Assignment 12, Discrete Mathematics

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1 Problems

1.1 Problem 1

Let M be the set of all relations over $A = \{1, 2, 3\}$.

1. How many members does M have?
2. Let S be a set of relations over M, defined as follows:
$$S = \{R_1 R_2 | R_1, R_2 \in M \wedge R_1 R_2 = R_2 R_1\}.$$
Show that S is not an equivalence relation.

1.1.1 Answer 1

The number of elements of M is $2^{|A|2} = 2^{3^2} = 2^9 = 512$. This can be proved from a partial sum of a recurence: $a_{n+1} = a_n + 2n - 1$, which describes the maximum number of ordered pairs possible to create from n elements and $b_n = 2b_{n-1}$ recurrence which describes the number of elements of a powerset. Since one can see that the number of relations over a set is exactly the number of ways to subset the set of all ordered pairs possible to create from that set, the final answer is the composition of both recurrences: $c_n = b_n \circ a_n$.

Below I've implemented the counting algorithm in Prolog:

```
powerset(Set, Result) :-
    powerset_helper(Set, [[]], Result).
powerset_helper([], X, X) :- !.
powerset_helper([X | Xs], In, Result) :-
    findall(Z, (member(Y, In), append([X], Y, Z)), Z),
    append(In, Z, Next),
    powerset_helper(Xs, Next, Result).

cross(Set, (A, B)) :- member(A, Set), member(B, Set).

pairs(Set, Pairs) :- findall(X, cross(Set, X), Pairs).

question_1(A) :-
    pairs(A, Pairs),
    powerset(Pairs, M),
    length(M, Len),
    format("$\\left|M\\right| = ~p$~n", [Len]).
```

$|M| = 512$

1.1.2 Answer 3

I will prove that S is not an equivalence using the definition of equivalence which states that a relation is an equivalence if it is **relfexive**, **symmetric** and **transitive**. It is easy to see that the definition of reflexivity requires that *all* members of M be present in the relation, but, for example, $\{(1,2)\}$ is absent from S. Suppose, for contradiction, it was present in S, then it would imply that there exists a pair $\{(a,b)\} \in M$ s.t. $(\{(1,2)\}, \{(a,b)\}) \in S$ and $(\{(a,b)\}, \{(1,2)\}) \in S$. By looking at the first and the last members of the two members of S we know that $\{(1,2)\} \circ \{(a,b)\} = \{(1,b)\}$ and $\{(a,b)\} \circ \{(1,2)\} = \{(a,2)\}$, in other words it is necessary that: $\{(1,2)\} \circ \{(a,b)\} = \{(1,b)\} = \{(1,2)\} = \{(a,2)\} = \{(a,b)\} \circ \{(1,2)\}$, which in turn, means that $b = 2$ and $a = 1$. Plugging these values back into original equilaty gives $\{(1,2)\} \circ \{(1,2)\} = \{(1,2)\}$, which is obviously false. Hence, by contradiction, the proof is complete.