Assignment 14, Infinitesimal Calculus

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Contents

1 Problems

1.1 Problem 1

1. Find the domain of f defined as $f(x) = \sqrt{\tan x - 1}$.
2. Find all values of x in segment $[0, \pi]$, for which $|\tan x| \leq \sin 2x$.

1.1.1 Answer 1

$\tan x > 1$ where $\frac{\pi}{4} < x \bmod \frac{\pi}{2} < \frac{\pi}{2}$. Since we assume that f is real-valued, we cannot extract roots from negative numbers.

Hence $\mathrm{Dom}(f) = \{x \in \mathbb{R} \mid \frac{\pi}{4} < x \bmod \frac{\pi}{2} < \frac{\pi}{2}\}$.

```
programmode: false;
print (plot2d (tan(x),
    [x, -2 * %pi, 2 * %pi], [y, -3, 3],
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     "set term cairolatex standalone pdf size 16cm,10.5cm"],
    [gnuplot_postamble,
     "set arrow from pi/4,-3 to pi/4,3 nohead
      set arrow from pi*5/4,-3 to pi*5/4,3 nohead
      set arrow from -2*pi,1 to 2*pi,1 nohead"],
    [ytics, 1, 1, 1], [xtics, 0, 1, 0],
    [label, ["${\\displaystyle \\frac{\\pi}{4}}$", 1.4, 2.4],
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```

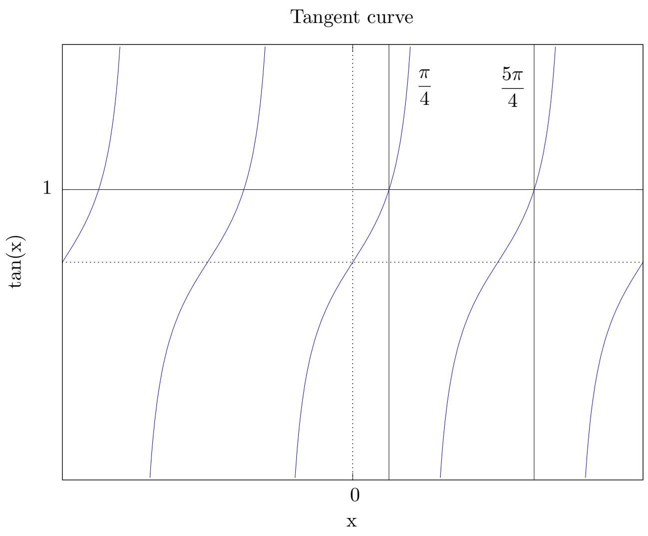

1.1.2 Answer 2

Both functions attain the same values at 0 and $\frac{\pi}{4}$. But sine is a concave function and tanget is a convex function, thus tangent must be less than sine at this interval. Tangent keeps increasing until $\frac{\pi}{2}$, while sine will be decreasing until $\frac{3\pi}{2}$, thus, on this interval tangent is greater than sine. The functions meet again at $x = \pi$.

```
programmode: false;
gnuplot_pdf_command: %command;
print(plot2d ([abs(tan(x)), sin(2 * x)],
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      set key spacing 1.8 left bottom"],
    [xtics, 0, 1, 0], [ytics, 1, 1, 1],
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            ["${\\displaystyle \\pi}$", 3.4, 2.1]],
    [title, "Tangent curve intersecting with sine curve"],
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```

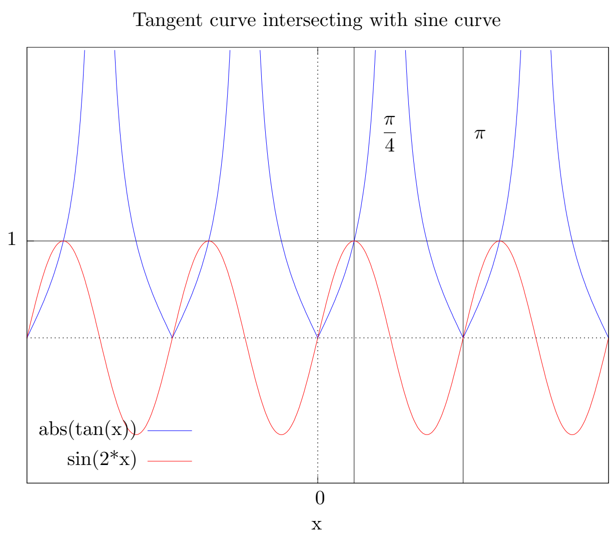

1.2 Problem 2

Let f, g and h be functions from $\mathbb{R}$ to $\mathbb{R}$.

