Assignment 11, Introduction to Statistics

Oleg Sivokon

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1.1 Problem 1

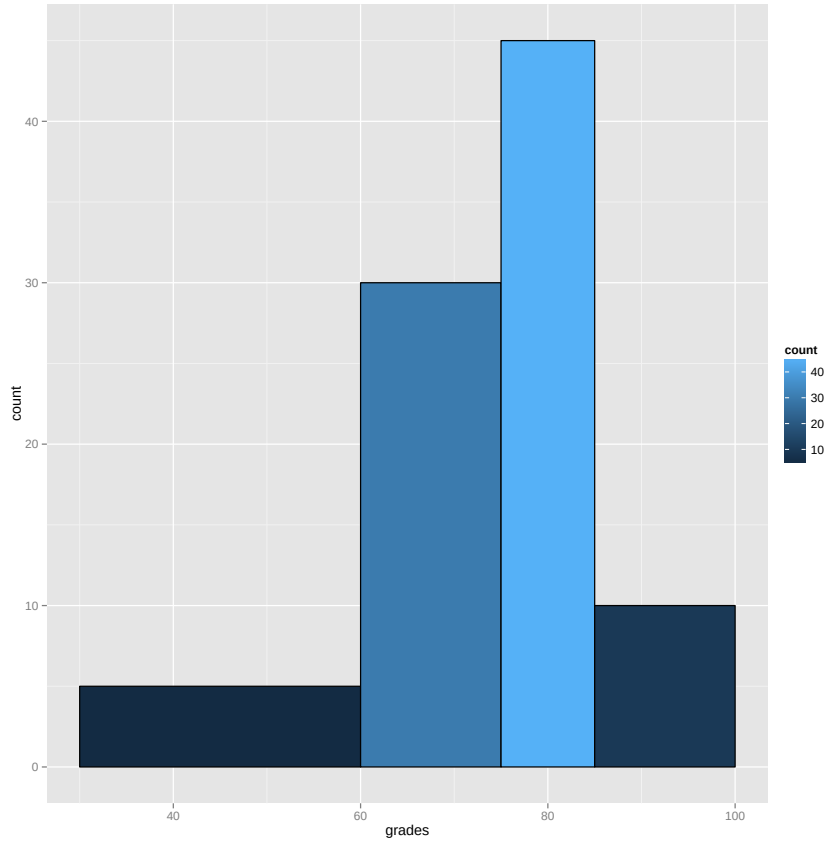
Given the grades in an engineering faculty were as follows:

lower	higher	graded
30	60	15
60	75	45
75	85	45
85	100	15

1. Present the data using a histogram.
2. Calculate mode, median, algebraic average and variance.
3. Calculate the number of students who earned at least 82 points.
4. Given the following data for the preceding year for 80 students where the average grade was 70 and variance was 200, find the average and the variance for two years combined.

1.1.1 Answer 1

```
library(ggplot2)
tbl$avg <- (tbl$lower + tbl$higher) / 2
tbl$density <- 10 * tbl$graded / (tbl$higher - tbl$lower)
ggplot(tbl) +
    geom_histogram(
        aes(x = avg, weight = density, fill = ..count..),
        breaks = unique(append(tbl$lower, tbl$higher)),
        position = "identity", colour = "black") +
            xlab("grades")
```



1.1.2 Answer 2

As easy to see from the diagram, the **mode** is in the (75, 85] range.

