Assignment 12, Linear Algebra 1

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1 Problems

1.1 Problem 1

Given two systems of linear equations O homogenous and M nonhomogenous.

$$\left.\begin{aligned} ax+by+cz &= 0 \\ fx+gy+hz &= 0 \end{aligned}\right\} = O \quad \left.\begin{aligned} ax+by+cz &= d \\ fx+gy+hz &= k \end{aligned}\right\} = M$$

$(1,0,1)$ and $(-1,1,1)$ are known to be the solutions of O and $(2,-3,1)$ is a solution of M.

Find general solution for O.

1.1.1 Answer 1

1.2 Problem 2

Given matrices

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = B \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & 3 & -3 \end{bmatrix} = A$$

1. Use elementary operations to obtain C such that $CC^{-1} = I$ and $B = CA$.
2. Write C as a product of elementary matrices.

1.2.1 Answer 2

It is easier to start answering from (2). Since we can see that all we need is to find two elementary matrices, one such that it would reduce the second row by twice the first row and one that would diminish the third row by $\frac{1}{3}$, we can readily represent these operations as multiplication of two elementary matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = C_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = C_2$$

The product of $C_1 \times C_2 = C$ gives us the matrix C we were asked to find.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Now, let's find C^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2=R_2+2R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3=3R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right]$$

Finally $CC^{-1} = I$:

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.3 Problem 3

Let A be a square matrix of the n-th order. Assume $A^2 + A + I = 0$ holds.

1. Prove that A is invertible and that $A^2 = A^{-1}$.
2. Prove $A^2 - A + I = 0$ is invertible.

Let A and B be of the order of $n \times n$. Prove that if $AB^2 - A$ is invertible, so is $AB - A$.

1.3.1 Answer 3

First, show that A is invertible:

$$\begin{aligned}
A^2 + A + I = 0 &\iff && \text{given} \\
AA + A = -I &\iff && \text{by simple algebra} \\
A(A + I) = -I &\iff && \text{by associativity} \\
A(-(A + I)) = I &\implies && \text{multiplication by scalar} \\
& && A \text{ is invertible}
\end{aligned}$$

Then, suppose A^2 is the inverse of A, it must be then:

$$\begin{aligned}
A^{-1} + A + I = 0 &\iff && \text{by assumption } A^2 = A^{-1} \\
A^{-1} + A = -I &\iff && \text{move } I \text{ to the right} \\
AA^{-1} + AA = A(-I) &\iff && \text{multiply both sides by } A \\
I + A^2 = -A &\iff && \text{simplifying} \\
I + A^2 + A = 0 &\iff && \text{move } -A \text{ to the left} \\
A^2 + A + I = 0 & && \text{completes the proof}
\end{aligned}$$

1.3.2 Answer 4

Now, we show that $A^2 - A + I$ is invertible. First, observe that $A = (A^{-1})^2$ (since we already proven $A^2 = A^{-1}$, for complete proof, see below). Next, we'll use the $A^2 + A + I = 0$ to write the following equation: $A^2 - A + I = -2A$. This reduces the proof to proving that A is invertible, but it is because A^2 is, for extended proof see below.

$$\begin{aligned}
A^2 B = I &\iff && \text{by definition of invertibility} \\
(AA)B = I &\iff && \text{by elementary algebra} \\
A(AB) = I &\iff && \text{by associativity} \\
& && A \text{ is invertible}
\end{aligned}$$

$$\begin{aligned}
A^2 = A^{-1} &\iff && \text{proven earlier} \\
(AA)^{-1} = A^{-1}A^{-1} &\iff && \text{by definition of invertibility} \\
(A^{-1})^{-1} = A^{-1}A^{-1} &\iff && \text{since } AA = A^2 = A^{-1} \\
A = A^{-1}A^{-1} & && \text{inverse of inverse}
\end{aligned}$$