1. If $f \circ g = f \circ h$, does it follow $g = h$?

2. If $f \circ g = f \circ h$, and f is one-to-one, does it follow $g = h$?
3. If $f \circ g = f \circ h$, and f is onto, does it follow $g = h$?
4. If $f \circ g$ is increasing, and f is decreasing, does it follow that g is increasing?
5. If $f \circ g$ is increasing, and f is one-to-one, does it follow that g is monotonic?

1.2.1 Answer 3

No, g and h are not necessarily equal. Whenever co-domain of f doesn't contain some real number, g and h may differ in that input. For example, let $f(x) = 2x$, $g(x) = (x \bmod 2) + x$ and $h(x) = (x \bmod 2) * 2 + x$. Because f in this example will only generate even numbers, the $(x \bmod 2)$ term will always be zero, thus $f \circ g = f \circ h$, but, obviously, $g \neq h$.

1.2.2 Answer 4

No, it isn't sufficient for f to be one-to-one to ensure right-cancellation property under composition. The example given in 1.2.1 is applicable in this case too since whenever $f(x) = f(y)$ so is $x = y$ (since multiplication does have the cancellation property).

1.2.3 Answer 5

Yes, if f is onto, then the composition is right-cancellable. Suppose, for contradiction it wasn't, then for some y $g(y) \neq h(y)$, but $y = f(x)$ (since by definition of a total function, every element in its co-domain has an element in its domain). Hence $g(f(x)) \neq h(f(x))$, but we are given that $g \circ f = h \circ f$, which is a contradiction. Hence functions are equal.

1.2.4 Answer 6

No, g doesn't need to be increasing. Put $g(x) = f(x) = -x$, both f and g are decreasing but $f \circ g = Id$, which is an increasing function.

1.2.5 Anser 7

No, g is not necessarily monotonic. Put $f(x) = x(-1)^x$ and $g(x) = |x|$. Then $(f \circ g)(x) = \big|x(-1)^x\big| = x\big|(-1)^x\big| = x$. $f \circ g$ is increasing, f is one-to-one, but g

isn't monotonic: it decreases whenever x is negative and increases whenever x is positive.

1.3 Problem 3

1. Prove from $\epsilon - \delta$ definition of limit that $\lim_{x\to 2}\sqrt{3x-2} = 2$.
2. Prove from $\epsilon - M$ definition of limit that $\lim_{x\to\infty}\frac{x}{x+\sin x} = 1$.

1.3.1 Answer 8

Recall the definition:

> For all $\epsilon > 0$ there exists $\sigma > 0$ s.t. for all x in $\mathrm{Dom}(f(x))$ which satisfy $0 <|x - x_0| < \sigma$ the inequality $\left|f(x) - L\right| < \epsilon$ holds.

Let ϵ be arbitrary real, put

$$
\begin{aligned}
&\left|f(x) - L\right| && < \epsilon \iff \\
&\left|\sqrt{3x-2} - 2\right| && < \epsilon \\
&\textit{Suppose } \sqrt{3x-2} - 2 > 0 && \\
&0 < \sqrt{3x-2} - 2 && < \epsilon \iff \\
&0 < \sqrt{3x-2} && < \epsilon + 2 \iff \\
&0 < 3x - 2 && < (\epsilon+2)^2 \iff \\
&0 < 3x - 2 && < \epsilon^2 + 4\epsilon + 4 \iff \\
&0 < 3x - 6 && < \epsilon^2 + 4\epsilon \iff \\
&0 < x - 2 && < \frac{\epsilon^2 + 4\epsilon}{3} \\
&\textit{Similarly for } \sqrt{3x-2} - 2 < 0 && \\
&0 > \sqrt{3x-2} - 2 && > -\epsilon \iff \\
&0 > \sqrt{3x-2} && > -\epsilon + 2 \iff \\
&0 > 3x - 2 && > (-\epsilon+2)^2 \iff \\
&0 > 3x - 2 && > \epsilon^2 - 4\epsilon + 4 \iff \\
&0 > 3x - 6 && > \epsilon^2 - 4\epsilon \iff \\
&0 > x - 2 && > \frac{\epsilon^2 - 4\epsilon}{3}
\end{aligned}
$$

Hence, we can choose δ to be $\frac{\epsilon^2-4\epsilon}{3}$ whenever $x < x_0$ and $\frac{\epsilon^2+4\epsilon}{3}$ whenever $x > x_0$, which completes the proof.

