Assignment 13, Linear Algebra 1

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Contents

1 Problems

1.1 Problem 1

1. Find all solutions of $z^3 + 3i\overline{z} = 0$.

2. Let z_1, z_2 be complex numbers. Prove that unless $z_1 z_2 = 1$ and $|z_1| = |z_2| = 1$, then $\frac{z_1+z_2}{1+z_1 z_2}$ is a real number.

1.1.1 Answer 1

First, note that zero is a solution of this equation. Other roots can be found as follows:

$$z = r \operatorname{cis} \theta$$

$$\begin{aligned}
r^3 \operatorname{cis}(3\theta) + 3i \operatorname{cis}(-\theta) = 0 &\iff \\
r^3 \operatorname{cis}(3\theta) + 3(i \cos(-\theta) + i^2 \sin(-\theta)) = 0 &\iff \\
r^3 \operatorname{cis}(3\theta) + 3(i \sin(\theta) + \cos(\theta)) = 0 &\iff \\
r^3 \operatorname{cis}(3\theta) + 3 \operatorname{cis}(\theta) = 0 &\iff \\
r^3 \operatorname{cis}(3\theta) = -3 \operatorname{cis}(\theta)& \\
\textit{Equating radius and angle:}& \\
r^3 = -3 &\iff \\
r = -3^{\frac{1}{3}}& \\
3\theta = \theta \mod 2\pi &\iff \\
2\theta = 0 \mod 2\pi &\iff \\
\theta = \pi \mod 2\pi &.
\end{aligned}$$

Hence, other roots are:

$$\begin{aligned}
z_1 &= -3^{\frac{1}{3}} \operatorname{cis}(0) \\
z_2 &= -3^{\frac{1}{3}} \operatorname{cis}(\pi)
\end{aligned}$$

1.1.2 Answer 3

Since $|z_1|^2 = z_1\overline{z_1} = 1$, we have that $z_1 = \overline{z_1} = \frac{1}{z_1}$, and similarly $z_2 = \overline{z_2} = \frac{1}{z_2}$. Further using the properties of complex conjugate we have:

$$\begin{aligned}\frac{z_1 + z_2}{1 + z_1 z_2} &= \frac{\frac{1}{z_1} + \frac{1}{z_2}}{1 + \frac{1}{z_1 z_2}} \\ &= \frac{\overline{z_1} + \overline{z_2}}{1 + \overline{z_1 z_2}} \\ &= \overline{\frac{z_1 + z_2}{1 + z_1 z_2}}\end{aligned}$$

And since $z = \overline{z} \implies z \in \mathbb{R}$, we have that the given expression is real.

1.2 Problem 2

Let $\mathbb{Q}$ denote the field of rational numbers. And K defined as follows:

$$K = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \mid a, b \in \mathbb{Q} \right\}.$$

Is K a field under matrix addition and multiplication?

1.2.1 Answer 3

Yes, K is a field, following is the illustration of field axioms satisfied by K.

1. Closure under addition:

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} + \begin{bmatrix} c & 2d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & 2(b+d) \\ b+d & a+c \end{bmatrix}$$

where $(a+c) \in \mathbb{Q} = e$, $(b+d) \in \mathbb{Q} = f$, results in a general matrix:

$$\begin{bmatrix} e & 2f \\ f & e \end{bmatrix} \in K$$

2. Closure under multiplication:

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \times \begin{bmatrix} c & 2d \\ d & c \end{bmatrix} = \begin{bmatrix} ac + 2db & 2ad + 2bc \\ bc + ad & 2db + ac \end{bmatrix}$$

where $(ac + 2db) \in \mathbb{Q} = e$ and $2(ad + 2bc) \in \mathbb{Q} = f$ and, similarly:

$$\begin{bmatrix} e & 2f \\ f & e \end{bmatrix} \in K$$

3. Associativity of addition:

$$\left(\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} + \begin{bmatrix} c & 2d \\ d & c \end{bmatrix}\right) + \begin{bmatrix} e & 2f \\ f & e \end{bmatrix} = \begin{bmatrix} a + c & 2(b + d) \\ b + d & a + c \end{bmatrix} + \begin{bmatrix} e & 2f \\ f & e \end{bmatrix} =$$

$$\begin{bmatrix} a + c + e & 2(b + d + f) \\ b + d + f & a + c + e \end{bmatrix} =$$

