Assignment 11, Infinitesimal Calculus

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Contents

1 Problems

1.1 Problem 1

1. Prove that for any natural number n it holds that:
$$4^n \geq \binom{2n}{n}.$$

2. Prove by induction, or in any other way, that for any natural number n it holds that:
$$\binom{2n}{n} \geq \frac{4^n}{2n+1}.$$

1.1.1 Answer 1

Discussion: The idea behind the proof is to show that $\binom{2n}{n}$ is a term in the binomial expansion representing 4^n. Since all other terms in this expansion will be non-negative, then 4^n will be at least as big as $\binom{2n}{n}$.

Proof: Put $4^n = (1+1)^{2n}$, using binomial formula obtains:

$$\begin{aligned}(1+1)^{2n} &= \sum_{k=0}^{2n} \binom{2n}{k} 1^{2n} 1^{2n-k} \\ &= \binom{2n}{n} + \sum_{k=0, k\neq n}^{2n} \binom{2n}{k}.\end{aligned}$$

We know that $\binom{2n}{n}$ is a term of binomial expansion because we know that $k_i \leq 2n$, which implies that since k_i, n are natural numbers, there exists $k_i = n$. Besides, there might exist other terms in binomial expansion which are guaranteed to be non-negative. Hence, $4^n \geq \binom{2n}{n}$.

1.1.2 Answer 2

Discussion: In order to make the proof a little less verbose, I will prove a stronger claim, viz. $\binom{2n}{n} \geq \frac{4^n}{2n}$. Since n is positive, $2n < 2n+1$, hence $\frac{4^n}{2n} > \frac{4^n}{2n+1}$. The proof will proceed by induction on n. First I will find a factor s.t. multiplying it with S_{n-1} I will obtain S_n, and then multiply it with the $\frac{4^n}{2n}$ to show that it will necessary be at leas as large as $\frac{4^{n+1}}{2(n+1)}$.

Proof: Using mathematical induction, let's first prove the base step, where $n = 1$:

$$\begin{aligned}\binom{2*1}{1} &\geq \frac{4^1}{2*1} \iff \\ \frac{2!}{1!(2-1)!} &\geq \frac{4}{2} \iff \\ \frac{2}{1} &\geq 2 \iff \\ 2 &\geq 2.\end{aligned}$$

Now, to the inductive step (for $n > 1$), some useful simplification first:

$$\begin{aligned}\binom{2n}{n} &= \frac{(2n)!}{n!(2n-n)!} \\ &= \frac{(2n!)}{n!n!}.\end{aligned} \tag{1}$$

Invoking inductive hypothesis $\binom{2(n+1)}{(n+1)} \geq \frac{4^n}{2n}$:

$$\begin{aligned}\binom{2(n+1)}{n+1} &= \frac{(2(n+1))!}{(n+1)!(2(n+1)-n+1)!} \\ &= \frac{(2n!)(2n+1)(2n+2)}{n!(n+1)n!(n+1)} \\ &= \frac{(2n!)(2n+1)2(n+1)}{n!(n+1)n!(n+1)} \\ &= \frac{(2n!)(2n+1)2}{n!n!(n+1)}.\end{aligned} \tag{2}$$

Dividing 2 by 1 gives us the factor $\frac{(2n+1)2}{(n+1)}$. Thus:

$$\frac{(2n+1)2}{n+1} \times \frac{4^n}{2n} \geq \frac{4^{n+1}}{2(n+1)}$$
$$\frac{(2n+1)2 * 4^n}{2n(n+1)} \geq \frac{4^{n+1}}{2(n+1)}$$
$$\frac{(2n+1)2 * 4^n}{n} \geq 4^{n+1}$$
$$\frac{(2n+1)2 * 4^n}{n} \geq 4^n * 4$$
$$\frac{(2n+1)2}{n} \geq 4$$
$$\frac{4n+2}{n} \geq 4$$
$$4 + \frac{2}{n} \geq 4$$

Since $n > 1$, $\frac{2}{n}$ is positive, hence the inequality holds. This completes the inductive step. Hence, by using mathematical induction the proof is complete.

