with endpoints X and Y.

Applications

Applying `ua` from univalence

Example (Monoids)

$$M_1 \equiv (\mathbb{N}, (m,n) \mapsto m+n, 0)$$

and

$$M_2 \equiv (\mathbb{N}_0, (m,n) \mapsto m+n-1, 1)$$

- are isomorphic by
$$\lambda n \to n+1$$
- (path-) equal in CTT

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Encoding the structures in Agda

```
notZero n = Σ ℕ (λ m → (n ≡ (m + 1)))
ℕ₀ = Σ ℕ (λ n → notZero n)
```

Figure: Definition of $\mathbb{N}_0$ with sum type.

```
M₂ : Algebra.Magma _ _
M₂ = record {
  Carrier = ℕ₀ ;
  _≈_ = (_≡_) ;
  _•_ = op₂ ;
  isMagma = ... ,
  }
```

Figure: Problem reduced to magmas.

Equality of sets $\mathbb{N} = \mathbb{N}_0$

```
f : ℕ → ℕ₀
f n = (suc n , ( n , refl ) )
```

Figure: Bijections are equivalences for (set-like) types

univalence/`ua` returns equality `ℕ ≡ ℕ₀`

```
fEquiv : ℕ ≃ ℕ₀
fEquiv = (f ,  isoToIsEquiv (iso f g l' r'))

fEq : ℕ ≡ ℕ₀
fEq = ua fEquiv
```

Figure: The equality

Equality of structures $M_1 = M_2$

If $M : M_1 = M_2$, then $\forall i \in [0,1]$, $M(i)$ magma $\Rightarrow$ point-wise definition:

- carrier set $M(i)$ given by `fEq i`
- operator $op_i : (m,n) \mapsto m +_i n$ on $M(i)$ is defined with transport of arguments m, n from `fEq i` to $\mathbb{N}$

proofs of magma properties can be transported over M: commutativity, etc.

HoTT/univalence/`ua` *gives topological interpretation to invariance of algebraic properties ...*

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Topological transport of arguments

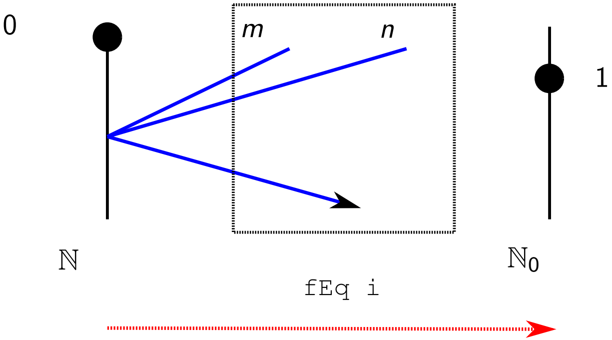

Figure: Transport of the arguments

Conclusion

Licata, Harper, Cavallo, Orton et al.

- HoTT redefines equality
- CTT implements HoTT
- HoTT gives formulism for equivalent structures.

Other introductions to cubical type theory: [Hub16] and [Ort19]
Recent interesting publications:

- computational type theory is an alternative implementation [AHH18]
- composition operation CTT may not be too strong [CM19]
- modelling $\widehat{\square}$ and `Glue` with language of topoi or other axioms to simplify CTT and composition operations [OP17], [Ort19]

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For Further Reading I

Thanks for watching!

Carlo Angiuli, Robert Harper, and Kuen-Bang Hou, *Cartesian cubical computational type theory: Constructive reasoning with paths and equalities*, 27th EACSL Annual Conference on Computer Science Logic (CSL 2018), Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018, Available on `https://www.cs.cmu.edu/~rwh/papers/cartesian/paper.pdf`.

Evan Cavallo and Anders Mörtberg, *A unifying cartesian cubical type theory*, Available on `http://www.cs.cmu.edu/~ecavallo/works/unifying-cartesian.pdf`.

Simon Huber, *Cubical interpretations of type theory*, Ph.D. thesis, University of Gothenburg, Gothenburg, Sweden, November 2016, Available on `http://www.cse.chalmers.se/~simonhu/misc/thesis.pdf`.

For Further Reading II

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