Assignment 12, Infinitesimal Calculus

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Contents

1 Problems

1.1 Problem 1

1. Prove from definition $\epsilon - N$:

$$\lim_{n\to\infty} \frac{n^2}{n^2-1} = L.$$

2. Prove from definition $M - N$:

$$\lim_{n\to\infty} \frac{n^2-n}{n+2} = \infty.$$

3. Formulate $\lim_{n\to\infty} a_n = \infty$ in terms of $M - N$ definition.

4. Prove using the formulation from (3) that:

$$\lim_{n\to\infty} (n\sqrt{2} + (-1)^n \lfloor \sqrt{n} \rfloor) = \infty.$$

1.1.1 Answer 1

Recall the definition:

> Let $(a_n)_{n=1}^{\infty}$ be a sequence, L be a real number. The **sequence** (a_n) **converges to** L if for all $\epsilon > 0$ almost all elements of the sequence are in the ϵ-neighborhood of L. In such a case we will say L is the **limit of sequence** (a_n).

To be honest, I don't know what this question is asking me to do. How can I prove something which is already given in the description of a question? Anyways, I followed the examples in the book, and here is what I think the answer might look like. Sorry, if this is not what you expect.

$$\begin{aligned}
\epsilon > 1 &\implies && \textit{From definition of reals} \\
\epsilon > \frac{n^2-1}{n^2-1} &\implies && \textit{Provided } n \neq 1 \\
\epsilon' > \frac{1}{n^2-1} &\implies && \textit{From Archimedian property} \\
\epsilon + \epsilon' > \frac{n^2}{n^2-1} &\implies && \textit{Provided } \epsilon'' = \epsilon' + \epsilon \\
\epsilon'' > \left| \frac{n^2}{n^2-1} - L \right| & && \textit{Provided } 0 \leq L \leq 1
\end{aligned}$$

Which completes the "proof".

1.1.2 Answer 2

First, recall the definition:

> We will say that the sequence $(a_n)_{n=1}^{\infty}$ **tends to infinity**, and write $\lim_{n\to\infty} a_n = \infty$, if for every real number M there exists a natural number N such that for all $n > N$ it holds that $a_n > M$.

We can start by finding a more convenient formulat to work with, for instance:

$$\begin{aligned}
M &= \frac{n^2-n}{n+2} \\
&= \frac{n(n-1)}{n+2} \\
&= \frac{n(n+2)}{n+2} - \frac{3n}{n+2} \\
&< n - \frac{3n}{n} \\
&= n - 3.
\end{aligned}$$

Now, put $N = M + 3$. Thus for every M, a_n is greater than M whenver $n \geq M + 3$.

1.1.3 Answer 3

I'm confused by this question in the same measure I'm confused by the previous one. I don't know what do you want me to prove here. There can be different reasons for why some sequence doesn't tend to infinity. It could be because it is a divergent series, or because it converges to some real limit, or because it tends to negative infinity. I can formally negate the statement but I don't think this negation is useful.

$$\begin{aligned} \neg\forall M \in \mathbb{R} : \exists N \in \mathbb{N} : \forall n \in \mathbb{N} : (n > N \implies a_n > M) \\ \textit{Alternatively:} \\ \exists M \in \mathbb{R} : \forall N \in \mathbb{N} : \exists n \in \mathbb{N} : (n > N \implies a_n \leq M) \end{aligned}$$

1.1.4 Answer 4

It is easy to see that the given sequence is divergent. It has two limit points, where one is at infinity and another one is at zero. Since I need to reuse the $M - N$ rule, I will restate this clame in terms of this rule. There exists a real number M, such that for every natural number N, there exists a natural number n, such that whenever $n > N$, $a_n \leq M$. Put $M = 2$, then $a_n = 2(\sqrt{2} + \lfloor\sqrt{2}\rfloor)$. Now, no matter the N, we can always choose n to be odd, which will give us:

$$\begin{aligned} 2\sqrt{2} + \lfloor 2\sqrt{2}\rfloor \geq n\sqrt{2} - \lfloor n\sqrt{2}\rfloor \\ \textit{For every } n > 2 \\ 2 * 2 + \lfloor 2 * 2\rfloor \geq n2 - \lfloor n * 2\rfloor \\ \textit{By axiom of order} \\ 8 \geq 0. \end{aligned}$$