1.2 Problem 2

1. Given $k, l \in \mathbb{N}$, prove that $a = k + l\sqrt{2}$ is irrational.
2. Prove that for every natural number n it holds that:

$$\sum_{i=0}^{n} \sqrt{2}^i$$

is irrational.

1.2.1 Answer 3

Discussion: One way to see that summation of rational with irrational cannot produce a rational number is through invoking field axioms: summation must send the sum to the field of rationals, which would imply that the inverses of summands must be rationals too. This would also require summands to be rationals, but that's not possible.

Proof: Suppose, for contradiction that $k + l\sqrt{2} = a$ is rational, then

$$\begin{aligned}
k + l\sqrt{2} &= a && \text{Given} \\
k + (-k) + l\sqrt{2} &= a - k && k \text{ must have additive inverse in } \mathbb{Q} \\
0 + l\sqrt{2} &= a - k \\
l^{-1}l\sqrt{2} &= l^{-1}(a - k) && l \text{ must have multiplicative inverse in } \mathbb{Q} \\
1\sqrt{2} &= l^{-1}(a - k) \\
\sqrt{2} &= l^{-1}(a - k).
\end{aligned}$$

l^{-1}, q and k are all rationals, rationals are closed under multiplication and addition, hence $l^{-1}(q - k)$ must be rational, but $\sqrt{2}$ is not. Contradiction. Hence $a = k + l\sqrt{2}$ is irrational.

1.2.2 Answer 4

Discussion: The way to see that this statement is true is to divide the sequence into odd and even terms. All even terms will produce rationals (even poverse of square root of two will be rational). While all odd terms will produce irrational numbers (a product of even number of square roots of two will give a rational, but them multiplied with an irrational number will give an irrational). Since n must be at least one (and thus we are guaranteed to have at least one odd term in this sequence), the sum of the sequence will always be irrational.

Proof: Let's rewrite this sum as two sums of the form:

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \sqrt{2}^{2i} + \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} \sqrt{2}^{2i+1}.$$

It is easy to see that the first term is a sum of powers of 2, viz: $\sqrt{2}^2 +$

$\sqrt{2}^4 + \ldots + \sqrt{2}^{\lfloor n/2 \rfloor}$, which is just $2 + 4 + \ldots + 2^{\lfloor n/2-1 \rfloor}$. Similarly, the terms of the other sum can be expressed as $\sqrt{2}^1 + \sqrt{2}^3 + \ldots + \sqrt{2}^{\lfloor (n+1)/2 \rfloor}$.

Let's give names to the sequences we outlined: $S_1 = \sum_{i=0}^{\lfloor n/2 \rfloor} \sqrt{2}^{2i}$ and $S_2 = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} \sqrt{2}^{2i+1}$. Now, for contradiction, assume S_2 to be rational. Then let's define $\div$ to be the element-wise division of sequences, i.e. for sequences A and B of length n, $\div$ is defined to be:

$$\sum_{i=0}^{n} \frac{A_i}{B_i}.$$

It's easy to see that if all elements of A and B are rational, then $A \div B$ is rational too (because we only used addition and multiplication, which are known to be closed over rationals).

Observe that $S_2 \div S_3 = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} \sqrt{2}$, i.e. a sum of $n/2$ square roots of 2, which is the same as $n/2 * \sqrt{2}$, but we've just showed that the product of a rational and irrational cannot be rational (in previous question). Hence S_2 must be irrational, contrary to assumed. Hence it must be that $\sum_{i=0}^{n} \sqrt{2}^i$ is irrational. This completes the proof.