1.3.2 Answer 9

Recall the definition:

> For all $\epsilon > 0$ there exists $M > 0$ s.t. for all x in $\mathrm{Dom}(f(x))$ $x > M$ implies $\left|f(x) - L\right| < \epsilon$.

Let $\epsilon > 0$, then look for appropriate value for x:

$$\begin{aligned}
\left|\frac{x}{x+\sin x} - 1\right| &< \epsilon \iff \\
\left|\frac{x}{x+\sin x} - \sin^2 x - \cos^2 x\right| &< \epsilon \iff \\
\left|\frac{x - \sin^2 x(x+\sin x) - \cos^2 x(x+\sin x)}{x+\sin x}\right| &< \epsilon \iff \\
\left|\frac{x - x\sin^2 x - \sin^3 x - x\cos^2 x - \cos^2 x \sin x}{x+\sin x}\right| &< \epsilon \iff \\
\left|\frac{x(1 - \sin^2 - \cos^2 x) - \sin x(\sin^2 x - \cos^2)}{x+\sin x}\right| &< \epsilon \iff \\
\left|\frac{x(1-1) - \sin x(1)}{x+\sin x}\right| &< \epsilon \iff \\
\left|\frac{-\sin x}{x+\sin x}\right| &< \epsilon
\end{aligned}$$

Assume $x > 0$:

$$\begin{aligned}
\frac{-\sin x}{x+\sin x} &< \epsilon \iff \\
-\sin x &< \epsilon(x+\sin x) \iff \\
-\sin x &< \epsilon x + \epsilon \sin x \iff \\
-\epsilon x &< \sin x + \epsilon \sin x \iff \\
-x &< \frac{\sin x + \epsilon \sin x}{\epsilon} \iff \\
x &> \frac{\sin x(1+\epsilon)}{\epsilon}
\end{aligned}$$

Put $M = \max\left(0, \frac{\sin x(1+\epsilon)}{\epsilon}\right)$. If $x > M$, then $x > 0$ and $x > \frac{\sin x(1+\epsilon)}{\epsilon}$, hence:

$$x > \frac{\sin x(1+\epsilon)}{\epsilon}$$

$\ldots$ *Reverse the calculations above*

$$\frac{-\sin x}{x+\sin x} < \epsilon$$

$\ldots$

$$\left|\frac{x}{x+\sin x} - 1\right| < \epsilon.$$

Which completes the proof.

1.4 Problem 4

1. Let f be a function defined in the neighborhood of x_0. Express "f doesn't have a limit at x_0" using:
   - $\epsilon - \sigma$ language.
   - Using Heine definition of limit (for sequences).
2. Prove that $f(x) = \frac{x}{x-\lfloor x \rfloor}$ doesn't have a finite limit at $x \to 0$ in the following ways:
   - By using $\epsilon - \sigma$ definition given above.
   - By using Heine definition of limit (also given above).

1.4.1 Answer 10

Recall the $\epsilon - \delta$ definition:

The limit of $f(x)$ at x_0 is defined to be L s.t. for every $\epsilon > \left|f(x) - L\right| > 0$ we can find $\delta > |x - x_0| > 0$.

To negate this is to say that there exists such ϵ for which we can't find a positive δ larger than the distance from x to x_0.

Heine defines limit to be L whenever for every sequence (x_n) which convergest to x_0, every sequence of function values $f(x_n)$ converges to L.

To negate this definition we claim that there exists a sequence (x_m), which convergest to x_0, however $f(x_m)$ doesn't converge to L.

1.4.2 Answer 11

I'll do the Heine first, because it's easier. We can choose sequences $(x_n) = -\frac{\pi}{2nx}$ and $(x_m) = \frac{\pi}{2mx}$, both are immediately reducible to the limit of a fraction as x approaches zero, hence, the limit for both is zero. Now, if we plug them back into f, we get:

$$\begin{aligned}
\lim_{x\to 0} \frac{x}{x - \lfloor \sin x \rfloor} &= \lim_{x\to 0} \frac{x}{x-0} \\
&\lfloor \sin x \rfloor = 0 \textit{ whenever } 0 < x < \frac{\pi}{2} \\
&= 1 \ .
\end{aligned}$$

Similarly:

$$\begin{aligned}
\lim_{x\to 0} \frac{x}{x - \lfloor \sin x \rfloor} &= \lim_{x\to 0} \frac{x}{x+1} \\
&\lfloor \sin x \rfloor = -1 \textit{ whenever } 0 > x > -\frac{\pi}{2} \\
&= 0 \ .
\end{aligned}$$

