Assignment 17, Infinitesimal Calculus

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Contents

1 Problems

1.1 Problem 1

1. Compute

$$\lim_{n\to\infty}\left(1+\sqrt{n}\sin\frac{1}{n}\right)^{\sqrt{n}}$$

2. Compute

$$\lim_{x\to 0}\left(1+\frac{1-\cos x}{x}\right)^{\frac{1}{x}}$$

1.1.1 Answer 1

At first step we can replace $m=\sqrt{n}$. This gives us:

$$\lim_{n\to\infty}\left(1+m\sin(m^{-2})\right)^{m}$$

Now, notice that $\lim_{n\to\infty} m=\infty$, and that the formula obtained is very similar to $\lim_{b\to\infty}(1+a)^b=e$, whre a tends to zero. Now, if $m\sin(m^{-2})$ tends to zero, we know the limit to be e. So, all we need to show is that $\lim_{m\to\infty} m\sin(m^{-2})=0$.

$$\begin{aligned}
&\textit{Define support variable } k\\
&k=\frac{1}{m}\\
&\lim_{m\to\infty} m\sin(m^{-2})=\\
&\lim_{k\to 0}\frac{\sin k^2}{k}=\\
&\textit{Using L'Hospital's rule}\\
&\lim_{k\to 0}\frac{(-\sin(k^2))\cdot 2k}{1}=\\
&\textit{Sinus is defined and is continuous at 0, simply substitute}\\
&\lim_{k\to 0}\frac{0\cdot 2\cdot 0}{1}=0\ .
\end{aligned}$$

Since we showed $\lim_{m\to\infty} m\sin(m^{-2})=0$, it follows that $\lim_{n\to\infty}\left(1+\sqrt{n}\sin\left(\frac{1}{n}\right)\right)^{\sqrt{n}}=e$.

1.1.2 Answer 2

Similar to the previous answer, we will at first define a helper variable: $y = \frac{1}{x}$, then we will search for the solution of equivalent problem:

$$\lim_{y\to\infty}\left(1+\frac{1-\cos\left(\frac{1}{y}\right)}{\frac{1}{y}}\right)^{y} =$$

$$\lim_{y\to\infty}\left(1+y\left(1-\cos\left(\frac{1}{y}\right)\right)\right)^{y} =$$

$$\lim_{y\to\infty}\left(1+y\left(1-\cos\left(\frac{1}{y}\right)\right)\right)^{\frac{1}{y(1-\cos(\frac{1}{y}))}\cdot y^2(1-\cos(\frac{1}{y}))} =$$

$$\lim_{y\to\infty} e^{y^2(1-\cos(\frac{1}{y}))}$$

Now, since the limit of the exponent is the exponent of the limits, we may limit ourselves to finding the limit of $y^2(1-\cos(\frac{1}{y}))$. Again, define a helper variable $z = \frac{1}{y}$ and search for the limit as z approaches zero:

$$\lim_{z\to 0}\frac{1-\cos(z)}{z^2} =$$

Using L'Hospital

$$\lim_{z\to 0}\frac{\sin(z)}{2z} =$$

Using L'Hospital again

$$\lim_{z\to 0}\frac{\cos(z)}{2} =$$

Cosine is defined at 0 and is continuous, substituting

$$\lim_{z\to 0}\frac{1}{2} = \frac{1}{2}\ .$$

Substituting the intermediate result back gives: $e^{\frac{1}{2}} = \sqrt{e}$.

1.2 Problem 2

Let $f(x) = e^{-x} + \cos x$.

1. Prove $\lim_{n\to\infty} f(\pi + 2\pi n) = -1$.
2. Prove $\inf f([0,\infty)) = -1$.
3. Prove that for all $-1 < c < 2$ there exists a solution for $f(x) = c$ in $[0,\infty)$.