1.3 Problem 3

1. Given real numbers a and b prove that if
$$\frac{|a|}{2} > \left| b - \frac{a}{2} \right|,$$
then
$$|b - a| < |a|.$$

1.3.1 Answer 5

Discussion: The proof will be based on invariance of order under multiplication, in particular, it will rely on the fact that $|x| < |y| \iff x^2 < y^2$. This will allow us to solve the inequality without splitting it into several cases.

Proof: First, let's simplify the expression:

$$
\begin{aligned}
\frac{|a|}{2} &> \left|b - \frac{a}{2}\right| && \text{Given}\\
|a| &> 2\left|b - \frac{a}{2}\right| && \text{Invariance of order under multiplication}\\
|a| &> |2b - a| && \text{Distributivity of multiplication over addition}\\
|2b - a| &< |a| \\
|2b - a|^2 &< |a|^2 && \text{Invariance under exponentiation}\\
(2b - a)^2 &< a^2 && \text{Squares are always positive}\\
4b^2 - 4ba + a^2 &< a^2 \\
4b^2 - 4ba &< 0 \\
4b^2 &< 4ba \\
b^2 &< ba && \text{Invariance under multiplication}
\end{aligned}
$$

Next, let's perform similar operations on $|b - a| < |a|$

$$
\begin{aligned}
|b - a| &< |a| && \text{Given}\\
|b - a|^2 &< |a|^2 && \text{Invariance under exponentiation}\\
(b - a)^2 &< a^2 && \text{Squares are always positive}\\
b^2 - 2ba + a^2 &< a^2 \\
b^2 - 2ba &< 0 \\
b^2 &< 2ba.
\end{aligned}
$$

From transitivity of order it follows that $b^2 < ba < 2ba$, i.e. $b^2 < 2ba$, but this is exactly the condition we set out to prove in the very beginning. Thus, the proof is complete.

1.4 Problem 4

Given $a, b, c \in \mathbb{R}$,

1. Prove that if $a > 0$ and $a + b > a + c$, then $b > c$.

2. Prove that if $a > 0$ and $ab > ac$, then $b > c$.
3. Prove that $|a| > |b|$ iff $a^2 > b^2$.
4. Prove that if $b > c$ and $|a - b| > |a - c|$, then $b > a$.
5. Show (my means of example) that from $b > c$ and $b > a$ it doesn't follow that $|a - b| > |a - c|$.

1.4.1 Answer 6

Proof: The proof is simple algebra relying on invariance of order under addition:

$$\begin{aligned} a + b &> a + c && \text{Given} \\ -a + a + b &> -a + a + c && \text{Invariance under addition} \\ 0 + b &> 0 + c && \text{Devinition of inverse} \\ b &> c && \text{Devinition of inverse} \end{aligned}$$

1.4.2 Answer 7

Proof: The proof is simple algebra relying on invariance of order under multiplication:

$$\begin{aligned} ab &> ac && \text{Given} \\ a^{-1}ab &> a^{-1}ac && \text{Invariance under multiplication} \\ 1b &> 1c && \text{Devinition of inverse} \\ b &> c && \text{Devinition of inverse} \end{aligned}$$

1.4.3 Answer 8

Proof: first I will prove the implies part, i.e. $|a| > |b| \implies a^2 > b^2$, and then the $|a| > |b| \impliedby a^2 > b^2$:

$$
\begin{aligned}
|a| &> |b| && \text{Given} \\
|a||a| &> |a||b| && \text{Invariance under multiplication} \\
a^2 &> |ab| && \text{Simple algebra} \\
|a| &> |b| && \text{Reiteration of the given} \\
|a||b| &> |b||b| && \text{Invariance under multiplication} \\
|ab| &> b^2 && \text{Simple algebra} \\
a^2 > |ab| &> b^2 && \text{Reusing derivation from step 3} \\
a^2 &> b^2 && \text{By transitivity of order}
\end{aligned}
$$

