

Assignment 14, Infinitesimal Calculus

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Contents

1	Problems	3
1.1	Problem 1	3
1.1.1	Answer 1	3
1.1.2	Answer 2	4
1.2	Problem 2	4
1.2.1	Answer 3	5
1.2.2	Answer 4	5
1.2.3	Answer 5	5
1.2.4	Answer 6	5
1.2.5	Anser 7	5
1.3	Problem 3	6
1.3.1	Answer 8	6
1.3.2	Answer 9	7
1.4	Problem 4	8

1.4.1	Answer 10	8
1.4.2	Answer 11	8
1.5	Problem 5	8
1.5.1	Answer 12	9
1.5.2	Answer 13	9
1.5.3	Answer 14	10
1.5.4	Answer 15	10
1.5.5	Answer 16	11

1 Problems

1.1 Problem 1

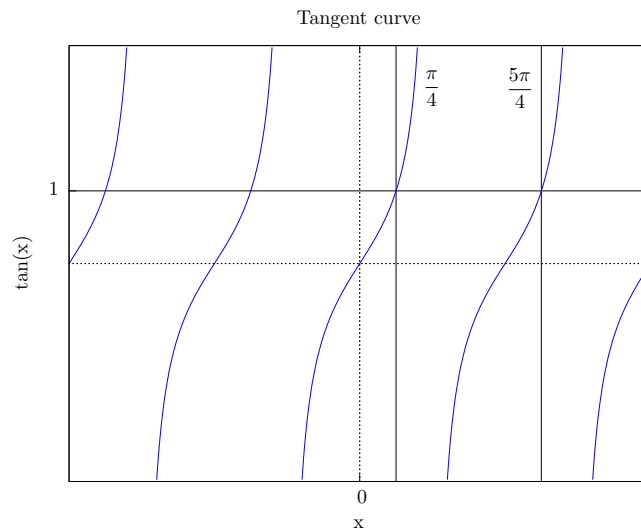
1. Find the domain of f defined as $f(x) = \sqrt{\tan x - 1}$.
2. Find all values of x in segment $[0, \pi]$, for which $|\tan x| \leq \sin 2x$.

1.1.1 Answer 1

$\tan x > 1$ where $\frac{\pi}{4} < x \bmod \frac{\pi}{2} < \frac{\pi}{2}$. Since we assume that f is real-valued, we cannot extract roots from negative numbers.

Hence $\text{Dom}(f) = \{x \in \mathbb{R} \mid \frac{\pi}{4} < x \bmod \frac{\pi}{2} < \frac{\pi}{2}\}$.

```
programmode: false;
print (plot2d (tan(x),
  [x, -2 * %pi, 2 * %pi], [y, -3, 3],
  [gnuplot_pdf_term_command,
    "set term cairolatex standalone pdf size 16cm,10.5cm"],
  [gnuplot_postamble,
    "set arrow from pi/4,-3 to pi/4,3 nohead
    set arrow from pi*5/4,-3 to pi*5/4,3 nohead
    set arrow from -2*pi,1 to 2*pi,1 nohead"],
  [ytics, 1, 1, 1], [xtics, 0, 1, 0],
  [label, ["${\\displaystyle \\frac{\\pi}{4}}$", 1.4, 2.4],
    ["${\\displaystyle \\frac{5\\pi}{4}}$", 3.2, 2.4]],
  [title, "Tangent curve"], [pdf_file, %out]));
```



