Assignment 15, Authomata Theory

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Contents

1 Problems

1.1 Problem 1

Given context-free grammar $V = \{S, M, N, W, X, Y, Z\}$ s.t. $T = \{1, 0\}$

$$\begin{aligned}
&S \to M \mid XN \mid W \mid 0N \mid 1Z1 \\
&M \to 0M0 \mid N \\
&N \to N0 \mid 0 \\
&W \to 0W \mid 00W0 \\
&X \to 0X1 \mid 0 \mid 0Y0 \\
&Z \to W \;.
\end{aligned}$$

1. Is V ambiguous?
2. Give a normalized grammar equivalent to V.

1.1.1 Answer 2

It is easier to normalize the grammar first and then to look for ambiguities, thus the answers are in reverse order.

1. Any derivation containing W cannot terminate, and so does Z.
2. Further, we can eliminate the rule $M \to N$.
3. Y has no derivation rules, thus we can also remove it.

Thus obtaining:

$$\begin{aligned}
&S \to M \mid XM \mid 0M \\
&M \to 0M0 \mid 0 \mid 0M \\
&X \to 0X1 \mid 0 \;.
\end{aligned}$$

1. It is easy to see that M derives number of zeros greater than one, thus $M \to 0M0$ is redundant. Subsequently, $S \to 0M$ is already covered by $S \to M$.

What remains is:

$$\begin{aligned} S &\to M \mid XM \\ M &\to 0 \mid 0M \\ X &\to 0 \mid 0X1\ . \end{aligned}$$

1.1.2 Answer 1

Now it is easy to see that the string 00 can be derived in two different ways:

- $S \to M$, $M \to 0M$, $M \to 0$.
- $S \to XM$, $X \to 0$, $M \to 0$.

Hence V is ambiguous.

1.2 Problem 2

Given context-free grammar $V = \{S, M, N, W, X, Y, Z\}$ s.t. $T = \{1, 0\}$

$$\begin{aligned} S &\to 0W11 \mid 0X1 \mid 0Y \\ W &\to S \mid Z \\ X &\to S \mid W \\ Y &\to 1 \\ Z &\to X\ . \end{aligned}$$

1. Bring V to Chomsky's normal form.
2. What is the language of V?

1.2.1 Answer 3

1. We can easily eliminate Y variable, thus removing $Y \rightarrow 1$ rule, and adding $S \rightarrow 01$ rule.
2. We can eliminate Z variable by removing $Z \rightarrow X$ and $W \rightarrow Z$ rules and adding $W \rightarrow X$ rule.
3. We can eliminate X variable by removing $X \rightarrow S \mid W$ and $S \rightarrow 0X1$ rules, and adding: $S \rightarrow 0S1$ rule.
4. Finally, we can eliminate W variable by removing $W \rightarrow S$ and $S \rightarrow 0W11$ rules and adding $S \rightarrow 0S11$ rule.

The resulting grammar will be:

$$S \rightarrow 0S11 \mid 0S1 \mid 01 \ .$$

Since this is still not CNF, I introduce an extra variable: X and derivation rules $X \rightarrow 0$, $Y \rightarrow 1 \mid 11$ and $Z \rightarrow SY$ thus obtaining:

$$\begin{aligned} S &\rightarrow XZ \mid 01 \\ X &\rightarrow 0 \\ Y &\rightarrow 1 \mid 11 \\ Z &\rightarrow SY \ . \end{aligned}$$

Which is in CNF.

1.2.2 Answer 4

Using the results from the previous answer it is easy to see that the language $L(V) = \{0^n 1^k \mid n \leq k \wedge n, k > 0\}$.