1.2.1 Answer 3

Limit of the sum is the sum of the limits, hence:

$$\lim_{x \to 0} e^{-x} + \cos x =$$
$$\lim_{x \to 0} e^{-x} + \lim_{x \to 0} \cos x =$$
Regardless of how we sample x, since e^{-x} is monotonicaly decreasing
$$0 + \lim_{x \to 0} \cos x =$$
Since cosine is continuous everywhere substituting
$$\lim_{x \to 0} \cos(\pi + 2\pi n) = -1 \ .$$

1.2.2 Answer 4

The proof amounts to showing that e^{-x} is monotonically decreasing, and has its infimum at zero. Since infimum of cosine is at -1, this would complete the proof. (Recall that the sum of infima of two functions is no less then their sum.) To show that e^{-x} is monotonically decreasing, and thus must be bounded from below by its limit we claim that for any $x_0 < x_1$ it is also the case that $e^{-x_0} > e^{-x_1}$ (the clame we made without a proof in the previous answer.) But this is immediate from definition of exponentiation. Therfore the proof is complete.

1.2.3 Answer 5

The sum of two continuous functions is continuous, therefore f is continuous in the same range where e^{-x} and cosine are continuous, and in particular in the range $(-1, 2)$. By intermediate value theorem, we are guaranteed that f attains the value c in the specified range if we can show that it is defined at the edges. In the previous answers we found that f has a limit point at -1, in other words, we can make it as close to -1 as we like. Solving for $c = 2$ is tricky, but we can pick a larger value, without harming the claim, for example, pick $x = -\frac{\pi}{2}$. This gives $e^{\frac{\pi}{2}} + \cos(\frac{\pi}{2}) = e^{\frac{\pi}{2}} + 0 > 4 > 2$.

1.3 Problem 3

1. Prove $\lim_{n \to \infty}(\ln(2\pi n + \frac{\pi}{2}) - \ln(2\pi n)) = 0$.
2. Prove that $f(x) = \sin(e^x)$ is not uniformly continuous.

1.3.1 Answer 6

Using the properties of ln, viz. $ln(x) - ln(y) = ln(x/y)$ obtains:

$$\lim_{n\to\infty}\left(\ln(2\pi n + \frac{\pi}{2}) - \ln(2\pi n)\right) =$$
$$\lim_{n\to\infty} \ln\left(\frac{2\pi n + \frac{\pi}{2}}{2\pi n}\right) =$$
Using limit of function composition
$$\ln\left(\lim_{n\to\infty}\frac{2\pi n + \frac{\pi}{2}}{2\pi n}\right) =$$
Using L'Hospital's rule
$$\ln\left(\lim_{n\to\infty}\frac{2\pi}{2\pi}\right) =$$
$$\ln(1) = 0 \; .$$

1.3.2 Answer 7

We are going to use the definition of uniform continuity which requires that if a limit of a difference of two sequences is equal to zero, then the limit of the difference of sequences of function's values at these sequences must be zero too.

Let $(x_n) = \ln(n+2)$ and $(y_n) = \ln(n)$. The proof of the limit of their difference being equal to zero is identical to the one given in the previous answer.

Now consider these two sequences $(x_{f(n)}) = e^{\ln(n+2)}$ and $(y_{f(n)}) = e^{\ln(n)}$. From definition of uniform continuity, it follows that:

$\lim_{n\to\infty}\left(e^{\ln(n+2)} - e^{\ln(n)}\right) = 0$ too.

$$\lim_{n\to\infty}\left(e^{\ln(n+2)} - e^{\ln(n)}\right) =$$
Since $e^{\ln(x)} = x$
$$\lim_{n\to\infty}(n+2-n) = 2 \; .$$

Contrary to assumed. Hence $f(x) = e^x$ is not uniformly continuous.

1.4 Problem 4

For all functions given below find their domain of definition, domain of continuity, and domain of differentiability. Find a definition for every point in the differentiability domain *(I have no idea what this is supposed to mean)*.

1.

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & \text{if} x \neq 0 \\ 0 & \text{if} x = 0 \; . \end{cases}$$

2. $f'(x)$ for f defined in previous question.

3.

$$f(x) = x^2 D(x) = \begin{cases} x^2 & \text{if} x \in \mathbb{Q} \\ 0 & \text{if} x \notin \mathbb{Q} \; . \end{cases}$$

1.5 Problem 5

Prove that:

1.

$$\lim_{x \to 0^-} \frac{e^{\frac{1}{x}}}{x} = 0 \; .$$

2. Prove that the function:

$$f(x) = \begin{cases} e^{\frac{1}{x}} + \sin x & \text{if } x < 0 \\ \ln(1 + x) & \text{if } x \geq 0 \end{cases}$$

is differentiable at 0.

