Assignment 11, Linear Algebra 1

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Contents

1 Problems

1.1 Problem 1

Solve given systems of linear equations:

$$\left.\begin{aligned} 2x - y \quad + 4z &= 1 \\ x \quad + 2y - 3z &= 6 \\ x \quad - y \quad + z \quad &= 3 \end{aligned}\right\} \qquad x, y, z \in \mathbb{R}$$

$$\left.\begin{aligned} 2x + 2y + 8z + w \quad &= 0 \\ 3x + 3y + 3z + 13w &= 0 \\ 2x + 2y + 4z + 3w \quad &= 0 \end{aligned}\right\} \qquad x, y, z, w \in \mathbb{R}$$

Which variables are bound, which are free?

1.1.1 Answer 1

```
programmode: false;
linsystem: [  2*x -   y + 4*z = 1,
                x + 2*y - 3*z = 6,
                x -   y +   z = 3];
print(linsolve(linsystem, [x, y, z]));
```

```
Solution:
                                   17
                               x = --
                                   5
                                    7
                               y = - -
                                    5
                                    9
                               z = - -
                                    5
[%t1, %t2, %t3]
```

First, we subtract third equation from the second in order to express y in terms of z:

$$
\begin{aligned}
(x+2y-3z)-(x-y+z) &= 6-3 \iff \\
3y-4z &= 3 \iff \\
y &= \frac{3+4z}{3} \ .
\end{aligned}
$$

Next, we do the same for third and first equation to express x in terms of z:

$$
\begin{aligned}
(2x-y+4z)-(x-y+z) &= 1-3 \iff \\
x+3z &= 2 \iff \\
x &= -2-3z \ .
\end{aligned}
$$

Then we substitute x and y into the third equation:

$$
\begin{aligned}
-2-3z-\frac{3+4z}{3}+z &= 3 \iff \\
-2z-1-\frac{4}{3}z &= 5 \iff \\
-\frac{10}{3}z &= 6 \iff \\
z &= -\frac{9}{5} \ .
\end{aligned}
$$

Similarly, we find $x = \frac{17}{5}$ and $y = -\frac{7}{5}$.

Substituting the results into original system gives:

$$
\left.
\begin{array}{l}
2\frac{17}{5} - -\frac{7}{5} + -4\frac{9}{5} = 1 \\
\frac{17}{5} + -2\frac{7}{5} - -3\frac{9}{5} = 6 \\
\frac{17}{5} - -\frac{7}{5} + -\frac{9}{5} = 3
\end{array}
\right\} \iff
$$

$$
\left.
\begin{array}{l}
34+7 \quad -36 = 5 \\
17-14+27 = 30 \\
17+7 \quad -9 \quad = 15
\end{array}
\right\} \iff
$$

$$
\left.
\begin{array}{l}
5 \quad = 5 \\
30 = 30 \\
15 = 15
\end{array}
\right\} .
$$

If we convert the given system to a matrix and bring the matrix to the row-echelon form we get:

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 2 & -3 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_3=R_3-R_2} \begin{bmatrix} 2 & -1 & 4 \\ 1 & 2 & -3 \\ 0 & -3 & 4 \end{bmatrix} \xrightarrow{R_1=R_1-2R_2}$$
$$\begin{bmatrix} 0 & -3 & 10 \\ 1 & 2 & -3 \\ 0 & -3 & 4 \end{bmatrix} \xrightarrow{R_1=R_2, R_2=R_1} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 10 \\ 0 & -3 & 4 \end{bmatrix} \xrightarrow{R_3=R_3-R_2} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 10 \\ 0 & 0 & -6 \end{bmatrix}$$

We can see that all columns have leading variables, thus there are no free variables.

