# Assignment 12, Infinitesimal Calculus

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*<2015-04-03 Fri>*

## Contents

# 1 Problems

## 1.1 Problem 1

1. Prove from definition $\epsilon - N$:

$$\lim_{n\to\infty} \frac{n^2}{n^2-1} = L.$$

2. Prove from definition $M - N$:

$$\lim_{n\to\infty} \frac{n^2-n}{n+2} = \infty.$$

3. Formulate $\lim_{n\to\infty} a_n = \infty$ in terms of $M - N$ definition.

4. Prove using the formulation from (3) that:

$$\lim_{n\to\infty} (n\sqrt{2} + (-1)^n \lfloor \sqrt{n} \rfloor) = \infty.$$

### 1.1.1 Answer 1

Recall the definition:

> Let $(a_n)_{n=1}^{\infty}$ be a sequence, $L$ be a real number. The **sequence** $(a_n)$ **converges to** $L$ if for all $\epsilon > 0$ almost all elements of the sequence are in the $\epsilon$-neighborhood of $L$. In such a case we will say $L$ is the **limit of sequence** $(a_n)$.

To be honest, I don't know what this question is asking me to do. How can I prove something which is already given in the description of a question? Anyways, I followed the examples in the book, and here is what I think the answer might look like. Sorry, if this is not what you expect.

$$\begin{aligned}
\epsilon > 1 &\implies && \textit{From definition of reals}\\
\epsilon > \frac{n^2-1}{n^2-1} &\implies && \textit{Provided } n \neq 1\\
\epsilon' > \frac{1}{n^2-1} &\implies && \textit{From Archimedian property}\\
\epsilon + \epsilon' > \frac{n^2}{n^2-1} &\implies && \textit{Provided } \epsilon'' = \epsilon' + \epsilon\\
\epsilon'' > \left|\frac{n^2}{n^2-1} - L\right| & && \textit{Provided } 0 \leq L \leq 1
\end{aligned}$$

Which completes the "proof".

### 1.1.2 Answer 2

First, recall the definition:

> We will say that the sequence $(a_n)_{n=1}^{\infty}$ **tends to infinity**, and write $\lim_{n\to\infty} a_n = \infty$, if for every real number $M$ there exists a natural number $N$ such that for all $n > N$ it holds that $a_n > M$.

We can start by finding a more convenient formulat to work with, for instance:

$$\begin{aligned}
M &= \frac{n^2-n}{n+2}\\
&= \frac{n(n-1)}{n+2}\\
&= \frac{n(n+2)}{n+2} - \frac{3n}{n+2}\\
&< n - \frac{3n}{n}\\
&= n - 3.
\end{aligned}$$

Now, put $N = M + 3$. Thus for every $M$, $a_n$ is greater than $M$ whenver $n \geq M + 3$.

### 1.1.3 Answer 3

I'm confused by this question in the same measure I'm confused by the previous one. I don't know what do you want me to prove here. There can be different reasons for why some sequence doesn't tend to infinity. It could be because it is a divergent series, or because it converges to some real limit, or because it tends to negative infinity. I can formally negate the statement but I don't think this negation is useful.

$$\begin{aligned} \neg \forall M \in \mathbb{R} : \exists N \in \mathbb{N} : \forall n \in \mathbb{N} : (n > N \implies a_n > M) \\ \textit{Alternatively:} \\ \exists M \in \mathbb{R} : \forall N \in \mathbb{N} : \exists n \in \mathbb{N} : (n > N \implies a_n \leq M) \end{aligned}$$

### 1.1.4 Answer 4

It is easy to see that the given sequence is divergent. It has two limit points, where one is at infinity and another one is at zero. Since I need to reuse the $M - N$ rule, I will restate this clame in terms of this rule. There exists a real number $M$, such that for every natural number $N$, there exists a natural number $n$, such that whenever $n > N$, $a_n \leq M$. Put $M = 2$, then $a_n = 2(\sqrt{2} + \lfloor\sqrt{2}\rfloor)$. Now, no matter the $N$, we can always choose $n$ to be odd, which will give us:

$$\begin{aligned} 2\sqrt{2} + \lfloor 2\sqrt{2} \rfloor \geq n\sqrt{2} - \lfloor n\sqrt{2} \rfloor \\ \textit{For every } n > 2 \\ 2 * 2 + \lfloor 2 * 2 \rfloor \geq n2 - \lfloor n * 2 \rfloor \\ \textit{By axiom of order} \\ 8 \geq 0. \end{aligned}$$

