Assignment 14, Linear Algebra 1

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1 Problems

1.1 Problem 1

Given f, g, h are functions from $\mathbb{R}$ to $\mathbb{R}$, check that all of them are linearly independent when:

1. $f(x) = \sin x,\ g(x) = \cos x,\ h(x) = x \cos x.$
2. $f(x) = x(x-1),\ g(x) = x(x-2),\ h(x) = (x-1)(x-2).$
3. $f(x) = \sin^2 x,\ g(x) = \cos^2 x,\ h(x) = 3.$

1.1.1 Answer 1

Assuming interval is $(-\infty, \infty)$, using Wronskian determinant:

$$D = \begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ -\sin x & -\cos x & -\sin x - \sin x - x\cos x \end{vmatrix} = \begin{vmatrix} \sin x & \cos x & x\cos x \\ \cos x & -\sin x & \cos x - x\sin x \\ 0 & 0 & -2\sin x \end{vmatrix} =$$

$$-2\sin x \times \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -2\sin x(-\sin^2 x - \cos^2 x) = 2\sin x\ .$$

Since determinant depends on x, it is certanly not zero for all x, hence f, g, h are linearly independent.

1.1.2 Answer 2

Assuming interval is $(-\infty, \infty)$, using Wronskian determinant:

$$D = \begin{vmatrix} x^2 - x & x^2 - 2x & x^2 - 3x - 3 \\ 2x - 1 & 2x - 2 & 2x - 3 \\ 2 & 2 & 2 \end{vmatrix} =$$

$$2(x^2 - x)(2x - 2 - 2x + 3) - 4(x^2 - 2x)(2x - 1 - 2x - 2) - 2(x^2 - 3x - 3) =$$

$$2x^2 - 2x - 4x^2 + 8x - 2x^2 + 6x + 6 =$$

$$-4x^2 + 12x + 6\ .$$

Since determinant depends on x, it is certanly not zero for all x, hence f, g, h are linearly independent.

1.1.3 Answer 3

Assuming interval is $(-\infty, \infty)$, $a\sin^2(x) + b\cos^2(x) + 3c = 0$ has a solution when $a = b = 1$ and $c = -\frac{1}{3}$. Thus f, g, h are linearly dependent.

1.2 Problem 2

Given the following subsets of $\mathbb{R}^4$:

$$U = \{(x, y, z, t) \in \mathbb{R}^4 \mid x - y + z = 0 \land x - y - 2t = 0\}$$
$$W = \mathrm{Sp}\{(1, 0, 1, 1), (0, 1, 0, -1), (1, 0, 1, 0)\}$$

1. Prove that U and W are subspaces of $\mathbb{R}^4$.
2. Find basis for U, W and $U + W$.
3. Find basis for $U \cap W$.
4. Find subspace T of $\mathbb{R}^4$ s.t. $U \oplus T = \mathbb{R}^4$.

1.2.1 Answer 4

W is a subspace of $\mathbb{R}^4$ immediately from definition of span. U is a subspace because it can be represented by $\mathrm{Sp}\{(1, 0, 0, 0), (1, 1, -1, -2), (0, -1, 1, 2), (0, -\frac{1}{2}, \frac{1}{2}, 1)\}$ by noting that $x = y-2t$ and $z = 2t$, which is again, by definition, a subspace of $\mathbb{R}^4$.

1.2.2 Answer 5

Span of U is also its basis since all vectors in it are linearly independent. Span of W contains linearly dependent vectors, t is a multiple of z and both of them are multiples of linear combination of x and y, hence its basis has only two vectors, for example $\{(1, 0, 0, 0), (0, 1, -1, 2)\}$. We can find the basis of $U+W$ by adjoining both bases and performing Gaussian elimination:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is the standard basis for $\mathbb{R}^4$.

1.2.3 Answer 6

A generic vector $\vec{v} \in U \cap W$ must be representable as $\vec{v} = a(1,0,1,1) + b(0,1,0,-1) + c(1,0,1,0)$ and $\vec{v} = d(1,0,0,0) + e(0,1,-1,2)$. Equating both parts gives: $a(1,0,1,1) + b(0,1,0,-1) + c(1,0,1,0) - d(1,0,0,0) - e(0,1,-1,2) = 0$. Solving for (a,b,c,d,e) will give us the kernel space of $U \cap W$

$$\begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Since the only column that doesn't have a pivot element is the last one, the basis contains just one vector. That is any solution to $a(1,0,1,1) + b(0,1,0,-1) + c(1,0,1,0) - d(1,0,0,0) - e(0,1,-1,2) = 0$ will give us the basis of $U \cap W$.

$$\begin{aligned} 3(1,0,1,1) + 1(0,1,0,-1) - 4(1,0,1,0) + 1(1,0,0,0) - 1(0,1,-1,2) = 0 \\ \vec{v} = 3(1,0,1,1) + 1(0,1,0,-1) - 4(1,0,1,0) = \\ (3,0,3,3) + (0,1,0,-1) - (4,0,4,0) = (3-4, 1, 3-4, 3-1) = (-1,1,-1,2) \end{aligned}$$

Hence, the basis of $U \cap W = \{(-1,1,-1,2)\}$.