The **median** is the value at the 60'th studen, which is easy to see from

the table as being the 75 points.

$$
\begin{aligned}
Md &= \frac{\frac{n}{2} - F(x_{m-1})}{f(x_m)} * (L_1 - L_0) + L_0 \\
Md &= \frac{\frac{120}{2} - 60}{45} * (85 - 75) + 75 \\
Md &= \frac{60 - 60}{45} * (85 - 75) + 75 \\
Md &= 75
\end{aligned}
$$

The **average** is given by the formula:

$$
\begin{aligned}
\frac{15 * \frac{30+60}{2} + 45 * \frac{60+75}{2} + 45 * \frac{75+85}{2} + 15 * \frac{85+100}{2}}{15 + 45 + 45 + 15} &= \\
\frac{15 * 90 + 45 * 135 + 45 * 160 + 15 * 185}{2 * 120} &= \\
\frac{17400}{240} &= \\
72.5
\end{aligned}
$$

And the **variance**:

$$
\begin{aligned}
S^2 &= \frac{\sum_1^n (x - \overline{x})^2 * f(x)}{n} \\
S^2 &= \frac{\begin{array}{c}(45 - 72.5)^2 * 15 + (67.5 - 72.5)^2 * 45 \\ + (80 - 72.5)^2 * 45 + (92.5 - 72.5)^2 * 15\end{array}}{120} \\
S^2 &= \frac{756.25 * 15 + 25 * 45 + 56.25 * 45 + 400 * 15}{120} \\
S^2 &= \frac{21000}{120} \\
S^2 &= 175
\end{aligned}
$$

1.1.3 Answer 3

Using a slightly altered formula for the median, we can calculate the 82'nd percentile. It is easy to see that the 82'nd percentile falls in the third group,

viz. [75, 85) interval. Assuming values are uniformly distribute in this interval, $\frac{7}{10}$ of these are below 82 and $\frac{3}{10}$ are above. In other words, we need to take 0.3 of the 45 students in this category, i.e. 13.5, together with 15 students who earned more points this gives roughly 19 students.

1.1.4 Answer 4

New average is just the weighted average of both averages: $\frac{70*80+72.5*120}{80+120} = 71.5$.

The total variance is calculated using

$$\begin{aligned}
s_1 &= n(S_1 + \overline{x}_1^2) \\
s_1 &= 120 * (175 + 72.5^2) \\
s_1 &= 651750. \\
s_2 &= n(S_2 + \overline{x}_2^2) \\
s_2 &= 80 * (200 + 70^2) \\
s_2 &= 408000.
\end{aligned}$$

$$\begin{aligned}
S^2 &= \frac{s_1 + s_2}{n_1 + n_2} - \overline{x}^2 \\
S^2 &= \frac{651750 + 408000}{200} - 71.5^2 \\
S^2 &= \frac{651750 + 408000}{200} - 5112.25 \\
S^2 &= 186.5
\end{aligned}$$

1.2 Problem 2

Given the number of assignments submitted in 2009 (shown below):

assignments	students
0	11
1	18
2	28
3	22
4	15
5	16

1. Draw a bar chart representing the data.
2. Find mode, median, algebraic average and variance.
3. In addition to the number of assignments submitted, students also received final grades. Let X be the number of assignments submitted, let Y be the grade the student received. Given also that Pearson coefficient is $r = 0.75$.