## 1.2 Problem 2

Calculate the limits of the expression given above, or prove that the limtes don't exists:

$$\lim_{n\to\infty} \sqrt{1+\frac{a}{n}} \tag{1}$$
$$\lim_{n\to\infty} \sqrt{n+a} * \sqrt{n+b} - n \tag{2}$$
$$\lim_{n\to\infty} \frac{n^7 - 2n^4 - 1}{n^4 - 3n^6 + 7} \tag{3}$$
$$\lim_{n\to\infty} \Big(\frac{n}{n+1} \sum_{k=1}^{n} \frac{k}{k+1}\Big) \tag{4}$$

### 1.2.1 Answer 5

Using "sandwich" rule we can show that:

$$\sqrt{1} < \sqrt{1+\frac{a}{n}} < 1 + \frac{a}{n}$$

Provided $a$ is non-negative and isn't equal to $n$. The limit of $\sqrt{1}$ is trivially 1. The limit of $1+\frac{a}{n}$ is the sum of the limits of 1 and $\frac{a}{n}$, where the former is the limit of a constant (1), and the later is zero. Thus the value is "sandwiched" between 1 and 1, hence it must be 1.

### 1.2.2 Answer 6

Let's generalize the expression to reuse the previous case:

$$\lim_{n\to\infty} \sqrt{n+a} * \sqrt{n+b} - n =$$
$$\lim_{n\to\infty} \sqrt{n(1+\frac{a}{n})} * \sqrt{n(1+\frac{b}{n})} - n =$$
$$\lim_{n\to\infty} n\sqrt{1+\frac{a}{n}} * \sqrt{1+\frac{b}{n}} - n =$$
$$\lim_{n\to\infty} n\Big(\sqrt{1+\frac{a}{n}} * \sqrt{1+\frac{b}{n}} - 1\Big)$$

Now, recall that $\lim_{n\to\infty} \sqrt{1+\frac{a}{n}} = 1$, this gives us:

$$\lim_{n\to\infty} n\Big(\sqrt{1+\frac{a}{n}} * \sqrt{1+\frac{b}{n}} - 1\Big) =$$
$$\lim_{n\to\infty} n * \lim_{n\to\infty} \sqrt{1+\frac{a}{n}} * \lim_{n\to\infty} \sqrt{1+\frac{b}{n}} - \lim_{n\to\infty} 1 =$$
$$lim_{n\to\infty} n * (1 * 1 - 1) =$$
$$lim_{n\to\infty} n * 0 = 0.$$

### 1.2.3 Answer 7

### 1.2.4 Answer 8

I will show that the given sequence tentds to infinity. Some notation first. I will use $H_n$ to denote the *nth* harmonic numbers. Let's at first, simplify the sum:

$$\sum_{k=1}^{n} \frac{k}{k+1} =$$
$$\sum_{k=1}^{n} \frac{k+1-1}{k+1} =$$
$$\sum_{k=1}^{n} \frac{k+1}{k+1} - \frac{1}{k+1} =$$
$$\sum_{k=1}^{n} \frac{k+1}{k+1} - \sum_{k=1}^{n} \frac{1}{k+1} =$$
$$n - \sum_{k=1}^{n} \frac{1}{k+1} =$$
$$n - \sum_{k=2}^{n+1} \frac{1}{k} =$$
$$n - H_{n+1} + 1$$

Now, plug this back into our formula:

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n}{n+1}(n - H_{n+1} + 1) &= \\
\lim_{n\to\infty} \frac{n}{n+1}(n - H_{n+1}) + \lim_{n\to\infty} 1 &= \\
\lim_{n\to\infty} \frac{n}{n+1}(n - H_{n+1}) + 1 &= \\
\lim_{n\to\infty} \left(\frac{n^2}{n+1} - \frac{n}{n+1} H_{n+1}\right) + 1 &= \\
\lim_{n\to\infty} \left(\frac{n^2+1}{n+1} - \frac{1}{n+1} - \frac{n}{n+1} H_{n+1}\right) + 1 &= \\
\lim_{n\to\infty} \left(\frac{n(n+1)}{n+1} - \frac{1}{n+1} - \frac{n}{n+1} H_{n+1}\right) + 1 &= \\
\lim_{n\to\infty} \left(n - \frac{1}{n+1} - \frac{n}{n+1} H_{n+1}\right) + 1 &= \\
\lim_{n\to\infty} n - \lim_{n\to\infty} \frac{1}{n+1} - \lim_{n\to\infty} \frac{n}{n+1} H_{n+1} + 1 &= \\
\lim_{n\to\infty} n - 0 - \lim_{n\to\infty} 1 * H_{n+1} + 1 &= \\
\lim_{n\to\infty} n &= \infty.
\end{aligned}
$$

The last step is allowed becasue for all harmonic numbers greater than one it holds that $n > H_{n+1}$.