Assignment 15, Discrete Mathematics

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1 Problems

1.1 Problem 1

Given the alphabet $\{0, 1, 2\}$ and the grammar rules which disallow sequences $\langle 00 \rangle$ and $\langle 01 \rangle$ of the sequence $\{a_n\}$ which counts the possible products of the grammar:

1. Write a_0, a_1, a_2, write recursive definition of the sequence.
2. Find a closed-form solution for $\{a_n\}$.

1.1.1 Answer 1

- a_0 is the number of sequences of zero length, there's only one such sequence.
- a_1 is the number of sequences of length one, there's exactly three such sequences: $\langle 0 \rangle$, $\langle 1 \rangle$, $\langle 2 \rangle$.
- a_2 is the number of sequences of length two, here, for the first time we encounter the restrictions imposed by the grammar. Without restriction, the number of sequences possible to form from an alphabet of size 3 is $3^2 = 9$, since two of these are not allowed, the total number of sequences we can form is 7.

Now, let's construct the recursive definition. Consider these two cases: either the first element of the sequence is 2, or not. If it is 2, then our grammar doesn't restrict us, thus the number of possible grammar products is to be a_{n-1}. Conversely, when the first number is either 0 or 1, the further products can be only selected in a way that they don't contain zero, i.e. there are only two possible ways to continue the sequence, which gives us $2 \cdot 2a_{n-2}$.

In conclusion, the recursive definition is $a_n = a_{n-1} + 4a_{n-2}$. Verifying:

$$\begin{aligned}
a_0 &= 1 \\
a_1 &= 3 \\
a_2 &= a_1 + 4a_0 = 3 + 4 \cdot 1 = 7 \;.
\end{aligned}$$

1.1.2 Answer 2

In order to find the closed form solution, let's assume there exists λ s.t. $a_n = \lambda^n$. Substituting our assumption back into recursive formula gives:

$$\begin{aligned}
\lambda^n &= \lambda^{n-1} + 4\lambda^{n-2} && \iff \\
\lambda^2 &= \lambda + 4 && \iff \\
\lambda^2 - \lambda - 4 &= 0 && \iff \\
\lambda &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot (-4) \cdot 1}}{2 \cdot 1} \\
&= \frac{-1 \pm \sqrt{17}}{2} \;.
\end{aligned}$$

From here we see that there are two different terms contributing to the function defining the sequence. Lets denote the contribution these terms make to the value as α and β. Thus:

$$\begin{aligned}
a_n &= \alpha \frac{-1 + \sqrt{17}}{2} + \beta \frac{-1 - \sqrt{17}}{2} \\
&\alpha + 4\beta = 0 \\
&\alpha \frac{-1 + \sqrt{17}}{2} + 4\beta \frac{-1 - \sqrt{17}}{2} = 3 \;.
\end{aligned}$$

Solving these simultaneous equations gives: $\alpha = -3, \beta = -\frac{3}{4}$. Hence, the direct formula is:

$$a_n = 3 \left(\frac{-1 - \sqrt{17}}{2} \right)^n - \frac{3}{4} \left(\frac{-1 + \sqrt{17}}{2} \right)^n \;.$$

1.2 Problem 2

Find the number of solutions for $x_1 + x_2 + x_3 + x_4 + x_5 = 24$ provided two of the unknowns are odd natural numbers, the rest are even natural numbers and neither unknown is equal to 0 or 1.

1.2.1 Answer 3

First, a simple way of counting the solutions. We must deal at least a 3 to two of the unknows and 2 to the other 3 of the unknowns. This gives us $2 \cdot 3 + 3 \cdot 2$ of "mandatory allocations". After these have been allocated, we must distribute the remaining 12, but we can only distribute it in pairs, so, all in all we end up with $\binom{6+5-1}{6} = 210$ ways to allocate the "points". Having done this, we count the number of ways the solutions can be permuted: $\frac{5!}{2!3!} = 10$. Hence, the final answer, the number of solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 24$ is $10 \cdot 210 = 2100$.

