Assignment 13, Infinitesimal Calculus

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1 Problems

1.1 Problem 1

Given the sequence (a_n) defined as $0 < a_1 < 6$ and $a_{n+1} = \sqrt{6a_n}$ for all n.

1. Prove that the sequence converges.
2. Find $\lim_{n\to\infty} a_n$.

1.1.1 Answer 1

First, observe that every next element of the sequence (a_n) is bigger than the previous. Using mathematical induction we can show that:

Base step: $\sqrt{a_1} < \sqrt{a_2}$ because $a_1 < 6$ and thus $\sqrt{a_1} < \sqrt{6}$, consequently (from order preservation under multiplication) $\sqrt{a_1} \times \sqrt{a_1} < \sqrt{6} \times \sqrt{a_1}$ and hence $\sqrt{a_1} < \sqrt{a_2}$.

Inductive step:

$$\begin{aligned}
a_k &< \sqrt{6 \times a_{k+1}} \\
&\quad \textit{Using induction hypothesis, } k = n - 1 \\
\sqrt{6 \times a_k} &< \sqrt{6 \times a_{k+1}} \\
\sqrt{6} \times \sqrt{a_k} &< \sqrt{6} \times \sqrt{a_{k+1}} \\
&\quad \textit{Order preservation under multiplication} \\
\sqrt{a_k} &< \sqrt{a_{k+1}} \\
a_k &< a_{k+1}
\end{aligned}$$

Hence, by mathematical induction, the sequence (a_n) is an increasing one.

Now we demonstrate that this sequence is bounded above and it has a supremum. Again, using induction, we can choose 6 to be the upper bound.

Base step: $a_1 < 6$ by definition.

Inductive step:

$$\begin{aligned}
a_k &< 6 \\
&\quad \textit{Using induction hypothesis, } k = n - 1 \\
\sqrt{6 \times a_k} &< \sqrt{6 \times 6} \\
\sqrt{6} \times \sqrt{a_k} &< \sqrt{6} \times \sqrt{6} \\
&\quad \textit{Order preservation under multiplication} \\
\sqrt{a_k} &< \sqrt{6} \\
a_k &< 6
\end{aligned}$$

Hence, by mathematical induction, (a_n) has an upper bound. It is trivial to see that this is also the supremum, since a_1 is defined to be strictly less than 6, so no smaller upper bound can exist.

Having showed both these properties, we can now use the monotone sequence theorem to claim that this sequence converges. Accidentally, we also found the limit of this sequence, which wasn't due until the next question.

1.1.2 Answer 2

We have already found the limit of (a_n) as n approaches infinity. This is given by a corollary from the monotone sequence theorem: the supremum is also the limit of a monotone sequence. Thus the answer is 6.

1.2 Problem 2

Find the limits of:

1. $$\lim_{n\to\infty} \frac{(-5)^n - 2^n + 2}{3^n + (-2)^n - 2}.$$

2.

$$\lim_{n\to\infty} \frac{3^n + (-2)^n - 2}{(-5)^n - 2^n + 2}.$$

3.

$$\lim_{n\to\infty} (\lfloor 2n \rfloor - 2\lfloor n \rfloor).$$

4.

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}.$$

1.2.1 Answer 3

We can see that the denominator of this limit will be a positive integer for $n > 1$. The numerator will alternate between positive and negative values, where the difference will only increase with time since $(-5)^2 n > 2^n$ and $(-5)^{2n-1} > 2^n$, thus this expression has two limit points: ∞ and $-\infty$.

1.2.2 Answer 4

The same reasoning we put forward in 1.2.1 applies here, but now the numerator will be some positive integer, smaller than the denominator. Thus this expression will have a single limit at 0.

1.2.3 Answer 5

Here I will assume n to be real, otherwise the limit is trivially 0. In this case we can split the sequence into the following subsequences:

$$(a_n) = \{\lfloor x \rfloor | x \in \mathbb{R} : x \pmod 2 < 0.5\}$$
$$(b_n) = \{\lfloor x \rfloor | x \in \mathbb{R} : x \pmod 2 \geq 0.5\}$$

It's easy to see that $(a_n) \cup (b_n)$ gives the initial set. These subsets have constant limits $\{0, 1\}$ as is given by their generating formulae: $\lfloor x \pmod 2 <$

$0.5\rfloor$ and $\lfloor x \pmod 2) < 0.5\rfloor$ resp. But the whole expression doesn't have a single limit point, since we can always find as many points as we like that would not be an ϵ distance from one of the limit points found.

