Assignment 16, Introduction To Mathematics

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Contents

1 Problems

1.1 Problem 1

Given a system of axioms concerning "points", "lines" and a relation "on" *(point on a line)*:

1. Given two distinct points A and B, and two distinct lines ℓ_1 and ℓ_2 such that A and B are on both of them.
2. For every line ℓ there is a point P which is not on ℓ.
   - Prove that the system is consistent.
   - Prove that the system is not categorical.
   - Prove that the system is independant.
   - Prove that the system entails theorem "There exist at least four points".

1.1.1 Answer 1

In order to show consistency it is enough to find a model for the system. Let's find such a model. If I consider lines as sets, then I can imagine a system with lines $\ell_1 = \{A, B, P_1\}$ and $\ell_2 = \{A, B, P_2\}$. By construction points A, B are on both ℓ_1 and ℓ_2, $P_1 \notin \ell_2$ and $P_2 \notin \ell_1$. This satisfies both axioms, hence the model is consistent.

1.1.2 Answer 2

In order to show that the system is not categorical I need to find at least two models not isomorphic to each other. Observe that we can easily extend the model given in 1.1.1 by appending points to either of two lines without violating any of the two axioms. Thus, for example, a model $\ell_1 = \{A, B, P_1, P_3\}, \ell_2 = \{A, B, P_2\}$ would still satisfy both axioms but would not be isomorphic to the system I extended because the first one didn't have point P_3 in it.

1.1.3 Answer 3

To show independence, I need to find such models where one of the axioms in the system in question doesn't hold, but other do. An example of such system could be modeled using the model from the 1.1.1 with a line ℓ_3 appended to it such that $\ell_3 = \{A, B, P_1, P_2\}$. This would allow for the first axiom to hold, while the second axiom would be violated (as there would be no points that are not on ℓ_3.

1.1.4 Answer 4

I am given at least two points by definition, now I need to establish that there must be at least two more. Observe that it is required by the second axiom that there be a point P not on some line. Neither A nor B can satisfy this condition, since both of them are on both lines, thus there must be at least one more point for both ℓ_1 and ℓ_2 (since these are the only lines warranted to exist by the first axiom). Thus we have that there will be at least four points in this system.

1.2 Problem 2

Given the system of axioms which defins "point", "line", an "on" relation and axioms:

1. There are at least two lines.
2. There are exactly seven points.
3. There are exactly three points on each line.
4. For every two lines there's exactly one point that is on both of them.
   - Prove that the system is consistent.
   - Prove that the system is independent.
   - Prove or disporve the statement: the system is categorical.

Given following axioms:

1. Every two points are on one and only line.
2. Every three points are on one and only line.

For axioms 5 and 6, prove or disprove: after appending either one of them to the original system, the system will remain consistent.

1.2.1 Answer 5

To prove consistency I will need to show a model for the system. I will start by naming the points: A, B, C, D, E, F, G. Suppose now, I built four three lines: $\ell_1 = \{A, B, C\}$, $\ell_2 = \{A, D, E\}$, $\ell_3 = \{B, D, F\}$, $\ell_4 = \{C, D, G\}$.

First axiom holds since there are at least two lines (four actually). Second axiom holds since there are exactly seven points. Third axiom holds since ther are exactly three points on each line. Fourth axiom holds since there is exactly one point where each two lines intersect. Let's list them in a table:

	ℓ_1	ℓ_2	ℓ_3	ℓ_4
ℓ_1				
ℓ_2	A			
ℓ_3	B	D		
ℓ_4	C	D	D	

(note that we only need to fill half the table, since it's symmetrical)

1.2.2 Answer 6

To prove independence, I need to show a model that is inconsisten with all modles, but is consistent with the system lacking either one of the axioms. First axiom warrants us the existence of lines, if it wasn't for the first axiom, the system could have no lines, and all other axioms would hold vacuously. If we haven't been given that thre must be exactly seven points, we could construct a system which would have fewer points but satisfy all other conditions, for example: $\ell_1 = \{A, B, C\}, \ell_2 = \{C, D, E\}$, which has at least two lines, exactly three points on each line and C is the only line where ℓ_1 and ℓ_2 intersect. If we remove the fourth axiom, then we could build a model, where some lines don't intersect, for example: $\ell_1 = \{A, B, C\}, \ell_2 = \{C, D, E\}, \ell_3 = \{E, F, G\}$.

