Assignment 16, Discrete Mathematics

Oleg Sivokon

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Contents

1 Problems

1.1 Problem 1

Graph $G = (V, E)$ is defined as follows: vertex $v \in V$ must be a subset of cardinality exactly 3 taken from the set $S = \{1, 2, 3, 4, 5, 6, 7\}$. An edge $e \in E, e = \{v_i, v_j\}$ exists only when $\left|v_i \cap v_j\right| = 1, v_i, v_j \in V$.

1. Prove that G is a connected graph.
2. Prove that G is not a bipartite graph.
3. Prove or disprove: G is an Eulerian graph.
4. Prove that G is Hamiltonian.

1.1.1 Answer 1

Suppose, for contradiction that G is not connected, then it would have to have two edges, call them v_1 and v_2 such that there isn't a path between them. From the definition of the graph it follows that each vertex connects to the number of vertices that can be calculated as follows: for each of the elements of the set assigned to the edge v_1, call them a, b and c, we must select such other vertices that they only contain a, but not b or c. There are clearly $3 \cdot \binom{7-2}{3} = 30$ of such edges. Now we argue, by pigeonhole principle, that it will not be possible to have two disconnected nodes v_1 and v_2 with fan-out of 30 in a graph containing all in all $\binom{7}{3} = 35$ vertices.

1.1.2 Answer 2

Recall that a bipartite graph cannot contain an odd-length cycle. Thus, illustrating such cycle would prove our claim. And indeed there are such cycles in G, for instance: $(\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\})$.

1.1.3 Answer 3

From definition we know that G is regular and from 1.1.1 we know that it is connected and of the degree 30. This is sufficient to show that the graph is Eulerian, since it is connected and every vertex has an even degree.

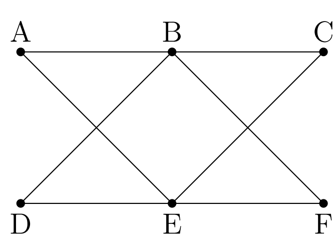
1.1.4 Answer 4

Using Dirac's theorem, which states that a simple graph with n vertices ($n \geq 3$) is Hamiltonian if every vertex has degree $\frac{n}{2}$ or greater, we claim that G is Hamiltonian. G has $35 > 3$ vertices and every vertex has a degree $30 > \frac{35}{2}$. From definition of G, G is simple since its vertices are defined to be connected if they share only one set element, thus they can't connect to themselves (form loops), nor can they connect to other vertices more than once.

1.2 Problem 2

1. Draw an Eulerian graph with even number of vertices which doesn't have a perfect matching.
2. Prove that if G is Hamiltonian with an even number of vertices, then it has a perfect matching.

1.2.1 Answer 5



The graph is Eulerian because it has even fan-out at every vertex, it also consists of six vertices, thus matching the requirement of having an even number of vertices. There is no perfect matching in this graph, which is easy to get convinced of if you notice that it has two vertices with degree 4 viz. B and E, which don't connect to each other. In other words, both these vertices would have to connect to every one of the remaining vertices, thus selecting only two of the remaining four (since neithr of them can be connected by the perfect matching to more than one vertex). All remaining vertices have degree 2, meaning that they don't interconnect. Thus there is no way to connect them pairwise without also connecting them to the vertices with degree 4.

1.2.2 Answer 6

Having a Hamiltonian graph means having a cycle. If that cycle also happens to have an even number of vertices, it will certainly have an odd number of edges. We can always order the edges in the same order they are connected in the graph, selecting only the oddly numbered edges from this ordering will necessarily produce a perfect matching. It's easy to see this, because, all vertices would be included by construction and the matching number will be exactly half of the number of vertices, which is the definition of perfect matching.

1.3 Problem 3

1. How many perfect matchings there are in K_5 graph?
2. How mahy perfect matchings there are in bipartite graph $K_{5,5}$?
3. Suppose we removed 4 edges from bipartite graph $K_{5,5}$, how many perfect matchings there are in this graph?

1.3.1 Answer 7

There are no perfect matchings in K_5. By definition, the number of edges in the perfect matching must be half of the number of nodes in the graph, but it also must be a natural number, while 5 isn't evenly divisible by 2.

1.3.2 Answer 8

Bipartite graph where each vertex in the both independent sets connects to each vertex in the other set can be perfectly matched in the same number of ways as many one-to-one and onto functions we can define where the cardinality of domain and co-domain are both 5. Or, in other words, it is the same as the number of permutations of a sequence of length 5, viz 5!.

1.3.3 Answer 9

Similar to 1.3.2, we first count the edge between the node that has only one edge leading to the opposite independent set, and the rest of the nodes will only be able to choose among four remaining nodes in the opposite independent set, thus the final answer would be $1 + 4!$.

1.4 Problem 4

Let P be a simple path with ten nodes (i.e. a graph with two leafs and eight nodes in between). Graph G is defined by taking P and adding two new nodes to it, v and u, such that every node in P now connects to both v and u and there's an edge between v and u.

1. Show G is planar.
2. Add one more node, w and connect it to v, u, and to the both leafs of P. Call this graph H. Prove H isn't planar.

1.4.1 Answer 10

In order to show that the graph isn't planar, we only need to show that it doesn't embed neither K_5 graph, nor $K_{3,3}$ graph. Note that G doesn't contain K_5 because there are two nodes in it with degree 11 and the rest are either 4 or 3, but no three nodes with degree 4 form a K_3. In other words, if we order the nodes of P starting at one leaf and adding the node connected to the node we just counted, then no two evenly numbered nodes are connected, neither two oddly numbered nodes are connected. In order to show that $K_{3,3}$ is not a minor of G, note that there are no three vertices in G such that they would connect to three same other vertices. Each of the vertices in P connects to at most two of its neighbours. As for the v and u, which would be the only likely candidates, we are one vertex short to complete to $K_{3,3}$. Since neither K_5 nor $K_{3,3}$ are G's minors, G must be blanar.

1.4.2 Answer 11

Continuing the argument for the embedding of K_5 as a minor of G in 1.4.1, we see that once we add one more node and edges between the leafs of P, v and u, we can now find a minor $\{x, y, v, u, w\}$ which is a K_5 (x and y being the leafs of P). The edges between v, u, w are a given. We are also given that x and y link to any of v, u, w. We thus only need to show that it is possible to connect x and y in a way that doesn't involve any of v, u or w. But this is immediate from definition of P. Hence H embeds K_5 as a minor, hence it cannot be planar.

1.5 Problem 5

1. What is the chromatic indes of P from 1.4?
2. What is the chromatic indes of G from the previous problem?
3. What is the chromatic indes of H from the previous problem?

1.5.1 Answer 12

Since P is a path, we only need 2 colors to paint its edges.

1.5.2 Answer 13

The highest degree in G is 11, thus we would need at least so many colors to paint it.

1.5.3 Answer 14

By similar argument, we would need at least 12 colors to paint the edges of H.