Assignment 14, Discrete Mathematics

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Contents

1 Problems

1.1 Problem 1

1. Develop the identity $(3-2)^n = 1$ using binom of Newton formula:

$$\sum_{i=0}^{n} \binom{?}{?} 3^{?} \cdot (??)^{?} = 1$$

   And verify the identity for the case $n = 4$.

2. Let number of ways to distribute k identical balls between 10 boxes is $D(10, k)$. Paint three boxes green and the remaining seven—red. Derive:

$$D(10, k) = \sum_{i=0}^{k} ???,$$

   and verify for the case $k = 3$.

1.1.1 Answer 1

$$\sum_{i=0}^{n} \binom{i}{n} 3^{i} \cdot (-2)^{n-i} = 1.$$

Solution: *(using Maxima)*

```
n: 4;
tex(sum(binomial(n, i) * 3^i * (-2)^(n - i), i, 0, n));
```

1

(hand-made)

$$\sum_{i=0}^{4} \binom{i}{4} 3^i \cdot (-2)^{4-i}$$
$$= 1 \cdot 3^0 \cdot (-2)^4 + 4 \cdot 3^1 \cdot (-2)^3 + 6 \cdot 3^2 \cdot (-2)^2 + 4 \cdot 3^3 \cdot (-2)^1 + 1 \cdot 3^4 \cdot (-2)^0$$
$$= 16 - 96 + 216 - 216 + 81$$
$$= 1$$

1.1.2 Answer 2

1.2 Prolbem 2

How many permutations of a string $AAABBCCDD$ can you form s.t. they don't contain subsequences AAA, BB, CC or DD?

1.2.1 Answer 3

The total number of ways in which we can arrange the sequence $AAABBCCDD$ is $n(\Omega) = \frac{9!}{3!2!2!2!} = 7560$. Then we find all permutations which contain sequences of consequtive letters, AAA, BB and so on.

$$\begin{aligned}\frac{9!}{3!2!^3} - \left(\frac{7!}{2!^3} + 3 \cdot \frac{8!}{3!2!^2}\right) + \left(3 \cdot \frac{6!}{2!^2} + 3 \cdot \frac{7!}{3!2!}\right) - \left(3 \cdot \frac{5!}{2!} + \frac{6!}{3!}\right) + 4! \\ = 7560 - (630 + 5040) + (540 + 1260) - (180 + 120) + 24 \\ = 3414 .\end{aligned}$$

We count in stages: first we find all permutations of the string containing AAA or duplicated characters, divided by the internal orderings of the remaining duplicates. Some of these permuations will also intersect with each other, thus, we want to subtract duplicates such as $AAA \cup BB$, but now we subtracted some of the duplicates twice, so we need to add them back. Those which we counted twice are those containing three subsequences, and so on. Finally:

```
is_prefix([], _).
is_prefix(_, []) :- fail.
is_prefix([X | Xs], [X | Ys]) :- is_prefix(Xs, Ys).

not_allowed([a, a, a]).
not_allowed([b, b]).
not_allowed([c, c]).
not_allowed([d, d]).

prefix_allowed(Sofar) :-
    not_allowed(Bad), is_prefix(Bad, Sofar).

valid_seqence(X, [], X).
valid_seqence(Sofar, Pool, Result) :-
    select(E, Pool, Rem), Next = [E | Sofar],
    \+prefix_allowed(Next),
    valid_seqence(Next, Rem, Result).
valid_seqence(X) :-
    valid_seqence([], [a, a, a, b, b, c, c, d, d], X).

sans_repetitions :-
    findall(X, valid_seqence(X), X),
    list_to_set(X, Y),
    length(Y, Result),
    format('$$~p$$', [Result]).
```

$$3414$$

1.3 Problem 3

Four families (all distinct) went out to barbecue. They took 8 steaks and 10 kebabs. In how many ways is it possible to distribute the food to the families, while every family has to have at least one meal?

1.3.1 Answer 4

We can distribute all meals in the following way: at first we will count the total number of ways in which meals can be distributed, this is given by $\binom{10+4-1}{10} \cdot \binom{8+4-1}{8}$. Now, we need to subtract the combinations where at least one family didn't get any food, add combinations, where at least two families didn't get any food and subtract the combinations where three familiies didn't get any food.

$$
\begin{aligned}
&\binom{4}{4} \cdot \binom{10+4-1}{10} \cdot \binom{8+4-1}{8} \\
-&\binom{4}{3} \cdot \binom{10+3-1}{10} \cdot \binom{8+3-1}{8} \\
+&\binom{4}{2} \cdot \binom{10+2-1}{10} \cdot \binom{8+2-1}{8} \\
-&\binom{4}{1} \cdot \binom{10+1-1}{10} \cdot \binom{8+1-1}{8} \\
=&286 * 165 - 4 * 66 * 45 + 6 * 11 * 9 - 4 \\
=&35900 \;.
\end{aligned}
$$

```
:- use_module(library(clpfd)).

feed_families([(S, K), (S1, K1), (S2, K2), (S3, K3)]) :-
    Steaks = [S, S1, S2, S3],
    Kebabs = [K, K1, K2, K3],
    Steaks ins 0..8, sum(Steaks, #=, 8),
    Kebabs ins 0..10, sum(Kebabs, #=, 10),
    append([Steaks, Kebabs], Meals),
    maplist(indomain, Meals),
    Total is (S + K) * (S1 + K1) * (S2 + K2) * (S3 + K3),
    Total > 0.

barbecue :-
    findall(X, feed_families(X), X),
    length(X, Result),
    format('$$~p$$', [Result]).
```

35900

1.4 Problem 4

Rami and Dina play a game where Dina selects 8 numbers in the $10 \leq n \leq 36$ range. Rami has to find a way to form two sums from the numbers Dina selected. The conditions for Rami are that he can use anywhere from 1 to 8 disctinct numbers to form the sum. The numbers themselves, however, can repeat in the other sum.

1.4.1 Answer 5

The number of sums one can form from numbers in the range $[10, 36]$ is $\sum_{n=36}^{2} 8n - 10 + 1 = 260 - 9 = 254$. Now, if we can compute the number of combinations we can form from 8 numbers and establish that the number is larger than $254 \times 2 = 508$, then we showed (by pigeonhole principle) that we can indeed produce two required sums (Rami has a winning strategy). The number of combinations we are talking about is the cardinality of the powerset of cardinality 8. Let A be the set of numbers selected by Dina, then $\left|P(A)\right| = 2^{|A|} = 2^8 = 256$. Since $256 > 254$ it follows that there will always be a sum produced by at least two subsets of the numbers selected by Dina.