Assignment 15, Introduction To Mathematics

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Contents

1 Problems

1.1 Problem 1

1. Let A be a set and f, g be functions defined over A. Prove that if f isn't surjective, so is the $f \circ g$.
2. Let f and g be functions from $\mathbb{N}$ to $\mathbb{N}$, defined as follows:

$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ 1 & \text{otherwise} \end{cases}, \qquad g(n) = 2n - 1$$

   Prove that g is not surjective, and yet $f \circ g$ is.
3. Let A be a set and f, g be functions defined on it. Prove that if g is not surjective and f is bijective, then $f \circ g$ is not surjective either.

1.1.1 Answer 1

I will proceed proving this following this intuition: if the element is not in the domain of f then no matter what you compose f with on the left side, that element will not be in the domain of f anyway.

Let's formalize this. For a function to *not* be surjective means that there exists an element (let's call it x) in the domain of function which has no match in the function's co-domain. Now, suppse g could produce any element in the co-domain of f, there would still be no element in the domain of f for x.

1.1.2 Answer 2

Some observation first: It is easy to see that g is the function that generates all odd numbers. All even numbers in the domain of f as well as odds have an element in its co-domain. So, if we prove the equivalent statement: all odd numbers in the co-domain of $f \circ g$ have a match in the domain of $f \circ g$ we had proved the original statement as well.

But before we proceed, let's establish that g is not surjective (as required by the problem statement). Clearly no even number in the co-domain of g has a match in its domain. This contradicts the definition of surjection.

Now, let's substitute g into the definition of f to obtain $f \circ g$:

$$(f \circ g)(n) = \begin{cases} \frac{2n-1+1}{2} & \text{if } n \text{ is odd} \\ 1 & \text{otherwise} \end{cases} = \begin{cases} n & \text{if } n \text{ is odd} \\ 1 & \text{otherwise} \end{cases}$$

Clearly, this function matches every even element with 1. That being proved we proved the problem statement as well.

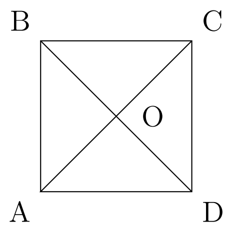
1.1.3 Answer 3

The intuition for the proof is that if elements of the domain of g are missing some particular element, then, if f is a bijection it would have to be missing the element which it matches to the missing one (and it has to assign a *unique* element to it, because it is a bijection).

I will proceed by contradiction. I will assume that for some element x in the domain of g there is no match in the co-domain of g. Now since f is a bijectin, it follows that it matches x with x' in its domain. Since f is a function, it can only match one element to x, i.e. there is at most one x', and it necessary that x' exists, otherwise f wouldn't be a bijection. So, it is necessary that if x' is in A, so should be x (by definition of f) in the domain of g, but we started this proof by assuming that x is not in the domain of g. Having arrived at contradiction I conclude that the initial assumption must be true.

1.2 Problem 2

Let $ABCD$ be a rectangle with its center in O (as seen in the image below).



And let f be an isometry such that $\{A, B, C, D\}$ is its fixed set, while $ABCD$ is a square with its center in O. f is an isometry defined on $\{A, B, C, D\}$ such that $f(C) = A$.

1. Prove that $f(A) = C$ and that O is the fixed point of f.
2. Prove that if $f(B) = B$, then f is a reflection.
3. Prove that if $f(B) = C$, then f is a rotation.
4. Prove that f cannot send B neither to A nor to C.

1.2.1 Answer 3

As the first step, I shall narrow down possible candidates for this isometry: We can immediately discard a possibility that f is a translation - because, were it a translatin, $f(B)$ would not be positioned on either of the vertices of $ABCD$, and then by pigeonhole principle, we would not be able to produce enough fixed points using this transformation. In other words, if f was a translation, it would have l equal to the diagonal of $ABCD$ and α equal $\frac{\pi}{2}$ relative to the diagonal BD. Applying the same transformation to B clearly won't put it neither into A not into D.

