Assignment 12, Introduction To Mathematics

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Contents

1 Problems

1.1 Problem 1

1. Given $A = \{1, \{1\}\}$ and $B = \{1, \emptyset\}$ write down:
   $P(A)$, $P(B)$, $P(A) \setminus B(B)$, $P(B) \setminus B$, $P(A) \setminus \{A\}$.
2. Given set C, prove or disprove:
   (i) $P(C) \cap C = \emptyset$
   (ii) $P(C) \cap C \neq \emptyset$

1.1.1 Answer 1

$$\begin{aligned}
&P(A) = \{\emptyset, \{1\}, \{\{1\}\}, \{1, \{1\}\}\} \\
&P(B) = \{\emptyset, \{1\}, \{\emptyset\}, \{1, \emptyset\}\} \\
&P(A) \setminus B(B) = \{\{\{1\}\}, \{1, \{1\}\}\} \\
&P(A) \setminus \{A\} = \{\emptyset, \{\{1\}\}, \{1, \{1\}\}\}
\end{aligned} \tag{1}$$

1.1.2 Answer 2

(i) is clearly false if only because constructing a powerset would had to be an identity operation. Since a single example suffices to disprove a claim, I will proceed by constructing an example:

Assume $C = \{\emptyset\}$, then:

$$\begin{aligned}
&P(C) = \{\emptyset, \{\emptyset\}\} \\
&P(C) \cap C = \{\emptyset\}
\end{aligned} \tag{2}$$

We assumed $P(C) \cap C$ to be $\emptyset$, but discovered it to be $\{\emptyset\}$, which is a contradiction. Therefore the inital claim is false.

1.1.3 Answer 3

(ii) is flase, we can certainly have sets where intersection of that set with the set of its subsets will produce no results. In fact any set that contains no other sets will produce an empty result under intersection. For example:

Assume $C = \{1\}$, then:

$$\begin{aligned} &P(C) = \{\emptyset, \{1\}\} \\ &P(C) \cap C = \emptyset \end{aligned} \tag{3}$$

We assumed $P(C) \cap C \neq \emptyset$, but discovered that it is also possible that $P(C) \cap C = \emptyset$, which is a contradiction. Therefore the initial claim is false.

1.2 Problem 2

Given sets A, B, and C prove that:

1. $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.
2. $A \setminus (B \setminus C) \subseteq (A \setminus B) \setminus C \implies A \cap C = \emptyset$.

Prove or disprove:

1. $P(A) \subseteq B \implies \{\emptyset, A\} \in P(B)$.
2. $\{\emptyset, A\} \in P(B) \implies P(A) \subseteq B$.

1.2.1 Answer 4

I will prove this using a method similar to proving the distributivity property of union over intersection.

To prove by contradiction, we need to examine two cases:

1. There is $x \in (A \cup B) \setminus C$ such that $x \notin ((A \setminus C) \cup (B \setminus C))$.
2. There is $x \in ((A \setminus C) \cup (B \setminus C))$ such that $x \notin (A \cup B) \setminus C$.

(1) For x to be member of a difference of sets would mean it has to be the member of minuend (by definition of subtraction). We find that minuend is the union of sets, which means that, by definition of union, x could come from either one of the unioned sets. So, for x not to be in the left-hand side of the formula it would mean that it must be the case that it is either not in the union of A and B, or that it is in C. Neither of which is true, which is easy to become convinced of by observing the two parts of the right-hand side of the formula: if $x \notin (A \setminus C)$ then it absolutely has to be in B and not in C, which is exactly what the rest of the formula negates. Thus we arrive at contradiction. It must be the case that whenever $x \in (A \cup B) \setminus C$ it is also the case that $x \in ((A \setminus C) \cup (B \setminus C))$.

(2) By symmetrical argument we show that if $x \in ((A \setminus C) \cup (B \setminus C))$ then it is the case that $x \in (A \cup B) \setminus C$.

Having proved both cases we now conclude that union is distributive over subtraction.

1.2.2 Answer 5

To prove the claim, I'm going to show that there is no such x, for which both: $x \in (A \setminus (B \setminus C) \subseteq (A \setminus B) \setminus C)$ and $x \in (A \cap C)$ is true at the same time. Proving this is equivalent to proving the initial claim because if this was the case, then $A \cap C$ would not be an empty set (it would had to contain

x), or $A \setminus (B \setminus C) \subseteq (A \setminus B) \setminus C$ would be false (these are the only two cases how a material implication can fail, from definition of material implication).

For x to be a member of $A \cap C$ means that it is both in A and in C (by definition of intersection). This can only happen when we *don't* subtract C from the result (otherwise the $A \cap C$ would be an empty set). This only happens in $A \setminus (B \setminus C)$. This means that if the initial claim was true, there had to be such elements in $A \setminus (B \setminus C)$, which are not in $(A \setminus B) \setminus C$, but this contradicts the definition of set inclusion. Since we arrived at contradiction in our initial claim, which we set to disprove, we arrived at the proof.