1.2 Problem 2

Calculate the limits of the expression given above, or prove that the limtes don't exists:

$$\lim_{n\to\infty} \sqrt{1+\frac{a}{n}} \tag{1}$$

$$\lim_{n\to\infty} \sqrt{n+a} * \sqrt{n+b} - n \tag{2}$$

$$\lim_{n\to\infty} \frac{n^7 - 2n^4 - 1}{n^4 - 3n^6 + 7} \tag{3}$$

$$\lim_{n\to\infty} \Big(\frac{n}{n+1} \sum_{k=1}^{n} \frac{k}{k+1}\Big) \tag{4}$$

1.2.1 Answer 5

Using "sandwich" rule we can show that:

$$\sqrt{1} < \sqrt{1+\frac{a}{n}} < 1 + \frac{a}{n}$$

Provided a is non-negative and isn't equal to n. The limit of $\sqrt{1}$ is trivially 1. The limit of $1+\frac{a}{n}$ is the sum of the limits of 1 and $\frac{a}{n}$, where the former is the limit of a constant (1), and the later is zero. Thus the value is "sandwiched" between 1 and 1, hence it must be 1.

1.2.2 Answer 6

Let's generalize the expression to reuse the previous case:

$$\lim_{n\to\infty} \sqrt{n+a} * \sqrt{n+b} - n =$$

$$\lim_{n\to\infty} \sqrt{n(1+\frac{a}{n})} * \sqrt{n(1+\frac{b}{n})} - n =$$

$$\lim_{n\to\infty} n\sqrt{1+\frac{a}{n}} * \sqrt{1+\frac{b}{n}} - n =$$

$$\lim_{n\to\infty} n\Big(\sqrt{1+\frac{a}{n}} * \sqrt{1+\frac{b}{n}} - 1\Big)$$

Now, recall that $\lim_{n\to\infty} \sqrt{1+\frac{a}{n}} = 1$, this gives us:

$$\lim_{n\to\infty} n\Big(\sqrt{1+\frac{a}{n}} * \sqrt{1+\frac{b}{n}} - 1\Big) =$$
$$\lim_{n\to\infty} n * \lim_{n\to\infty} \sqrt{1+\frac{a}{n}} * \lim_{n\to\infty} \sqrt{1+\frac{b}{n}} - \lim_{n\to\infty} 1 =$$
$$lim_{n\to\infty} n * (1 * 1 - 1) =$$
$$lim_{n\to\infty} n * 0 = 0.$$

1.2.3 Answer 7

I know that this expression tends to negative infinity as n tends to infinity simply because the numerator has a power of n greater than the denominator and the greatest term in the denominator is negative. But I couldn't find a way to prove this formally.

1.2.4 Answer 8

I will show that the given sequence tentds to infinity. Some notation first. I will use H_n to denote the *nth* harmonic numbers. Let's at first, simplify the sum:

$$\sum_{k=1}^{n} \frac{k}{k+1} =$$
$$\sum_{k=1}^{n} \frac{k+1-1}{k+1} =$$
$$\sum_{k=1}^{n} \frac{k+1}{k+1} - \frac{1}{k+1} =$$
$$\sum_{k=1}^{n} \frac{k+1}{k+1} - \sum_{k=1}^{n} \frac{1}{k+1} =$$
$$n - \sum_{k=1}^{n} \frac{1}{k+1} =$$
$$n - \sum_{k=2}^{n+1} \frac{1}{k} =$$
$$n - H_{n+1} + 1$$

Now, plug this back into our formula:

$$\lim_{n\to\infty} \frac{n}{n+1}(n - H_{n+1} + 1) =$$
$$\lim_{n\to\infty} \frac{n}{n+1}(n - H_{n+1}) + \lim_{n\to\infty} 1 =$$
$$\lim_{n\to\infty} \frac{n}{n+1}(n - H_{n+1}) + 1 =$$
$$\lim_{n\to\infty} \left(\frac{n^2}{n+1} - \frac{n}{n+1} H_{n+1}\right) + 1 =$$
$$\lim_{n\to\infty} \left(\frac{n^2+1}{n+1} - \frac{1}{n+1} - \frac{n}{n+1} H_{n+1}\right) + 1 =$$
$$\lim_{n\to\infty} \left(\frac{n(n+1)}{n+1} - \frac{1}{n+1} - \frac{n}{n+1} H_{n+1}\right) + 1 =$$
$$\lim_{n\to\infty} \left(n - \frac{1}{n+1} - \frac{n}{n+1} H_{n+1}\right) + 1 =$$
$$\lim_{n\to\infty} n - \lim_{n\to\infty} \frac{1}{n+1} - \lim_{n\to\infty} \frac{n}{n+1} H_{n+1} + 1 =$$
$$\lim_{n\to\infty} n - 0 - \lim_{n\to\infty} 1 * H_{n+1} + 1 =$$
$$\lim_{n\to\infty} n = \infty.$$

The last step is allowed becasue for all harmonic numbers greater than one it holds that $n > H_{n+1}$.

