Assignment 11, Linear Algebra 1

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Contents

1 Problems

1.1 Problem 1

1. Given the system of linear equations below:

$$\left.\begin{aligned} x \quad &+ 2y \qquad + az = \; -3 - a \\ x \quad &+ (2-a)y - z \;\; = 1 - a \\ ax + &ay \qquad\qquad\;\; = 6 \end{aligned}\right\} \qquad a \in \mathbb{R}$$

1. What assignments to a produce no solutions?
2. What assignments to a produce single solution?
3. What assignments to a produce infinitely many solutions?

1.1.1 Answer 1

Before answering the questions, let's reduce the matrix of coefficients of the given system of linear equations to the echelon form. This will be useful when talking about its properties.

$$\left[\begin{array}{ccc|c} 1 & 2 & a & -3-a \\ 1 & 2-a & 1 & 1-a \\ a & a & 0 & 6 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & a & -3-a \\ 0 & -a & 1-a & 4 \\ a & a & 0 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & a & -3-a \\ 0 & -a & 1-a & 4 \\ a & a & 0 & 6 \end{array}\right] \xrightarrow{R_3 = R_3 - aR_1} \left[\begin{array}{ccc|c} 1 & 2 & a & -3-a \\ 0 & -a & 1-a & 4 \\ 0 & -a & -a^2 & 6+3a+a^2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & a & -3-a \\ 0 & -a & 1-a & 4 \\ 0 & -a & -a^2 & 6+3a+a^2 \end{array}\right] \xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & a & -3-a \\ 0 & -a & 1-a & 4 \\ 0 & 0 & -a^2+a-1 & 2+3a+a^2 \end{array}\right]$$

1. From just looking at the last equation in the system, we can conclude that $a = 0$ creates an inconsistent system because that would imply $0x + 0y + 0z = 6$, i.e. $0 = 6$, which is impossible.
2. In order for the system to have a single solution the matrix of its coefficients in its echelon form must have as many pivots as there are unknowns. a can't influence the pivot in the first row, we already know that $-a = 0$ leads to having no solutions and

by solving $-a^2 + a - 1$ we find that it has no real roots, but a is given to be real, thus every column has a pivot, which means that in case the matrix has solutions it must be unique.

3. This system could have less pivots than the rank of the matrix of its coefficients if either $-a = 0 \wedge 1 - a = 0$ or $-a^2 + a - 1 = 0$, first is clearly impossible and the second doesn't have any real roots (but it is given that a is real). So this system can never have more than one solution.

1.2 Problem 2

1. Given the system of linear equations below:

$$\left.\begin{aligned}
&x \quad + ay && + bz && + aw && = b \\
&x \quad + (a+1)y && + (a+b)z && + (a+b)w && = a+b \\
&ax + a^2y && + (ab+1)z && + (a+a^2)w && = b+ab \\
&2x + (2a+1)y && + (a+2b)z && + aw && = 2b-2a-ab
\end{aligned}\right\} \qquad a, b \in \mathbb{R}$$

1. What assignments to a and b produce no solutions?
2. What assignments to a and b produce single solution?
3. What assignments to a and b produce infinitely many solutions?

1.2.1 Answer 2

As before, let's first extract the coefficient matrix and by using Gaussian elimination bring it to echelon form:

$$\left[\begin{array}{cccc|c} 1 & a & b & a & b \\ 1 & a+1 & a+b & a+b & a+b \\ a & a^2 & ab+1 & a+a^2 & b+ab \\ 2 & 2a+1 & a+2b & a & 2b-2a-ab \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{cccc|c} 1 & a & b & a & b \\ 0 & 1 & a & b & a \\ a & a^2 & ab+1 & a+a^2 & b+ab \\ 2 & 2a+1 & a+2b & a & 2b-2a-ab \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & a & b & a & b \\ 0 & 1 & a & b & a \\ a & a^2 & ab+1 & a+a^2 & b+ab \\ 2 & 2a+1 & a+2b & a & 2b-2a-ab \end{array}\right] \xrightarrow{R_4 = R_4 - 2R_1} \left[\begin{array}{cccc|c} 1 & a & b & a & b \\ 0 & 1 & a & b & a \\ a & a^2 & ab+1 & a+a^2 & b+ab \\ 0 & 1 & a & 0 & 2a-ab \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & a & b & a & b \\ 0 & 1 & a & b & a \\ a & a^2 & ab+1 & a+a^2 & b+ab \\ 0 & 1 & a & 0 & 2a-ab \end{array}\right] \xrightarrow{R_3=R_3-aR_1} \left[\begin{array}{cccc|c} 1 & a & b & a & b \\ 0 & 1 & a & b & a \\ 0 & 0 & 1 & a & b \\ 0 & 1 & a & 0 & 2a-ab \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & a & b & a & b \\ 0 & 1 & a & b & a \\ 0 & 0 & 1 & a & b \\ 0 & 1 & a & 0 & 2a-ab \end{array}\right] \xrightarrow{R_4=R_4-R_2} \left[\begin{array}{cccc|c} 1 & a & b & a & b \\ 0 & 1 & a & b & a \\ 0 & 0 & 1 & a & b \\ 0 & 0 & 0 & -b & -3a-ab \end{array}\right]$$

