Assignment 13, Infinitesimal Calculus

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Contents

1 Problems

1.1 Problem 1

Given the sequence (a_n) defined as $0 < a_1 < 6$ and $a_{n+1} = \sqrt{6a_n}$ for all n.

1. Prove that the sequence converges.
2. Find $\lim_{n\to\infty} a_n$.

1.1.1 Answer 1

First, observe that every next element of the sequence (a_n) is bigger than the previous. Using mathematical induction we can show that:

Base step: $\sqrt{a_1} < \sqrt{a_2}$ because $a_1 < 6$ and thus $\sqrt{a_1} < \sqrt{6}$, consequently (from order preservation under multiplication) $\sqrt{a_1} \times \sqrt{a_1} < \sqrt{6} \times \sqrt{a_1}$ and hence $\sqrt{a_1} < \sqrt{a_2}$.

Inductive step:

$$
\begin{aligned}
a_k &< \sqrt{6 \times a_{k+1}} \\
&\quad \textit{Using induction hypothesis, } k = n - 1 \\
\sqrt{6 \times a_k} &< \sqrt{6 \times a_{k+1}} \\
\sqrt{6} \times \sqrt{a_k} &< \sqrt{6} \times \sqrt{a_{k+1}} \\
&\quad \textit{Order preservation under multiplication} \\
\sqrt{a_k} &< \sqrt{a_{k+1}} \\
a_k &< a_{k+1}
\end{aligned}
$$

Hence, by mathematical induction, the sequence (a_n) is an increasing one.

Now we demonstrate that this sequence is bounded above and it has a supremum. Again, using induction, we can choose 6 to be the upper bound.

Base step: $a_1 < 6$ by definition.

Inductive step:

$$\begin{aligned}
a_k &< 6 \\
&\quad \textit{Using induction hypothesis, } k = n - 1 \\
\sqrt{6 \times a_k} &< \sqrt{6 \times 6} \\
\sqrt{6} \times \sqrt{a_k} &< \sqrt{6} \times \sqrt{6} \\
&\quad \textit{Order preservation under multiplication} \\
\sqrt{a_k} &< \sqrt{6} \\
a_k &< 6
\end{aligned}$$

Hence, by mathematical induction, (a_n) has an upper bound. It is trivial to see that this is also the supremum, since a_1 is defined to be strictly less than 6, so no smaller upper bound can exist.

Having showed both these properties, we can now use the monotone sequence theorem to claim that this sequence converges. Accidentally, we also found the limit of this sequence, which wasn't due until the next question.

1.1.2 Answer 2

We have already found the limit of (a_n) as n approaches infinity. This is given by a corollary from the monotone sequence theorem: the supremum is also the limit of a monotone sequence. Thus the answer is 6.

1.2 Problem 2

Find the limits of:

1. $$\lim_{n\to\infty} \frac{(-5)^n - 2^n + 2}{3^n + (-2)^n - 2}.$$

2.
$$\lim_{n\to\infty} \frac{3^n + (-2)^n - 2}{(-5)^n - 2^n + 2}.$$

3.
$$\lim_{n\to\infty} (\lfloor 2n \rfloor - 2\lfloor n \rfloor).$$

4.
$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}.$$

1.2.1 Answer 3

We can see that the denominator of this limit will be a positive integer for $n > 1$. The numerator will alternate between positive and negative values, where the difference will only increase with time since $(-5)^2n > 2^n$ and $(-5)^{2n-1} > 2^n$, thus this expression has two limit points: ∞ and $-\infty$.

1.2.2 Answer 4

The same reasoning we put forward in 1.2.1 applies here, but now the numerator will be some positive integer, smaller than the denominator. Thus this expression will have a single limit at 0.

1.2.3 Answer 5

Here I will assume n to be real, otherwise the limit is trivially 0. In this case we can split the sequence into the following subsequences:

$$(a_n) = \{\lfloor x \rfloor | x \in \mathbb{R} : x \pmod 2 < 0.5\}$$
$$(b_n) = \{\lfloor x \rfloor | x \in \mathbb{R} : x \pmod 2 \geq 0.5\}$$

It's easy to see that $(a_n) \cup (b_n)$ gives the initial set. These subsets have constant limits $\{0, 1\}$ as is given by their generating formulae: $\lfloor x \pmod 2 <$

$0.5\rfloor$ and $\lfloor x \pmod 2 < 0.5\rfloor$ resp. But the whole expression doesn't have a single limit point, since we can always find as many points as we like that would not be an ϵ distance from one of the limit points found.

1.2.4 Answer 6

We can show this expression approaches 1 as n gets larger using the following algebraic transformations:

$$\begin{aligned}\frac{\sqrt[n]{n!}}{n} &= \sqrt[n]{\frac{n!}{n^n}} \\ &= \sqrt[n]{\prod_{k=1}^{n} \frac{n-k}{n}}\end{aligned}$$

Since $n - k < n$, $\frac{n-k}{n} < 1$. The product of arbitrary many terms all of which are less than one will be less than one. And will become smaller as the number of terms grows. However, the nth root will become progressively large as the fractional term becomes smaller, yet never exceeding 1. Thus the limit of this expression is 1.

1.3 Problem 3

Let (a_n) and (b_n) be sequences both bounded from above.

1. Prove $\sup\{a_n + b_n | n \in \mathbb{N}\} \leq \sup\{a_n | n \in \mathbb{N}\} + \sup\{b_n | n \in \mathbb{N}\}$.
2. Find such (a_n) and (b_n) which are equal under the condition given in (1).
3. Find such (a_n) and (b_n) for which the inequality given in (1) holds.

1.3.1 Answer 7

1.3.2 Answer 8

An example of $(a_n) = \frac{1}{n}$ whose supremum is 1 and $(b_n) = \frac{2}{n}$. Since they both reach their maximum output at $n = 1$, the sum of their suprema and the supremum of their sum is the same.

1.3.3 Answer 9

To contrast 1.3.2, $(a_n) = \frac{1}{n}$ and $(b_n) = \frac{1}{n-1}$ reach their suprema at different times, ((b_n) isn't even defined at the time (a_n) reaches its maximum), the supremum of their sum is thus 1.5, but the sum of their suprema is 2.

1.4 Problem 4

Let $(a_n) = n - \lfloor \sqrt{n} \rfloor^2$.

1. Prove that (a_n) is bounded from below.
2. Prove that 0 is a limit point of (a_n).
3. Find $\inf\{a_n | n \in \mathbb{N}\}$, $\liminf_{n\to\infty} a_n$. Establish whether (a_n) has minima.
4. Given natural number ℓ, prove that almost for all n it holds that $n < \sqrt{n^2 + \ell} < n + 1$.
5. Prove that every natural number is a limit point of (a_n).
6. Is (a_n) bounded from above?
7. Find $\limsup_{n\to\infty} a_n$.

1.4.1 Answer 10

1.4.2 Answer 11

1.4.3 Answer 12

1.5 Problem 5

Let (a_n) and (b_n) be sequences.

1. Assume $\lim_{n\to\infty}(a_n + b_n) = L$ for some finite L. Prove that if (a_n) is bounded then (b_n) is bounded too.
2. Assume $\lim_{n\to\infty}(a_n + b_n) = L$ for some finite L. Prove that if (a_n) has a limit point a, then $L - a$ is a limit point of (b_n).
3. Assume (a_n) has 20106 limit points and (b_n) has 20474 limit points. Prove that $(a_n + b_n)$ diverges.