1.3 Problem 3

Let (a_n) be a sequence such that $\lim_{n\to\infty}(a_{n+1} - a_n) = 0$.

1. Prove that if there exists a constant $c > 0$ s.t. for all n $|a_n| \geq c$, then almost all elements of the sequence are either negative or positive.
2. Show that the same doesn't hold for $|a_n| > 0$.
3. Show that if for all n it holds that $|a_n| \geq n$, then (a_n) converges in a general sense.

1.3.1 Answer 9

First, observe that the definition gives us a Cauchy sequence. Recall that Cauchy sequence is a sequence where intervals between subseqent members tend to zero as n tends to infinity. That said, from the definition of the limit, we can find $\epsilon < c$, such that almost all members of the sequence would be in the vicinity of c no farther away than ϵ. Thus, the sequence would converge to either a negative number or a positive number, but not zero. Since almost all members of a sequence can be showed to be in a vicinity of a number other than zero, we have proved the claim.

1.3.2 Answer 10

The example (a_n) such that doesn't match the requirement from the previous question is, for example, $a_n = \frac{(-1)^n}{2n}$. I.e. an alternating series where even members are positive and odd members are negative. The general form of $a_{n+1} - a_n$ will thus look like this:

$$\lim_{n\to\infty}\left(\frac{(-1)^{n+1} - 2(-1)^n}{2n}\right) = 0.$$

It's easy to see that this limit is zero: the nominator is a constant term because one raised to any power is either one or negative one, while denominator is growing with n. It is also easy to see that this is an alternating series, i.e. the sign alters with parity of n.

1.3.3 Answer 11

Recall 1.3.1, where we used the fact that (a_n) is described as a Cauchy sequence. This is enough to show that this sequence converges in a general sense, whether the equation $|a_n| \geq n$ holds or not is inconsequential. Perhaps it's worth saying that the above requirement makes it explicit that the sequence converges to infinity if any of its members is positive, and negative infinity otherwise.

1.4 Problem 4

Given sequences (a_n) and (b_n) s.t. $\lim_{n\to\infty}(a_n b_n) = 0$, prove or disprove the following:

1. Either $\lim_{n\to\infty} a_n = 0$ or $\lim_{n\to\infty} b_n = 0$.
2. If $\lim_{n\to\infty} b_n = 1$, then there exists $N > 0$ s.t. $a_n < \frac{1}{2}$ for all $n > N$.
3. If $\lim_{n\to\infty} b_n = 1$, then $\lim_{n\to\infty} a_n = 0$.
4. If almost all members of (b_n) are positive, then $\lim_{n\to\infty} a_n \neq \infty$.
5. If almost all members of (b_n) are positive, then $\lim_{n\to\infty} a_n = 0$.
6. If there exists a constant $c > 0$ s.t. almost all $b_n \geq c$, then $\lim_{n\to\infty} a_n = 0$.

1.4.1 Answer 12

The limit of a product is a product of limits (their existence is provisional). This gives us an equation in two unknowns: $ab = 0$. From this equation follows that either a or b must be zero Thus the claim is true.

1.4.2 Answer 13

As shown in 1.4.1, either one (or both) limits must be zero, thus if $\lim_{n\to\infty} a_n$ must be zero. From definition of the limit, there must be an ϵ, such that for any N, almost all $|a_n - L| < \epsilon$, for $n > N$. We can choose $\epsilon = \frac{1}{2}$, since L (the limit of (a_n)) is zero the inequality becomes $|a_n| < \frac{1}{2}$ since $a_n \leq |a_n| < \frac{1}{2}$ the claim is true.

1.4.3 Answer 14

The claim is true. This is simply a corollary of 1.4.1.

1.4.4 Answer 16

This is not true. Put $b_n = \frac{1}{n}$, the limit of (b_n) is 0. Even if (a_n) is a divergent series, for example, $a_n = \sqrt{n}$ the limit of a product is still zero.

1.4.5 Answer 17

Same example as 1.4.4. Limit of (a_n) isn't zero and yet the requirement is satisfied, thus the claim is false.

1.4.6 Answer 18

Existence of lower bound greater than zero precludes (b_n) from having zero as its limit. Assuming that there are members of a sequence which are smaller than the lower bound is in direct contradiction with the bound definition. Thus the limit of (a_n) must be zero (follows from what we proved in 1.4.1). Thus the claim is true.