Since I've tried to remove every axiom and was able to build a model inconsistent with the removed axiom, but consistent with the rest, the system must be independent.

1.2.3 Answer 7

The system is categorical. It has exactly seven points and all assignments of lines to these points are equivalent up to isomorphism. To prove this, suppose this was not the case. Suppose there was an assignment of lines to points such that would be different from the one modeled in 1.2.1. First, observe that this wouldn't have been possible to achieve with three lines, as this would require that provided there are three points on each line, either two of three lines don't intersect, or, there would be one spare point left. To see why this is true, let's count the elements of three sets where each contains exactly three elements such that it shares one of the elements with the other set. Since there are three of them, there must be two elements in each set that are shared with the other sets. I.e. each set contains one element it doesn't share with anyone, and two elements it shares with each other set. This, in turn gives us 3 non-shared elements, and three elements shared between pairs of sets. Which is one element shy of the requirement.

Observe also that this system cannot be modeled by five or more lines because five lines would require that lines with exactly three points intersect with every other four lines in exactly one point, which is not possible (by pigeonhole principle, some two lines would have to intersect in two points, or there would have to be more points in each line).

With this out of the way, we can now limit our proof to only proving that any assignment of four lines to seven points with initial constraints will be isomorphic. Obseve that without loss of generality, we could select any three points and draw a line through them. Let these be A, B and C for the ease of reference. We are now left with four points, we need to draw a line through each pair of these four points such that the third point would be then either A, B or C. Combinations of two from four give us three, all of them equivalent choices. These are also the only choices that we can make, since if we chose either three or one points from the D, E, F, G, then we will fail to satisfy either the requirement that all lines have a commont point or that that any two lines intersect in only one point. This gives us an assignment of lines to points unique up to renaming, which is another way of saying that the model is unique up to isomorphism.

1.2.4 Answer 8

Every two points are on one and only line—this follows from the system in question. Since we proved earlier that the system is categorical, we can use its model to prove simply by counting that every line has a combination of two unique points.

1.2.5 Answer 9

If, however, we claimed that every three points are on one and only line, the system would've become inconsistent. One way to see that is to, again, simply by looking at the model (and we are allowed to do so, since we know it's the only possible model). Another way is to simply count the combinations of three out of seven without repetition. This gives us that if this axiom was consistent with the system, there would have to be 35 lines, but we know ther to be only 4. Thus it would be a contradiction.

1.3 Problem 3

Given the axioms of group which define "element" and "binary operation".

1. Prove that the system is consistent.

2. Prove that the second axiom is not derivable from the rest.
3. Prove that the fourth axiom is not derivable from the rest.
4. With added fifth axiom: “there are exactly four elements”, let G be a set of functions $f, g, h, k : \{1,2,3,4\} \to \{1,2,3,4\}$ defined as: f is the identity function, $g(4) = 4, g(3) = 3, g(2) = 1, g(1) = 2$, $h(4) = 3, h(3) = 4, h(2) = 1, h(1) = 2$, $k = g \circ h$. Prove that G with function composition operation models the system $(1,2,3,4,5)$.
5. Prove that $(1,2,3,4,5)$ is not categorical.

1.3.1 Answer 10

To prove consistency I need to show a model of the system. Let’s recal the axioms:

Closure the result of application of group operation to any two elements of the group is an element of the group.

Associativity the order of application of group operation doesn’t matter.

Identity element there exists an element in the group, which, under group operation with any element of the group will give that same element.

Inverse for every group element there exists an element in the group which under group operation produces the identity element of that group.

Integers under addition form a group, which is a model of any group, thus the system is consistent.

1.3.2 Answer 11

I don’t know which one is second, but I’ll assume **associativity**. An example of a set with operation, which is closed, has identity element and has inverses one could consider matrix multiplication on the set of invertible matrices.

These will have inverses by construction, multiplication of any two invertible matrices is still an invertible matrix, the identity matrix is certainly the member of the set of all invertible matrices, but, in general, multiplication will not be associative. Thus, second axiom is not derivable from the rest.

