Assignment 11, Discrete Mathematics

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Contents

1 Problems

1.1 Problem 1

In all questions that follow, establish whether $x \in y$ or $x \subseteq y$.

1. $x = \{1, 2\}, y = \{1, 2, 3\}$.
2. $x = \{3\}, y = \{\{1\}, \{2\}, \{3\}\}$.
3. $x = \{1, 2\}, y = \{\{1, 2\}, 3\}$.
4. $x = \{1, 3\}, y = \{\{1, 2\}, 3\}$.
5. $x = \emptyset, y = \emptyset$.
6. $x = \{\emptyset\}, y = \{\emptyset\}$.
7. $x = \{1\}, y = \{1, 2\}$.
8. $x = \emptyset, y = P(\{1, 2, 3\})$.

1.1.1 Answer 1

1. $x \subseteq y$.
2. $x \in y$.
3. $x \in y$.
4. $x \notin y \wedge x \nsubseteq y$.
5. $x \subseteq y$.
6. $x \subseteq y$.
7. $x \subseteq y$.
8. $x \subseteq y$.

1.2 Problem 2

Prove or disprove:

1. $(A \setminus B) \setminus B = A \setminus B$.
2. $A \setminus (B \setminus A) = A$.
3. $A \subseteq P(A)$.

1.2.1 Answer 2

The statement is true. Suppose for contradiction that there is a $b \in A \setminus B$ s.t. $b \notin (A \setminus B) \setminus B$. Then, we know that $b \in A$ and $b \notin B$. But we assumed $b \notin A \vee b \in B$. This is a contradiction, hence $(A \setminus B) \setminus B = A \setminus B$. Additionally, it is easy to see that set subtraction is idempotent (i.e. if applied repeatedly, will yield the same result after first application).

1.2.2 Answer 3

The statement is true. Suppose for contradiction there was an $a \in A$, which is not in $A \setminus (B \setminus A)$. Then such a would have to be in A, but not in $(B \setminus A)$. I will now show that no elements of $(B \setminus A)$ is in A, thus subtracting this group would not remove any elements of A, i.e. $A \cap (B \setminus A) = \emptyset$. Suppose b was in A and in $(B \setminus A)$ at the same time, then it would have to be in A, but A is excluded from $(B \setminus A)$. This is in contradiction to the previous assumtion. Therefore $A \setminus (B \setminus A) = A$.

1.2.3 Answer 4

This statement isn't true in general. An example which disproves it would be an assignment: $A = \{1, 2\}$. $P(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. $A \not\subseteq P(A)$.

1.3 Problem 3

Prove the following identity using $A \setminus B = A \cap B'$: $(A_1 \cup A_2) \setminus (B_1 \cap B_2) = (A_1 \cup B_1) \cup (A_1 \cup B_2) \cup (A_2 \cup B_1) \cup (A_2 \cup B_2)$.

1.3.1 Answer 5

The proof doesn't require discussion. It is accomplished by simple set-theoretical manipulations.

$$
\begin{aligned}
(A_1 \cup A_2) \setminus (B_1 \cap B_2) &\iff && \text{Given} \\
A_1 \setminus (B_1 \cap B_2) \cup A_2 \setminus (B_1 \cap B_2) &\iff && \text{Union over subtraction} \\
A_1 \cap (B_1 \cap B_2)' \cup A_2 \cap (B_1 \cap B_2)' &\iff && \text{Required identity} \\
A_1 \cap (B_1' \cup B_2') \cup A_2 \cap (B_1' \cup B_2') &\iff && \text{De Morgan's law} \\
(A_1 \cap B_1') \cup (A_1 \cap B_2') \cup (A_2 \cap B_1') \cup (A_2 \cap B_2') &\iff && \text{Union over subtraction} \\
(A_1 \setminus B_1) \cup (A_1 \setminus B_2) \cup (A_2 \setminus B_1) \cup (A_2 \setminus B_2) &\iff && \text{Required identity}
\end{aligned}
$$

1.4 Problem 4

Given $\forall n \in \mathbb{N} : A_n = \{x \in \mathbb{R} | 4 \leq x \leq 2n+2\}$ and $\forall n \in \mathbb{N} : B_n = A_{n+1} \setminus A_n$. *For the purpose of this exercise* $\mathbb{N}$ *contains zero.*

1. Calculate A_0, A_1, A_2, A_3 and B_0, B_1, B_2.
2. Write closed-form formula for B_n.
3. Calculate $\bigcup_{2 \leq n \in \mathbb{N}} B_n$. Prove the result using set containment relation.
4. Using general rules for set subtraction and union as well as De Morgan laws as applied to universal and existential quantifiers, prove the generalized formulae for $\bigcup_{i \in I}(A')$ and $\bigcap_{i \in I}(A')$.
5. Let $D_n = \mathbb{R} \setminus B_n$. Calculate $\bigcup_{2 \leq n \in \mathbb{N}} D_n$.