1.2.4 Answer 7

One way to find such subspace of $\mathbb{R}^4$ we can simply complete the two vectors of the basis of U with two vectors from the standard basis to form an invertible matrix like so:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is an upper-triangular matrix, therefor invertible, therefore its row space spans $\mathbb{R}^4$. Thus $U \oplus \{(0,0,1,0), (0,0,0,1)\} = \mathbb{R}^4$.

1.3 Problem 3

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ and $\vec{w}$ be vectors in linear space V. Given $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent and that $\vec{w} \notin \mathrm{Sp}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, prove that $\vec{v_1} \notin \mathrm{Sp}\{\vec{v}_1 + \vec{w}, \vec{v}_2 + \vec{w}, \ldots, \vec{v}_k + \vec{w}\}$.

1.3.1 Answer 8

Suppose, for contradiction, $\vec{v}_1 = a(\vec{v}_1 + \vec{w}) + b(\vec{v}_2 + \vec{w}) + \cdots + c(\vec{v}_k + \vec{w})$ for some constants $a, b \ldots c$. Then we have $(1-a)\vec{v}_1 - b\vec{v}_2 - \cdots - c\vec{v}_k = -(a + b + \cdots + c)\vec{w}$. Notice that this says that $\vec{w}$ is a linear combination of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, which is a contradiction to the given. Hence, $\vec{v}_1 \notin \mathrm{Sp}\{\vec{v}_1 + \vec{w}, \vec{v}_2 + \vec{w}, \ldots, \vec{v}_k + \vec{w}\}$.

1.4 Problem 4

Let U and W be distinct linear subspaces of $\mathbb{R}^4$ of dimension 3. Suppose $(2,1,0,1), (1,0,1,1) \in U \cap W$, what is the dimension of $U + W$?

1.4.1 Answer 9

Spans of both U and W have a vector that the other doesn't have (otherwise they would be the same subspace). Since the span of their intersection has two distinct vectors, it means that their sum must contain all linear combinations of the vectors in the span of intersection and the remaining distinct vector. This leaves us the only option of $\dim(U + W) = 4$.

1.5 Problem 5

Let A and B be square matrices of size n, $n \geq 2$. Suppose A and B are of rank 1,

1. what are the possible ranks of $A + B$?
2. What is the possible rank of $A + B$ when they both are of rank 2?
3. Prove that it is possible to write any matrix of rank 2 as a sum of two matrices of rank 1.

1.5.1 Answer 10

It can be either zero, one or two.

1. It can be zero when $A = -B$. Since this is the zero matrix, its rank is 0.
2. It can be one when, for example, $A = B$, since it would imply $A + B = 2A$ and rank is preserved under scalar multiplication.

3. For matrix to have rank equal to one it means that all of its non-zero values are concentrated in either the same row, or the same column. Whenever a matrix has two or more rows (or columns), where one of the rows (or columns) is the zero vector, it is possible to find another matrix, which has in the corresponding position its non-zero entries. Thus, the rank can be also 2, but no more than that, since another matrix will necessarily have at most one non-zero row.

1.5.2 Answer 11

Similarly, for matrices of rank 2, we can have rank at leas 0 or at most 4, or anything in between.

1. Zeroth rank will result from adding A and B s.t. $A = -B$.
2. Rank of one will result from matrices s.t. the entries of A are $a_{i,j}$, the entries of B are $b_{i,j}$ and $\forall(0 \geq i \geq n, 0 \geq j \geq n) : a_{i,j} = -b_{i,j}$, unless $i = k$ and $j = m$ (some constants). In other words, when all but one entries in these two matrices are additive inverses of each other except for one entry.
3. The rank will be two when, for example, $A = B$ for the same reason as above.
4. Example of rank three sum:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
5. Rank can similarly be four:
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
But it cannot be more than four since both A and B will have at most two non-zero rows (or columns), which is only enough to produce four pivot elements in the total. Since rank is equal to the dimension of column (or row) span of the matrix, and the dimesnion of these spaces is determined by the number of pivots, it is thus impossible to have rank higher than four when adding A and B.