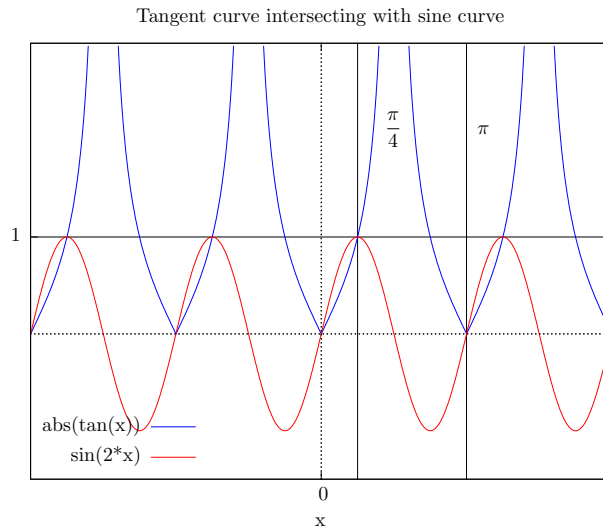
1.1.2 Answer 2

Both functions attain the same values at 0 and $\frac{\pi}{4}$. But sine is a concave function and tangent is a convex function, thus tangent must be less than sine at this interval. Tangent keeps increasing until $\frac{\pi}{2}$, while sine will be decreasing until $\frac{3\pi}{2}$, thus, on this interval tangent is greater than sine. The functions meet again at $x = \pi$.

```

programmode: false;
gnuplot_pdf_command: %command;
print(plot2d ([abs(tan(x)), sin(2 * x)],
  [x, -2 * %pi, 2 * %pi], [y, -1.5, 3],
  [gnuplot_pdf_term_command,
    "set term cairolatex standalone pdf size 16cm,10.5cm"],
  [gnuplot_postamble,
    "set arrow from pi/4,-1.5 to pi/4,3 nohead
    set arrow from pi,-1.5 to pi,3 nohead
    set arrow from -2*pi,1 to 2*pi,1 nohead
    set key spacing 1.8 left bottom"],
  [xtics, 0, 1, 0], [ytics, 1, 1, 1],
  [label, ["${\\displaystyle \\frac{\\pi}{4}}$", 1.4, 2.1],
    ["${\\displaystyle \\pi}$", 3.4, 2.1]],
  [title, "Tangent curve intersecting with sine curve"],
  [pdf_file, %out]));

```



1.2 Problem 2

Let f , g and h be functions from \mathbb{R} to \mathbb{R} .

1. If $f \circ g = f \circ h$, does it follow $g = h$?

2. If $f \circ g = f \circ h$, and f is one-to-one, does it follow $g = h$?
3. If $f \circ g = f \circ h$, and f is onto, does it follow $g = h$?
4. If $f \circ g$ is increasing, and f is decreasing, does it follow that g is increasing?
5. If $f \circ g$ is increasing, and f is one-to-one, does it follow that g is monotonic?

1.2.1 Answer 3

No, g and h are not necessarily equal. Whenever co-domain of f doesn't contain some real number, g and h may differ in that input. For example, let $f(x) = 2x$, $g(x) = (x \bmod 2) + x$ and $h(x) = (x \bmod 2) * 2 + x$. Because f in this example will only generate even numbers, the $(x \bmod 2)$ term will always be zero, thus $f \circ g = f \circ h$, but, obviously, $g \neq h$.

1.2.2 Answer 4

No, it isn't sufficient for f to be one-to-one to ensure right-cancellation property under composition. The example given in 1.2.1 is applicable in this case too since whenever $f(x) = f(y)$ so is $x = y$ (since multiplication does have the cancellation property).

1.2.3 Answer 5

Yes, if f is onto, then the composition is right-cancellable. Suppose, for contradiction it wasn't, then for some y $g(y) \neq h(y)$, but $y = f(x)$ (since by definition of a total function, every element in its co-domain has an element in its domain). Hence $g(f(x)) \neq h(f(x))$, but we are given that $g \circ f = h \circ f$, which is a contradiction. Hence functions are equal.

1.2.4 Answer 6

No, g doesn't need to be increasing. Put $g(x) = f(x) = -x$, both f and g are decreasing but $f \circ g = Id$, which is an increasing function.

1.2.5 Answer 7

No, g is not necessarily monotonic. Put $f(x) = x(-1)^x$ and $g(x) = |x|$. Then $(f \circ g)(x) = |x(-1)^x| = x|(-1)^x| = x$. $f \circ g$ is increasing, f is one-to-one, but g

isn't monotonic: it decreases whenever x is negative and increases whenever x is positive.

1.3 Problem 3

1. Prove from $\epsilon - \delta$ definition of limit that $\lim_{x \rightarrow 2} \sqrt{3x - 2} = 2$.
2. Prove from $\epsilon - M$ definition of limit that $\lim_{x \rightarrow \infty} \frac{x}{x + \sin x} = 1$.

1.3.1 Answer 8

Recall the definition:

For all $\epsilon > 0$ there exists $\sigma > 0$ s.t. for all x in $\text{Dom}(f(x))$ which satisfy $0 < |x - x_0| < \sigma$ the inequality $|f(x) - L| < \epsilon$ holds.