1.2.4 Answer 6

We can show this expression approaches 1 as n gets larger using the following algebraic transformations:

$$\begin{aligned}\frac{\sqrt[n]{n!}}{n} &= \sqrt[n]{\frac{n!}{n^n}} \\ &= \sqrt[n]{\prod_{k=1}^{n} \frac{n-k}{n}}\end{aligned}$$

Since $n - k < n$, $\frac{n-k}{n} < 1$. The product of arbitrary many terms all of which are less than one will be less than one. And will become smaller as the number of terms grows. However, the nth root will become progressively large as the fractional term becomes smaller, yet never exceeding 1. Thus the limit of this expression is 1.

1.3 Problem 3

Let (a_n) and (b_n) be sequences both bounded from above.

1. Prove $\sup\{a_n + b_n | n \in \mathbb{N}\} \leq \sup\{a_n | n \in \mathbb{N}\} + \sup\{b_n | n \in \mathbb{N}\}$.
2. Find such (a_n) and (b_n) which are equal under the condition given in (1).
3. Find such (a_n) and (b_n) for which the inequality given in (1) holds.

1.3.1 Answer 7

Suppose, for contradiction, that there is some such sequences (a_n) and (b_n), for which $\sup\{a_n + b_n | n \in \mathbb{N}\} > \sup\{a_n | n \in \mathbb{N}\} + \sup\{b_n | n \in \mathbb{N}\}$. But we know that every element of (a_n) is at least as large as $\sup(a_n)$, and the same is true of (b_n). Let's call the supremum of the sum of the sequences s_1, the supremum of (a_n)—s_2 and the supremum of (b_n)— s_3. Since $s_1 > s_2 + s_3$, either (a_n) or (b_n) must contain an element which is as large as $s_2 + s_1 - s_2$ (and similarly for (b_n).) But this is a contradiction, since s_2 is the largest element of (a_n) (and similarly for (b_n).) Hence, the initial assumption is false, hence the supremum of the element-wise sum of two sequences cannot be greater than the sum of the suprema of these sequences.

1.3.2 Answer 8

An example of $(a_n) = \frac{1}{n}$ whose supremum is 1 and $(b_n) = \frac{2}{n}$. Since they both reach their maximum output at $n = 1$, the sum of their suprema and the supremum of their sum is the same.

1.3.3 Answer 9

To contrast 1.3.2, $(a_n) = \frac{1}{n}$ and $(b_n) = \frac{1}{n-1}$ reach their suprema at different times, ((b_n) isn't even defined at the time (a_n) reaches its maximum), the supremum of their sum is thus 1.5, but the sum of their suprema is 2.

1.4 Problem 4

Let $(a_n) = n - \lfloor \sqrt{n} \rfloor^2$.

1. Prove that (a_n) is bounded from below.
2. Prove that 0 is a limit point of (a_n).

3. Find $\inf\{a_n | n \in \mathbb{N}\}$, $\liminf_{n\to\infty} a_n$. Establish whether (a_n) has minima.
4. Given natural number ℓ, prove that almost for all n it holds that $n < \sqrt{n^2 + \ell} < n + 1$.
5. Prove that every natural number is a limit point of (a_n).
6. Is (a_n) bounded from above?
7. Find $\limsup_{n\to\infty} a_n$.

1.4.1 Answer 10

I will claim that the lower bound on this sequence is 0. The term $\lfloor\sqrt{n}\rfloor^2$ can be either equal to n (when n is a perfect sqare), or it can be smaller than n. Assume, for contradiction, that $\lfloor\sqrt{n}\rfloor^2 > n$, then it gives $\lfloor\sqrt{n}\rfloor > \sqrt{n}$, but since n is positive, this is impossible. Hence contradiction. Hence, the original claim stands.

1.4.2 Answer 11

We can choose the subsequence of (a_n) such that it consists only of the perfect squares. In which case the $\lim_{m\to\infty}(a_m) = 0$ because $m = n * n$, and subsequently $m = \lfloor\sqrt{m}\rfloor^2$.

