Assignment 12, Linear Algebra 1

Oleg Sivokon

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Contents

1 Problems

1.1 Problem 1

Let A be a square matrix of order 3×3 s.t. $A^3 = 0$, but $A^2 \neq 0$.

1. Prove that there exists a vector $\vec{v} \in \mathbb{R}^3$ s.t. $A\vec{v} \neq 0$.
2. Prove that there exits vectors $\vec{v} \in \mathbb{R}^3$ s.t. $\{\vec{v}, \vec{v}A, \vec{v}A^2\}$ are the basis of $\mathbb{R}^3$.

1.1.1 Answer 1

Notice that A itself is made of the column vectors, call them r_1, r_2, r_3. All of which are in $\mathbb{R}^3$. Suppose, for contradiction, that there is no vector in $\vec{v} \in \mathbb{R}^3$ satisfying $A\vec{v} \neq 0$. In particular, none of the $\vec{r}_1, \vec{r}_2, \vec{r}_3$ satisfies the above condition. In other words, $A\vec{r}_1 = 0$, $A\vec{r}_2 = 0$, $A\vec{r}_3 = 0$. (where 0 means zero matrix). On the other hand, $A\vec{r}_1 + A\vec{r}_2 + A\vec{r}_3 = A^2 \neq 0$. Contradiction. Hence, there exists $\vec{v} \in \mathbb{R}^3$ s.t. $A\vec{v} \neq 0$.

1.1.2 Answer 2

1.2 Problem 2

Given square matrices A, B, C, D of order $n \times n$ s.t. $ABCD = I$, prove that $ABCD = DABC = CDAB = BCDA = I$.

1.2.1 Answer 3

The proof is immediate from the definition of inverse: $XX^{-1} = I$ and associativity of matrix multiplication. In other words:

$$
\begin{aligned}
ABCD = I &\iff \\
A(BCD) = I &\iff \\
A^{-1} = BCD &\iff \\
DABC = I &\iff \\
AB(CD) = I &\iff \\
(AB)^{-1} = CD &\iff \\
CDAB = I &\iff \\
A^{-1}A = I &\iff \\
I = BCDA &.
\end{aligned}
$$

1.3 Problem 3

Let A be a square matrix of order $m \times m$, let B be a matrix of order $m \times n$. Prove in two different ways that if A is invertible, then $B\vec{x} = 0$ and $AB\vec{x} = 0$ has the same solution space.

1.3.1 Answer 4

1.4 Problem 4

Given matrix A of a general form:

$$
\begin{bmatrix}
0 & a_1 & \dots & 0 & 0 \\
0 & 0 & a_2 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & a_{n-1} \\
a_n & 0 & 0 & \dots & 0
\end{bmatrix}
$$

Prove that it is invertible, show A^{-1}.

1.4.1 Answer 5

Performing elementary operations: $R_n \to R_1$ and $R_k \to R_{k+1}, 1 \leq k < n$ gives us diagonal matrix. This matrix is invertible since it has a pivot element in each of its columns.

The inverse of A will, in general look like this:

$$\begin{bmatrix} 0 & 0 & \dots & 0 & \frac{1}{a_n} \\ \frac{1}{a_1} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{a_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{a_{n-1}} & 0 \end{bmatrix}$$

Notice that for each row of A, we will be matching the column of A^{-1}. We need to make sure that the only non-zero element of A_c was matched by the only non-zero element of A_r^{-1} (where c stands for column index and r stands for row index). In order to obtain a diagonal with all ones (i.e. the identity matrix), we need to also make sure that $A_{c,i} \times A_{r,j}^{-1} = 1$. In other words, we need to match a_1 with $\frac{1}{a_1}$, a_2 with $\frac{1}{a_2}$, and so on.

1.5 Problem 5

Let A and B be square matrices of the order 3×3 s.t. $B^2A = -2B^3$ and $B^3 + AB^2 = 3I$.

Prove that A and B are invertible and express A^{-1} and B^{-1} in terms of B.

1.5.1 Answer 6

Using some matrix algebra we obtain: $B^{-1} = -\frac{1}{3}B^2$ and $A^{-1} = (-2B)^{-1}$.

$$
\begin{aligned}
B^2A = -2B^3 &\iff \\
B^2A = B^2(-2I)B &\iff \\
A = -2B &
\end{aligned}
$$

substituting into second equation:

$$
\begin{aligned}
B^3 + AB^2 = 3I &\iff \\
B^3 - 2B^3 = 3I &\iff \\
-B^3 = 3I &\iff \\
B(-B^2) = 3I &\iff \\
B(-\frac{1}{3}B^2) = I &\iff \\
B^{-1} = -\frac{1}{3}B^2 &
\end{aligned}
$$

A is invertible because it is similar to B

$$
\begin{aligned}
-2B &= (\sqrt{2}I)B(\sqrt{2}I^{-1}) \\
A^{-1} &= (-2B)^{-1}\ .
\end{aligned}
$$