Assignment 11, Infinitesimal Calculus

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Contents

1 Problems

1.1 Problem 1

1. Prove that for any natural number n it holds that:
$$4^n \geq \binom{2n}{n}.$$

2. Prove by induction, or in any other way, that for any natural number n it holds that:
$$\binom{2n}{n} \geq \frac{4^n}{2n+1}.$$

1.1.1 Answer 1

Discussion: The idea behind the proof is to show that $\binom{2n}{n}$ is a term in the binomial expansion representing 4^n. Since all other terms in this expansion will be non-negative, then 4^n will be at least as big as $\binom{2n}{n}$.

Proof: Put $4^n = (1+1)^{2n}$, using binomial formula obtains:

$$\begin{aligned}(1+1)^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} 1^{2n} 1^{2n-k} \\ &= \binom{2n}{n} + \sum_{k=0, k\neq n}^{2n} \binom{2n}{k}.\end{aligned}$$

We know that $\binom{2n}{n}$ is a term of binomial expansion because we know that $k_i \leq 2n$, which implies that since k_i, n are natural numbers, there exists $k_i = n$. Besides, thre might exist other terms in binomial expansion which are guaranteed to be non-negative. Hence, $4^n \geq \binom{2n}{n}$.

1.1.2 Answer 2

Discussion: In order to make the proof a little less verbose, I will prove a stronger claim, viz. $\binom{2n}{n} \geq \frac{4^n}{2n}$. Since n is positive, $2n < 2n + 1$, hence $\frac{4^n}{2n} > \frac{4^n}{2n+1}$. The proof will proceed by induction on n. First I will find a factor s.t. multiplying it with S_{n-1} I will obtain S_n, and then multiply it with the $\frac{4^n}{2n}$ to show that it will necessary be at leas as large as $\frac{4^{n+1}}{2(n+1)}$.

Proof: Using mathematical induction, let's first prove the base step, where $n = 1$:

$$\begin{aligned}\binom{2*1}{1} &\geq \frac{4^1}{2*1} \iff \\ \frac{2!}{1!(2-1)!} &\geq \frac{4}{2} \iff \\ \frac{2}{1} &\geq 2 \iff \\ 2 &\geq 2.\end{aligned}$$

Now, to the inductive step (for $n > 1$), some useful simplification first:

$$\begin{aligned}\binom{2n}{n} &= \frac{(2n)!}{n!(2n-n)!} \\ &= \frac{(2n!)}{n!n!}.\end{aligned} \tag{1}$$

Invoking inductive hypothesis $\binom{2(n+1)}{(n+1)} \geq \frac{4^n}{2n}$:

$$\begin{aligned}\binom{2(n+1)}{n+1} &= \frac{(2(n+1))!}{(n+1)!(2(n+1)-n+1)!} \\ &= \frac{(2n!)(2n+1)(2n+2)}{n!(n+1)n!(n+1)} \\ &= \frac{(2n!)(2n+1)2(n+1)}{n!(n+1)n!(n+1)} \\ &= \frac{(2n!)(2n+1)2}{n!n!(n+1)}.\end{aligned} \tag{2}$$

Dividing 2 by 1 gives us the factor $\frac{(2n+1)2}{(n+1)}$. Thus:

$$\begin{aligned}
\frac{(2n+1)2}{n+1} \times \frac{4^n}{2n} &\geq \frac{4^{n+1}}{2(n+1)} \\
\frac{(2n+1)2 * 4^n}{2n(n+1)} &\geq \frac{4^{n+1}}{2(n+1)} \\
\frac{(2n+1)2 * 4^n}{n} &\geq 4^{n+1} \\
\frac{(2n+1)2 * 4^n}{n} &\geq 4^n * 4 \\
\frac{(2n+1)2}{n} &\geq 4 \\
\frac{4n+2}{n} &\geq 4 \\
4 + \frac{2}{n} &\geq 4
\end{aligned}$$

Since $n > 1$, $\frac{2}{n}$ is positive, hence the inequality holds. This completes the inductive step. Hence, by using mathematical induction the proof is complete.

1.2 Problem 2

1. Given $k, l \in \mathbb{N}$, prove that $a = k + l\sqrt{2}$ is irrational.
2. Prove that for every natural number n it holds that:

$$\sum_{i=0}^{n} \sqrt{2}^{i}$$

is irrational.

1.2.1 Answer 3

1.2.2 Answer 4

1.3 Problem 3

1. Given real numbers a and b prove that if

$$\frac{|a|}{2} > \left| b - \frac{a}{2} \right|,$$

then

$$|b - a| < |a| .$$

1.3.1 Answer 5

1.4 Problem 4

Given $a, b, c \in \mathbb{R}$,

1. Prove that if $a > 0$ and $a + b > a + c$, then $b > c$.
2. Prove that if $a > 0$ and $ab > ac$, then $b > c$.
3. Prove that if $|a| > |b|$ iff $a^2 > b^2$.
4. Prove that if $b > c$ and $|a - b| > |a - c|$, then $b > a$.
5. Show (my means of example) that from $b > c$ and $b > a$ it doesn't follow that $|a - b| > |a - c|$.

1.4.1 Answer 6

1.4.2 Answer 7

1.4.3 Answer 8

1.4.4 Answer 9

1.4.5 Answer 10

1.5 Problem 5

Solve the equation:

$$\lfloor |x+1| - |x-1| \rfloor = x.$$

1.5.1 Answer 11

1.6 Problem 6

Definition: set A of real numbers is called **dense in interval** I if for every $x, y \in I$ s.t. $x < y$ there exists $a \in A$ such that $x < a < y$.

1. Let A be dense in interval $[0,1]$, prove that set $B = \{na | a \in A, n \in \mathbb{N}\}$ is dense in interval $[0, \infty)$.
2. Let $A = \mathbb{R}$, prove that A isn't dense in I iff exists an open interval (x, y) in I, such that $A \cap (x, y) = \emptyset$.
3. Let A be the real numbers in interval $[0,1]$, prove that the set $C = \{\frac{a+1}{n^2} | a \in A, n \in \mathbb{N}\}$ isn't dense in $[0,1]$.

1.6.1 Answer 12

1.6.2 Answer 13

1.6.3 Answer 14