1. The only case there would be no solution to this system is when $b = 0 \wedge -3a - ab \neq 0$. Otherwise we'd have that some real number not equal to zero equals to zero. Suppose now that $b = 0$, then if $-3a \neq 0$ the system has no solutions. Which amounts to that whenever $a \neq 0 \wedge b = 0$ the system has no solutions.

2. In order for the system to have single solution the rank of the coefficient matrix needs to be equal to the number of unknowns of the system. The only way for this system to not have that property is if b is zero and $-3a - ab = 0$. As discussed above, if the second condition doesn't hold, the system has no solutions, so we are only interested in all which remains, i.e. the cases when $b \neq 0$.

3. Conversely, if $b = 0 \wedge -3a - ab = 0$ then we have a free variable in this system, and hence infinite solutions.

1.3 Problem 3

Solve the system of linear equations:

$$\left.\begin{aligned} \frac{1}{x} + \frac{2}{y} - \frac{4}{z} &= 1 \\ \frac{2}{x} + \frac{3}{y} + \frac{8}{z} &= 0 \\ \frac{1}{x} + \frac{9}{y} - \frac{10}{z} &= 5 \end{aligned}\right\} \qquad x, y, z \in \mathbb{R}$$

1.3.1 Answer 3

Because writing coefficient matrix as reciprocals to the system unknowns will make this unwieldy, we'll perform Gaussian elimination directly on the equations given.

$$\left.\begin{array}{l}\frac{1}{x}+\frac{2}{y}-\frac{4}{z}=1\\ \frac{2}{x}+\frac{3}{y}+\frac{8}{z}=0\\ \frac{1}{x}+\frac{9}{y}-\frac{10}{z}=5\end{array}\right\}\xrightarrow{R_2=R_2-2R_1}\left.\begin{array}{l}\frac{1}{x}+\frac{2}{y}-\frac{4}{z}=1\\ 0+-\frac{1}{y}+\frac{16}{z}=-2\\ \frac{1}{x}+\frac{9}{y}-\frac{10}{z}=5\end{array}\right\}$$

$$\left.\begin{array}{l}\frac{1}{x}+\frac{2}{y}-\frac{4}{z}=1\\ 0-\frac{1}{y}+\frac{16}{z}=-2\\ \frac{1}{x}+\frac{9}{y}-\frac{10}{z}=5\end{array}\right\}\xrightarrow{R_3=R_3+R_1}\left.\begin{array}{l}\frac{1}{x}+\frac{2}{y}-\frac{4}{z}=1\\ 0-\frac{1}{y}+\frac{16}{z}=-2\\ 0+\frac{11}{y}-\frac{6}{z}=6\end{array}\right\}$$

$$\left.\begin{array}{l}\frac{1}{x}+\frac{2}{y}-\frac{4}{z}=1\\ 0-\frac{1}{y}+\frac{16}{z}=-2\\ 0+\frac{11}{y}-\frac{6}{z}=6\end{array}\right\}\xrightarrow{R_3=R_3+11R_2}\left.\begin{array}{l}\frac{1}{x}+\frac{2}{y}-\frac{4}{z}=1\\ 0-\frac{1}{y}+\frac{16}{z}=-2\\ 0+0+\frac{102}{z}=-16\end{array}\right\}$$

$$\left.\begin{array}{l}\frac{1}{x}+\frac{2}{y}-\frac{4}{z}=1\\ 0-\frac{1}{y}+\frac{16}{z}=-2\\ 0+0+\frac{102}{z}=-16\end{array}\right\}\xrightarrow{R_1=R_1+2R_2}\left.\begin{array}{l}\frac{1}{x}+0+\frac{28}{z}=-3\\ 0-\frac{1}{y}+\frac{16}{z}=-2\\ 0+0+\frac{102}{z}=-16\end{array}\right\}$$

