Assignment 13, Linear Algebra 1

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1 Problems

1.1 Problem 1

1. Given:

$$
\begin{aligned}
w &= 1 - i \\
t &= \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}
\end{aligned}
$$

Solve:

$$z^3 = \frac{w}{\bar{t}}$$

1. Let $z_1, z_2, ..., z_n$ be all solutions of an equation $z^n = 1$ in $\mathbb{C}$.
   Prove that $z_1 \times z_2 \times ... \times z_n = 1$.

1.1.1 Answer 1

Let's first represent w in polar form:

$$
\begin{aligned}
w &= re^{\theta i} \\
r &= \sqrt{1^2 - i^2} = \sqrt{1+1} = \sqrt{2} \\
\tan\theta &= \frac{-1}{1} \\
\theta &= \tan^{-1} -1 = \frac{\pi}{4} \\
w &= \sqrt{2}e^{\frac{\pi}{4}}
\end{aligned}
$$

Next, substitute both w and $\bar{t}$ into equation:

$$
\begin{aligned}
z &= \left(\frac{\sqrt{2}e^{\frac{i\pi}{4}}}{e^{\frac{-i4\pi}{3}}}\right)^{\frac{1}{3}} \\
&= \left(\sqrt{2}e^{\frac{i\pi-(-3i\pi)}{4}}\right)^{\frac{1}{3}} \\
&= (\sqrt{2}e^{\pi})^{\frac{1}{3}} \\
&= \sqrt[2\times 3]{2}e^{i\pi\times\frac{1}{3}} \\
&= \sqrt[6]{2}e^{\frac{i\pi}{3}}
\end{aligned}
$$

Which is the reduced polar form.

Now, let's verify:

```
(expt
 (/ (- 1 #c(0 -1))
    (- (cos (/ (* 3 pi) 4))
       (* #c(0 1) (sin (/ (* 3 pi) 4)))))
      1/3)
```

```
#C(0.5612310241546866d0 0.9720806486198328d0)
```

```
(* (expt 2 1/6) (expt (exp 1) (* #c(0 1) (/ pi 3))))
```

```
#C(0.5612310480260124d0 0.9720806185750182d0)
```

By fundamental theorem of algebra, this equation must have two more results. So, the general solution to this equation must be given by

$$
z = \sqrt[6]{2}e^{\frac{i\pi+2\pi K}{3}}
$$

Where $k = 1, 2, 3$. *(this is probably wrong)*

1.1.2 Answer 2

Unfortunately, the conjecture is false, thus no proof is possible. For example, for the case $n = 2$, it gives $z^2 = 1$ with two roots: $z = 1$ and $z = -1$, where $1 \times (-1) = -1$.

However, with some refinements, it is possible to prove this conjecture for all non-real roots, to do so let a_i, $i = 1...n$ be the solution of $z^n - 1 = 0$. Then:

$$z - 1^n = (z - a_1)(z - a_2)...(z - a_n)$$

After we open parenthesis and multiply the terms, we will get:

$$(-1)^n a_1...a_n$$

When we equate the coefficients, we get that:

$$a_1...a_n = (-1)^{n+1}$$

Now, observe that one n is odd, then the only real solution is 1, then the rest of the solutions would give us $(-1)^{n+1} = 1$. And when n is even, we will have two real solutions: 1 and -1. Then the product of all non-real solutions would be: $\frac{(-1)^{n+1}}{-1} = 1$.

1.2 Problem 2

Given $A = \{(x, 1)|x \in \mathbb{R}\}$ is a subset of $\mathbb{R}^2$ and operations over A defined as follows:

- Addition, $\oplus$: $\forall x, y \in \mathbb{R} : (x, 1) \oplus (y, 1) = (x + y, 1)$.
- Multiplication by scalar, $\odot$: $\forall k, y \in \mathbb{R} : k \odot (x, 1) = (k^2 x, 1)$.
- Multiplication, $*$: $\forall x, y \in \mathbb{R} : (x, 1) * (y, 1) = (x * y, 1)$.

1. Is $(A, \oplus, \odot)$ a vector space over $\mathbb{R}$?
2. Is $(A, \oplus, *)$ a field?

1.2.1 Answer 3

No, $(A, \oplus, \odot)$ is not a vector space over $\mathbb{R}$. One of requirements of a vector field is that multiplication with scalar give a *unique* solution in A, while every pair of any member of A and its additive inverse violates this requirement, for instance:

$$-1 \odot (x, 1) = ((-1)^2 x, 1) = (x, 1) = (1^2 x, 1) = 1 \odot (x, 1)$$

1. Vector addition is commutative:

$$(x, 1) + (y, 1) = (x + y, 1) = (y + x, 1) = (y, 1) + (x, 1)$$

1. Vector addition is associative:

$$(x, 1) + ((y, 1) + (z, 1)) = (x, 1) + (y + z, 1) = (x + y + z, 1) = (x + y, 1) + (z, 1)$$

1. There exists additive identity 0: As the previous two, this follows from addition in $\mathbb{R}$ having additive identity.
2. 1 is the multiplicative idenity: Since $1^2 = 1$, we can reuse the multiplicative identity defined on reals.
3. Every element has an additive inverse: Again, we are using the addition in real numbers, so we are guaranteed to have additive inverses for every such number.
4. Scalar multiplication is associative:

$$r(k \odot (x, 1)) = r(k^2 x, 1) = (r^2 k^2 x, 1) = ((rk)^2 x, 1) = (rk)(x, 1)$$

1. Scalar multiplication is distributive:

$$k \odot ((x, 1) + (y, 1)) = k \odot (x + y, 1) = (k^2(x + y), 1) = (k^2 x, 1) + (k^2 y, 1) = k \odot (x, 1) + k \odot (y, 1)$$

1.3 Problem 3

1. Establish which of the given sets are linear vector fields over $\mathbb{F}$ under normal *(what is considered "normal" addition of tuples of complex numbers?)* operations.