1.2.3 Answer 6

Both A and $\emptyset$ are certainly the members of $P(A)$, so if B contains at least all the elements of $P(A)$, it certainly contains both the A and the $\emptyset$. Then, of course, $P(B)$ must contain a set containing two elements of $P(A)$, $\{A, \emptyset\}$ among them.

1.2.4 Answer 7

This is a false claim, here's an assignment that holds for antecedent, but doesn't for consequent:

$$
\begin{aligned}
&A = \{a, b\} \\
&P(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \\
&B = \{\emptyset, \{a, b\}\} \\
&P(B) = \{\emptyset, \{\emptyset\}, \{\{a, b\}\}, \{\emptyset, \{a, b\}\}\} \\
&\{A, \emptyset\} \in P(B) = \{\emptyset, \{a, b\}\} \in P(B) = \{\emptyset, \{a, b\}\} \in \{\emptyset, \{\emptyset\}, \{\{a, b\}\}, \{\emptyset, \{a, b\}\}\} \\
&P(A) \not\subseteq B = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \not\subseteq \{\emptyset, \{a, b\}\}
\end{aligned}
\tag{4}
$$

I.e. it is possible to construct such set B that its powerset would satisfy the left-hand side of the formula, but fail the right-hand side.

1.3 Problem 3

1. Given a binary operation * on even numbers $A = \{2x | x \in \mathbb{N}\}$, for all $x, y \in A$ the following holds:

$$x * y = \frac{(x-2)(y-2)}{2} + 2 \tag{5}$$

   Verify whether A is closed under *, that * is associative, that there exists an identity element and that every element has its inverse under this operation.

2. Assume now that A is defined as $A = \{2x | x \in \mathbb{Q}\}$, give the characteristics of properties outlined in (1).

1.3.1 Answer 8

In order to show that A is closed under *, observe that adding 2 to any even number will not make it odd. We can thus disregard the addition. Next, observe that in order for division by 2 to produce an even number the divident must be divisible by 4. Our dividen is a product of two numbers. If we can show that both of them are even (that is each of them has a factor of two), then their product must be a multiple of 4. This problem is thus equivalent to showing that even numbers are closed under subtraction of 2. Given our definition of A, it is possible to produce 0 as the result of the $(x-2)(y-2)/2$, if either x or y are equal to 2, however, the $+2$ part saves us from ever obtaining a final result less then 2. Thus, I conclude that A is closed under *.

To prove $(x * y) * z = x * (y * z)$ observe that the problem scales down to proving associativity of multiplication (since other function terms have no effect on associativity property). I.e. we actually need to prove $((x-2)(y-2))(z-2) = (x-2)((y-2)(z-2))$ which is true, so long multiplication on natural numbers is associative.

The definition of identity element is $ix = x$. And there is such element, namely 4. Since the function behaves similarly to multiplication, I looked for the transformation I'd had to perform on the multiplication identity element in order to find one here. But, I wouldn't know of a proof of it being the identity element.

Not all elements have their inverses in A. Inverse is defined to be $x * y = i$, where i is the identity element, whose existence we just established. But and since this function behaves equivalently to multiplication, we can assume that its values only increase, therefore $6 * 2 = 2$, $6 * 4 = 6$, $6 * 6 = 10$, ... I.e. there would be no inverse for 6. Another way to think about the problem is to notice that 4, the identity element has no odd factors, but some even numbers do have such factors. By using only multiplication and no rationals less then 1, we cannot possibly arrive from a number with odd factors to a number which doesn't have them.

1.3.2 Answer 9

The closure property will hold if we substitute $\mathbb{Q}$ for $\mathbb{N}$. Even though we now allow negative numbers, we are no longer constrained by natural numbers range. Using this function we never create irrational numbers, because all its input and its terms are integers.

Associativity holds just as it did before, we didn't rely on the properties of integers to show the associativity the first time.

Consequently, the identity elements remains the same element.

However, now we have a chance of finding inverses for all elements!

It looks like the function:

$$x *' y = \frac{1}{2} - \frac{2}{(x+2)(y+2)} \tag{6}$$

Will inverse the effect of the original definition, but it doesn't, becuase when either $x = -2$ or $y = -2$ it results in division by zero.

2 Exercises

Given $A = \{a\}$, and a binary operation defined on it such that $a * a = a$. $B = \{a, b\}$ and binary operation $+$ defined on B as follows:

$+$	a	b
a	a	b
b	b	b

$C = \{a, b, c\}$ and binary operation % defined on C as follows:

%	a	b	c
a	a	b	c
b	b	a	c
c	c	a	a

D is a non-empty set, $n, m \in \mathbb{N}$, exp is the exponentiation function defined on $\mathbb{N}$. f defined as follows:

$$f(x, y) = xy \bmod 10 \tag{7}$$

G is a group, with $\circ$ being its group operation, e being its identity element, having a, b and c as it members (it is also possible that G has other members), $Commutative(S, O)$ predicate meaning that set S, its argument, is commutative under operation O.