1.5.3 Answer 12

Let C be arbitrary matrix of rank 2. Since rank is preserved under multiplication with fully-ranked matrix, we can perform Gaussian elimination on C to obtain a general-form matrix C'

lookin like this:

$$C' = \begin{bmatrix} 1 & \dots & \dots \\ 0 & 1 & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

Which, clearly, can be decomposed into two matrices of rank 1. From distributivity of matrix multiplication over addition we can show that by multiplying by elementary matrices we can restore the obtained sum to match C (the original matrix). In other words: $C \times c_1 E \times c_2 E \times \ldots \times c_n E = C'$, where Es are some elementary matrices and c_i are some constants, hence $(A + B) \times c_1 E \times c_2 E \times \ldots \times c_n E = C$, provided $A + B = C'$.

1.6 Problem 6

Given bases $B = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ and $C = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ both in $\mathbb{R}^3$ s.t.

$$\begin{aligned} \vec{u}_1 &= (2, 1, 1) \\ \vec{u}_2 &= (2, -1, 1) \\ \vec{u}_3 &= (1, 2, 1) \\ \vec{v}_1 &= (3, 1, -5) \\ \vec{v}_2 &= (1, 1, -3) \\ \vec{v}_3 &= (-1, 0, 2) \end{aligned}$$

1. Write the matrix of change of basis from B to C and its inverse.
2. Compute the coordinate vector $[\vec{w}]_B$ where $\vec{w} = (-5, 8, -5)$.
3. Similarly, compute $[\vec{w}]_C$.

1.6.1 Answer 13

The change of basis matrix from B to C can be found as follow:

1. Undo the transformation from the standard basis to basis B by multiplying the coordinates vector by B^{-1}.
2. Multiply by C to perform transformation.
3. Multiply by B to represent the resulting coordinates with respect to B.

Or, in other words, $T_{B\to C} = C^{-1}BC$:

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -5 & -3 & 2 \end{bmatrix}^{-1} \times \begin{bmatrix} 2 & 2 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -5 & -3 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -5 & -3 & 2 \end{bmatrix}^{-1} \times \begin{bmatrix} 6+1-1 & 6-1-1 & 3+2-1 \\ 2+1+0 & 2-1+0 & 1+2+0 \\ -10-3+2 & -10+3+2 & -5-6+3 \end{bmatrix} =$$

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -5 & -3 & 2 \end{bmatrix}^{-1} \times \begin{bmatrix} 6 & 4 & 4 \\ 3 & 1 & 3 \\ -11 & -5 & -8 \end{bmatrix}$$

Now, let's find C^{-1}:

$$\left[\begin{array}{cccccc} 3 & 1 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -5 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & -1 & 1 & -3 & 0 \\ 0 & 2 & 2 & 0 & 5 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right]$$

finally:

$$\begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 0 \\ -5 & -3 & 2 \end{bmatrix}^{-1} \times \begin{bmatrix} 6 & 4 & 4 \\ 3 & 1 & 3 \\ -11 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 6 & 4 & 4 \\ 3 & 1 & 3 \\ -11 & -5 & -8 \end{bmatrix} =$$

$$\begin{bmatrix} 6-\frac{3}{2}-\frac{11}{2} & 4-\frac{1}{2}-\frac{5}{2} & 4-\frac{3}{2}-4 \\ -6+\frac{3}{2}+\frac{11}{2} & -4+\frac{1}{2}+\frac{5}{2} & -4+\frac{3}{2}+4 \\ 6+6-11 & 4+2-5 & 4+6-8 \end{bmatrix} = \begin{bmatrix} -1 & 1 & \frac{3}{2} \\ 1 & 1 & \frac{3}{2} \\ 1 & 1 & 2 \end{bmatrix}$$

1.6.2 Answer 14

$[\vec{w}]_B$ is just $B^{-1}w$. First, compute B^{-1}:

$$\begin{bmatrix} 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -1 & 0 & 2 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -1 & 0 & 2 \end{bmatrix}$$

$$\textit{Hence } B^{-1} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -1 & 0 & 2 \end{bmatrix}$$

Thus, $[\vec{w}]_B$ can be computed via:

$$\begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{5}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -1 & 0 & 2 \end{bmatrix} \times \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{15}{2} + 4 + \frac{25}{2} \\ \frac{5}{2} - 4 - \frac{15}{2} \\ 5 - 10 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ -5 \end{bmatrix}$$

1.6.3 Answer 15

Similarly, $[\vec{w}]_C$ is just $C^{-1}w$ (we've already computed C^{-1} in the previous answer.)

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix} = \begin{bmatrix} -5 - 4 - \frac{5}{2} \\ 5 + 4 + \frac{5}{2} \\ -5 + 16 - 5 \end{bmatrix} = \begin{bmatrix} -\frac{23}{2} \\ \frac{23}{2} \\ 6 \end{bmatrix}$$