The converse proof:

$$
\begin{aligned}
a^2 > b^2 |a||a| &> |b||b| && \text{Simple algebra} \\
|a||a| - |b||b| &> 0 && \text{Simple algebra} \\
|a| (|b| + c) - |b||b| &> 0 && \text{Define } c = |a| - |b| \\
|a||b| + |a| c - |b||b| &> 0 \\
(|a| - |b|)|b| + |a| c &> 0 \\
(|a| - |b|)|b| &> -|a| c \\
(|a| - |b|)|b| &> -|a| (|a| - |b|) && \text{Recall } c = |a| - |b| \\
|b| &> -|a| && \text{Invariance under multiplication} \\
-|a| &< |b| \\
|a| &> |b| && \text{Multiplying by } -1
\end{aligned}
$$

Both the "if" part and its converse have been proved, thus the proof is complete.

1.4.4 Answer 9

Proof: For the sake of diversity, this proof will rely on axioms rather than algebraic manipulations. Perusing the first axiom of order, we know that either one of the three holds: $c < a$, $c = a$ or $c > a$. Consider $c < a$. We are given that the distance from a to b is greater than the distance from a to c. This means that b cannot lie between a and c (recall the triangle inequality), neither can it lie between c and a if $a = c$, hence $b > a$.

Consider now $c > a$, we are given that $b > c$, thus, by transitivity of order we conclude that $b > c > a$ and in particular $b > a$. This completes the proof.

1.4.5 Answer 10

Any combination of a, b and c s.t. $c > a > b$ and $a - c > b - a$ will satisfy the requirement, and in particular $c = 1, a = 3, b = 4$ as is easy to see: $|a - b| > |a - c| \implies |3 - 4| > |3 - 1| \implies 1 > 2$. A way to see why this is true is to picture the number line, where the distance is measured from the central point a to two points on the left and on the right of it. We aren't constrained in any way as to which side should be bigger, thus we can certainly obtain the one that fits the requirement.

1.5 Problem 5

Solve the equation:

$$\lfloor |x + 1| - |x - 1| \rfloor = x.$$

1.5.1 Answer 11

Discussion: First, observe that flooring on the left side guarantees that x is an integer. Having said that, we can drop the flooring operation altogether. The solution will examine three cases of $x < 0$, $x = 0$ and $x > 0$ warranted by the first axiom of order.

Case $x = 0$:

$$\begin{aligned} |x+1| - |x-1| - x &= 0 \\ 0 + 1 - 0 - 1 - 0 &= 0 \\ 0 &= 0 \end{aligned}$$

Substituting 0 back into original formula proves to give a correct identity: $|0+1| - |0-1| - 0 = 1 - 1 + 0 = 0$.

Case $x > 0$:

$$\begin{aligned} |x+1| - |x-1| - x &= 0 \\ x + 1 - |x-1| - x &= 0 \\ 1 - |x-1| &= 0 \\ 1 - x + 1 &= 0 \\ x &= 2 \end{aligned}$$

Substituting 2 back into equation gives $|2+1| - |2-1| = 3 - 1 = 2$, thus we obtained additional solution.

Case $x < 0$:

$$\begin{aligned} |x+1| - |x-1| - x &= 0 \\ x + 1 - |x-1| - x &= 0 \\ 1 - |x-1| &= 0 \\ 1 - x + 1 &= 0 \\ x &= 2 \end{aligned}$$

But we started with the assumpion that $x < 0$, thus there are no solutions for $x < 0$. Hence the only solutions are 0 and 2.

1.6 Problem 6

Definition: set A of real numbers is called **dense in interval** I if for every $x, y \in I$ s.t. $x < y$ there exists $a \in A$ such that $x < a < y$.

1. Let A be dense in interval $[0,1]$, prove that set $B = \{na | a \in A, n \in \mathbb{N}\}$ is dense in interval $[0, \infty)$.

2. Let $A = \mathbb{R}$, prove that A isn't dense in I iff exists an open interval (x, y) in I, such that $A \cap (x, y) = \emptyset$.