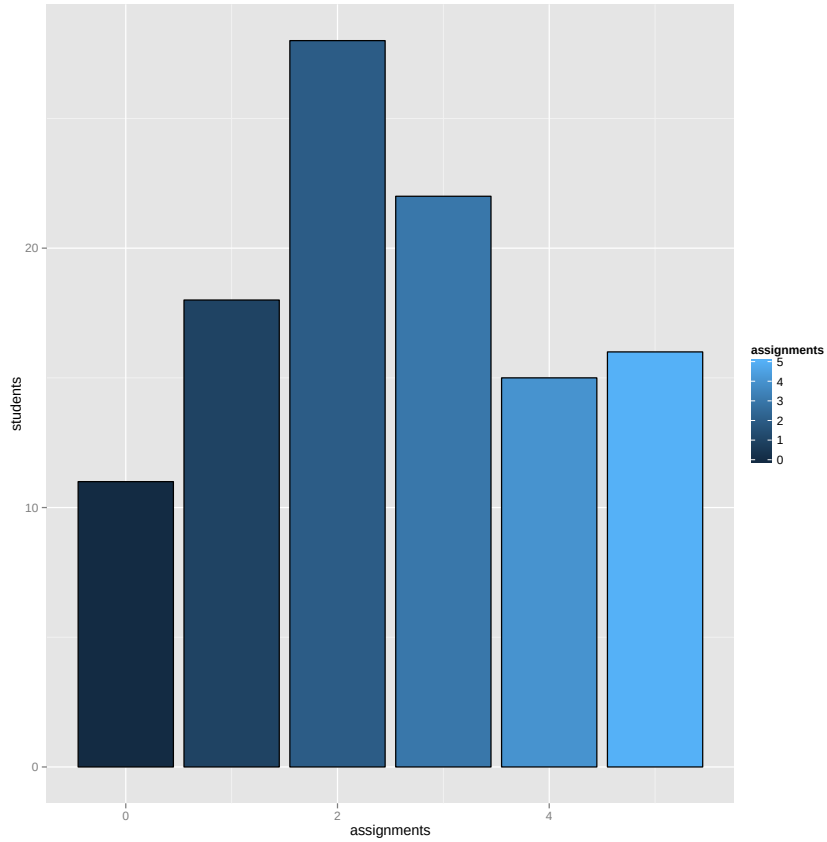
   Prove or disprove:

   (a) $Y = 0.75X + 96.25$.

   (b) The number of assignments submitted negatively correlates with the final grade they received.
4. An investigation found data on 10 more students. 5 of them didn't submit any assignment and 5 of them submitted 5 assignments each. Describe what will happen to each metric calculated in question 2 relying on the previously obtained values.

1.2.1 Answer 5

```
library(ggplot2)
ggplot(data = tbl,
       aes(x = assignments, y = students, fill = assignments)) +
           geom_bar(colour = "black", stat = "identity")
```



1.2.2 Answer 6

Mode for the assignments data is 2, i.e. most of the students submitted only two assignments.

Median is $2 + (3 - 2)/2 = 2.5$ (because there is an even number of bins).

Weighted **average** can be obtained using $\frac{0*11+1*18+2*28+3*22+4*15+5*16}{110} = \frac{280}{110} = 2.45(45)$.

Variance can be obtained using:

$$S^2 = \frac{(0*11)^2 + (1*18)^2 + (2*28)^2 + (3*22)^2 + (4*15)^2 + (5*16)^2}{110} - 2.45^2$$
$$S^2 = \frac{17816}{110} - 6.0025$$
$$S^2 = 161.963636364$$
$$S^2 \simeq 162$$

1.2.3 Answer 7

While it seems appealing, it isn't really possible to have a determination coefficient predict the value of another variable with absolute certainity unless the coefficient is equal to one. Thus $Y = 0.75X + 96.25$ is at best a lucky coincidence.

Positive coefficient means that there exists positive correlation between two variables. In particular, it means that roughly in three fourth of all cases the number of assignments submited perfectly predicted the final grade. So the claim is obviously false.

1.2.4 Answer 8

After adding ten more observations the **mode** will not change as the observations fall into the bins which aren't as dense as the most dense one, i.e. the first bin will contain 11+5=16 students *(fewer than 28 of the densest bin)* and 6+5=11 in the last bin.