   - $U = \{(z, w) \in \mathbb{C}^2 \mid 2z = 3w\}$, $\mathbb{F} = \mathbb{C}$.
   - $W = \{f : \mathbb{R} \to \mathbb{R} \mid f(x + 1) = f(x) + 1, x \in \mathbb{R}\}$, $\mathbb{F} = \mathbb{R}$.
   - $M = \{p(x) \in \mathbb{R}^4[x] \mid p(x) = p(x - 1)\}$, $\mathbb{F} = \mathbb{R}$.
   - $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R}) \mid ad = 0 \right\}$, $\mathbb{F} = \mathbb{R}$.

2. For every vector space found, write its finite spanning set.

1.3.1 Answer 4

U is a vector space, assuming addition is point-wise. Addition properties follow from the same properties of addition of complex numbers. Similarly for multiplication with field's identity element and multiplicatin by scalar.

W is certainly not a vector space. Put $f(0) = -3$, then $f(1) = -2$ and $f(2) = -1$. Now $4 \times f(1+2) = 4 \times f(3) = 4 \times 0 = 0$, while $4 \times f(1) + 4 \times f(2) = 4 \times -2 + 4 \times -1 = -8 - 4 = -12$. Clearly $-12 \neq 0$.

M is a vector space since. To see this let's first eamine what kinds of polinomials are represented by W:

First, let's find the zero polinomial. Looking at $p(1)$ would be also interesting because it will immediately show what kinds of polinomials are possible in this field.

$$p(0) = \alpha_0 + \alpha_1 \times 0 + \alpha_2 \times 0^2 + \alpha_2 \times 0^3$$
$$p(1) = \alpha_0 + \alpha_1 \times 1 + \alpha_2 \times 1^2 + \alpha_2 \times 1^3$$

Which, in turn implies that since $p(1) = p(1-1) = p(0)$, only the constant term of these two polinomials matters, in other words, $p(0) = \alpha_0 = p(1)$ implies that other terms of these polinomials must be zero.

This leaves us with polinomials of the form $p(x) = \alpha_0$, adding such polinomials will be equivalent to addition defined for real numbers, same goes for multiplication.

S is not a vector space. Adding any two matrices where $a \neq d$ and $a' \neq a$ would give us a matrix where both a and d are non-zero, producing a matrix outside the valid range.

1.3.2 Answer 5

1. Example spanning set for U would be $Sp(i, \frac{3}{2}i)$.
2. Example spanning set for M would be $Sp(p(x) = 1)$.

1.4 Problem 4

1. Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in linear vector field V over $\mathbb{F}$. Does it then hold that: $Sp\{u, v, w\} = Sp\{u + v - w, u - v + 2w, v + w\}$?
2. Let $U = Sp\{(1, 2, 5), (1, 1, 3)\}$ and $W = Sp\{(1, 0, 1), (0, 1, 1)\}$. Are $U = W$?

1.4.1 Answer 6

Let's equate two expressions and try to see if equality holds:

$$\begin{aligned} \text{u+v+w} &= \lambda_0(u+v-w)+\lambda_1(u-v+2w)+\lambda_2(v+w) \\ &= \text{u}(\lambda_0+\lambda_1)+v(\lambda_0-\lambda_1+\lambda_2)+w(-\lambda_0+2\lambda_1+\lambda_2) \end{aligned}$$

From here we can already see that we can make coefficients arbitrary large. In other words, the coefficients of the vectors are linearly independant:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

Since every column of this matrix has a pivot, the vectors represented by its columns are linearly independent. Hence, both spans are quivalent.

1.4.2 Answer 7

Assuming equality of spanning sets means that its the spaces spanned by them must be equal and not the spanning sets themselves (otherwise the answer would be obviously negative). In order to find out whether the span of the space is the same, we could adjoin a vecotr from one set to the vectors from another set and see if we obtain linearly dependant set. If the set is linearly dependent, it would mean that the vector from another set may be generated from the first set, and if this is true for all vectors in the other set, then it would mean that two spans are the same. However, knowing that a vector from a spanning set cannot be generated by applying elementary operations to the set of vectors, we would know that it is not possible to generate it using the spanning set, thus they must be different.

We will adjoin (1, 0, 1), (0, 1, 1) and (1, 1, 3):

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the rank of this matrix is 3, it represents a linearly independent combination of vectors, hence $U \neq W$.

1.5 Problem 5

Let U and W be sub-spaces of the linear vector space V s.t. $U \oplus W = V$. Let $S \subseteq U$ and $T \subseteq W$ be two finite linearly independent sets. Prove that $S \cup T$ is linearly independent.

1.5.1 Answer 8

Since we know that S and T are both linearly independant, they are also spans, they either span the entire subspace, from which they are taken, or a subspace of that subspace. Now, suppose their union was linearly dependant, this, would also imply that the subspaces from which they were taken had common members other than the zero vector (those would be exactly the vectors that must have been common to the union of S and T. Since by the definition of direct sum this is not possible, it must be that union of S and T is linearly independant.