$$\left.\begin{array}{l}\frac{1}{x}+0+\frac{28}{z}=-3\\ 0-\frac{1}{y}+\frac{16}{z}=-2\\ 0+0+\frac{102}{z}=-16\end{array}\right\}\xrightarrow{R_2=R_2-\frac{14}{51}R_3}\left.\begin{array}{l}\frac{1}{x}+0+\frac{28}{z}=-3\\ 0-\frac{1}{y}+0=\frac{26}{51}\\ 0+0+\frac{102}{z}=-16\end{array}\right\}$$

$$\left.\begin{array}{l}\frac{1}{x}+0+\frac{28}{z}=-3\\ 0-\frac{1}{y}+0=\frac{26}{51}\\ 0+0+\frac{102}{z}=-16\end{array}\right\}\xrightarrow{R_1=R_1-\frac{8}{51}R_3}\left.\begin{array}{l}\frac{1}{x}+0+0=\frac{71}{51}\\ 0-\frac{1}{y}+0=\frac{26}{51}\\ 0+0+\frac{102}{z}=-16\end{array}\right\}$$

Now we can extract the variables:

$$x=\frac{51}{71},\quad y=-\frac{51}{26},\quad z=-\frac{102}{16}$$

Let's verify:

$$\begin{aligned}
\frac{1}{x} + \frac{2}{y} - \frac{4}{z} &= 1 \\
\frac{71}{51} - \frac{52}{51} + \frac{64}{102} &= 1 \\
\frac{19}{51} - \frac{32}{51} &= 1 \\
\frac{51}{51} &= 1
\end{aligned}$$

Similarly for other cases.

1.4 Problem 4

Given $U = \{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}\}$ is a linearly independant set of vectors in $\mathbb{R}^5$ and vectors:

$$\begin{aligned}
v_1 &= 8au_1 + \quad 2u_2 + u_3 \\
v_2 &= \qquad\quad 16au_2 + u_4 \\
v_3 &= u_1 - \quad \frac{1}{2}u_3 + au_4 \\
a &\in \mathbb{R}
\end{aligned}$$

1. Find all a such that $V = \{v_1, v_2, v_3\}$ is linearly dependent.
2. For every a found in (1), write v_2 as linear combination of v_1 and v_3.
3. Is it possible to adjoin the vectors v_i to U such that $U \cup \{v_i\}$ would become a basis in $\mathbb{R}$?

1.4.1 Answer 4

First we will arrange all coefficients describing vectors v_i as rows of the matrix. Since in order to find a linearly dependent combination of rows we need the matrix to be homogenous, the last row of the matrix is the zero vector. Thus, I'll only write the "interesting" columns. I will reduce this matrix to the echelon form in order to find possible contradictions (possible contradictions are rows containing single coefficient). These raws will yield equations, which, if solved, will give values of a required for the system to have solutions. This will be equivalent to finding values of a s.t. they make linear combination of vectors v_i linearly dependant.

$$\begin{bmatrix} 8a & 0 & 1 \\ 2 & 16a & 0 \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & a \end{bmatrix} \xrightarrow{R_1=R_2,R_2=R_1} \begin{bmatrix} 2 & 16a & 0 \\ 8a & 0 & 1 \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & a \end{bmatrix} \xrightarrow{R_2=R_2+4aR_1} \begin{bmatrix} 2 & 16a & 0 \\ 0 & 64a^2 & 1 \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & a \end{bmatrix}$$

$$\begin{bmatrix} 2 & 16a & 0 \\ 0 & 64a^2 & 1 \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & a \end{bmatrix} \xrightarrow{R_3=2R_3} \begin{bmatrix} 2 & 16a & 0 \\ 0 & 64a^2 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & a \end{bmatrix} \xrightarrow{R_3=R_3-R_1} \begin{bmatrix} 2 & 16a & 0 \\ 0 & 64a^2 & 1 \\ 0 & -16a & -1 \\ 0 & 1 & a \end{bmatrix}$$

$$\begin{bmatrix} 2 & 16a & 0 \\ 0 & 64a^2 & 1 \\ 0 & -16a & -1 \\ 0 & 1 & a \end{bmatrix} \xrightarrow{R_3=R_2,R_2=R_3} \begin{bmatrix} 2 & 16a & 0 \\ 0 & -16a & -1 \\ 0 & 64a^2 & 1 \\ 0 & 1 & a \end{bmatrix} \xrightarrow{R_3=R_3-4aR_2} \begin{bmatrix} 2 & 16a & 0 \\ 0 & -16a & -1 \\ 0 & 0 & 1-4a \\ 0 & 1 & a \end{bmatrix}$$

$$\begin{bmatrix} 2 & 16a & 0 \\ 0 & -16a & -1 \\ 0 & 0 & 1-4a \\ 0 & 1 & a \end{bmatrix} \xrightarrow{R_4=16R_4} \begin{bmatrix} 2 & 16a & 0 \\ 0 & -16a & -1 \\ 0 & 0 & 1-4a \\ 0 & 16a & 16a^2 \end{bmatrix} \xrightarrow{R_4=R_4-R_2} \begin{bmatrix} 2 & 16a & 0 \\ 0 & -16a & -1 \\ 0 & 0 & 1-4a \\ 0 & 0 & 16a^2-1 \end{bmatrix}$$