3. Let A be the real numbers in interval $[0, 1]$, prove that the set $C = \{\frac{a+1}{n^2} | a \in A, n \in \mathbb{N}\}$ isn't dense in $[0, 1]$.

1.6.1 Answer 12

Discussion: One way to see why this statement is true, is to observe that multiplication is one-to-one and it preserves order, so for any pair of numbers chosen from the set A the order will remain the same after multiplication. But I will give an algebraic proof to reinforce this idea by simple manipulations on formulae. The observation used in the proof is essentially the same as the discussion given above, but it is less general—it chooses a concrete number between two arbitrary chosen x and y, exactly, $\frac{x+y}{2}$, but it would work for any number between x and y, provided $x < y$.

Proof: First, I will restate the problem more formally to allow mechanical manipulations:

$$A = \{a | 0 \leq a \leq 1\}, \forall x, y \in A : x < y \implies \exists z \in A : x < z < y$$
$$B = \{na | a \in A, n \in \mathbb{N}\}$$

Now, for contradiction, assume:

$$\exists x, y \in B : x < y \implies \forall z \in B(z \leq x \vee z \geq y)$$

I.e. we assumed that there exist some pair x, y in B, s.t. $x < y$ implies that for every z in B z is either smaller or equal to x or greater or equal to y. Choose $z = \frac{x+y}{2}$, let's first show that such z exists. By definition of B, $z = \frac{nx_1 + y_1 n}{2}$ where $x_1, y_1 \in A$ and $x_1 < y_1$. By construction of A we know that there are such elements in it (A isn't empty).

Now, suppose $z \leq x$. Then

$$\begin{aligned} z &= \frac{x+y}{2} \\ 2z &= x + y \\ 2z - y &= x \\ z - \frac{y}{2} &= \frac{x}{2} \end{aligned}$$

but since $\frac{y}{2} > \frac{x}{2}$ $\frac{z}{2} > \frac{x}{2}$, which in turn implies $z > x$, contrary to assumed $z \leq x$.

The argument for $z \geq y$ is symmetrical to the one above. In other words, we found a z s.t. contradicts the basic assumtion of this proof, hence, by contradiction, the proof is complete.

1.6.2 Answer 13

Discussion: The proof relies on the axiom of completeness of real numbers and equivalence between biconditional and its negation. (The later isn't true in all logics, but I shall assume a more traditional first-order logic). So, the basic idea is to claim that there are no such intervals in $\mathbb{R}$, hence there are no such A which isn't dense.

Proof: The textbook has a proof of density property of $\mathbb{R}$, which I will not repeat here, so, I will teat it as a fact. This means that existence of I in $\mathbb{R}$, s.t. it is not a subset of $\mathbb{R}$ is a negation of completeness of reals axiom. Stated more formally $\exists I \iff \neg Complete(\mathbb{R})$, using equivalence of biconditional and its negation: $\neg(\exists I) \iff \neg\neg Complete(\mathbb{R})$ simplifying gives: $\neg(\exists I) \iff Complete(\mathbb{R})$.

1.6.3 Answer 14

Discussion: I am not sure whether the question has any deeper meaning, though to disprove the statement exactly as stated, it suffices to find x, y

outside the range of I, for them there would be no $z \in I$ to satisfy the density requirement. In particular, I can choose $x = \frac{0.5+1}{1} = 1.5$, $y = \frac{1+1}{1} = 2$. Then, by transitivity of order, there is no $a \in A$ s.t. $x < a < y \wedge x < y$.

Proof: As discussed above, let $x = 1.5$ and $y = 2$. 1 is the maximal element of $[0, 1]$, hence $\forall z \in [0, 1] : z < x$, but in order for C to be dense in $[0, 1]$ there has to be $z \in [0, 1]$ s.t. $z > x$. Since no such z exists, the proof is complete.