Assignment 12, Discrete Mathematics

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1 Problems

1.1 Problem 1

Let M be the set of all relations over $A = \{1, 2, 3\}$.

1. How many members does M have?
2. Let S be a set of relations over M, defined as follows:

$$S = \{R_1R_2 | R_1, R_2 \in M \wedge R_1R_2 = R_2R_1\}.$$

   Show that S is not an equivalence relation.

1.1.1 Answer 1

The number of elements of M is $2^{|A|^2} = 2^{3^2} = 2^9 = 512$. This can be proved from a partial sum of a recurence: $a_{n+1} = a_n + 2n - 1$, which describes the maximum number of ordered pairs possible to create from n elements and $b_n = 2b_{n-1}$ recurrence which describes the number of elements of a powerset. Since one can see that the number of relations over a set is exactly the number of ways to subset the set of all ordered pairs possible to create from that set, the final answer is the composition of both recurrences: $c_n = b_n \circ a_n$.

Below I've implemented the counting algorithm in Prolog:

```prolog
powerset(Set, Result) :-
    powerset_helper(Set, [[]], Result).
powerset_helper([], X, X) :- !.
powerset_helper([X | Xs], In, Result) :-
    findall(Z, (member(Y, In), append([X], Y, Z)), Z),
    append(In, Z, Next),
    powerset_helper(Xs, Next, Result).

cross(Set, (A, B)) :- member(A, Set), member(B, Set).

pairs(Set, Pairs) :- findall(X, cross(Set, X), Pairs).

question_1(A) :-
    pairs(A, Pairs),
    powerset(Pairs, M),
    length(M, Len),
    format("$\\left|M\\right| = ~p$~n", [Len]).
```

$|M| = 512$

1.1.2 Answer 2

I will prove that S is not an equivalence using the definition of equivalence which states that a relation is an equivalence if it is **relfexive**, **symmetric** and **transitive**. It is easy to see that the definition of reflexivity requires that *all* members of M be present in the relation, but, for example, $\{(1,2)\}$ is absent from S. Suppose, for contradiction, it was present in S, then it would imply that there exists a pair $\{(a,b)\} \in M$ s.t. $(\{(1,2)\}, \{(a,b)\}) \in S$ and $(\{(a,b)\}, \{(1,2)\}) \in S$. By looking at the first and the last members of the two members of S we know that $\{(1,2)\} \circ \{(a,b)\} = \{(1,b)\}$ and $\{(a,b)\} \circ \{(1,2)\} = \{(a,2)\}$, in other words it is necessary that: $\{(1,2)\} \circ \{(a,b)\} = \{(1,b)\} = \{(1,2)\} = \{(a,2)\} = \{(a,b)\} \circ \{(1,2)\}$, which in turn, means that $b = 2$ and $a = 1$. Plugging these values back into original equilaty gives $\{(1,2)\} \circ \{(1,2)\} = \{(1,2)\}$, which is obviously false. Hence, by contradiction, the proof is complete.

1.2 Problem 2

Given a set $A = \{1,2,3\}$ and M a set of all reations over A. Let also $s : M \to M$ be a function assigning to each $R \in M$ its transitive closure (R^+). Prove or disprove:

1. s is one-to-one.
2. s is onto.
3. For all $R_1, R_2 \in M$ it holds that $s(R_1R_2) = s(R_1)s(R_2)$.
4. For all $R \in M$ it holds that $s(R) = s(s(R))$.

1.2.1 Answer 3

A function is one-to-one when if $s(x) = s(y)$ then it must also be that $x = y$. From definition of transitive closure it follows that it may be closed over more than one set (i.e. it must be its own subset but may have more subsets which are not transitive relations). Thus it's not true that $s(x) = s(y) \implies x = y$. To be more concrete, suppose this example: $R_1 = \{(1,2),(1,3)\}$ and $R_2 = \{(1,2),(2,3)\}$. Then $s(R_1) = \{(1,2),(1,3),(2,3)\}$ and $s(R_2) = \{(1,2),(2,3),(1,3)\}$. Thus s is not one-to-one.

1.2.2 Answer 4

Recall the definition of a function which is onto: for each element in the function's co-domain there exists an element in its domain. Using the definition of transitive closure, every such closure contains at least one relation (itself), thus s has an element in its domain for each element in its co-domain. Thus the claim is true.

1.2.3 Answer 5

This is not in general true, consider R_1 to be $\{(1,1),(1,2),(1,3)\}$ and $R_2 = \{(3,3)\}$. Then:

$$\begin{aligned} s(R_1R_2) &= s(\{(1,3)\}) &&= \{(1,3)\} \\ s(R_1)s(R_2) &= \{(1,1),(1,2),(2,3),(1,3)\} \circ \{(3,3)\} &&= \{(2,3),(1,3)\}. \end{aligned}$$

1.2.4 Answer 6

It is true that s is *idempotent*. This is so because for any transitive relation its transitive closure is equal to it and the co-domain of s is defined to contain only transitive relations.

1.3 Problem 3

Let F be the set of functions from $\mathbb{N}$ to itself. Define relation K over F s.t. $f, g \in F, (f,g) \in K \iff \forall n \in \mathbb{N} f(n) \leq g(n)$.