1.1.2 Answer 2

```
programmode: false;
linsystem: [  2*x + 2*y + 8*z +    w = 0,
              3*x + 3*y + 3*z + 13*w = 0,
              2*x + 2*y + 4*z +   3*w = 0];
print(linsolve(linsystem, [x, y, z, w]));
```

```
Solution:
                                   x = - %r1
                                     z = 0
                                     w = 0
                                    y = %r1
[%t1, %t2, %t3, %t4]
```

Similarly to the 1.1.1, we first express w in terms of z:

$$\begin{aligned} 2x + 2y + 8z + w - 2x - 2y - 4z - 3w = 0 &\iff \\ 4z - 2w = 0 &\iff \\ w == 2z \; . \end{aligned}$$

Now we can rewrite the system as:

$$\left.\begin{aligned} 2x + 2y + 10z = 0 \\ 3x + 3y + 29z = 0 \\ 2x + 2y + 10z = 0 \end{aligned}\right\} \qquad x, y, z \in \mathbb{R}$$

which is essentially the same as:

$$\left.\begin{aligned} 2x + 2y + 10z = 0 \\ 3x + 3y + 29z = 0 \end{aligned}\right\} \qquad x, y, z \in \mathbb{R}$$

Expressing x in terms of y and z gives:

$$\begin{aligned} 3x + 3y + 29z - 2x - 2y - 10z = 0 \iff \\ x + y + 19z = 0 \iff \\ x = -y - 19z \ . \end{aligned}$$

Substituting it back into firxt equation to solve for y:

$$\begin{aligned} 3(-y - 19z) + 3y + 29z = 0 \iff \\ 3y - 3y - 57z + 29z = 0 \iff \\ 28z = 0 \iff \\ z = 0 \ . \end{aligned}$$

Now we substitute this result back into our description of x, thus obtaining:

$$\begin{aligned} & x = -y - 19 \times 0 \iff \\ & x = -y \ . \end{aligned}$$

Which is the solution for the given system of linear equations.

We'll bring the matrix corresponding to this system to the row-echelon form to find the free and the bound variables.

$$\begin{bmatrix} 2 & 2 & 8 & 1 \\ 3 & 3 & 3 & 13 \\ 2 & 2 & 4 & 3 \end{bmatrix} \xrightarrow{R_3=R_3-R_1} \begin{bmatrix} 2 & 2 & 8 & 1 \\ 3 & 3 & 3 & 13 \\ 0 & 0 & -4 & 2 \end{bmatrix} \xrightarrow{R_2=2R_2}$$

$$\begin{bmatrix} 2 & 2 & 8 & 1 \\ 6 & 6 & 6 & 26 \\ 0 & 0 & -4 & 2 \end{bmatrix} \xrightarrow{R_2=R_2-3R_1} \begin{bmatrix} 2 & 2 & 8 & 1 \\ 0 & 0 & -18 & 23 \\ 0 & 0 & -4 & 2 \end{bmatrix} \xrightarrow{R_3=9R_3}$$

$$\begin{bmatrix} 2 & 2 & 8 & 1 \\ 0 & 0 & -18 & 23 \\ 0 & 0 & -36 & 18 \end{bmatrix} \xrightarrow{R_3=R_3-2R_2} \begin{bmatrix} 2 & 2 & 8 & 1 \\ 0 & 0 & -18 & 23 \\ 0 & 0 & 0 & -28 \end{bmatrix}$$

Since the second column doesn't have a pivot element, I conclude that y is free in this linear system, while the rest of the variables are bound.

1.2 Problem 2

For the given system:

$$\left.\begin{aligned} &x \quad + ay \quad + z = 1 \\ &ax + a^2y + z = 2 + a \\ &ax + 3ay + z = 2 - t \end{aligned}\right\} \qquad a, t, x, y, z \in \mathbb{R}$$

1. Find a, t s.t. the system has a unique solution.
2. Find a, t s.t. the system has infinitely many solutions.
3. Find a, t s.t. the system has no solutions.

1.2.1 Answer 3

We could first reduce the matrix representing this system to the row-echelon form:

$$\begin{bmatrix} 1 & a & 1 \\ a & a^2 & 1 \\ a & 3a & a \end{bmatrix} \xrightarrow{R_3=R_3-R_2} \begin{bmatrix} 1 & a & 1 \\ a & a^2 & 1 \\ 0 & 3a-a^2 & a \end{bmatrix} \xrightarrow{R_2=R_2-aR_2} \begin{bmatrix} 1 & a & 1 \\ 0 & 0 & 1-a \\ 0 & 3a-a^2 & a \end{bmatrix}$$

From which we conclude that whenever $1-a \neq 0$ and $3a-a^2 \neq 0$ there would be a pivot element in every column, thus ensuring the system has exactly one solution.