1.3.3 Answer 12

Again, I'll assume that the fourth axiom is the **inverse element**. An example of an operation on a set which has closure, associativity, identity element but not inverses is the set of natural numbers under multiplication. 1 is the identity element, associativy holds by the definition of multiplication and similarly for closure, however, in general, natural numbers don't have inverses under multiplication in the set of natural numbers. (Only identity element has an inverse—itself).

1.3.4 Answer 13

Let's prove G is a group, since the fifth axiom holds trivially by construction.

G has identity element, again, by definition, it is f. So, the **identity** axiom holds. Now, observe that under function composition $g \circ g = I$, $f \circ f = I$, $h \circ h = I$, $k \circ k = I$. This warrants us that every element has an **inverse** (itself). By looking through all results of all functions, it is easy to see that none of the given functions never produces anything other than 1, 2, 3 or 4, so G is **closed** under function composition.

In order to prove **associativity** I will have to build the operation table. Given the table is symmetrical along its diagonal, the operation would have to be associative:

	f	g	h	k
f	$(1,2,3,4)$	$(2,1,3,4)$	$(1,2,4,3)$	$(2,1,4,3)$
g	$(2,1,3,4)$	$(1,2,3,4)$	$(2,1,4,3)$	$(1,2,4,3)$
h	$(2,1,3,4)$	$(2,1,4,3)$	$(1,2,3,4)$	$(2,1,3,4)$
k	$(2,1,4,3)$	$(1,2,4,3)$	$(2,1,3,4)$	$(1,2,3,4)$

(Note that the results are given as tuples, where order matters)

Since the table is symmetrical along its diagonal, the operation is associative. This completes the proof.

1.4 Problem 4

Given system of axioms which defines "point", "line" and "on" relationship where:

1. There are exactly four points.
2. Every two points belong to the one and only line.
3. For every line ℓ and every point P, which is not on ℓ there exists the only line such that P is on it and this (other) line has no points in common with ℓ.
4. Prove that the system is consistent.
5. Prove that the system isn't categorical.
6. Prove that the system is not complete, in other words, that there exists a theorem, which, when added to the system doesn't make it contradictory.
7. Prove that this system entails "there doesn't exist a line which has exactly three points on it."

1.4.1 Answer 14

To prove consistency, I will demonstrate the model for this system. One such model can be constructed from $\ell_1 = \{A, B\}, \ell_2 = \{B, C\}, \ell_3 = \{C, D\}, \ell_4 = \{D, A\}, \ell_5 = \{A, C\}\ell_6 = \{B, D\}$.

(1) holds because there are exactly four points: A, B, C and D. (2) holds because every two points belong to distinct lines. (3) holds because for ℓ_1

there's ℓ_3 which satisfies the condition for C and D, which are not on ℓ_1, and, symmetrically, for all lines other than ℓ_5 and ℓ_6. ℓ_6 satisfies third axiom using ℓ_5 and points D and B, ℓ_5 is symmetrical to ℓ_6.

1.4.2 Answer 15

To prove that the system isn't categorical, I will build a model which also satisfies the three axioms, but is not isomorophic to the model given in 1.4.1. The model is very simple: $\ell = \{A, B, C, D\}$.

(1) holds by construction. (2) holds since any two points are on ℓ (there are no other lines they can belong to). (3) holds vacuously since there are no points which are not on ℓ, we are allowed to conclude that the condition is satisfied.

1.4.3 Ansswer 16

There could be plenty of axioms added to this system such that they will neither derive from the rest nor contradict the rest. As seen in 1.4.2 I could add a requirement that there be only one line. Since in the 1.4.1 I just demonstrated a model which is consistent with the original model, but is inconsistent with the new model, the added axiom is independentent of the first three. The argument for consistency was given in 1.4.2.

1.4.4 Answer 17

In order to prove that the statement is not entailed by the system I will add this statement to the system, and will prove inconsistency.

So, suppose there was line with exactly three points. Without loss of generality, let's name these points A, B and C. The remaining point D would, by the third axiom require that we be able to draw a line through it, which has no common points with ℓ ($\ell = \{A, B, C\}$). But we are also given that every two ponts must be on a line. So, D must be on a line with some other point. Since we are given that there are only four of them, we are left with just three choices, and all of them are on ℓ. Thus D must be on ℓ. But, by assumption, D is not on ℓ. This is a contradiction, hence system with the fourth axiom added is inconsistent, hence it does not entail it.