1.3 Problem 3

Build a PDA accepting the language L by emptying the stack.

$$\begin{aligned} L = \{a^{i_1}b^{j_1}a^{i_2}b^{j_2}\dots a^{i_m}b^{j_m} &\mid m \geq 1 \\ &\mid \forall k : i_k \geq j_k \geq 1 \\ &\mid \exists k : i_k > j_k\} \end{aligned}$$

1.3.1 Answer 5

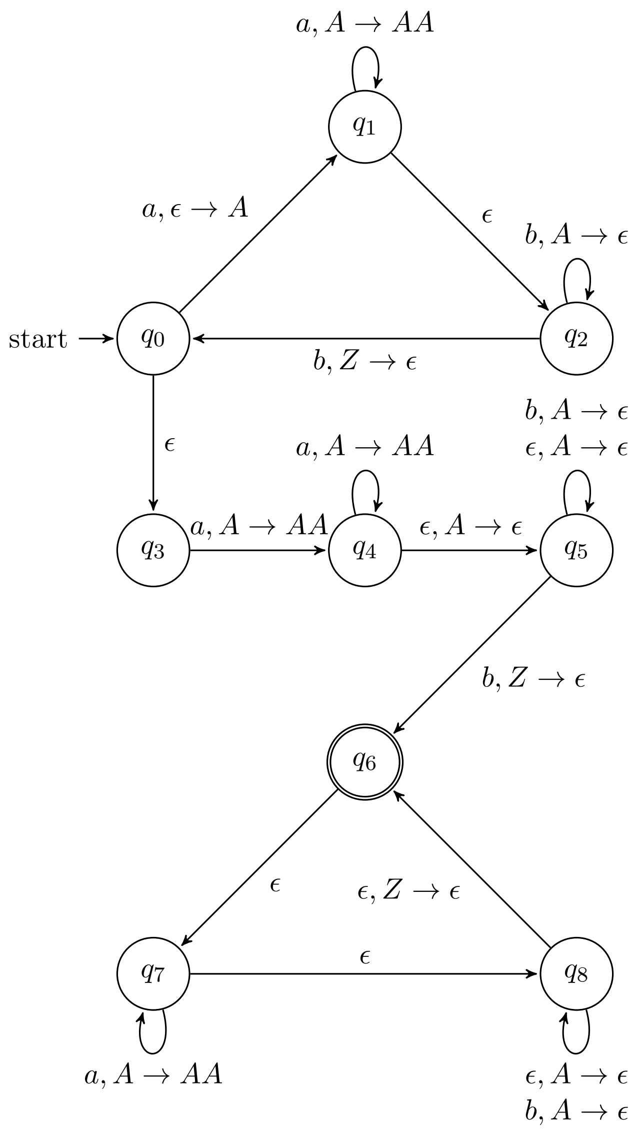
First, I'll write the grammar for L (because it's easier to do):

$$\begin{aligned} S &\to aSb \mid K \mid SS \\ K &\to aKb \mid aaXb \\ X &\to aXb \mid aaXb \mid \epsilon\ . \end{aligned}$$

1. X generates strings of the form $\{a^nb^m \mid n \geq m\}$.
2. Similarly, K generates strings of the form $\{a^nb^m \mid n > m\}$.
3. The derivatin of S can only terminate when it eventually derives K. It can repeat as many times as needed to accept the entire string. Where the repeated element is, again, of the form of either $\{a^nb^m \mid n \geq m\}$, or $\{a^nb^m \mid n > m\}$.

Thus, at least informally, we are convinced the grammar generates L.

Now, the automaton:



The idea behind this diagram is as follows:

1. Loop as many times as needed (possibly zero) over strings $a^n b^n$, where $n \geq 1$.
2. Nondeterministically parse a string $a^n b^m$ where $n > m$.

3. Loop as many times as needed (possibly zero) over strings $a^n b^m$ where $n \geq m$.
4. Before accepting state, keep discarding A until none remain for as long as you don't see an a.
5. Accept the string.

