Assignment 14, Linear Algebra 1

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Contents

1 Problems

1.1 Problem 1

Let W and U be subspaces of $\mathbb{R}^4[x]$:

$$\begin{aligned} U &= \mathrm{Sp}\{x^3+4x^2-x+3, x^3+5x^2+5, 3x^3+10x^2+5\} \\ W &= \mathrm{Sp}\{x^3+4x^2+6, x^3+2x^2-x+5, 2x^3+2x^2-3x+9\} \end{aligned}$$

1. Find dimension, basis of U, W and $U+W$.
2. What is the dimension and basis of $U \cap W$?
3. Find a subspace T such that $T \oplus W = \mathbb{R}^4[x]$.

1.1.1 Answer 1

In order to find dimension, I can first find the basis, and then simply count the number of vectors in it. Since basis by definition has to be linearly independent, I will adjoin vectors of U to a matrix, will do the same for vectors of W, triangularize these matrices and remove the zero rows to obtain the basis.

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 1 & 5 & 0 & 5 \\ 3 & 10 & 0 & 5 \end{bmatrix} \xrightarrow{R_1=R_2, R_2=R_1} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 1 & 2 & -1 & 3 \\ 3 & 10 & 0 & 5 \end{bmatrix} \xrightarrow{R_3=R_3-3R_1} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 1 & 2 & -1 & 3 \\ 0 & -5 & 0 & -10 \end{bmatrix}$$

$$\xrightarrow{R_2=R_3, R_3=R_2} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -5 & 0 & -10 \\ 1 & 2 & -1 & 3 \end{bmatrix} \xrightarrow{R_3=R_3-R_1} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -5 & 0 & -10 \\ 0 & -7 & -1 & -2 \end{bmatrix} \xrightarrow{R_3=R_3-R_2}$$

$$\begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -5 & 0 & -10 \\ 0 & -2 & -1 & 8 \end{bmatrix} \xrightarrow{R_3=5R_3} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -5 & 0 & -10 \\ 0 & -10 & -5 & 40 \end{bmatrix} \xrightarrow{R_3=R_3-2R_2} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -5 & 0 & -10 \\ 0 & 0 & -5 & 20 \end{bmatrix}$$

Hence the basis of U is $\{x^3+5x^2+5, -5x^2-10, -5x+20\}$. And its dimension is 3.

Similarly, for W:

$$\begin{bmatrix} 1 & 4 & 0 & 6 \\ 1 & 2 & -1 & 5 \\ 2 & 2 & -3 & 9 \end{bmatrix} \xrightarrow{R_1=R_2, R_2=R_1} \begin{bmatrix} 1 & 4 & 0 & 6 \\ 0 & -2 & -1 & -1 \\ 0 & 6 & -3 & -3 \end{bmatrix} \xrightarrow{R_3=R_3-3R_1} \begin{bmatrix} 1 & 4 & 0 & 6 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence the basis of W is $\{x^3+4x^2+6, -2x^2-x-1\}$, and its dimension is 2.

Now, let's find the basis of $V+U$. Again, adjoin the bases of U and W, triangularize and remove all zero vectors if any.

$$\begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -5 & 0 & -10 \\ 0 & 0 & -5 & 20 \\ 1 & 4 & 0 & 6 \\ 0 & -2 & -1 & -1 \end{bmatrix} \xrightarrow{R_1=R_2, R_2=R_1} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -5 & 0 & -10 \\ 0 & 0 & -5 & 20 \\ 0 & -1 & 0 & -1 \\ 0 & -2 & -1 & -1 \end{bmatrix} \xrightarrow{R_3=R_3-3R_1} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -5 & 0 & -10 \\ 0 & 0 & -5 & 20 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3=R_3-3R_1} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -5 & 20 \\ 0 & -5 & 0 & -10 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3=R_3-3R_1} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -5 & 20 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3=R_3-3R_1}$$

$$\begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & -5 & 20 \end{bmatrix} \xrightarrow{R_3=R_3-3R_1} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 15 \end{bmatrix} \xrightarrow{R_3=R_3-3R_1} \begin{bmatrix} 1 & 5 & 0 & 5 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $\mathrm{Dim}(V)=4$ and $\mathrm{Dim}(U)+\mathrm{Dim}(W)=5$ it follows that $\mathrm{Dim}(U\cap W)$ has to be 1. This follows from the formula of sum of dimensions of subspaces, which says $\mathrm{Dim}(V)=\mathrm{Dim}(S)+\mathrm{Dim}(T)-\mathrm{Dim}(S\cap T)$.

1.2 Problem 2

Let U and W be subspaces of $\mathbb{R}^4$ such that $\mathrm{Dim}(U)>\mathrm{Dim}(W)$. Provided that $W\cap U = \mathrm{Sp}\{(1,2,3,4),(1,1,1,1),(-1,0,1,2)\}$, and $(0,0,1,0)\notin U+W$. Find the dimension of $U+W$ and its basis.

1.3 Problem 3

Given the following subspaces of $\mathbb{R}^4$:

$$U=\mathrm{Sp}\{(a,a-1,a,4),(2,2,1,-3)\}$$

$$W=\mathrm{P}\left(\left.\begin{array}{r} x+\ y+z \ =0 \\ y+2z-2t=0 \end{array}\right\}\right)$$

Where $\mathrm{P}(X)$ is the vector space of all solutions of linear system X.

Find values of a for which holds that $\mathrm{Dim}(U\cap W)=1$. Show the basis of $U\cap W$ in this case.

1.4 Problem 4

Prove or disprove each of the following statements:

1. If $B = \{\vec{v_1}, \ldots, \vec{v_n}\}$ is the basis of V and $U \subseteq V$ is a subspace of dimension k, $k \leq n$, then there are k vectors in B spanning U.
2. If V is a vectors pace of dimension n and if $m \leq n, m \in \mathbb{N}$, then exists sub-space U of V with dimension equal to m.

1.5 Problem 5

In field $\mathbb{F}$ are given members $a_1, a_2, \ldots, a_m$, not all zero and, similarly, $b_1, b_2, \ldots, b_n$ not all zero. What is the dimension of the matrix given by:

$$M = (m_{ij})_{1<i<m, 1<j<n}$$
$$m_{ij} = a_i b_j$$

1.6 Problem 6

Let V be a vector space over $\mathbb{R}$ of dimension 3, and let B be its basis. Given vectors $\vec{v_1}, \vec{v_2}, \vec{v_3}$ and $\vec{w}$ in V:

$$[v_1]_B = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, [v_2]_B = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, [v_3]_B = \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}, [w]_B = \begin{pmatrix} 5 \\ 5 \\ 16 \end{pmatrix}$$

Prove that $C = (\vec{v_1}, \vec{v_2}, \vec{v_3})$ is the basis of V and find $[w]_C$.