Second equation factors as $a(3-a)$, thus its roots are $a=0$ and $a=3$. Subsequently, whenever $a \neq 1$ and $a \neq 0$ and $a \neq 3$ the system has a unique solution.

1.2.2 Answer 4

If we put $a=3, t=-3$ then the system has infinitely many solutions since the second and the third its equations become multiples of each other:

$$\begin{aligned}
3x + 3^2 y + z &= 2 + 3 \\
&\textit{while, at the same time} \\
3x + 3 \times 3 + z &= 2 - (-3) \\
&\textit{simplifying both parts} \\
3x + 9y + z &= 5 \\
&\textit{and} \\
3x + 9y + z &= 5\ .
\end{aligned}$$

1.2.3 Answer 5

It is easy to see that whenever $a=1$, no matter the value of t, the system is inconsistent:

$$\begin{aligned}
x + 1 \times y + z &= 1 \\
&\textit{while, at the same time} \\
1 \times x + 1^2 y + z &= 2 + 1 \\
&\textit{subtracting both parts} \\
x + y + z - x - y - z &= 1 - 3 \iff \\
0 &= -2\ .
\end{aligned}$$

Another case when the system becomes inconsisten is when $a=0$ and $t \neq 0$, since the third and the second equations would become inconsistent:

$$0x + 0^2 y + z = 2 + 0$$
while, at the same time
$$0x + 3 \times 0 + z = 2 - t$$
simplifying both parts
$$z = 2$$
and
$$z = 2 - t \; .$$

1.3 Problem 3

Given that vectors $\vec{v} = (4, -2, -2, 4)$ and $\vec{u} = (-2, 4, 4, -2)$ are solutions to the system of linear equations M with four unknowns. Also known that $(2, 2, 2, 2)$ isn't a solution of M.

1. Prove that the system isn't homogeneous.
2. Prove that $(0, 2, 2, 0)$ is also a solution of the system.

1.3.1 Answer 6

Suppose, for contradiction, M is homogeneous. Then it must be the case that any linear combination of $\vec{v}$ and $\vec{u}$ is also a solution to the system. In particular, $\vec{v} + \vec{u}$ is such a solution, but $\vec{v} + \vec{u} = (2, 2, 2, 2)$, contrary to the given.

Hence, by contradiction, M is not homogeneous.

1.3.2 Answer 7

Since the set of all solutions to the linear system is closed under multiplication by a scalar, it is possible that $\vec{v}$, $\vec{u}$, or their linear combination multiplied by a scalar will result in $(0, 2, 2, 0)$, and indeed, $\frac{1}{3}(\vec{v} + 2\vec{u}) = (0, 2, 2, 0)$.

Hence, $(0, 2, 2, 0)$ is a solution of M.

1.4 Problem 4

Let $\{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}\}$ be a basis in $\mathbb{R}^4$.

$$\begin{aligned}
\vec{v_1} &= k\vec{u_1} - \vec{u_3} + \vec{u_4} \\
\vec{v_2} &= \vec{u_1} + \vec{u_2} - \vec{u_4} \\
\vec{v_3} &= 4\vec{u_2} + k\vec{u_3} - 6\vec{u_4} \\
\textit{where } k &\in \mathbb{R}
\end{aligned}$$

1. For what values of k vectors $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3}$ are linearly (in-)dependent?
2. Whenever the above vectors are linearly dependent, write $\vec{v_3}$ as a combination of $\vec{v_1}$ and $\vec{v_2}$.
3. What are the values of k for which the set $\{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{v_1}, \vec{v_2}\}$ spans $\mathbb{R}^4$?