By the similar reasoning f can't be a reflection with translation. I.e. if we chose to move A to C accross any line that isn't parallel with diagonal BD, we would end up with B not lending on any of the fixed points of the $ABCD$ after we applied f to it.

Trivially, this can't be an identity transformation, since it doesn't send A to A.

Thus, we are left with two possibilities:

1. f is a rotation.
2. f is a reflection.

We can now simply examine all possible cases of translation of the remaining points. Since f is a bijection and a surjection, we know that there is only one point it can send any point to. We also just proved that it can't send A to itself (if it did the $\overline{AC} \neq \overline{f(A)f(C)}$. Hence, the possibilities are:

1. A is sent to C.
2. A is sent to B.
3. A is sent to D.

It is easy to see that in order to preserve the distance $\overline{AC}$ we must choose to send A to C. Otherwise $f(A)$ ends up connected to C by the side of the square, which is not equal to the original diagonal of the same square.

Now, let's prove that O is indeed the fixed point of f. Since we already konw that $f(A) = C$, we are only left with two possible translations of B and D:

1. $f(B) = B$, $f(D) = D$.
2. $f(B) = D$, $f(D) = B$.

In case 1, this is a reflection along the diagonal BD. This is so because only reflection (of all the options that we are left with) can have more than one fixed point (observe that B and D under f, if it is a reflection are fixed). Since O lies on the diagonal BD by construction, it is, by definition of reflection is its fixed point.

In case 2, this is a rotation by π around O. The readers may convince themselves of this in the following way: By construction, $OA = OB =$

$OC = OD$, $\angle AOC = \angle DOB = \angle COB = \angle BOD$. These are the ohly possible angles, given O is a center of isometry f, provided f is a rotation, with the radius being haf the diagonal of $ABCD$.

Thus, O is the fixed point of f and $f(A) = C$.

1.2.2 Answer 4

I had proved this eventually during 1.2.1. Just to restate the proof briefly: $f(B) = B \implies f(D) = D$, but only reflection (of the remaining possible options) can afford multiple fixed points.

1.2.3 Answer 5

Again, I have showed this already during 1.2.1, but to make this more comprehensive: I listed all possible angles and radii that would have been produced by such isometry, and, comparing them to each other become convinced that they satisfy the requirement for rotation.

1.2.4 Answer 6

Sending B to either A or C would require the distance $\overline{AB}$ or $\overline{BC}$ to be preserved under this transformation (otherwise it would not be an isometry). But this is not possible because $\overline{f(A)f(B)} = \overline{AA} = 0$, but $\overline{AB} \neq 0$. The proof for $\overline{BC}$ is identical.

1.3 Problem 3

Let f and g be isometries on a surface, s.t. $f' = g \circ f \circ g^{-1}$.

1. Prove that f' is an isometry and that f' preserves direction iff so does f.

2. Prove that if A is a fixed point of isometry f, then $g(A)$ must be a fixed point of isometry f'. And if B is a fixed point of f', then $g^{-1}(B)$ is a fixed point of f.

3. Prove that f and f' are of the same kind.

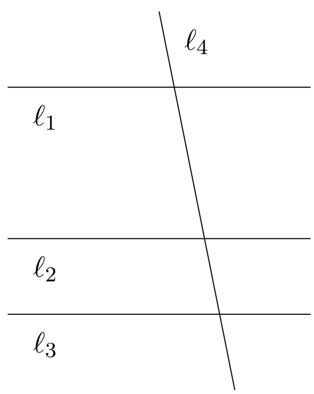
1.3.1 Answer 7

1.3.2 Answer 8

1.3.3 Answer 9

1.4 Problem 4

Shown in the picture below are three lines: ℓ_1, ℓ_2, ℓ_3 all parallel and ℓ_4 which intersects with them.



1. Prove that $S_{\ell_4} \circ S_{\ell_3} \circ S_{\ell_2} \circ S_{\ell_1}$ is a rotation.

2. Prove that $S_{\ell_4} \circ S_{\ell_3} \circ S_{\ell_2} \circ S_{\ell_1} = S_{\ell_4} \circ S_{\ell_1} \circ S_{\ell_2} \circ S_{\ell_3}$.