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} + \begin{bmatrix} c + e & 2(d + f) \\ d + f & c + e \end{bmatrix} = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} + \left(\begin{bmatrix} c & 2d \\ d & c \end{bmatrix} + \begin{bmatrix} e & 2f \\ f & e \end{bmatrix}\right)$$

4. Associativity of multiplication:

$$\left(\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \times \begin{bmatrix} c & 2d \\ d & c \end{bmatrix}\right) \times \begin{bmatrix} e & 2f \\ f & e \end{bmatrix} = \begin{bmatrix} ac + 2db & 2ad + 2bc \\ bc + ad & 2db + ac \end{bmatrix} \times \begin{bmatrix} c & 2d \\ d & c \end{bmatrix} =$$

$$\begin{bmatrix} e(ac + 2db) + f(2ad + 2bc) & 2f(ad + 2db) + e(2ad + 2bc) \\ e(bc + ad) + f(2db + ac) & 2f(bc + ad) + e(2db + ac) \end{bmatrix} =$$

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \times \begin{bmatrix} ec + 2df & 2fc + 2ed \\ ed + fc & 2fd + ec \end{bmatrix} = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \times \left(\begin{bmatrix} c & 2d \\ d & c \end{bmatrix} \times \begin{bmatrix} e & 2f \\ f & e \end{bmatrix}\right)$$

5. Commutativity of addition:

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} + \begin{bmatrix} c & 2d \\ d & c \end{bmatrix} = \begin{bmatrix} a + c & 2(b + d) \\ b + d & a + c \end{bmatrix} = \begin{bmatrix} c & 2d \\ d & c \end{bmatrix} + \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$$

6. Commutativity of multiplication:

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \times \begin{bmatrix} c & 2d \\ d & c \end{bmatrix} = \begin{bmatrix} ac + 2db & 2ad + 2bc \\ bc + ad & 2db + ac \end{bmatrix} = \begin{bmatrix} c & 2d \\ d & c \end{bmatrix} \times \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}$$

7. Additive identity is the zero matrix (from matrix addition properties).

8. Multiplicative identity is the identity matrix (again, from matrix multiplication properties).

9. General inverse under addition:

$$\left(\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}\right)^{-1} = \begin{bmatrix} c & 2d \\ d & c \end{bmatrix}$$

$$where\ c + a = 0, d + b = 0$$

$$i.e.\ c = -a, d = -b$$

$$\left(\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}\right)^{-1} = \begin{bmatrix} -a & -2b \\ -b & -a \end{bmatrix}$$

10. General inverse under multiplication. First, we will find the determinant of a generic matrix in K:

$$D = \begin{vmatrix} a & 2b \\ b & a \end{vmatrix} = aa - 2bb = a^2 - 2b^2$$

Since 2 appears without a pair in the expression $2b^2$, it means that the prime factorization of this expression contains an odd number of twos. Hence, it is not possible for a^2 to be equal to $2b^2$, unless both $a = 0$ and $b = 0$. Hence, the only element of K which doesn't have an inverse is the zero matrix. For every other element its inverse is:

$$\left(\begin{bmatrix} a & 2b \\ b & a \end{bmatrix}\right)^{-1} = \frac{1}{D}\begin{bmatrix} a & -2b \\ -b & a \end{bmatrix} = \frac{1}{a^2 - 2b^2}\begin{bmatrix} a & -2b \\ -b & a \end{bmatrix} = \begin{bmatrix} \frac{a}{a^2-2b^2} & \frac{-2b}{a^2-2b^2} \\ \frac{-b}{a^2-2b^2} & \frac{a}{a^2-2b^2} \end{bmatrix}$$

As before, $\frac{a}{a^2+2b^2} \in \mathbb{Q} = e$ and $\frac{-b}{a^2+2b^2} \in \mathbb{Q} = f$, hence:

$$\begin{bmatrix} e & 2f \\ f & e \end{bmatrix} \in K$$

11. Finally, distributivity of multiplication over addition:

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \times \left(\begin{bmatrix} c & 2d \\ d & c \end{bmatrix} + \begin{bmatrix} e & 2f \\ f & e \end{bmatrix}\right) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \times \begin{bmatrix} c+e & 2(d+f) \\ d+f & c+e \end{bmatrix} =$$

$$\begin{bmatrix} a(c+e) + 2b(d+f) & 2b(c+e) + 2a(d+f) \\ a(d+f) + b(c+e) & 2b(d+f) + a(c+e) \end{bmatrix} =$$

$$\begin{bmatrix} ac + 2bd & 2ad + 2bc \\ bc + ad & 2bd + ac \end{bmatrix} + \begin{bmatrix} ae + 2bf & 2af + 2be \\ be + af & 2bf + ae \end{bmatrix} =$$

$$\begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \times \begin{bmatrix} c & 2d \\ d & c \end{bmatrix} + \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \times \begin{bmatrix} e & 2f \\ f & e \end{bmatrix}$$

