

Assignment 12, Linear Algebra 1

Oleg Sivokon

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1 Problems

1.1 Problem 1

Let A be a square matrix of order 3×3 s.t. $A^3 = 0$, but $A^2 \neq 0$.

1. Prove that there exists a vector $\vec{v} \in \mathbb{R}^3$ s.t. $A\vec{v} \neq 0$.
2. Prove that there exists vector $\vec{v} \in \mathbb{R}^3$ s.t. $\{\vec{v}, \vec{v}A, \vec{v}A^2\}$ are the basis of \mathbb{R}^3 .

1.1.1 Answer 1

Notice that A itself is made of the column vectors, call them c_1, c_2, c_3 . All of which are in \mathbb{R}^3 . Suppose, for contradiction, that there is no vector in $\vec{v} \in \mathbb{R}^3$ satisfying $A\vec{v} \neq 0$. In particular, none of the $\vec{c}_1, \vec{c}_2, \vec{c}_3$ satisfies the above condition. In other words, $A\vec{c}_1 = 0, A\vec{c}_2 = 0, A\vec{c}_3 = 0$. (where 0 means zero matrix). On the other hand, a matrix $(A\vec{c}_1, A\vec{c}_2, A\vec{c}_3) = A^2 \neq 0$. Contradiction. Hence, there exists $\vec{v} \in \mathbb{R}^3$ s.t. $A\vec{v} \neq 0$.

1.1.2 Answer 2

Suppose for contradiction there was no such vector \vec{v} . This would in turn imply that all vectors $\vec{x} \in \mathbb{R}^3$ would be sent by $A^2(\vec{x})$ to its kernel, i.e. $\forall \vec{x} \in \mathbb{R}^3 : A^2(\vec{x}) = [0]$. But we just proved the opposite in the previous answer. Hence, by contradiction, there must exist \vec{v} s.t. $\{\vec{v}, A\vec{v}, A^2\vec{v}\}$ is the span (and hence the basis) of \mathbb{R}^3 .

1.2 Problem 2

Given square matrices A, B, C, D of order $n \times n$ s.t. $ABCD = I$, prove that $ABCD = DABC = CDAB = BCDA = I$.

1.2.1 Answer 3

The proof is immediate from the definition of inverse: $XX^{-1} = I$ and associativity of matrix multiplication. In other words:

$$\begin{aligned}
ABCD &= I \iff \\
A(BCD) &= I \iff \\
A^{-1} &= BCD \iff \\
DABC &= I \iff \\
AB(CD) &= I \iff \\
(AB)^{-1} &= CD \iff \\
CDAB &= I \iff \\
A^{-1}A &= I \iff \\
I &= BCDA.
\end{aligned}$$

1.3 Problem 3

Let A be a square matrix of order $m \times m$, let B be a matrix of order $m \times n$. Prove in two different ways that if A is invertible, then homogeneous systems $B\vec{x} = 0$ and $AB\vec{x} = 0$ have the same solution space.

1.3.1 Answer 4

1. Since A is invertible, we can write:

$$\begin{aligned}
Bx &= \vec{0} \iff \\
ABx &= A\vec{0} \iff \\
ABx &= \vec{0} \\
&\text{Alternatively:} \\
ABx &= \vec{0} \iff \\
A^{-1}ABx &= A^{-1}\vec{0} \iff \\
Bx &= \vec{0}.
\end{aligned}$$

2. Alternatively, let $T_A(x) = Ax$ and $T_B(x) = Bx$ be linear transformations. Since $T_A(x)$ is one-to-one and onto (due to A being invertible), then $(T_A \circ T_B)(x)$ has the same kernel space as $T_B(x)$. In other words, $ABx = Bx$ when $Bx = 0$.

1.4 Problem 4

Given matrix A of a general form:

$$\begin{bmatrix} 0 & a_1 & \dots & 0 & 0 \\ 0 & 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n-1} \\ a_n & 0 & 0 & \dots & 0 \end{bmatrix}$$

Prove that it is invertible, show A^{-1} .

1.4.1 Answer 5

Performing elementary operations: $R_n \rightarrow R_1$ and $R_k \rightarrow R_{k+1}, 1 \leq k < n$ gives us diagonal matrix. This matrix is invertible since it has a pivot element in each of its columns.

The inverse of A will, in general look like this:

$$\begin{bmatrix} 0 & 0 & \dots & 0 & \frac{1}{a_n} \\ \frac{1}{a_1} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{a_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{a_{n-1}} & 0 \end{bmatrix}$$

Notice that for each row of A , we will be matching the column of A^{-1} . We need to make sure that the only non-zero element of A_c was matched by the only non-zero element of A_r^{-1} (where c stands for column index and r stands for row index). In order to obtain a diagonal with all ones (i.e. the identity matrix), we need to also make sure that $A_{c,i} \times A_{r,j}^{-1} = 1$. In other words, we need to match a_1 with $\frac{1}{a_1}$, a_2 with $\frac{1}{a_2}$, and so on.