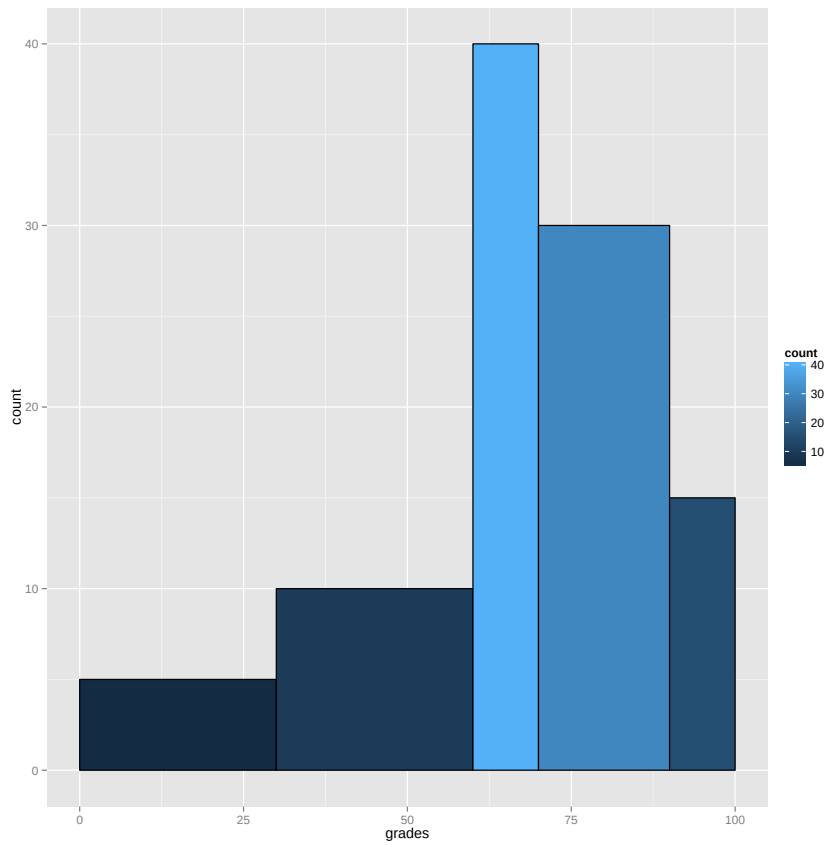
The **median** will not change either because we are adding observations to the outermost bins in an equal measure.

The **average** will almost not change because the added observations will "cancel out", however, it will shift very slightly towards the upper end, since we increased the relative weight of the bin of those who submitted the most assignments.

Since **variance** is affected by the mean, it is hard to tell without doing full recalculation whether it will or will not change. However, since we added more borderline observations, which are most likely to be far away from the mean, we'd expect the variance to grow.

1.3 Problem 3

Given the histogram below:



1. Write the spread of the diagram.
2. Calculate the mode, median, average and standard deviation.

3. Students Paz, Or and Nadav received the following grades:

   Paz Has standard score of 0.75.

   Or Was awarded 75 points.

   Nadav Is in the 80'th percentile.

   Rank Paz, Or and Nadav according to their grades from lowest to highest.

4. After the exam took place, the grader decided to award additional ten points to every student s.t. the resulting grade will not be greater than 100.

   Prove or disprove:

   (a) The new average is 72.5.

   (b) Standard deviation grew as the result of the change.

1.3.1 Answer 9

As can be infered from the diagram, the grades were divides as shown below:

low	high	students	F(students)	f(students)
0	30	15	15	15
30	60	30	45	45
60	70	40	85	65
70	90	60	145	80
90	100	15	160	95

1.3.2 Answer 10

The **mode**, as can be seen in the diagram is 40.

The **median** is in the middle of the third group (at position 80), which gives, using the formula:

$$
\begin{aligned}
Md &= \frac{\frac{n}{2} - F(x_{m-1})}{f(x_m)} * (L_1 - L_0) + L_0 \\
Md &= \frac{\frac{160}{2} - 45}{65} * (70 - 60) + 60 \\
Md &= \frac{80 - 45}{65} * 10 + 75 \\
Md &= \frac{70}{13} + 75 \\
Md &= 80.3846153846.
\end{aligned}
$$

The **average** can be obtained via $\frac{15*15+30*45+40*65+60*80+15*95}{160} = 65$.