1.4.1 Answer 8

Recall that matrix comprised of column vectors adjoined to the solution vector (zero in our case) will have single solution if the vectors are linearly independent. Hence, represent the $\vec{v_i}$ first in terms of $\vec{u_i}$, then in matrix form:

$$\begin{aligned}
a_1(k\vec{u_1} - \vec{u_3} + \vec{u_4}) + a_2(\vec{u_1} + \vec{u_2} - \vec{u_4}) + a_3(\vec{u_2} + k\vec{u_3} - 6\vec{u_4}) &= \vec{0} \iff \\
(a_1k + a_2)\vec{u_1} + (a_2 + 4a_3)\vec{u_2} + (ka_3 - a_1)\vec{u_3} + (a_1 - a_2 - 6a_3)\vec{u_4} &= \vec{0} \iff \\
\textit{has unique solution} &\begin{bmatrix} k & 1 & 0 \\ 0 & 1 & 4 \\ -1 & 0 & k \\ 1 & -1 & -6 \end{bmatrix}
\end{aligned}$$

It is important now to see what happens when $k = 0$ since this will affect the first pivot element:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 4 \\ -1 & 0 & 0 \\ 1 & -1 & -6 \end{bmatrix} \xrightarrow{R_1=R_4, R_4=R_1} \begin{bmatrix} 1 & -1 & -6 \\ 0 & 1 & 4 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3=R_3+R_1} \begin{bmatrix} 1 & -1 & -6 \\ 0 & 1 & 4 \\ 0 & -1 & -6 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3=R_3+R_2}$$

$$\begin{bmatrix} 1 & -1 & -6 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_4=R_4-R_2} \begin{bmatrix} 1 & -1 & -6 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & -4 \end{bmatrix} \xrightarrow{R_4=R_4+2R_3} \begin{bmatrix} 1 & -1 & -6 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

I.e. when $k = 0$, the system has unique solution. Otherwise:

$$\begin{bmatrix} k & 1 & 0 \\ 0 & 1 & 4 \\ -1 & 0 & k \\ 1 & -1 & -6 \end{bmatrix} \xrightarrow{R_3=R_3+R_4} \begin{bmatrix} k & 1 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & k-6 \\ 1 & -1 & -6 \end{bmatrix} \xrightarrow{R_4=kR_4-R_1} \begin{bmatrix} k & 1 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & k-6 \\ 0 & -k-1 & -6 \end{bmatrix} \xrightarrow{R_3=R_3+R_2}$$

$$\begin{bmatrix} k & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & k-2 \\ 0 & -k-1 & -6 \end{bmatrix} \xrightarrow{R_4=R_4+(k+1)R_2} \begin{bmatrix} k & 1 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & k-2 \\ 0 & 0 & 4k-2 \end{bmatrix}$$

We can see that the third and fourth equations are equivalent, and the system only has unique solution whenever $k \neq \frac{1}{2}$ and $k \neq 2$.

1.4.2 Answer 9

In the way similar to the 1.4.1, we can write a system of symultaneous equations:

$$
\left.\begin{aligned}
kx_1 \quad + x_2 &= 0 \\
x_2 &= 4y \\
- x_1 \qquad &= ky \\
- x_1 - x_2 &= 6y
\end{aligned}\right\} \iff
$$

$$
\begin{aligned}
-x_1 &= 6y + x_2 & \iff \\
ky &= 6y + x_2 & \iff \\
(k+6)y &= 4y & \iff \\
k + 6 &= 4 \lor y = 0 \\
& \textit{assume } y \neq 0 \\
k &= -2 \ .
\end{aligned}
$$

Verifying:

$$
\begin{aligned}
-2x_1 + x_2 &= 0 \land \\
x_2 &= 4y \land \\
-x_1 &= -2y \land \\
-x_1 - x_2 &= 6y \\
\textit{is consitent } \iff & \\
x_1 &= \frac{1}{2}x_2 \land \\
x_2 &= 4y \land \\
-\frac{1}{2}x_2 &= -2y \land \\
-\frac{1}{2}x_2 - x_2 &= -\frac{3}{2}x_2 = 6y \ .
\end{aligned}
$$

Hence whenever $k = -2$, $y(v_1 + v_2) = v_3$.