1.3 Problem 3

Verify that V is a vectors space over field $\mathbf{F}$:

1. $\mathbf{F} = \mathbb{C}, V = \mathbb{M}^{\mathbb{C}}_{n\times n}$ and addition defined to be the regular addition, while multiplication is defined to be $\lambda \times A = |\lambda| \times A$.
2. $\mathbf{F} = \mathbb{R}, V = \{(x, y) \in \mathbb{R}^2 \mid y = 2x\}$, with addition being the addition of $\mathbb{R}^2$ and multiplication $\lambda \times (x, y) = (\lambda x, 0)$.

1.3.1 Answer 5

1. Distributivity of scalar addition prevents V from being a field over $\mathbf{F}$. Consider this example:

$$|-1+i| \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + |1-i| \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\sqrt{(-1)^2 + 1^2} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sqrt{1^2 + (-1)^2} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix}$$

   while, at the same time:

$$\big|(-1+i) + (1-i)\big| \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sqrt{0^2 + 0^2} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

   Obviously,

$$\begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & 2\sqrt{2} \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. Of course V is not a vector space over $\mathbf{F}$, almost none of scalar multiples are in V, since they are of the form (x, y), where $y = 0$ and $y = 2x$, but this is only true when $x = 0$ as well. Any other value of x will not satisfy closure under multiplication requirement.

1.4 Problem 4

Given sets:

1. $K = \{(x, y, z, t) \in \mathbb{R}^4 \mid x + y - z + t = 0 \wedge 2x + y + z - 3t = 0\}$.

2. $L = \{f \in V \mid f\left(\frac{1}{2}\right) > f(2)\}$, V is the vector space of all real-valued functions.
3. $M = \{p(x) \in \mathbb{R}^4[x] \mid p(-3) = 0\}$.
4. $R = \{(x, y) \in \mathbb{R}^3 \mid x^2 + y^2 = 0\}$.
5. $R = \{(x, y) \in \mathbb{R}^3 \mid x^2 - y^2 = 0\}$.

Fore each of sets given, assert them being vector spaces. In case they are, prove this in two different ways.

1.4.1 Answer 6

1. By substitution find that $x = 4t-2z$, $y = 3z-5t$. This gives us vectors $\vec{v}_1 = (4, -5, 0, 1)^T$ and $\vec{v}_2 = (-2, 3, 1, 0)^T$ which span K. In other words, K is a vector space over the field of real numbers with the operations of vector addition and multiplication.
2. L is not a vector space. For example, it doesn't have additive inverse. Suppose for contradiction that there exists additive inverse in L, then $f(x)+f'(x) = 0$, in particular, $f(\frac{1}{2}) + f'(\frac{1}{2}) = 0$ and $f(2) + f'(2) = 0$. We know that $f(\frac{1}{2}) > f(2)$. Let $f(\frac{1}{2}) = n$ and $f(2) = m$. Then $f'(\frac{1}{2}) = -n$ and $f'(2) = -m$. However, if $n > m$, then $-n < -m$. Contradiction. Hence, L is not a vector space.
3. M is a vector space with the span $P = \text{Sp}\{(1, 0, 0, (-3)^3), (0, 1, 0, (-3)^2), (0, 0, 1, (-3)^1), (0, 0, 0, 0)\}$. To see why these vectors span M suppose there was a polinomial $p(x)$ s.t. $p(-3) = 0$, but it is not a linear combination of P. However, $p(x)$ must be representable as follows $(\alpha(-3)^3 - c_\alpha) + (\beta(-3)^2 - c_\beta) + (\gamma(-3)^1 - c_\gamma) = 0$, with c_i some constants. Now, note that each of the summands individually can be represented by the elements of P, hence, contrary to assumed, $p(x)$ is a linear combinaton of P. Hence, P spans M.
4. R is a vector space, if you allow vector spaces with just one element: the condition $x^2 + y^2 = 0$ in real numbers can only be satisfied when $x = y = 0$, since squares of real numbers are non-negative. This space would be the $\text{Sp}\{(0, 0)\}$.
5. S is a vector space defined by $\text{Sp}\{(1, 1), (0, 0)\}$, it is equivalent to just the real numbers.