Now, the $\epsilon - \sigma$ approach:

Since we can pick arbitrary ϵ, put $\epsilon = \frac{1}{2}$. Now, we could try to find the limit withing π distance from zero. Assuming thus $x \in (\pi, -\pi)$.

$$\begin{aligned}
&-\frac{1}{2} < \frac{\sigma}{\sigma - \lfloor \sin \sigma \rfloor} - L < \frac{1}{2} \\
&-\frac{1}{2} < \frac{\sigma}{\sigma - 0} - L < \frac{1}{2} \\
&-\frac{1}{2} < 1 - L < \frac{1}{2} \\
&-\frac{3}{2} < -L < -\frac{1}{2} \\
&\frac{3}{2} > L > \frac{1}{2} \\
&\textit{Similarly:} \\
&-\frac{1}{2} < \frac{-\sigma}{\lfloor -\sin \sigma \rfloor - \sigma} - L < \frac{1}{2} \\
&-\frac{1}{2} < \frac{-\sigma}{-\sigma + 1} - L < \frac{1}{2}
\end{aligned}$$

$$-\frac{1}{2} - \frac{-\sigma}{-\sigma+1} < -L < \frac{1}{2} - \frac{-\sigma}{-\sigma+1}$$
$$-\frac{-\sigma+1+2\sigma}{-2\sigma+2} < -L < \frac{1}{2} - \frac{-\sigma}{1-\sigma}$$
$$-\frac{1+\sigma}{-2\sigma+2} < -L < \frac{1}{2} - \frac{-\sigma}{1-\sigma}$$
$$\frac{-1-\sigma}{-2\sigma+2} < -L < \frac{1}{2} - \frac{-\sigma}{1-\sigma}$$
$$\frac{-1(1+\sigma)}{-1(2\sigma-2)} < -L < \frac{1}{2} - \frac{-\sigma}{1-\sigma}$$
$$\frac{1+\sigma}{2\sigma-2} < -L < \frac{1}{2} - \frac{-\sigma}{1-\sigma}$$
$$\mathit{Since}\ \sigma > 0$$
$$\frac{1+\sigma}{2\sigma+2} < -L < \frac{1}{2} - \frac{-\sigma}{1-\sigma}$$
$$\frac{1}{2} < -L < \frac{1}{2} - \frac{-\sigma}{1-\sigma}$$
$$\mathit{Contradiction:}\ L > \frac{1}{2}\ \mathit{and}\ L < \frac{1}{2}.$$

```
programmode: false;
gnuplot_pdf_command: %command;
print(plot2d (x / (x - floor(sin(x))),
    [x, -2 * %pi, 2 * %pi], [y, -2, 2],
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    [ylabel, "${\\displaystyle \\frac{x}{x - \\lfloor \\sin x \\rfloor}}$"],
    [title, concat("${\\displaystyle \\lim_{x \\to 0}",
               "\\frac{x}{x - \\lfloor \\sin x \\rfloor}}$")],
    [pdf_file, %out]));
```

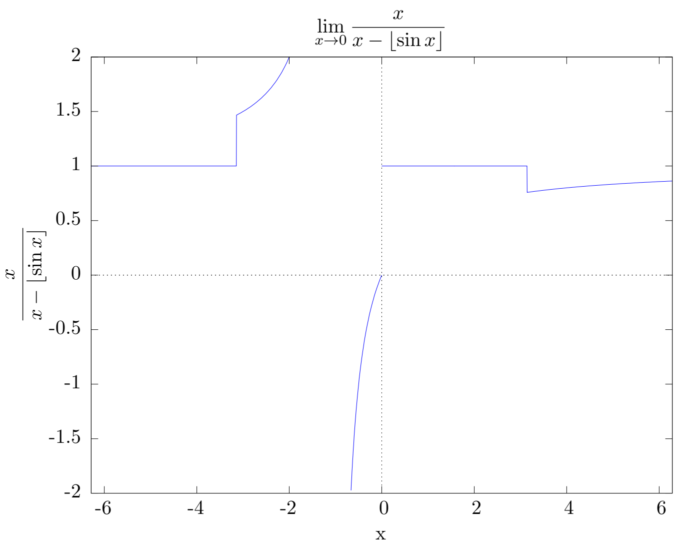

1.5 Problem 5

Find limits of:

1. $\lim_{x\to 0} \frac{1-\cos x}{x \sin x}$.
2. $\lim_{x\to 0} \frac{x+7x^3}{x^3-2x^4}$.
3. $\lim_{x\to 0} \frac{x^2-1}{2x^3-x^2-x}$.
4. $\lim_{x\to 0}(\sqrt{1+x+x^2} - \sqrt{1-x+x^2})$.
5. $\lim_{x\to k} \lfloor x \rfloor \tan \frac{\pi x}{2}$, $k = 0, 1, 2$.