Let ϵ be arbitrary real, put

$$\begin{aligned} |f(x) - L| &< \epsilon \iff \\ \left| \sqrt{3x - 2} - 2 \right| &< \epsilon \end{aligned}$$

Suppose $\sqrt{3x - 2} - 2 > 0$

$$\begin{aligned} 0 < \sqrt{3x - 2} - 2 &< \epsilon \iff \\ 0 < \sqrt{3x - 2} &< \epsilon + 2 \iff \\ 0 < 3x - 2 &< (\epsilon + 2)^2 \iff \\ 0 < 3x - 2 &< \epsilon^2 + 4\epsilon + 4 \iff \\ 0 < 3x - 6 &< \epsilon^2 + 4\epsilon \iff \\ 0 < x - 2 &< \frac{\epsilon^2 + 4\epsilon}{3} \end{aligned}$$

Similarly for $\sqrt{3x - 2} - 2 < 0$

$$\begin{aligned} 0 > \sqrt{3x - 2} - 2 &> -\epsilon \iff \\ 0 > \sqrt{3x - 2} &> -\epsilon + 2 \iff \\ 0 > 3x - 2 &> (-\epsilon + 2)^2 \iff \\ 0 > 3x - 2 &> \epsilon^2 - 4\epsilon + 4 \iff \\ 0 > 3x - 6 &> \epsilon^2 - 4\epsilon \iff \\ 0 > x - 2 &> \frac{\epsilon^2 - 4\epsilon}{3} \end{aligned}$$

Hence, we can choose δ to be $\frac{\epsilon^2 - 4\epsilon}{3}$ whenever $x < x_0$ and $\frac{\epsilon^2 + 4\epsilon}{3}$ whenever $x > x_0$, which completes the proof.

1.3.2 Answer 9

Recall the definition:

For all $\epsilon > 0$ there exists $M > 0$ s.t. for all x in $\text{Dom}(f(x))$ $x > M$ implies $|f(x) - L| < \epsilon$.

Let $\epsilon > 0$, then look for appropriate value for x :

$$\begin{aligned}
 \left| \frac{x}{x + \sin x} - 1 \right| &< \epsilon \iff \\
 \left| \frac{x}{x + \sin x} - \sin^2 x - \cos^2 x \right| &< \epsilon \iff \\
 \left| \frac{x - \sin^2 x(x + \sin x) - \cos^2 x(x + \sin x)}{x + \sin x} \right| &< \epsilon \iff \\
 \left| \frac{x - x \sin^2 x - \sin^3 x - x \cos^2 x - \cos^2 x \sin x}{x + \sin x} \right| &< \epsilon \iff \\
 \left| \frac{x(1 - \sin^2 - \cos^2 x) - \sin x(\sin^2 x - \cos^2)}{x + \sin x} \right| &< \epsilon \iff \\
 \left| \frac{x(1 - 1) - \sin x(1)}{x + \sin x} \right| &< \epsilon \iff \\
 \left| \frac{-\sin x}{x + \sin x} \right| &< \epsilon
 \end{aligned}$$

Assume $x > 0$:

$$\begin{aligned}
 \frac{-\sin x}{x + \sin x} &< \epsilon \iff \\
 -\sin x &< \epsilon(x + \sin x) \iff \\
 -\sin x &< \epsilon x + \epsilon \sin x \iff \\
 -\epsilon x &< \sin x + \epsilon \sin x \iff \\
 -x &< \frac{\sin x + \epsilon \sin x}{\epsilon} \iff \\
 x &> \frac{\sin x(1 + \epsilon)}{\epsilon}
 \end{aligned}$$

Put $M = \max\left(0, \frac{\sin x(1+\epsilon)}{\epsilon}\right)$. If $x > M$, then $x > 0$ and $x > \frac{\sin x(1+\epsilon)}{\epsilon}$, hence:

$$x > \frac{\sin x(1 + \epsilon)}{\epsilon}$$

... Reverse the calculations above

$$\begin{aligned} & \frac{-\sin x}{x + \sin x} < \epsilon \\ & \dots \\ & \left| \frac{x}{x + \sin x} - 1 \right| < \epsilon. \end{aligned}$$

Which completes the proof.

1.4 Problem 4

- Let f be a function defined in the neighborhood of x_0 . Express “ f doesn’t have a limit at x_0 ” using:
 - $\epsilon - \sigma$ language.
 - Using Heine definition of limit (for sequences).
- Prove that $f(x) = \frac{x}{x - \lfloor x \rfloor}$ doesn’t have a finite limit at $x \rightarrow 0$ in the following ways:
 - By using $\epsilon - \sigma$ definition given above.
 - By using Heine definition of limit (also given above).