Answer the following questions using:

- **a** if only the first statement is correct.
- **b** if only the second statement is correct.
- **c** if both statements are correct.
- **d** if neither statement is correct.

2.1 Exercise 1

1. A is closed under *.
2. * is not associative becuase A has less then three members.

Answer: **a**

2.2 Exercise 2

1. a is the identity element of A under *.
2. A is a group under *.

Answer: **c**

2.3 Exercise 3

1. B is closed under $+$.
2. $+$ is associative.

Answer: **c**

2.4 Exercise 4

1. b is the identity element in B.
2. B is a group under $+$.

Answer: **d**

2.5 Exercise 5

1. % is associative.
2. There is identity element in C under %.

Answer: **b**

2.6 Exercise 6

1. b is the inverse of c under %.
2. c is the inverse of c under %.

Answer: **b**

2.7 Exercise 7

1. % is commutative.
2. Every member of A has an inverse under %.

Answer: **c**

2.8 Exercise 8

1. $P(D)$ is closed under $\cap$ (set intersection).
2. $\cap$ is associative in $P(D)$.

Answer: **c**

2.9 Exercise 9

1. $\emptyset$ is the identity element of $P(D)$ under $\cap$.
2. D has an inverse in $P(D)$ under $\cap$.

Answer: **b**

2.10 Exercise 10

1. $P(D)$ is closed under $\setminus$ (set subtraction).
2. $\setminus$ is associative in $P(D)$.

Answer: **a**

2.11 Exercise 11

1. $\emptyset$ is the identity element of $P(D)$ under $\setminus$.
2. $\emptyset$ is the identity element of $P(D)$ under $\cup$.

Answer: **c**

2.12 Exercise 12

1. exp is associative.
2. exp has an identity element.

Answer: **b**

2.13 Exercise 13

1. $\{2, 4, 6, 8\}$ is a group under f.
2. $\{1, 3, 5, 7, 9\}$ is a group under f.

Answer: **a**

2.14 Exercise 14

1. $(x \in G \land x \circ x = x) \implies (x = e)$.
2. $(x \in G \land x \circ x = e) \implies (x = e)$.

Answer: **b**

2.15 Exercise 15

1. $(a \circ b = b \circ a) \implies Commutative(G, \circ)$.
2. $(a = b^{-1}) \implies (b = a^{-1})$.

Answer: **d**

2.16 Exercise 16

1. $\forall x, y, z \in G((x \circ y = x \circ z) \implies (y = z))$.
2. $\forall x, y, z \in G((x \circ y = y \circ z) \implies (x = z))$.

Answer: **d**

2.17 Exercise 17

1. The diagonal of the table describing $\circ$ on G all members of G must appear.
2. a appears exactly once in every column and every row of the table dscribing $\circ$ on G.

Answer: **d**

2.18 Exercise 18

1. $(a \circ b \circ c)^{-1} = a^{-1} \circ b^{-1} \circ c^{-1}$.
2. $((a \circ b)^{-1} \neq a^{-1} \circ b^{-1}) \implies \neg Commutative(G, \circ)$.

Answer: **a**

2.19 Exercise 19

1. $(a \circ b = e) \implies (b \circ a = e)$.
2. If G has exactly four members and $a \circ b = c$, then $b \circ a = c$.

Answer: **d**

2.20 Exercise 20

1. Any group with one, two or three members is commutative.
2. Any group is commutative.

Answer: **a**

2.21 Exercise 21

Let A be a group with three members.

1. There should exist a binary operation that conforms to all requirements of group operation sans associativity.
2. There exists a binary operation on A which doesn't support associativity but has the cancellation property.

Answer: **b**

2.22 Exercise 22

Let A be a set with defined binary operation *, which conforms to all group properties sans inverse element.

1. If * is commutative, then A is a group under *.
2. If * supports the cancellation property, then A is a group under *.

Answer: **d**

2.23 Exercise 23

We define a binary operation on $\mathbb{N} \cup \{\frac{1}{2}\}$ thus: $(\forall x, y \in \mathbb{N} \cup \{\frac{1}{2}\}) : x \Delta y = 2xy$.

1. Δ has the cancellation property.
2. $\mathbb{N} \cup \{\frac{1}{2}\}$ is a group under Δ.

Answer: **c**

2.24 Exercise 24

Let G be something defined elsewhere.

1. $(\exists x, y, z \in G) : x \circ y = y \circ z$, but $x \neq z$.
2. $(\forall x \in G) : x \circ x = I \vee x \circ x \circ x = I$.

Answer: **not a foggiest idea**