1.4.3 Answer 10

Since u_i are the basis, none of them is a linear combination of the others. Hence v_1 and v_2 must "compensate" for the loss of u_4. In other words, whenever u_4 is a linear combination of v_1 and v_2, the set $\{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{v_1}, \vec{v_2}\}$ spans $\mathbb{R}^4$.

More formally, whenever:

$$a_1(k\vec{u_1} - \vec{u_3} + \vec{u_4}) + a_2(\vec{u_1} + \vec{u_2} - \vec{u_4}) = \vec{u_4} \iff$$
$$(ka_1 + a_2)\vec{u_1} + a_2\vec{u_2} - a_1\vec{u_3} - (a_1 - a_2 - 1)\vec{u_4} = 0 \iff$$
$$\begin{bmatrix} k & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \textit{has solution}$$

The above set spans $\mathbb{R}^4$.

As before, we need to solve for $k = 0$ and when it doesn't.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 \textit{ is redundatn}} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_1 = R_2, R_2 = R_1} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_1}$$
$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In conclusion, whenever $k = 0$, we can represent $\vec{u_4}$ as a linear combination of $\vec{v_1}$ and $\vec{v_2}$.

$k \neq 0$ case:

$$\begin{bmatrix} k & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{R_4 = R_4 + R_3} \begin{bmatrix} k & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 = kR_3 + R_1} \begin{bmatrix} k & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_2 \textit{ is redundant}}$$
$$\begin{bmatrix} k & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{bmatrix} k & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, independent of k, we will always be able to represent $\vec{u_4}$ as a linear combination of vectors $\vec{v_1}$ and $\vec{v_2}$. Another way to see this is to notice that both $\vec{v_1}$ and $\vec{v_2}$ have a component from $\vec{u_4}$ and this component cannot be cancelled by any other vector, otherwise those other vectors wouldn't have formed a basis of $\mathbb{R}^n$.

1.5 Problem 5

Let $\vec{v}$, $\vec{u_1}, \dots, \vec{u_k}$ be vectors in $\mathbb{R}^n$. $\vec{v}$ has a unique representation as a linear combination of vectors $\vec{u_1}, \dots, \vec{u_k}$.

For questions (2) and (3) assume that for some $w \in \mathbb{R}$, $w = x_1\vec{u_1} + \cdots + x_k\vec{u_k}$ has no solutions.

1. Prove $\vec{u_1}, \dots, \vec{u_k}$ are linearly independent.
2. Prove $k < n$.
3. Prove $\{w, \vec{u_1}, \dots, \vec{u_k}\}$ is linearly independant.

1.5.1 Answer 11

Assume, for contradiction, $\vec{u_1}, \dots, \vec{u_k}$ are linearly dependent. Then, there exist some $\vec{u_n}$ s.t. for some $x_1\vec{u_1} + \cdots + x_k\vec{u_k} = \vec{u_n}$. Then, since $\vec{u_n}$ is used in the representation of $\vec{v}$, we can write this represenation in two distinct ways: one that involves $\vec{u_n}$ and the other one which doesn't. However, we are given the representation is unique.

Hence, by contradiction, $\vec{u_1}, \dots, \vec{u_k}$ are linearly independant.

1.5.2 Answer 12

Observe that k is at most n, otherwise $\vec{u_1}, \dots, \vec{u_k}$ would be linearly dependent. (We proved this in 1.5.1.)

Assume, for contradiction $k = n$, then $\vec{u_1}, \dots, \vec{u_k}$ spans $\mathbb{R}^n$, hence, every vector in $\mathbb{R}^n$ is representable as a linear combination of $\vec{u_1}, \dots, \vec{u_k}$. However, we are given that w is not representable as a linear combination of these vectors.

Hence, by contradition, $k < n$.

1.5.3 Answer 13

The proof is immediate from the definition. $w \neq x_1\vec{u_1} + \cdots + x_k\vec{u_k}$, hence w is not a linear combination of $\vec{u_1}, \dots, \vec{u_k}$, hence $\{w, \vec{u_1}, \dots, \vec{u_k}\}$ are linearly independent.