1.4.1 Answer 10

1.4.2 Answer 11

1.5 Problem 5

Find limits of:

- $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sin x}.$
- $\lim_{x \rightarrow 0} \frac{x + 7x^3}{x^3 - 2x^4}.$
- $\lim_{x \rightarrow 0} \frac{x^2 - 1}{2x^3 - x^2 - x}.$
- $\lim_{x \rightarrow 0} (\sqrt{1 + x + x^2} - \sqrt{1 - x + x^2}).$
- $\lim_{x \rightarrow k} \lfloor x \rfloor \tan \frac{\pi x}{2}, k = 0, 1, 2.$

1.5.1 Answer 12

```
tex(limit((1 - cos(x)) / (x * sin(x)), x, 0));
```

$$\frac{1}{2}$$

Proof:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x \sin x} &= \lim_{x \rightarrow 0} \frac{(1 - \cos x) * (1 + \cos x)}{x \sin x (1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x \sin x (1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x \sin x (1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x (1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} * \lim_{x \rightarrow 0} \frac{1}{1 + \cos x} \\ &= 1 * \lim_{x \rightarrow 0} \frac{1}{1 + \cos x} \\ &= \lim_{x \rightarrow 0} \frac{1}{1 + 1} \\ &= \frac{1}{2} \end{aligned}$$

1.5.2 Answer 13

```
tex(limit((x + 7 * x^3) / (x^3 - 2 * x^4), x, 0));
```

$$\infty$$

Proof:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x + 7x^3}{x^3 - 2x^4} &= \lim_{x \rightarrow 0} \frac{1 + 7x^2}{x^2 - 2x^3} \\ &= \lim_{x \rightarrow 0} \frac{1 - 4x^2}{x^2 - 2x^3} + \frac{11x^2}{x^2 - 2x^3} \\ &= \lim_{x \rightarrow 0} \frac{(1 - 2x)(1 + 2x)}{x^2(1 - 2x)} + \frac{11}{1 + 2x} \end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow 0} \frac{1+2x}{x^2} + \frac{11}{1+2x} \\
&= \lim_{x \rightarrow 0} \frac{1}{x^2} + \frac{2x}{x^2} + \frac{11}{1+2x} \\
&= \lim_{x \rightarrow 0} \frac{1}{x^2} + \frac{2}{x} + \frac{11}{1+2x} \\
&= \lim_{x \rightarrow 0} \frac{1}{x^2} + \lim_{x \rightarrow 0} \frac{2}{x} + \lim_{x \rightarrow 0} \frac{11}{1+2x} \\
&= \infty + \infty + 11 \qquad \text{Using ifinite limits addition} \\
&= \infty
\end{aligned}$$

1.5.3 Answer 14

```
tex(limit((x^2 - 1) / (2 * x^3 - x^2 - x), x, 0));
```

infinity

Proof:

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{x^2 - 1}{2x^3 - x^2 - x} &= \lim_{x \rightarrow 0} \frac{(x-1)(x+1)}{x^2(x-1) + x(x^2-1)} \\
&= \lim_{x \rightarrow 0} \frac{(x-1)(x+1)}{x^2(x-1) + x(x-1)(x+1)} \\
&= \lim_{x \rightarrow 0} \frac{(x-1)(x+1)}{(x-1)(x^2 + x(x+1))} \\
&= \lim_{x \rightarrow 0} \frac{x+1}{x^2 + x(x+1)} \\
&= \lim_{x \rightarrow 0} \frac{x+1}{x^2 + x^2 + x} \\
&= \lim_{x \rightarrow 0} \frac{x+1}{2x^2 + x} \\
&= \lim_{x \rightarrow 0} \frac{x+1}{x(2x+1)} \\
&= \lim_{x \rightarrow 0} \frac{x+1}{x} * \lim_{x \rightarrow 0} \frac{1}{2x+1} \\
&= \lim_{x \rightarrow 0} \frac{x+1}{x} * 1 \\
&= \infty
\end{aligned}$$

1.5.4 Answer 15

```
tex(limit(sqrt(1 + x + x^2) - sqrt(1 - x + x^2), x, 0));
```

0

Proof: this function is continuous at $x = 0$ since square root is continuous at 1 and this function is offset by one. From definition of continuity we know that the limit of the function coincides with its value, hence the limit is $\sqrt{1 + 0 + 0^2} - \sqrt{1 - 0 + 0^2} = 1 - 1 = 0$.

1.5.5 Answer 16

```
for i : 0 thru 2 do
  tex(limit(floor(x) * tan((%pi * x) / 2), x, i));
```

0

und

0