1.5 Problem 5

Let A and B be square matrices of the order 3×3 s.t. $B^2A = -2B^3$ and $B^3 + AB^2 = 3I$.

Prove that A and B are invertible and express A^{-1} and B^{-1} in terms of B .

1.5.1 Answer 6

Using some matrix algebra we obtain: $B^{-1} = -\frac{1}{3}B^2$ and $A^{-1} = (-2B)^{-1}$.

$$\begin{aligned} B^2A &= -2B^3 \iff \\ B^2A &= B^2(-2I)B \iff \\ A &= -2B \end{aligned}$$

substituting into second equation:

$$\begin{aligned} B^3 + AB^2 &= 3I \iff \\ B^3 - 2B^3 &= 3I \iff \\ -B^3 &= 3I \iff \\ B(-B^2) &= 3I \iff \\ B(-\frac{1}{3}B^2) &= I \iff \\ B^{-1} &= -\frac{1}{3}B^2 \end{aligned}$$

A is invertible because it is similar to B

$$\begin{aligned} -2B &= (\sqrt{2}I)B(\sqrt{2}I^{-1}) \\ A^{-1} &= (-2B)^{-1}. \end{aligned}$$

1.6 Problem 6

1. Compute the determinant:

$$D = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 2 \\ 1 & 1 & \dots & 1 & 3 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & n-1 & \dots & 1 & 1 \\ 1 & n & 1 & \dots & 1 & 1 \end{vmatrix}$$

2. Prove that given:

$$\Delta = \begin{vmatrix} a_1 & a_1^2 & \dots & a_1^{n-1} & 1 + a_1^n \\ a_2 & a_2^2 & \dots & a_2^{n-1} & 1 + a_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1} & a_{n-1}^2 & \dots & a_{n-1}^{n-1} & 1 + a_{n-1}^n \\ a_n & a_n^2 & \dots & a_n^{n-1} & 1 + a_n^n \end{vmatrix}$$

and

$$\Delta_1 = \begin{vmatrix} a_1 & a_1^2 & \dots & a_1^{n-1} & 1 \\ a_2 & a_2^2 & \dots & a_2^{n-1} & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1} & a_{n-1}^2 & \dots & a_{n-1}^{n-1} & 1 \\ a_n & a_n^2 & \dots & a_n^{n-1} & 1 \end{vmatrix}$$

where $a_1, a_2, \dots, a_n \in \mathbb{R}$. When $\Delta = 0$ and $\Delta_1 \neq 0$, $\prod_{i=1}^n a_i = -1$.

1.6.1 Answer 7

Firs, notice that we can row-reduce this matrix without affecting the value of the determinant by subtracting the top row from other rows. Once this is done, we will swap rows in order to bring this matrix to upper-diagonal form. To do this, we will need to swap $\frac{n}{2}$ times if n is even and $\frac{n-1}{2}$ times when n is odd. Thus, we can compute D using the formula below:

$$D = \begin{cases} n!, & \text{if } \lfloor \frac{n}{2} \rfloor \equiv 0 \pmod{2} \\ -1 \times n!, & \text{if } \lfloor \frac{n}{2} \rfloor \equiv 1 \pmod{2} \end{cases}$$

Note that we get $n!$ term by multiplying the diagonal entries, the -1 appears due to the odd number of row swaps.

1.6.2 Answer 8

Let $\Delta = \det(A)$, $\Delta_1 = \det(B)$, $C = (A_0, A_1, \dots, [a_1^n, a_2^n, \dots, a_n^n])$ s.t. $|C| + |B| = |A|$. Or, in other words, $|C| = -|B| = (-1)^n |B|$. However, since we know that the determinant of C is a multiple of determinant of B , and they differ in one row, we can conclude that these rows are multiples of each other. In other words, $(-1)^n [1, 1, \dots, 1] = [a_1^n, a_2^n, \dots, a_n^n]$. Hence $\prod_{i=1}^n a_i = (-1)^n 1 = (-1)^n$.

1.7 Problem 7

Given antisymmetric matrix $A \in \mathbb{M}^{3 \times 3}$ and any matrix $B \in \mathbb{M}^{3 \times 3}$,

1. prove $(A^2 B) \vec{x} = \vec{0}$ has non-trivial solution.
2. Assume B is antisymmetric, and both A and B are non-zero, is $(A + B)^2$ symmetric, invertible?

1.7.1 Answer 9

Any antisymmetric matrix of odd degree is not invertible since $\det(A) = \det(A^T) = \det(-A^T) = (-1)^n \det(A^T)$. Thus, in particular, A^2B is not invertible (product of invertible and non-invertible matrix is not invertible). Non-invertible means free variable, hence infinitely many solutions.

1.7.2 Answer 10

The matrix $(A - B)^2$ is not in general invertible or symmetric, or antisymmetric. For example, let:

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

then:

$$(A - B)^2 = \left(\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \right)^2 =$$
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Symmetric, but not invertible. While for:

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$(A - B)^2 = \left(\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right)^2 =$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

which isn't symmetric or antisymmetric, but is invertible.