1.5.1 Answer 12

```
tex(limit((1 - cos(x)) / (x * sin(x)), x, 0));
```

$\frac{1}{2}$

Proof:

$$
\begin{aligned}
\lim_{x\to 0} \frac{1-\cos x}{x \sin x} &= \lim_{x\to 0} \frac{(1-\cos x) * (1+\cos x)}{x \sin x (1+\cos x)} \\
&= \lim_{x\to 0} \frac{1-\cos^2 x}{x \sin x (1+\cos x)} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{x \sin x (1+\cos x)} \\
&= \lim_{x\to 0} \frac{\sin x}{x(1+\cos x)} \\
&= \lim_{x\to 0} \frac{\sin x}{x} * \lim_{x\to 0} \frac{1}{1+\cos x} \\
&= 1 * \lim_{x\to 0} \frac{1}{1+\cos x} \\
&= \lim_{x\to 0} \frac{1}{1+1} \\
&= \frac{1}{2}
\end{aligned}
$$

1.5.2 Answer 13

```
tex(limit((x + 7 * x^3) / (x^3 - 2 * x^4), x, 0));
```

∞

Proof:

$$\begin{aligned}
\lim_{x\to 0} \frac{x+7x^3}{x^3-2x^4} &= \lim_{x\to 0} \frac{1+7x^2}{x^2-2x^3} \\
&= \lim_{x\to 0} \frac{1-4x^2}{x^2-2x^3} + \frac{11x^2}{x^2-2x^3} \\
&= \lim_{x\to 0} \frac{(1-2x)(1+2x)}{x^2(1-2x)} + \frac{11}{1+2x} \\
&= \lim_{x\to 0} \frac{1+2x}{x^2} + \frac{11}{1+2x} \\
&= \lim_{x\to 0} \frac{1}{x^2} + \frac{2x}{x^2} + \frac{11}{1+2x} \\
&= \lim_{x\to 0} \frac{1}{x^2} + \frac{2}{x} + \frac{11}{1+2x} \\
&= \lim_{x\to 0} \frac{1}{x^2} + \lim_{x\to 0} \frac{2}{x} + \lim_{x\to 0} \frac{11}{1+2x} \\
&= \infty + \infty + 11 && \textit{Using ifinite limits addition} \\
&= \infty
\end{aligned}$$

1.5.3 Answer 14

```
tex(limit((x^2 - 1) / (2 * x^3 - x^2 - x), x, 0));
```

infinity

Proof:

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2-1}{2x^3-x^2-x} &= \lim_{x\to 0} \frac{(x-1)(x+1)}{x^2(x-1)+x(x^2-1)} \\
&= \lim_{x\to 0} \frac{(x-1)(x+1)}{x^2(x-1)+x(x-1)(x+1)} \\
&= \lim_{x\to 0} \frac{(x-1)(x+1)}{(x-1)(x^2+x(x+1))} \\
&= \lim_{x\to 0} \frac{x+1}{x^2+x(x+1)} \\
&= \lim_{x\to 0} \frac{x+1}{x^2+x^2+x}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{x\to 0} \frac{x+1}{2x^2+x} \\
&= \lim_{x\to 0} \frac{x+1}{x(2x+1)} \\
&= \lim_{x\to 0} \frac{x+1}{x} * \lim_{x\to 0} \frac{1}{2x+1} \\
&= \lim_{x\to 0} \frac{x+1}{x} * 1 \\
&= \infty
\end{aligned}
$$

1.5.4 Answer 15

```
tex(limit(sqrt(1 + x + x^2) - sqrt(1 - x + x^2), x, 0));
```

0

Proof: this function is continuous at $x = 0$ since square root is continuous at 1 and this function is offset by one. From definition of continuity we know that the limit of the function coincides with its value, hence the limit is $\sqrt{1+0+0^2} - \sqrt{1-0+0^2} = 1 - 1 = 0$.

1.5.5 Answer 16

```
for i : 0 thru 2 do
    tex(limit(floor(x) * tan((%pi * x) / 2), x, i));
```

0 *und* 0