1. Prove that K is a partially ordered.
2. Prove that K isn't totally ordered.
3. Are there maximal members in K? Is there a largest member in K?
4. Are there minimal members in K? Is there a smallest member in K?
5. Prove that for any $f \in F$ exists $g \in F$ bounding it from above. Prove there are more than one such g.

1.3.1 Answer 7

Partially ordered sets are defined to be reflexive, antisymmetrical and transitive. Let's verify all these properties:

Reflexitivty $f(n) \leq f(n)$ is true because the co-domains of f and g are the set of natural numbers, for which this property also holds, besides, we required that the inequality holds for each value of the functions taken pair-wise.

Antisymmetry whenever $f(n) \leq g(n)$ and $f(n) \leq g(n)$ then $f(n) = g(n)$. By the same reasoning as above, K is antisymmetrical.

Transitivity if $f(n) \leq g(n)$ and $q(n) \leq f(n)$, then $q(n) \leq g(n)$. And, again, since the co-domain of all these functions is the natural numbers transitivity holds.

Thus K is partially ordered.

1.3.2 Answer 8

To say that K isn't totally ordered is to say that there exists a pair of functions f' and g' such that for them neither $f'(n) \leq g'(n)$ nor $g'(n) \leq f'(n)$ for all $n \in \mathbb{N}$. Any two functions, whose graphs cross will do the job (before the interesection point one of the functions will be greater than the other and after the intersection the relation will change sides). So, for example functions $g'(n) = n^2$ and $f'(n) = n + 2$ are $g'(n) < f'(n)$ when $n < 2$ and $g'(n) > f'(n)$ otherwise.

1.3.3 Answer 9

No, there isn't a maximal element in K since there isn't a maximal element in the natrual numbers (and natural numbers are the co-domain of the functions used to construct K). Consequently, there isn't a largest element in K.

1.3.4 Answer 10

However, there is a minimal element in K (and subsequently the smallest one). It is $(x(n) = 0, x(n) = 0)$. It is easy to see that this element is in K, since it is a function from $\mathbb{N}$ to itself, also both elements of the pair adhere to the condition that they'd be no greater for all $n \in \mathbb{N}$. It is also easy to see that there is no pair which is not strictly greater than this element. Suppose, for contradiction, there was such a pair, then for some natural number n, the $x'(n) \leq x(n) = 0$, but 0 is the smallest element of $\mathbb{N}$, contrary to assumed. Thus there is a smallest element and the set of minimal elements (consisting of the smallest element).

1.3.5 Answer 11

Suppose for contradiction there was an $x(n)$ which wasn't bounded above by any $y(n)$. Then its value at n would be some $m \in \mathbb{N}$, but we can construct $y(n) = x(n) + 1$, in contradiction to our initial assumption. Thus every element in K must be bounded from above. Observe also that the bounds themselves are members of K, and since they are, then each bound has its own bound, and by transitivity of the relation "the upper bound of" we can see that every member of K has infinitely many such bounds.

1.4 Problem 4

1. Prove by induction that the following function definitions are equivalent: $f(0) = 0$, $f(1) = 10$, $f(n+2) = f(n+1) + 6f(n)$. And $f(n) = 2 * 3^n + (-2)^{n+1}$.

2. Is the function defined above onto?

1.4.1 Answer 12

Using mathematical induction, let's first verify the base step: $f(0) = 0$ and $f(0) = 2 * 3^0 + (-2)^1 = 2 - 2 = 0$.

Inductive step will first establish the relation between three subsequent terms of the sequence, and then verify that the same relation holds for both definition of f.

$$\begin{aligned}
f(n+3) &= f(n+2) + 6f(n+1) \\
&= f(n+1) + 6f(n) + 6f(n+1) \\
&= 7f(n+1) + 6f(n) \\
f(n+3) - f(n+2) &= 7f(n+1) + 6f(n) - f(n+1) - 6f(n) \\
&= 6f(n)
\end{aligned}$$

Now, I will use the induction hypothesis to demonstrate that $f(n+2) - f(n+1) = f(n)$:

$$\begin{aligned}
&f(n+2) - f(n+1) - 6f(n) = 0 \\
&2 * 3^{n+2} + (-2)^{n+3} - 2 * 3^{n+1} - (-2)^{n+2} - 6 * 2 * 3^n - (-2)^{n+1} = 0 \\
&2 * 3^{n+2} - 2 * 3^{n+1} - 6 * 2 * 3^n + (-2)^{n+3} - (-2)^{n+2} - (-2)^{n+1} = 0 \\
&2 * 3^{n+2} - 6 * 3^n - 6 * 2 * 3^n + \\
&\qquad + (-2) * (-2) * (-2)^{n+1} - (-2)(-2)^{n+1} - 6 * (-2)^{n+1} = 0 \\
&2 * 3^{n+2} - 3^n(6 + 6 * 2) + (-2)^{n+1}(4 - (-2) - 6) = 0 \\
&2 * 3^{n+2} - 3^n * 2 * 9 + (-2)^{n+1} * 0 = 0 \\
&0 + 0 = 0.
\end{aligned}$$

Having showed that the induction step holds too, using the principle of mathematical induction the proof is complete.

1.4.2 Answer 13

Recall that in order to demonstrate that the function is not onto it is enough to find an element in its co-domain, which is not in its domain. Suppose this function had a value in range $(0, 10)$, but since multiplication is preserving inequality and is non-decreasing, the value would have to be between zero and one, but there are no natural numbers between zero and one, hence the function is not onto.