1.5.1 Answer 8

Define helper variable $y = \frac{1}{x}$, then we need to solve an equivalent problem: $\lim_{y \to \infty} \frac{1}{ye^y}$. But the solution is immediate since $e^n \geq 1$ whenever $n > 0$. In other words, this is: $\lim_{y \to \infty} \frac{1}{y \cdot 1} = 0$.

1.5.2 Answer 9

To show that f is differentiable we need to show that both limits exist and that they agree at 0, furthermore, that the limits are finite. In other words:

$$\lim_{x \to 0^-} \left(e^{\frac{1}{x}} + \sin x \right) = \lim_{x \to 0^+} \ln(1 + x)$$

Logarithm is continuous at 1, substituting value for x:

$$\lim_{x \to 0^-} \left(e^{\frac{1}{x}} + \sin x \right) = 0$$

$$\lim_{x \to 0^-} \left(e^{\frac{1}{x}} \right) + \lim_{x \to 0^-} (\sin x) = 0$$

Sine is continuous at 0, substituting value for x:

$$\lim_{x \to 0^-} \left(e^{\frac{1}{x}} \right) + 0 = 0 \; .$$

In order to find the later limit, we could define a helper variable: $y = \frac{1}{x}$ and solve an equivalent problem:

$$\lim_{y \to \infty} \left(\frac{1}{e^y} \right) = 0$$

Since e^y is monotonically increasing and has no upper bound.

Having showed that, we showed that both left and right limits exist and that they agree at 0, hence f is differentiable at 0.

1.6 Problem 6

Let f be continuous at x_0. Prove that $g(x) = |x| \, f(x)$ is differentiable if and only if $f(0) = 0$.

1.6.1 Answer 10

The intuition for the proof is that additive identity in an ordered field is unique in that it is the only positive number, which doesn't preserve sign under multiplication. In other words, since the derivatives of absolute value function for number less than zero and for number greater than zero differ only in sign, the only way to ignore this discrepancy is to multiply them by zero. A more formal proof follows:

First, we will prove the "if" part, i.e. if $f(0) = 0$, then g is differentiable.

$$
\begin{aligned}
&\lim_{h\to 0} \frac{g(0+h) - g(0)}{h} \cdot f(h) \textit{ where } g(h) = |h| \\
&\lim_{h\to 0} \frac{|0+h| - |0|}{h} \cdot g(h) = \lim_{h\to 0} \frac{|h|}{h} \cdot g(h) \\
&\textit{Assume } h > 0 \textit{ then} \\
&\lim_{h\to 0^+} \frac{|h|}{h} = 1 \cdot g(h) \\
&\textit{Assume } f(h) = 0 \textit{ then} \\
&\lim_{h\to 0^+} 1 \cdot 0 = 0 \\
&\textit{else} \\
&\lim_{h\to 0^-} \frac{|h|}{h} = -1 \\
&\textit{Assume } f(h) = 0 \textit{ then, similarly} \\
&\lim_{h\to 0^-} -1 \cdot 0 = 0 \\
&0 = 0\,.
\end{aligned}
$$

Since both limits exists and agree, g is differentiable.

Now, we will prove the "only if" part. Suppose, for contradiction there existed some value, we define it later to be $y \neq 0$, exists, such that g would be differentiable at 0:

$$
\begin{aligned}
&\textit{Assume } \lim_{h\to 0^+} \frac{|h|}{h} \cdot f(0) = \lim_{h\to 0^-} \frac{|h|}{h} \cdot f(0) \\
&\lim_{h\to 0^+} 1 \cdot f(0) = \lim_{h\to 0^-} -1 \cdot f(0) \\
&\lim_{h\to 0^+} f(0) = - \lim_{h\to 0^-} f(0) \\
&\textit{Since } f \textit{ is continuous, its value at 0 is its limit at 0} \\
&\textit{Put } f(0) = y, y \neq 0 \textit{ then} \\
&- y = y \textit{ Contradiction!}
\end{aligned}
$$

Even though both limits exist, they don't agree, hence whenever g is not differentiable, $f(0) \neq 0$.