1.4.1 Answer 6

It is easy to do the calculation by hand, but to make it more interesting I wrote some Prolog code to do it. An intuitive way to see what is going on is to observe that every consequent A will contain the entire previous set and two more members, which are greater than the maximal element of the previously collected set.

```
:- use_module(library(clpfd)).

set_A(N, Set) :-
    High is N * 2 + 2,
    X in 4..High,
    findall(X, indomain(X), Set).

set_B(N, Set) :-
    set_A(N, A_n),
    N1 is N + 1,
    set_A(N1, A_n1),
    subtract(A_n1, A_n, Set).

all_sets(N, Pred, Answer) :-
    X in 1..N, indomain(X),
    call(Pred, X, Answer).

zip(X, Y, [X, Y]).

join(_, [X], X) :- !.
join(Sep, [X | Xs], S) :-
    join(Sep, Xs, Sx),
    string_concat(Sep, Sx, Sy),
    string_concat(X, Sy, S).

question_1(Na, Nb, As, Bs):-
    findall(As, all_sets(Na, set_A, As), As),
    findall(Bs, all_sets(Nb, set_B, Bs), Bs),
    X in 1..Na,
    Y in 1..Nb,
    findall(X, indomain(X), Nas),
    findall(Y, indomain(Y), Nbs),
    maplist(join(','), As, Jas),
    maplist(join(','), Bs, Jbs),
    maplist(zip, Nas, Jas, Zas),
    maplist(zip, Nbs, Jbs, Zbs),
    maplist(format('$A_~p=\\{~p\\}$~n~n'), Zas),
    maplist(format('$B_~p=\\{~p\\}$~n~n'), Zbs).
```

$A_1 = \{4\}$

$A_2 = \{4, 5, 6\}$

$A_3 = \{4, 5, 6, 7, 8\}$

$B_1 = \{5, 6\}$

$B_2 = \{7, 8\}$

$B_3 = \{9, 10\}$

1.4.2 Answer 7

$B = \{x \in \mathbb{R} | 2(n+1) < x \leq 2(n+2)\}$.

1.4.3 Answer 8

First, let me make the claim that $\bigcup_{2 \leq n \in \mathbb{N}} = A_{n+1}$. Below, is the proof that doesn't require division in two cases (it peruses the definition of B_n and the general technique of extraction of the last term of a sequence).

$$\begin{aligned}
\bigcup_{2 \leq n \in \mathbb{N}} B_n &= \bigcup_{2 \leq n \in \mathbb{N}} A_{n+1} \setminus A_n \\
&= \bigcup_{2 \leq n \in \mathbb{N}} A_{n+1} \setminus \bigcup_{2 \leq n \in \mathbb{N}} A_n \\
&= \bigcup_{2 \leq n \in \mathbb{N}} A_n \setminus \bigcup_{2 \leq n \in \mathbb{N}} A_n \cup A_{n+1} \\
&= A_{n+1}
\end{aligned}$$

1.4.4 Answer 9

First, let me reiterate De Morgan's law for first-order quantifiers:

$$\neg \forall x . \phi \iff \exists x . \neg \phi .$$

Combining it with the definitions of union of complement sets and intersection of complement sets gives us the following proof:

$$
\begin{aligned}
x \in \bigcap_{i \in I} (A_i)' &\iff \forall i (i \in I \implies x \in A') \\
&\iff \exists i (i \in I \implies x \in A)' \\
&\iff x \in (\bigcup_{i \in I} A_i)'.
\end{aligned}
$$

Note that the proof treats negation equivalently to complementation.

The proof for the union is symmetrical:

$$
\begin{aligned}
x \in \bigcup_{i \in I} (A_i)' &\iff \exists i (i \in I \implies x \in A') \\
&\iff \forall i (i \in I \implies x \in A)' \\
&\iff x \in (\bigcap_{i \in I} A_i)'.
\end{aligned}
$$

1.4.5 Answer 10

Reusing definitions and conclusions derived above gives:

$$D_n = \mathbb{R} \setminus B_n = \mathbb{R} \cap (B_n)'.$$

$$
\begin{aligned}
\bigcap_{2\leq n\in\mathbb{N}} D_n &= \bigcap_{2\leq n\in\mathbb{N}} \mathbb{R}\cap(B_n)' \\
&= \mathbb{R}\cap \bigcap_{2\leq n\in\mathbb{N}} (B_n)' \\
&= \mathbb{R}\cap (\bigcup_{2\leq n\in\mathbb{N}} B_n)' \\
&= \mathbb{R}\cap (\bigcup_{2\leq n\in\mathbb{N}} A_{n+1}\setminus A_n)' \\
&= \mathbb{R}\cap (\bigcup_{2\leq n\in\mathbb{N}} A_{n+1}\setminus \bigcup_{2\leq n\in\mathbb{N}} A_n)' \\
&= \mathbb{R}\cap (\bigcup_{2\leq n\in\mathbb{N}} A_{n+1}\cap (\bigcup_{2\leq n\in\mathbb{N}} A_n)')' \\
&= \mathbb{R}\cap (\bigcup_{2\leq n\in\mathbb{N}} A_{n+1})'\cup (\bigcup_{2\leq n\in\mathbb{N}} A_n)' \\
&= \mathbb{R}\cap (A_{n+1})' \\
&= \mathbb{R}\setminus A_{n+1}
\end{aligned}
$$