Which gives us two candidate equations: $1 - 4a = 0$ and $16a^2 - 1 = 0$ with respective roots $\frac{1}{4}$ and $-\frac{1}{4}$.

Now we can write v_2 as linear combination of v_1 and v_3 for $\frac{1}{4}$:

$$\begin{aligned}
(0, \frac{32}{4}, 0, 0) &= x(2,0,0,0) &&+ y(-1, 2, \frac{32}{4^2} - 2) \\
(0, 8, 0, 0) &= x(2,0,0,0) &&+ y(-1,2,0,0) \\
(0, 8, 0, 0) &= 4(-1,2,0,0) &&+ 2(2,0,0,0) \\
v_2 &= 4v_3 &&+ 2v_1
\end{aligned}$$

and similarly for $-\frac{1}{4}$:

$$\begin{aligned}
(0, -\frac{32}{4}, 0, 0) &= x(2,0,0,0) &&+ y(-1, 2, \frac{32}{-4^2} - 2) \\
(0, -8, 0, 0) &= x(2,0,0,0) &&+ y(-1,2,0,0) \\
(0, -8, 0, 0) &= -4(-1,2,0,0) &&+ -2(2,0,0,0) \\
v_2 &= -4v_3 &&+ -2v_1
\end{aligned}$$

No, it is not possible to create a basis from $u_i \cup v_i$ because none of v_i affects the fifth dimension of $\mathbb{R}^5$ and because everyone of v_i is a linear combination of u_i, none of u_i could have any effect on the fifth dimension either.

1.5 Problem 5

Given $\vec{a_1}, ..., \vec{a_k}$ and $\vec{b}$ all in $\mathbb{R}^n$. Also given that $\vec{b} \neq 0$ and all $\vec{a_1}, ..., \vec{a_k}$ are distinct. Assume also that the equation $x_1\vec{a}_1 + ... + x_k\vec{a}_k = \vec{b}$ has infinitely many solutions.

Prove or disprove:

1. If $k \geq n + 1$, then $\{\vec{a_1}, ..., \vec{a_k}\}$ spans $\mathbb{R}^n$.
2. $\{\vec{a_1}, ..., \vec{a_k}\}$ is linearly dependant.
3. Exists $\vec{c} \in \mathbb{R}^n$ s.t. $x_1\vec{a}_1 + ... + x_k\vec{a}_k = \vec{c}$ has unique solution.

1.5.1 Answer 5

(1) $\{\vec{a_1}, ..., \vec{a_k}\}$ doesn't necessary span $\mathbb{R}^n$. In order to span a field of a dimension n, this set has to have at least n pivot elements in its coefficient matrix. This can only happen when there are at least n linearly independant vectors (but we are only given that they are distinct, not necessarily independant). More so, we are given that there exists $\vec{b}$, which guarantees that the rank of the matrix of the coefficients will be at least one point short (but possibly more) of representing a spanning set.

To convince yourself this is actually possible, let's construct such vectors for $k = 3$, $n = 2$.

$$\begin{aligned} \vec{a_1} &= (0, 0) \\ \vec{a_2} &= (0, 1) \\ \vec{a_3} &= (0, 2) \\ \vec{b} &= (0, 3) \end{aligned}$$

The matrix of coefficients of this set of vectors would be:

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

This matrix has infinitely many solutions (because it lacks a pivot in the first column), so it satisfies the requirement, but the vectors used to construct its columns are clearly not a spanning set of $\mathbb{R}^2$ (becuase the first element of $\mathbb{R}^2$ is never assigned to).

(2) Yes, $\{\vec{a_1}, ..., \vec{a_k}\}$ is linearly dependant. This is warranted by infinite number of solutions to the equation describing the sum of the vectors, by Rouché–Capelli theorem.

(3) No, there can't be a $\vec{c}$ that would force this system to have a unique solution. The number of solutions of a system is the property of its augmented matrix and its coefficient matrix, neither of which include $\vec{c}$.