Assignment 14, Infinitesimal Calculus

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Contents

1 Problems

1.1 Problem 1

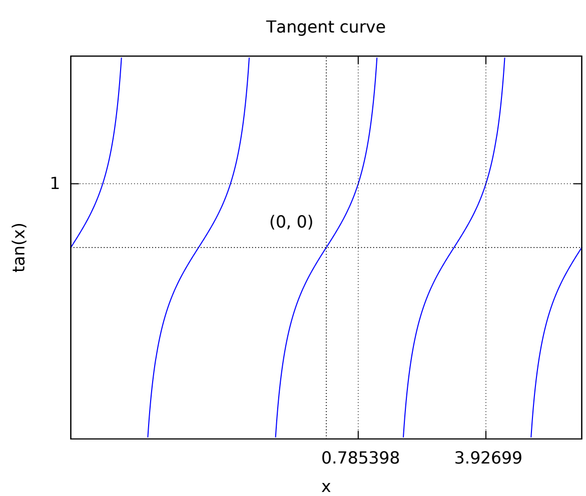
1. Find the domain of f defined as $f(x) = \sqrt{\tan x - 1}$.
2. Find all values of x in segment $[0, \pi]$, for which $|\tan x| \leq \sin 2x$.

1.1.1 Answer 1

$\tan x > 1$ where $\frac{\pi}{4} < x \bmod \frac{\pi}{2} < \frac{\pi}{2}$. Since we assume that f is real-valued, we cannot extract roots from negative numbers.

Hence $\mathrm{Dom}(f) = \{x \in \mathbb{R} \mid \frac{\pi}{4} < x \bmod \frac{\pi}{2} < \frac{\pi}{2}\}$.

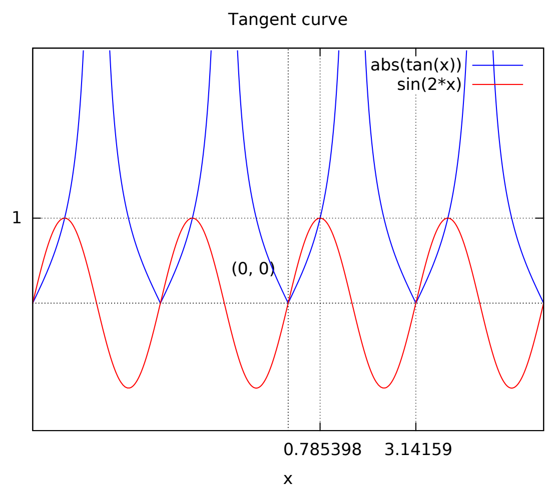
```
programmode: false;
gnuplot_pdf_command: %command;
print(plot2d (tan(x),
  [x, -2 * %pi, 2 * %pi], [y, -3, 3], grid2d,
  [xtics, 0.25 * %pi, %pi, 1.5 * %pi],
  [ytics, 1, 1, 1], [label, ["(0, 0)", -1.4, 0.4]],
  [title, "Tangent curve"], [pdf_file, %out]));
```



1.1.2 Answer 2

Both functions attain the same values at 0 and $\frac{\pi}{4}$. But sine is a concave function and tanget is a convex function, thus tangent must be less than sine at this interval. Tangent keeps increasing until $\frac{\pi}{2}$, while sine will be decreasing until $\frac{3\pi}{2}$, thus, on this interval tangent is greater than sine. The functions meet again at $x = \pi$.

```
programmode: false;
gnuplot_pdf_command: %command;
print(plot2d ([abs(tan(x)), sin(2 * x)],
  [x, -2 * %pi, 2 * %pi], [y, -1.5, 3], grid2d,
  [xtics, 0.25 * %pi, 0.75 * %pi, 1.5 * %pi],
  [ytics, 1, 1, 1], [label, ["(0, 0)", -1.4, 0.4]],
  [title, "Tangent curve"], [pdf_file, %out]));
```



1.2 Problem 2

Let f, g and h be functions from $\mathbb{R}$ to $\mathbb{R}$.

1. If $f \circ g = f \circ h$, does it follow $g = h$?
2. If $f \circ g = f \circ h$, and f is one-to-one, does it follow $g = h$?

3. If $f \circ g = f \circ h$, and f is onto, does it follow $g = h$?
4. If $f \circ g$ is increasing, and f is decreasing, does it follow that g is increasing?
5. If $f \circ g$ is increasing, and f is one-to-one, does it follow that g is monotonic?

1.2.1 Answer 3

No, g and h are not necessarily equal. Whenever co-domain of f doesn't contain some real number, g and h may differ in that input. For example, let $f(x) = 2x$, $g(x) = (x \bmod 2) + x$ and $h(x) = (x \bmod 2) * 2 + x$. Because f in this example will only generate even numbers, the $(x \bmod 2)$ term will always be zero, thus $f \circ g = f \circ h$, but, obviously, $g \neq h$.

1.2.2 Answer 4

No, it isn't sufficient for f to be one-to-one to ensure right-cancellation property under composition. The example given in 1.2.1 is applicable in this case too since whenever $f(x) = f(y)$ so is $x = y$ (since multiplication does have the cancellation property).

1.2.3 Answer 5

Yes, if f is onto, then the composition is right-cancellable. Suppose, for contradiction it wasn't, then for some y $g(y) \neq h(y)$, but $y = f(x)$ (since by definition of a total function, every element in its co-domain has an element in its domain). Hence $g(f(x)) \neq h(f(x))$, but we are given that $g \circ f = h \circ f$, which is a contradiction. Hence functions are equal.

1.2.4 Answer 6

No, g doesn't need to be increasing. Put $g(x) = f(x) = -x$, both f and g are decreasing but $f \circ g = Id$, which is an increasing function.

1.2.5 Anser 7

No, g is not necessarily monotonic. Put $f(x) = x(-1)^x$ and $g(x) = |x|$. Then $(f \circ g)(x) = \left|x(-1)^x\right| = x\left|(-1)^x\right| = x$. $f \circ g$ is increasing, f is one-to-one, but g isn't monotonic: it decreases whenever x is negative and increases whenever x is positive.