

# Assignment 12, Discrete Mathematics

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# 1 Problems

## 1.1 Problem 1

Let  $M$  be the set of all relations over  $A = \{1, 2, 3\}$ .

1. How many members does  $M$  have?
2. Let  $S$  be a set of relations over  $M$ , defined as follows:

$$S = \{R_1 R_2 \mid R_1, R_2 \in M \wedge R_1 R_2 = R_2 R_1\}.$$

Show that  $S$  is not an equivalence relation.

### 1.1.1 Answer 1

The number of elements of  $M$  is  $\exp(|A|^2)/e^2 = 2^{3^2} = 2^9 = 512$ . This can be proved from a partial sum of a recurrence:  $a_{n+1} = a_n + 2n - 1$ , which describes the maximum number of ordered pairs possible to create from  $n$  elements and  $b_n = 2b_{n-1}$  recurrence which describes the number of elements of a powerset. Since one can see that the number of relations over a set is exactly the number of ways to subset the set of all ordered pairs possible to create from that set, the final answer is the composition of both recurrences:  $c_n = b_n \circ a_n$ .

Below I've implemented the counting algorithm in Prolog:

```
powerset(Set, Result) :-
    powerset_helper(Set, [[]], Result).
powerset_helper([], X, X) :- !.
powerset_helper([X | Xs], In, Result) :-
    findall(Z, (member(Y, In), append([X], Y, Z)), Z),
    append(In, Z, Next),
    powerset_helper(Xs, Next, Result).

cross(Set, (A, B)) :- member(A, Set), member(B, Set).

pairs(Set, Pairs) :- findall(X, cross(Set, X), Pairs).

question_1(A) :-
```

```

pairs(A, Pairs),
powerset(Pairs, M),
length(M, Len),
format("$\\left|M\\right| = ~p$~n", [Len]).

```

$$|M| = 512$$

### 1.1.2 Answer 3

I will prove that  $S$  is not an equivalence using the definition of equivalence which states that a relation is an equivalence if it is **reflexive**, **symmetric** and **transitive**. It is easy to see that the definition of reflexivity requires that *all* members of  $M$  be present in the relation, but, for example,  $\{(1, 2)\}$  is absent from  $S$ . Suppose, for contradiction, it was present in  $S$ , then it would imply that there exists a pair  $\{(a, b)\} \in M$  s.t.  $(\{(1, 2)\}, \{(a, b)\}) \in S$  and  $(\{(a, b)\}, \{(1, 2)\}) \in S$ . By looking at the first and the last members of the two members of  $S$  we know that  $\{(1, 2)\} \circ \{(a, b)\} = \{(1, b)\}$  and  $\{(a, b)\} \circ \{(1, 2)\} = \{(a, 2)\}$ , in other words it is necessary that:  $\{(1, 2)\} \circ \{(a, b)\} = \{(1, b)\} = \{(1, 2)\} = \{(a, 2)\} = \{(a, b)\} \circ \{(1, 2)\}$ , which in turn, means that  $b = 2$  and  $a = 1$ . Plugging these values back into original equation gives  $\{(1, 2)\} \circ \{(1, 2)\} = \{(1, 2)\}$ , which is obviously false. Hence, by contradiction, the proof is complete.