Assignment 11, Authomata Theory

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Contents

1 Problems **2**

1.1 Problem 1 2

1.1.1 Answer 1 3

1.1.2 Answer 2 3

1.2 Problem 2 3

1.2.1 Answer 3 3

1.2.2 Anser 4 4

1.3 Problem 3 4

1.3.1 Answer 5 5

1.3.2 Answer 6 5

1 Problems

1.1 Problem 1

Given following languages over the alphabet $\{a, b\}$

- $L_1 = \emptyset$.
- $L_2 = \{\epsilon, aa\}$.
- $L_3 = \{\epsilon, a, aa, ab, abb\}$.
- $L_4 = \{aabb, aabbb, aa, aaa\}$.
- $L_5 = \{\epsilon, b, bbb, abab, abba, aabb\}$.
- $L_6 = \{\epsilon, bbbaa, baba, aaab, aabba, aa\}$.

1. What are the following languages:
   - $L_4 L_4$.
   - $(L_1 \cup L_2 \cup L_3)^R$.
   - $L_3 L_1 L_6$.
2. Define exponentiation as follows: $L^K = \{x \in L \mid \exists y \in K.(|y| = |x|)\}$. What are the languages $L_4^{L_5}$ and $L_6^{L_1}$.

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:- use_module(library(lists)).

concatentated_member(L1, L2, L3) :-
    member(M1, L1), member(M2, L2),
    string_concat(M1, M2, L3).

concatentated(L1, L2, L3) :-
    findall(X, concatentated_member(L1, L2, X), X),
    list_to_set(X, L3).

assignment_11a :-
    X = ["aabb", "aabbb", "aa", "aaa"],
    concatentated(X, X, Y),
    [First | Rest] = Y,
    write("$$\\{"),
    write(First),
    maplist(format(',\\allowbreak ~s'), Rest),
    write("\\}$$").
```

$\{aabbaabb, aabbaabbb, aabbaa, aabbaaa, aabbbaabb, aabbbaabbb, aabbbaa, aabbbaaa, aaaabb, aaaabbb, aaaa, aaaaa, aaaaabb, aaaaabbb, aaaaaa\}$

1.1.1 Answer 1

1. Concatenation of L_4 with itself gives: $L_4L_4 = \{aabbaabb, aabbaabbb, aabbaa, aabbaaa, aabbbaabb, aabbbaa, aabbbaaa, aaaabbb, aaaa, aaaaa, aaaaabb, aaaaabbb\}$
2. $(L_1 \cup L_2 \cup L_3)^R = \{\epsilon, a, aa, ba, bba\}$.
3. $L_3L_1L_6 = \emptyset$. This is so because there are no words in language L_1 to concatenate with.

1.1.2 Answer 2

1. $L_4^{L_5} = \{aaa, aabb\}$.
2. $L_6^{L_1} = \emptyset$.

1.2 Problem 2

Let L_1, L_2 and L_3 be languages over some alphabet Σ. Prove or disprove:

1. $(L_1 \cup L_2)L_3 = L_1L_3 \cup L_2L_3$.
2. $(L_1 \cap L_2)L_3 = L_1L_3 \cap L_2L_3$.

1.2.1 Answer 3

First, I will prove $(L_1 \cup L_2)L_3 \subset L_1L_3 \cup L_2L_3$. Assume to the contrary that there is $w \in (L_1 \cup L_2)L_3$ which is not in $L_1L_3 \cup L_2L_3$. Put $w = xy$ where $x \in (L_1 \cup L_2)$ and $y \in L_3$ (this implies $L_3 \neq \emptyset$ and at least one of $(L_1 \cup L_2) \neq$

$\emptyset$. Suppose x comes from L_1, then it has to be in $L_1L_3 \cup L_2L_3$ because it is in L$_1$ L$_3$\$, similartly if it originates in L_2. Suppose now $L_3 = \emptyset$, then there is an empty set on both sides of equation (by definition of concatenation). Suppose both L_1 and L_2 are empty, then, again, we have emtpy set on both sides of the equation. Thus we showed that it is impossible for w not to be in the $L_1L_3 \cup L_2L_3$, hence the original argument must be true.

Similarly, to prove $L_1L_3 \cup L_2L_3 \subset (L_1 \cup L_2)L_3$, assume there exists $w \in L_1L_3 \cup L_2L_3$, not a amember of $(L_1 \cup L_2)L_3$. Again, $w = xy$ where $y \in L_3$ and x may be a member of L_1, L_2 or both. Suppose, again, the sets aren't empty. If w came from L_1L_3, then x came from L_1, but it is a member of $(L_1 \cup L_2)$ and similarly if it came from L_2. Since $y \in L_3$ and L_3 is present on both sides, it is not possible for w to not be a member of $(L_1 \cup L_2)L_3$. As in previous case, whenever L_3 or $(L_1 \cup L_2)$ are empty, both sides of equation contain an empty set. Hence we proved both directions, hence the conjecture is true.

1.2.2 Anser 4

This conjecture isn't generally true. Suppose $L_1 = L_3 = \{a\}$ and $L_3 = \{\epsilon, aa\}$. Then:

1. $(L_1 \cap L_2)L_3 = \emptyset$.
2. $L_1L_3 \cap L_2L_3 = \{aa\}$.

I.e. both sides of equation are not equal. This completes the proof.

1.3 Problem 3

An equivalence relation over Σ^* will be called invariant from right if $\forall z \in \Sigma^*.(xRy \implies xzRyz)$. Answer for every relation in $\{a,b\}^*$ whether the relation is an equivalence relation and whether it is invariant from right.

1. $xRy \iff |x| \geq |y|$.
2. $xRy \iff (|x| = |y| = 0 \vee x = qz, y = pz, |z| \geq 1)$.

1.3.1 Answer 5

Total order relation is not symmetric. Suppose $x = a$ and $y = ab$, then $x \geq y$ but not $y \geq x$. Since this relation is not an equivalence, it cannot be right invariant either.

1.3.2 Answer 6

This relation is an equivalence. It is transitive because whenever $x = qz$, $y = pz$ and $w = vz$, all of the below hold: xRy, yRw, xRw since they all have the last letter in common. This also holds trivially in case the length is zero, since $x = y = w = \epsilon$ in that case.

The relation is reflexive because whenever every string is either empty or its last symbol is equal to itself, i.e. xRx is always true.

The relation is symmetric because whenever $x = qz$ and $y = pz$ then both xRy and yRx hold (again, becuase x and y have the final letter in common, or are both the empty string).

The relation is also invariant from the right. The proof will proceed by induction on the string's length.

Base step: $\epsilon R \epsilon \implies \epsilon z R \epsilon z$ because R is reflexive and $z = \epsilon z$.

Inductive step: suppose the inductive hypotesis $xRy \implies xzRyz$, then suppose we concatenate the same character c to both x and y. This character must be the same by definition of R. Then $xcRyc \implies xczRycz$ because we can simply rename $xc = x_1$ adn $yc = y_1$ and obtain the inductive hypothesis restated using new terms: $x_1Ry_1 \implies x_1zRy_1z$. This completes the inductive step, and hence the proof is completed.