The **standard deviation** can be obtained via:

$$
\begin{aligned}
\sigma &= \sqrt{\frac{\sum_{i=1}^{n}(x - \overline{x})^2}{n}} \\
\sigma &= \sqrt{\frac{\begin{array}{l} 15 * (15 - 65)^2 + 30 * (45 - 65)^2 \\ \quad + 40 * (65 - 65)^2 + 60 * (80 - 65)^2 + 15 * (95 - 65)^2 \end{array}}{160}} \\
\sigma &= \sqrt{\frac{76500}{160}} \\
\sigma &= 21.8660696057.
\end{aligned}
$$

1.3.3 Answer 11

First, we'll calculate **Paz**'s grade. Given the formula: $Z_x = \frac{x - \overline{x}}{S_x}$ obtains $x = Z_x S_x + \overline{x}$, substituting known values gives $x = 0.75 * 25 + 80.4 \simeq 98$.

Now, let's find **Nadav**'s grade. Using the percentile formula:

$$
\begin{aligned}
C_k &= \frac{\frac{nk}{100} - F(x_{m-1})}{n}(L_1 - L_0) + L_0 \\
C_k &= \frac{\frac{160*82}{100} - 85}{80}(90 - 70) + 70 \\
C_k &= \frac{1.6 * 82 - 85}{80} * 20 + 70 \\
C_k &= \frac{46.2}{4} + 70 \\
C_k &= 81.55.
\end{aligned}
$$

In conclusion, it looks like Or received the lowest grade (75), right after him was Nadav, with 82 points, and Paz was a clear leader, receiving a whooping 98 points.

1.3.4 Answer 12

One way to look at what has happened is to notice that each group of students had to lose a number of students due to them receiving higher grades (except for the last group), and each group would gain some students (those transfered from the one below it in the rating), except, again, for the first one. Assuming uniform distributin inside the bins, we can derive a formulat to calculate the number of students transfered: $\frac{L_1-10}{L_1-L_2}f(x)$, where L_1 is the lower bound on the group and L_2 is the higher bound. The table below provides a complete calculation:

low	high	students	out	in	new	avg
0	30	15	5	0	10	15
30	60	30	10	5	25	45
60	70	40	40	10	10	65
70	90	60	30	40	70	80
90	100	15	0	30	45	95

Which gives us the new average: $\frac{10*15+25*45+10*65+70*80+45*95}{160} = \frac{11800}{160} = 73.75$. Thus the answer it, no, new average is not 72.75 (but close).

Without recalculating the standard deviation, it is reasonable to assume that the margins of the distribution narrowed, but it doesn't hurt to calculate it, which gives:

$$\sigma = \sqrt{\frac{\begin{array}{l}10 * (15 - 73.75)^2 + 25 * (45 - 73.75)^2 \\ \quad + 10 * (65 - 73.75)^2 + 70 * (80 - 73.75)^2 + 45 * (95 - 73.75)^2\end{array}}{160}}$$

$$\sigma = \sqrt{\frac{79000}{160}}$$

$$\sigma = 22.2204860433.$$

Which is proves our initial assumption to be wrong, indeed the standard deviation grew as a result of this change.

