Assignment 13, Discrete Mathematics

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Contents

1 Problems

1.1 Problem 1

Demonstrate five distinct sets: A, B, $A \cup B$, $A \oplus B$, $A \setminus B$, all of which have the same cardinality.

1.1.1 Answer 1

There's a popular programming puzzle called "FizzBuzz". The taks is to print natural numbers according to the rule: if a number is divizible by three, the program prints `fizz`, if the number is divisible by five the program prints `buzz` (when both condition holds, the program thus prints `fizzbuzz`, hence the name). Otherwise the program prints the number itself.

It is easy to see that the description above perfectly matches the requirement:

$$\begin{aligned}
A &= \{x \in \mathbb{N} \mid x \bmod 3 = 0\} \\
B &= \{x \in \mathbb{N} \mid x \bmod 5 = 0\} \\
A \cup B &= \{x \in \mathbb{N} \mid x \bmod 3 = 0 \lor x \bmod 5 = 0\} \\
A \oplus B &= \{x \in \mathbb{N} \mid (x \bmod 3 = 0 \lor x \bmod 5 = 0) \land \neg(x \bmod 15 = 0)\} \\
A \setminus B &= \{x \in \mathbb{N} \mid x \bmod 3 = 0 \land \neg(x \bmod 5 = 0)\}.
\end{aligned}$$

The cardinality of all sets is $\aleph_0$ and neither one of them is equal to any other set.

1.2 Problem 2

1. Let n be a natural number. Show that the set of all subsets of $\mathbb{N}$ of cardinality exactly n is a countable set.
2. Show that the set of all finite subsets of $\mathbb{N}$ is countable.

3. Show that the set of all infinite subsets of $\mathbb{N}$ is not countable.
4. Find the cardinality of the set given in (3).
5. $\aleph_0 = \big|\{X \in P(\mathbb{N}) | |X| = n\}\big|$ is the restatement of (1). Write a similar statement for (2) and (4).

1.2.1 Answer 2

We can construct the set of all subsets of $\mathbb{N}$ by repeating cartesian product n times. Since $|\mathbb{N} \times \mathbb{N}| = \aleph_0$ so is the $|\mathbb{N}^n| = \aleph_0$ (because n is finite). Note that the requested sets in this case are given by the resulting ordered pairs. I.e. $(x_1, (x_2, \dots (x_{n-1}, x_n) \dots) \in \mathbb{N}^n$. In fact, we need only the unique tuples, of which there are even less.

1.2.2 Answer 3

This continues the argument from 1.2.1, but constructs the sets in question using recursive definition: $\bigcup_{i=1}^{n} \mathbb{N}^i$ since there are no infinte exponents involved in this construction, it must describe a countable set.

1.2.3 Answer 4

The set of all infinite sets can be described as $\mathbb{N}^{\mathbb{N}}$. We can substitute $\mathbb{N}$ for any infinite subset because any infinite subset of $\mathbb{N}$ is equivalent to $\mathbb{N}$, which follows from definition of infinite set). Because set exponentiation is non-decreasing in both arguments, this set has cardinality of at least $2^{\mathbb{N}}$, which is the cardinality of the powerset, which, by Cantor's theorem is infinitely larger than any countable set.

1.2.4 Answer 5

We just stated that the requested set is equivalent to $\mathbb{N}^{\mathbb{N}}$ in 1.2.3. Hence its cardinality is at least $\mathfrak{c}$ and no greater than $2^{\mathfrak{c}}$. Assuming (here and through the rest of the answers) continuity hypothesis is true, the cardinality of the set in question is $\mathfrak{c}$.

1.2.5 Answer 6

1. can be given by: $\aleph_0 = \big|\{X \in P(\mathbb{N}), n \in \mathbb{N} | |X| = n\}\big|$.
2. $\aleph_0 = \big|\{X \in P(\mathbb{N}) | |X| = |\mathbb{N}|\}\big|$.

1.3 Problem 3

Find the error in the following:

> We shall define simmetric set difference as follows: let k and m be cardinalities, not necessarily distinct. Define $k \oplus m$ to mean A and B sets for which $|A| = k$ and $|B| = m$, then the cardinality of $A \oplus B$ is the value of $k \oplus m$.

1.3.1 Answer 7

Since we are trying to define a *binary operation*, we must make sure that the result (or value) of this operation is uniquely defined. But it is easy to see it is not the case, since if $|A \cap B| \neq |A' \cap B'|$ while at the same time $|A| = |A'|$ and $|B| = |B'|$, we will get that $k \oplus m \neq k \oplus m$. Thus our attempt at defining a ring-sum operation will fail.

1.4 Problem 4

1. Let k_1, k_2, m_1, m_2 be cardinalities. Prove that if $k_1 \leq k_2$ and $m_1 \leq m_2$ then $k_1 \times m_1 \leq k_2 \times m_2$.
2. Prove that $\mathfrak{c} \times \aleph_0 = \mathfrak{c}$.
3. Prove that $\mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$.

1.4.1 Answer 8

Recall that the product of cardinalities is defined to be the cardinality of cartesian product. Also, recall that $|A| \leq |B|$, implies having an injective function from A to B. Now we can combine these two facts to build the proof. Suppose for contradiction that it was possible that $k_1 \times m_1 > k_2 \times m_2$. This would mean that for some sets A and B the cartesian product with cardinality $k_1 \times m_1$ defined by $A \times B$ contains a pair (a, b) with a property that for the chosen injective function from A to A' a is not the source of any element in A'. But we are given that there is an injective function from A' to A (because $k_1 \leq k_2$). The argument for b is identical. Recall that the function must be defined for each element in its domain, but this stands in contradiction to the earlier claim that a (or b) is not in the domain of this function. Since this is not possible, the proof is complete.

1.4.2 Answer 9

To prove $\mathfrak{c} \times \aleph_0 = \mathfrak{c}$ we can use the fact that there are no cardinal numbers between $\mathfrak{c}$ and $\aleph_2$. Also relying on the "sandwich" theorem, which says that if we can find a superset with cardinality n and a subset with the cardinality n, then the cardinality of the set "sandwiched" in between must be n. Consider $\mathfrak{c} \leq \mathfrak{c} \times \aleph_0 \leq \mathfrak{c} \times \mathfrak{c}$. We can show that $\mathfrak{c} \times \aleph_0$ is at least as big as $\mathfrak{c}$ since $\aleph_0$ is defined to be the multiplicative identity. We can also show that $\mathfrak{c} \times \mathfrak{c} = \mathfrak{c}$ by perusing Cantor's theorem which proves that the cartesian product of an infinite set with itself has the same cardinality as the initial set. Thus $\mathfrak{c} \times \aleph_0 = \mathfrak{c}$.

1.4.3 Answer 10

Observe that $\mathfrak{c}^{\mathfrak{c}}$ is the cardinality of the set of all functions from a set of cardinality $\mathfrak{c}$ to itself, while $2^{\mathfrak{c}}$ is the cardinality of the power-set of cardinality $\mathfrak{c}$. We know (from Cantor's diagonal argument) that $|X| < \big|P(X)\big|$ for all sets, this means that in particular $2^{\mathfrak{c}} < 2^{2^{\mathfrak{c}}}$. Since we know that cardinality of exponentiation is non-decreasing in both arguments it follows that $\mathfrak{c}^{\mathfrak{c}} < 2^{2^{\mathfrak{c}}}$. And because there are no cardinal numbers between X and $P(X)$ we can conclude that $\mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$.