Assignment 14, Introduction To Mathematics

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1 Problems

1.1 Problem 1

Given $A = \{1, 2\}$ and $B = \mathbb{N}$.

1. Describe all functions from A to B, which are not one-to-one (bijective).
2. Describe all functions from B to A which are not onto.
3. Prove or disprove:
   - There exist functions $f : A \to B$ and $g : B \to A$ such that $f \circ g$ has an inverse.
   - There exist functions $f : A \to B$ and $g : B \to A$ such that $g \circ f$ has an inverse.

1.1.1 Answer 1

Let's first give a name for any function $f : A \to B$. Now, any f, which sends either of the elements of A (of which there are two) to the same element of B would be non-bijective. We can describe the set of all such functions using this formula:

$$F = \{(a, b) | a \in A, b \in B, \exists a' \in A : a' \neq a \wedge f(a) = f(a')\}$$

To make this more concrete: all functions f which assign the same value in B to either 1 or 2 are non-bijections. This is assuming f is defined for all inputs.

1.1.2 Answer 2

Let's, again, give a name for all functions from B to A, viz. $g : B \to A$. The function is onto (surjective) iff every value in its codomain has an origin in its domain. We can describe all such functions using this formula:

$$G = \{(b, a) | a \in A, b \in B, \forall b' \in B : g(b) = g(b')\}$$

Or, in other words, all functions which send all elements of B to the same element of A (be it 1 or 2), are not surjective.

1.1.3 Answer 3

Yes, there exist such functions. For example, g may be defined to return 1 if the input is odd and 2, when the input is even. Then, if f sends its input to an odd number, given 1 and to an even number, given 2, the composition $f \circ g$ is simply the identity function.

1.1.4 Answer 4

We, however, cannot define a $g \circ f$ function with an inverse. The intuition behind this is that we lose information by sending elments from a larger domain to a smaller one. More formally, we can procede by pigeonhole principle, seeing how any function g (provided it is defined for all inputs) would have to send its input to the same element in A, while afterwards there would be no way of distinguishing that element from the one obtained by earlier applicatin of g.

1.2 Problem 2

Given $f : A \to B$ and $C \subseteq A$.

1. Prove that $C \subseteq f^{-1}(f(C))$.
2. Prove that if f is a bijection, then $C = f^{-1}(f(C))$.
3. Show sets A, B, C and function $f : A \to B$ such that it holds that $C \subset f^{-1}(f(C))$.

1.2.1 Answer 5

Let's assume by contradiction, that there exists d in $D = A \setminus C$, such that it is in the codomain of f^{-1}. Existence of sucn an element would in turn imply existence of element $b \in B$ such that $f^{-1}(b) = d$. This would in turn imply existence of $c \in C$ such that $f(c) = b$. But by the definition of inverse, $f^{-1}(b) = c$ and $c \in C$, while we assumed the contrary. Since our initial assumption failed, it must be the case that the elements of codomain of f^{-1} must come from C.

1.2.2 Answer 6

Since f is a bijection, it's inverse must be a bijection too (otherwise it would not be able to assign to all the elements in its image all the values in its domain). Composition of two bijections is necessarily a bijection, thus $C = f^{-1}(f(C))$.

1.2.3 Answer 7

Below is a minimal example of the inverse function being partial to f:

$$
\begin{aligned}
A &= \{1, 2, 3\} \\
B &= \{1, 2, 3\} \\
C &= \{1, 2\} \\
f(1) &= 1 \\
f(2) &= 1 \\
f^{-1}(1) &= 1
\end{aligned}
$$

1.3 Problem 3

Given functions f and g both are from $\mathbb{N}$ to $\mathbb{N}$. g is known to be onto and for all n in $\mathbb{N}$ it holds that

$$(f \circ g)(n) = 2n - 1$$

1. Prove that f is not surjective.
2. Prove that f is bijective.
3. Show such f and g that satisfy the formula given above.

1.3.1 Answer 8

Suppose, by contradiction, that f was surjective, this would imply that any even number, in particular, the number 2 in the image of f would have to have its origin in $\mathbb{N}$, but the origin of 2 is in $\mathbb{Q}$, but not in $\mathbb{N}$, i.e. $n = \frac{1}{2} \implies 2n - 1 = 2$. Because we know that g is surjective, it must be f, which is not surjective (the composition of surjective functions is surjective).

1.3.2 Answer 9

Suppose f is not bijective, this woul3d imply there exist $x, x' \in \mathbb{N}$ such that $x \neq x'$ while $x = 2n - 1$ and $x' = 2n - 1$, which is a contradiction.

1.3.3 Answer 10

The simplest example would be to take g equal to identity and f being $f(n) = 2n - 1$. The argument for f not being surjective is essentially the same as in 1.3.1.

1.4 Problem 4

Let G be a group with $*$ being the group operation. Let also $a \in G$. Given a function $f : G \to G$ defined as $\forall x \in G : f(x) = a^{-1} * x * a$.

1. Prove that f is surjective and bijective.
2. Find f^{-1}.
3. Prove that if $b, c \in G$ are each others inverses, then the same is true of $f(b)$ and $f(c)$.

1.4.1 Answer 11

Suppose f was not bijective, this would imply that there could be $a^{-1}*x*a = a^{-1} * x' * a$ for some $x \neq x'$. But using group cancellation property we could reduce $a^{-1} * x * a = a^{-1} * x' * a$ to $x = x'$, obtaining contradiction. Hence f is a bijection.

x is defined to be a member of G. Since G is closed under $*$, it means that there can't be any values in the domain of f which are not in its co-domain. In other words, suppose there was a $x' \in G$, which cannot be produced by

$f(x)$, that is, if we look at the operation table, each column of this table will have to have as many rows as there are elements. But no two rows in this column can repeat because this would defy the cancellation property (i.e. it would mean that for some p, q, r, s it would be that $p * q = r$ and $p * s = r$, r being the value, which repeats in the selected column and q, s are the elements for which it repeats. Hence f must be onto (surjective).

1.4.2 Answer 12

By definition of inverse, applying the group operation to inverse and the element it is inverse of will give us the identity element:

$$\begin{aligned}
(a^{-1} * x * a) * (a^{-1} * x * a)^{-1} &= a^{-1} * a && \text{given} \\
x * a * (a^{-1} * x * a)^{-1} &= a && \text{by group cancellation property} \\
x^{-1} * x * a * (a^{-1} * x * a)^{-1} &= x^{-1} * a && \text{again, by group cancellation property} \\
a * (a^{-1} * x * a)^{-1} &= x^{-1} * a && \text{by group operation on inverses} \\
a^{-1} * a * (a^{-1} * x * a)^{-1} &= a^{-1} * x^{-1} * a && \text{by cancellation property} \\
(a^{-1} * x * a)^{-1} &= a^{-1} * x^{-1} * a && \text{by group operation on inverses}
\end{aligned}$$

Hence $f^{-1}(x) = a^{-1} * x^{-1} * a$.

1.4.3 Answer 13

Provided $b = c^{-1}, c = b^{-1}$, the next equalities hold:

$$\begin{array}{lllllll}
f(b) & = & a^{-1} * b * a & = & a^{-1} * c^{-1} * a & = & f(c^{-1}) \\
f(c) & = & a^{-1} * c * a & = & a^{-1} * b^{-1} * a & = & f(b^{-1})
\end{array}$$

Which is what we were asked to prove.