1.3.3 Answer 5

We can represent the matrix we know to be invertible as a product of subtraction and addition (by distributivity of multiplication over addition). We are also guaranteed to have a C s.t. $A^2C - A = I$, thus:

$$\begin{aligned}
(AB^2 - A)C = I &\iff && \text{by definition of invertibility} \\
A(B^2 - I)C = I &\iff && \text{by distributivity of multiplication} \\
A(B - I)(B + I)C = I &\iff && \text{difference of squares } II = I \\
A(B - I) &\implies && (B - I) \text{ and } A \text{ are invertible} \\
(B - I)A &\implies && \text{is invertible (product of invertible matrices)} \\
BA - A &\implies && \text{is invertible}
\end{aligned}$$

1.4 Problem 4

Given $p(x) = a_k x^k + a_{k-1}x^{k-1} + ... + a_1 x + a_0$ is a polynomial and A is an $n \times n$ matrix. We will denote $p(A)$ the matrix $p(A) = a_k A^k + a_{k-1}A^{k-1} + ... + a_1 A + I_n$. Given $p(A) = 0$ and $p(0) \neq 0$.

1. Prove that A is invertible.
2. Prove that $g(A^{-1}) = 0$ when $g(x) = a_0 x^k + a_1 x^{k-1} + ... + a_{k-1}x + a_k$.

1.4.1 Answer 6

$p(0) \neq 0$ means that the last polynomial term isn't zero (which is even more obvious if we look at $p(A)$, where the last term is the identity matrix of the same shape as A. Once we know that the sum of polynomial terms, with the last term omitted amounts to the additive inverse of identity matrix, i.e. the $-I$, we obtain that the sum of other polynomial terms must produce $-I$, which is itself ivertible.

Next, we can employ the distributivity of multiplication over addition and rewrite the equation as:

$$(a_k A^{k-1} + a_{k-1}A^{k-2} + ... + a_1)A = -I_n$$

Since A is a factor that gives, multiplied by some other matrix an identity matrix, it is invertible (by definition of invertibility $BA = I$).

1.4.2 Answer 7

Define A^{-1} to be $-(a_k A^{k-1} + a_{k-1}A^{k-2} + ... + a_1)^{-1}$. (This immediately follows from 1.4.1.) Then observe that $g(x) - a_1$ multiplied with this term gives us $-I$. Since we already established $a_1 = I$, we obtain $-I + I = 0$, hence $g(A^{-1}) = 0$.

1.5 Problem 5

Calculate these determinants:

$$D_1 = \begin{vmatrix} a & b & 0 & \dots & \dots & 0 \\ 0 & a & b & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & a & b \\ b & 0 & 0 & \dots & 0 & a \end{vmatrix}$$

$$D_2 = \begin{vmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 2 & 3 & 4 & \dots & n-1 & n & n \\ 3 & 4 & 5 & \dots & n & n & n \\ \vdots & & & & & & \vdots \\ n-2 & n-1 & n & \dots & n & n & n \\ n-1 & n & n & \dots & n & n & n \\ n & n & n & \dots & n & n & n \end{vmatrix}$$

1.5.1 Answer 8

D_1 is the sum of two determinants, one of the identity matrix mutiplied by a, and the other is the full permutation matrix, multiplied by b, which has the same determinant as the identity matrix. Hence $D_1 = a^n + b^n$.

1.5.2 Answer 9

D_2 is zero for $n > 2$ since those matrices are singular. In order to get convinced they are singular, notice that when reducing such matrice to the row echelon form, the third row will always be the linear combination of the first and the second rows. By subtracting a multiple of the first row from the second, we obtain its two's-complement (i.e. the values of

R_2, which I'll denote $r_{2,i}$) will be calculated as $r_{2,i} = -(r_{1,i} - 1)$. The third row then will be $r_{3,i} = 3r_{1,i} + 2r_{2,i}$. Direct calculation of determinants for $n = 2$ gives us, using formula $D(A_{2\times 2}) = ad - bc$, $1 \times 3 - 2 \times 2 = -1$.