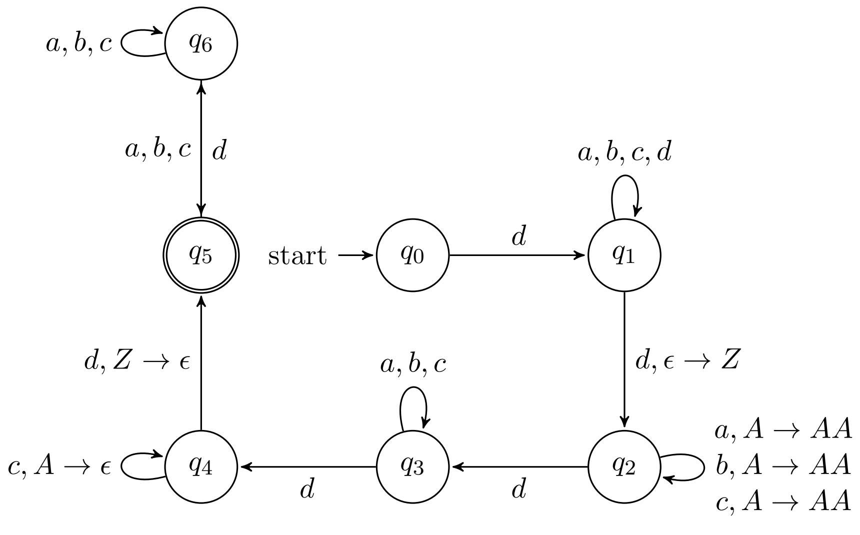
1.4 Problem 4

Construct a PDA for L defined as follows:

$$\begin{aligned} L = \{dw_1 dw_2 d \dots w_n d \mid{}& n \geq 3 \\ \mid{}& \forall k : w_k \in \{a, b, c\}^* \\ \mid{}& \exists k : 1 \leq k \leq n - 2 \wedge |w_k| = \#_c(w_{k+2})\} \end{aligned}$$

1.4.1 Answer 6

The automaton for L may look like this:



(Whenever stack symbols neither read, nor added nor removed, they are omitted from the diagram to make it easier to read)

The rationale for this diagram is as follows:

1. Read the first d.
2. Keep reading as many of as, bs, cs or ds as necessary.
3. Non-deterministically, upon reading d assume reading the "special" sequence of as, bs and cs, which must be equal in length to the sequence of cs to follow after. "Notice" this event by pushing Z on stack.
4. Read as, bs and cs pushing As on stack.
5. Read d.
6. Read any number of as, bs or cs.
7. Upon reading d start reading cs while discarding As.
8. Once you pop Z you also must read d, this will ensure the number of cs is the same as the number of as, bs and cs, read in step (4).
9. After this had happened, we've already found the "special" substring, now we just need to make sure to terminate when we read d.

1.5 Problem 5

Given language $L = \{a^k b^i c^j d^{j-i} e^k \mid 1 \leq i \leq j, k \geq 2\}$ build a deterministic PDA accepting it.

1.5.1 Answer 7

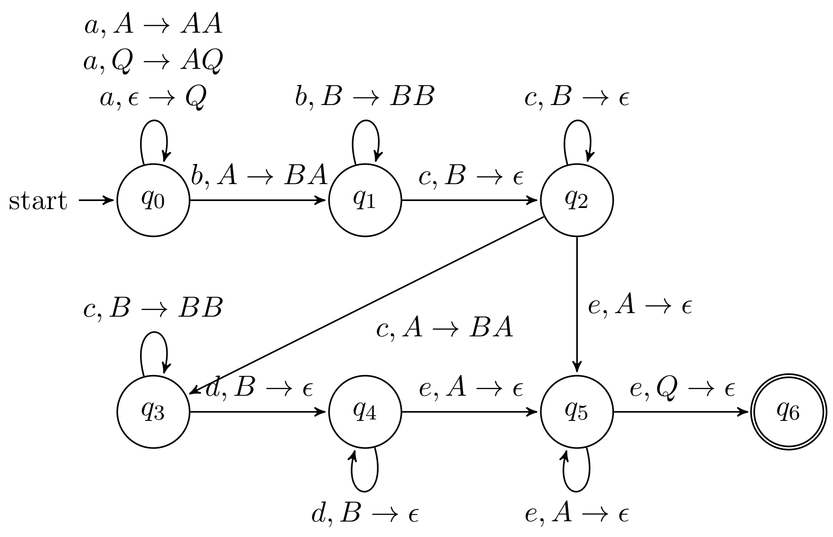
First, to better understand the language, I'll give its grammar:

$$
\begin{aligned}
S &\to aaXbb \\
X &\to aXb \mid YZ \\
Y &\to bYc \mid bc \\
Z &\to cZd \mid \epsilon \ .
\end{aligned}
$$

Informally, the language consists of three "major chunks":

1. An even number of *as* and *es* on the edges (which also, eventually, must be greater than four).
2. The middle "chunk" is split into two parts:
   - An even number of *bs* and *cs*, necessarily more than two characters long.
   - Followed by the optional third "chunk" of an even number of *cs* and *ds*.

Having realized this, we are now equipped to build an automaton:



In words, this automaton can be described as:

1. Read an a and put Q on stack.
2. Read another a and put A on stack (this is to satisfy $\left|a^k\right| \geq 2$.)
3. Keep reading as. While you read as push A on the stack.
4. Once you read b, start pushing Bs on the stack.
5. Once you read c, start popping Bs from the stack.
6. Once no Bs remain on the stack, if you read c, this means $j > i$, keep reading cs while pushing Bs on the stack.
    - When you encounter d, you need to stard popping Bs from the stack.
    - Once you read d, but you see A on the stack, move to (8).
7. Otherwise, $i = j$, hence you must be reading d. Skip to the next state.
8. Now you must be reading e. Keep reading es as long as there are As on the stack.
9. Once you encounter Q on the stack, this is also the final e. If the input ends here, you are done!

1.6 Problem 6

Given PDA M accepting language L by reaching an accepting state, construct PDA N, accepting language L^*.

1.6.1 Answer 8

Informally, all we need to do is to add transitions between the accepting states and the initial state of M to get N. There is a caveat: if we accept by reaching certain state, there might be symbols on the stack that will affect the way the automaton is restarded. To take care of these "leftovers", I suggest adding an extra state, which reads no input, and discards all stack symbols until it sees the initial stack symbol. Once it sees it, the automaton can restart.

More formally, let $M = \langle Q, \Sigma, \Gamma, \delta, q_0, Z, F\rangle$, where Q is the finite set of states, Σ is the input alphabet, Γ is the stack alphabet, δ is the relation $Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma \times Q \times \Gamma^*$ defining transitions of M. q_0 is the initial state, Z is the initial stack symbol, $F \subset Q$ is the set of all accepting states.

Then, $N = \langle Q_n, \Sigma, \Gamma, \delta_n, q_0, Z, F\rangle$, where $Q_n = Q \cup \{q_n\}$, q_n been a state I will describe shortly.

$$\begin{aligned}\delta_n = \delta \cup \{&(q_n, \epsilon, \gamma \in \Gamma \setminus \{Z\}, q_n, \epsilon),\\ &(q_n, \epsilon, Z, q_0, Z),\\ &(q_f \in F, \epsilon, \gamma \in \Gamma \setminus \{Z\}, q_n, \epsilon),\\ &(q_f \in F, \epsilon, Z, q_0, Z)\}\end{aligned}$$

reading as: when in state q_n reading no input, N sees γ on stack it must pop γ from stack and remain in q_n, otherwise, if it reads Z symbol on stack, it must transition to q_0, leaving the initial stack symbol unchanged. On top of it, for every accepting state q_f, there is a transition on any symbol, except the initial stack symbol to q_n. Finally, if the accepting state q_f already reads the initial stack symbol Z, it must tranzition to initial state q_0.