1.5 Problem 5

Given vector space V and $\vec{v}_1, \vec{v}_2, \vec{v}_3$ distinct vectors prove or disprove:

1. If $\text{Sp}\{\vec{v}_1, \vec{v}_2\} = \text{Sp}\{\vec{v}_1, \vec{v}_3\}$, then $\vec{v}_2$ is a multiple of $\vec{v}_3$.
2. If $\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 = \vec{0}$, then $\text{Sp}\{\vec{v}_1, \vec{v}_2\} = \text{Sp}\{\vec{v}_1, \vec{v}_3\}$.
3. If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, then $\text{Sp}\{\vec{v}_1, \vec{v}_2\} = \text{Sp}\{\vec{v}_1 + \vec{v}_3, \vec{v}_2 + \vec{v}_3\}$.

1.5.1 Answer 7

1. False, counterexample: $\vec{v}_1 = (1,0)^T$, $\vec{v}_2 = (1,1)^T$, $\vec{v}_3 = (0,1)^T$, but there doesn't exist λ s.t. $\lambda\vec{v}_2 = \vec{v}_3$.

2. True, take any vector generated by the first span:

$$
\begin{gathered}
\vec{x} = \alpha(2\vec{v}_2 - \vec{v}_3) + \beta\vec{v}_2 \\
\vec{y} = \gamma(2\vec{v}_2 - \vec{v}_3) + \delta\vec{v}_3 \\
\textit{Group the coefficients:} \\
\vec{x} = (2\alpha + \beta)\vec{v}_2 - \alpha\vec{v}_3 \\
\vec{y} = 2\gamma\vec{v}_2 - (\delta - \gamma)\vec{v}_3
\end{gathered}
$$

   Since both $\vec{x}$ and $\vec{y}$ are linear combinations of $\vec{v}_2$ and $\vec{v}_3$, they are in the same span. Hence $\mathrm{Sp}\{\vec{v}_1, \vec{v}_2\} = \mathrm{Sp}\{\vec{v}_1, \vec{v}_3\}$.

3. False, counterexample: $\vec{v}_1 = (0,0)^T$, $\vec{v}_2 = (1,0)^T$, $\vec{v}_3 = (0,1)^T$, but $\dim \mathrm{Sp}\{\vec{v}_1, \vec{v}_2\} = 1$ and $\dim \mathrm{Sp}\{\vec{v}_2, \vec{v}_3\} = 2$.

1.6 Problem 6

Given following subspaces of $\mathbb{R}^3$: $U = \mathrm{Sp}\{(1,1,2),(2,2,1)\}$ and $W = \mathrm{Sp}\{(1,3,4),(2,5,1)\}$, find spanning set of $U \cap W$.

1.6.1 Answer 8

Using linear space sum dimension theorem: $\dim(W+U) = \dim W + \dim U - \dim(W \cap U)$ we have that $\dim(W \cap U) = \dim W + \dim U - \dim(W+U)$. Now, let's find the summands:

$$\begin{aligned}
\dim U &= \dim\left(\begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\right) \\
&= \dim\left(\begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}\right) \\
&= 2\,. \\
\dim W &= \dim\left(\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 1 \end{bmatrix}\right) \\
&= \dim\left(\begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -7 \end{bmatrix}\right) \\
&= 2\,. \\
\dim(W+U) &= \dim\left(\begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -7 \\ 1 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}\right) \\
&= \dim\left(\begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -7 \\ 0 & -2 & 2 \\ 0 & 0 & -3 \end{bmatrix}\right) \\
&= \dim\left(\begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & 16 \\ 0 & 0 & -3 \end{bmatrix}\right) \\
&= \dim\left(\begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -7 \\ 0 & 0 & 16 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&= 3\,.
\end{aligned}$$

Imporatnt observation here is that the number of pivot elements in reduced echelon form is the dimension of the matrix.

Hence, $\dim(W \cap U) = 2 + 2 - 3 = 1$, or, in other words, U and W share one common axis.