Assignment 18, Infinitesimal Calculus

Oleg Sivokon

<2015-06-24 Wed>

Contents

1 Problems

1.1 Problem 1

Given $P(x) = x^4 + a_4 x^3 + a_3 x^2 + a_2 x + a_0$ is a polynomial with one non-zero real root, prove that it has at least one more real root.

1.1.1 Answer 1

Since we are given $P(x)$ has a root, let's call it r, then from factor theorem we also know that there exists $(x - r)D(x) = P(x)$ where $D(x)$ is a third degree polynomial with a non-zero leading coefficient. According to fundamental theorem of calculus, each root must have a conjugate pair. Since third degree polynomial has three roots, then by pigeonhole principle, one of these roots is equal to its conjugate, which is why this root must be real. Hence the original quadratic polynomial has at least one more real root (and four roots all in all).

1.2 Problem 2

Let f be a function differentiable at (a, b). Let $x_0 \in (a, b)$. Prove that there exists a sequence (x_n) s.t. $x_n \neq x_0$ for all n, $\lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} f'(x_n) = f'(x_0)$.

1.2.1 Answer 2

Let's pick x_0. From continuity of reals, we have that no matter where in (a, b) x_0 is, we can always find two intervlas (a, x_0) and (x_0, b). Let's pick a suitable derivative function. For example, let $f'(x) = f'(x_0)$, i.e. a constant function. Now we can construct the required sequence. $(x_n) = x_0 - \frac{x_0 - a}{n+2}$ meets our requirements. Since $\lim_{n\to\infty} x_0 = x_0$ and $\lim_{n\to\infty} \frac{x_0 - a}{n+2} = 0$.

1.3 Problem 3

Let f be differenetiable at $[0, \frac{\pi}{2}]$ s.t. $0 \leq f(x) \leq 1$ for all x in this interval. Prove that there exists a point x in this interval such that $f'(x) = \sin x$.

1.3.1 Answer 3

From Lagrange mean value theorem we are guaranteed a point c s.t.

$$f'(c) = \frac{f(a) - f(b)}{a - b}$$

Put $a = 0$, $b = \frac{\pi}{2}$. Then it follows that $0 < f'(c) < 1$ since $f(a) \geq a$ and $f(b) \leq b$. Since sine function is continuous at $[0, \frac{\pi}{2}]$ and it assumes all values in $[0, 1]$, then there must exist a point where $\sin x = f'(c)$ by intermediate value theorem.

1.4 Problem 4

1. Prove that $\sin x + \cos x \geq 1$ in the interval $[0, \frac{\pi}{2}]$, as well as in $[2\pi k, 2\pi k + \frac{\pi}{2}]$, where k is a natural number.
2. Prove that $f(x) = e^x \sin x$ isn't uniformly differentiable on $[0, \infty)$.
3. Prove that $f(x) = e^x \sin x$ is uniformly differentiable on $(-\infty, 0]$.

1.4.1 Answer 4

The proof is immediate from $\sin^2 x + \cos^2 x = 1$. Since $\sin x \leq 1$ and $\cos x \leq 1$, it follows that $\sin x \geq \sin^2 x$, and similarly for cosine whenever both of them are positive. Hence $\sin x + \cos x \geq 1$. Since both sine and cosine are periodical with the period equal to 2π, i.e. $\sin x = \sin 2k\pi x$, it follows that $\sin 2k\pi x + \cos 2k\pi x \geq 1$ for any natural k.

1.4.2 Answer 5

1.4.3 Answer 6

1.5 Problem 5

Let f be differentiable twice in interval (a, b). Let its second derivative be strictly positive. Prove that for every two points in the interval it holds that $f(x) \geq f(y) + f'(y)(x - y)$.

1.5.1 Answer 7

1.6 Problem 6

Let f be continuous in $\mathbb{R}$. Prove that between every two local maxima of f there exists a local minimum point.

1.6.1 Answer 8