The code to verify our calculations:

```
:- use_module(library(clpfd)).

even(N) :- 0 is N mod 2.

assignment_5_2_helper(Unknowns) :-
    length(Unknowns, 5),
    Unknowns ins 2..24, sum(Unknowns, #=, 24),
    maplist(indomain, Unknowns),
    include(even, Unknowns, Evens),
    length(Evens, 3).

assignment_5_2 :-
    findall(X, assignment_5_2_helper(X), X),
    length(X, N),
    format('$$~p$$', [N]).
```

2100

To find what a generating function for this would look like we consider the two groups of factors: the odd and the even ones. Odd will contribute two factors and even will contribute three, thus we have $Odds^2 \cdot Evens^3$. Now, odd factors can be written as $x^3 + x^5 + x^7 + \dots$ and, similarly even factors are $x^2 + x^4 + x^6 + \ldots$. From the argument above we know that there are 10 ways to arrange the unknowns, thus the final formula for generating function is $G(x) = 10(x^3 + x^5 + x^7 + \ldots)^2(x^2 + x^4 + x^6 + \ldots)^3$. The solution is given by the coefficient of x^{24}. Applying the following transformations:

$$
\begin{aligned}
G(x) &= 10(x^3 + x^5 + x^7 + \ldots)^2(x^2 + x^4 + x^6 + \ldots)^3 \\
&= 10x^2(x^2 + x^4 + x^6 + \ldots)^2(x^2 + x^4 + x^6 + \ldots)^3 \\
&= 10x^2(x^2 + x^4 + x^6 + \ldots)^5 \\
&= 10x^7(1 + x^2 + x^4 + \ldots)^5 \\
&= \frac{10x^7}{(1 - x^2)^5} \\
&= \binom{-5}{0} + \binom{-5}{1}10x^7 - \binom{-5}{2}10x^8 + \ldots
\end{aligned}
$$

We need the sixth term of these series, i.e. $\binom{-5}{6} = \binom{5+6-1}{6} = 210$, multiplied by 10 (number of arrangements) gives the same answer: 2100.

1.3 Problem 3

Joshua has to take pills:

- Against headache—at most 3 per day.
- Energy pill—at most 3 per day.
- Vitamin C—no restriction.
- Vitamin B—no restriction.

Joshua is also required to take exactly n pills each day. Let a_n be the number of ways the pills can be combined.

1. Find a generating function for a_n.
2. Find a closed-form formula for a_n.

1.3.1 Answer 4

We can create terms of the generating function in the following way: Since Joshua has to take either no pills against headache, one, two or three, this kind of pill will contribute the term $(1+x+x^2+x^3)$. Similarly, the energy pill. Vitamins will contribute infinite sum terms: $(1+x+x^2+x^3+\ldots)$. Cobining all together gives: $G(x) = (1+x+x^2+x^3)^2(1+x+x^2+x^3+\ldots)^2$.

1.3.2 Answer 5

To find the closed-form formula for $G(x)$ we can use the following identities:

$$\begin{aligned}(1+x+x^2+x^3) &= \frac{1-x^4}{1-x}\\ (1+x+x^2+x^3+\ldots) &= \frac{1}{1-x}\end{aligned}$$

Thus we can rewrite G as:

$$\begin{aligned}G(x) &= \left(\frac{1-x^4}{1-x}\right)^2 \cdot \left(\frac{1}{1-x}\right)^2\\ &= \frac{(1-x^4)^2 \cdot (1-x)^2}{(1-x)^2}\\ &= (1-x^4)^2\ .\end{aligned}$$

1.4 Problem 4

Find the coefficient of x^{2m} in both sides of the following identity.

$$\frac{(1-x^2)^n}{(1-x)^n} = (1+x)^n\ .$$

From this, derive the following idenity:

$$\sum_{k=0}^{?} ?? = \binom{n}{2m}\ .$$

Verify the answer for the assignments: $n = 5, m = 2$ and $n = 5, m = 3$.

1.4.1 Answer 6

First, I will establish that the identity holds:

$$\begin{aligned}\frac{(1-x^2)^n}{(1-x)^n} &= (1-x^2)^n(1+x+x^2+x^3+\ldots)^n \\ &= (1-x+(1-1)x^3+(1-1)x^4+\ldots)^n \\ &= (1+x)^n \ .\end{aligned}$$

Now, from binomial theorem:

$$(1+x)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^i \ .$$

Thus the coefficient of $x^{2m} = \binom{n}{m}$.