1.4 Problem 4

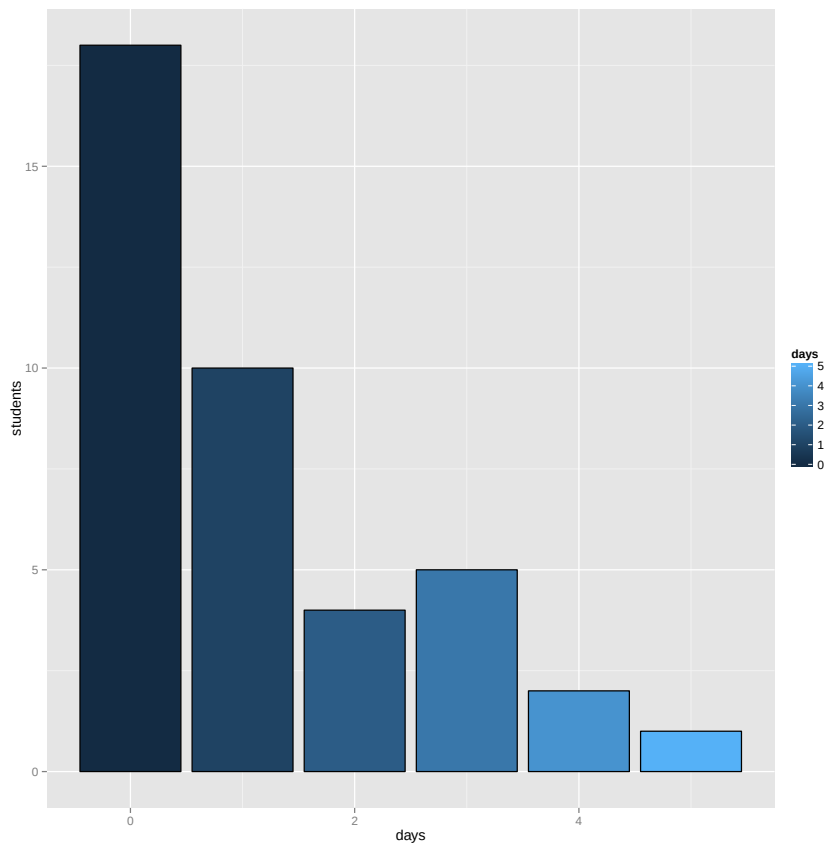
Given the table of overdue assignments:

days	students
0	18
1	10
2	4
3	5
4	2
5	1

1. Present the data using a bar chart.
2. Find mode, median, average and variance.
3. The professor decided to deduce 5 points for each day past the submission deadline. Provided the students could've been awarded at most 100 points before deduction, calculate the maximum average and maximum variance after deduction.
4. The professor forgot to add five more records of students who submitted their assignments even later than the rest. After these data are added, what will happen to the statistics calculated in (2)?

1.4.1 Answer 13

```
library(ggplot2)
ggplot(data = tbl,
       aes(x = days, y = students, fill = days)) +
           geom_bar(colour = "black", stat = "identity")
```



1.4.2 Answer 14

As is easy to see from the diagram, the **mode** is 0 (i.e. most students submitted their assignments on time).

The **median** is is betwen the 20'th and the 21'st students (since there

are in total 40 observations), and it is 1.

The **average** is $\frac{18*0+10*1+4*3+5*3+2*4+1*5}{40} = 1.25$.

The **variance** is thus:

$$\frac{\begin{array}{l}(18*(0-1.25)^2+10*(1-1.25)^2+4*(3-1.25)^2\\ \quad +5*(3-1.25)^2+2*(4-1.25)^2+1*(5-1.25)^2\end{array}}{40} = 2.1375.$$

1.4.3 Answer 15

It is easy to calculate the points deduced as a weighted sum of days, weighted by students, i.e. $5*(0*18+1*10+2*4+3*5+4*2+3*1) = 220$, while total number of points before deduction is $40*100 = 4000$, thus $\frac{4000-220}{40} = 94.5$ would be the **average** after deduction.

Using the average, we can now find **variance**

$$\frac{\begin{array}{l}18*(0*100)^2+1*(10*95)^2+2*(4*90)^2\\ \quad +3*(5*85)^2+4*(2*80)^2+3*(1*75)^2\end{array}}{40} - 94.5^2 =$$
$$\frac{1822850}{40} - 8930.25 = 36641.$$

1.4.4 Answer 16

After more observations are added, the **mode** will not change (it will still be the most common case that the most students submitted their assignments on time). The **median** will not change either, however now the median student will be the 23'rd one, but that student is still the one who submitted the assignment one day overdue. The average will slightly increase (since we added more students who are in a category far away from the old average). Finally, the variance will likely increase too, since we are adding observations which are far away from the average.