1.4.3 Answer 12

Following the argument similar to 1.4.1, the greatest lower bound on (a_n) is 0. Assume there was some positive x, a candidate for a greater lower bound, then $n - \lfloor\sqrt{n}\rfloor^2 > x$. Let's try $n = 1$ to show the contradiction. Substituting $n = 1$ gives $0 > x$, but we assumed $x > 0$. Hence, contradiction, hence the sequence has greates lower bound at zero.

1.4.4 Answer 13

Through some algebraic transformations we arrive at:

$$\begin{aligned}
&n < \sqrt{n^2 + \ell} < n + 1 \\
&n^2 < n^2 + \ell < n^2 + 2n + 1 \\
&\textit{First inequality is obviously true for } \ell > 0 \\
&n^2 + \ell < n^2 + 2n + 1 \\
&\ell < 2n + 1 \\
&\textit{Second inequality is true for } \ell < 2n + 1.
\end{aligned}$$

Thus the left side of inequality holds for $\ell > 0$ and the right side holds for $\ell < 2n+1$. Since ℓ is a constant, and n can be made as large as desired, the inequality holds for almost all n.

1.4.5 Answer 14

Pick a natural number x, without loss of generality we assume: $\lim_{m \to \infty}(a_m) = x$, where m is somehow choosen from the set of all values of n. We can define a selection rule to be "whenever $n + x$ is a perfect square", since there are infinitely many perfect squares, and there isn't a largest perfect square, x can be any number.

1.4.6 Anser 15

No, (a_n) doesn't have a supermum. Assume it had one, let's say N, then for all $n - \lfloor \sqrt{n} \rfloor^2 < N$. But we could think of $m = N^2 - 1$, which if we substitute back into our original formula:

$$
\begin{aligned}
&m - \lfloor \sqrt{m} \rfloor^2 < N \\
&N^2 - 1 - \lfloor \sqrt{N^2 - 1} \rfloor^2 < N \\
&N^2 - 1 - \sqrt{(N-1)^2}^2 < N \\
&N^2 - 1 \textit{ cannot be a perfect square} \\
&N^2 - 1 - (N-1)^2 < N \\
&N^2 - 1 - N^2 + 2N - 1 < N \\
&2N - 2 < N.
\end{aligned}
$$

Which is a contradiction (because N must be greater than 2). Hence no supremum exists.

1.4.7 Answer 16

Since we just showed that (a_n) diverges, the $\limsup_{n\to\infty}(a_n)$ doesn't exist in the narrow sense of the word, or is equal to ∞ in a more general sense.

1.5 Problem 5

Let (a_n) and (b_n) be sequences.

1. Assume $\lim_{n\to\infty}(a_n + b_n) = L$ for some finite L. Prove that if (a_n) is bounded then (b_n) is bounded too.
2. Assume $\lim_{n\to\infty}(a_n + b_n) = L$ for some finite L. Prove that if (a_n) has a limit point a, then $L - a$ is a limit point of (b_n).
3. Assume (a_n) has 20106 limit points and (b_n) has 20474 limit points. Prove that $(a_n + b_n)$ diverges.

1.5.1 Answer 17

Assume, for contradiction, (b_n) is unbounded, then $\lim_{n\to\infty}(b_n)$ is either ∞ or $-\infty$, but from the addition rules involving infinity we know that $\infty + L = \infty$ and $L - \infty = -\infty$, but we are given that the sequence converges, hence contradiction. Hence (b_n) must be bounded.

1.5.2 Answer 18

Since the sequence of sum converges, each one of its subsequences converges to the same limit. Hence both (a_n) and (b_n) have defined limits. We can use the limit sum law to show that $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty}(a_n) + \lim_{n\to\infty}(b_n) = L$. Since we are given $\lim_{n\to\infty}(a_n) = a$, it follows that $\lim_{n\to\infty}(b_n) = L - a$, which completes the proof.

1.5.3 Answer 19

By pigeonhole principle, there must be a limit point exclusive to (b_n). Existence of this limit point would violate what we proved in 1.5.1. Since the statement in previous answer is a biconditional, then, in particular, it follows that sequence $(a_n + b_n)$ diverges (has no finite limit).