Assignment 12, Linear Algebra 1

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Contents

1 Problems

1.1 Problem 1

1. Given the system of linear equations below:

$$\left.\begin{array}{llll} x & +2y & +az = & -3-a \\ x & +(2-a)y-z & =1-a & \\ ax+ay & & =6 & \end{array}\right\} \qquad a \in \mathbb{R}$$

- What assignments to a produce no solutions?
- What assignments to a produce single solution?
- What assignments to a produce infinitely many solutions?

1.1.1 Answer 1

First, express x, y and z through a:

```
programmode: false;
linsystem: [   x + 2*y        - a*z = -3 - a,
               x + (2 - a)*y -    z = 1  - a,
             a*x + a*y              = 6];
print(linsolve(linsystem, [x, y, z]));
```

```
Solution:
                                3       2
                               a  - 8 a  + 9 a - 12
                        x = - --------------------
                                    3    2
                                   a  - a  + a
                                   2
                                  a  + 3 a + 2
                             z = ------------
                                    2
                                   a  - a + 1
                                3       2
                               a  - 2 a  + 3 a - 6
                          y = -------------------
                                    3    2
                                   a  - a  + a
[%t1, %t2, %t3]
```

It's easy to see now that by solving $a^3 - a^2 + a = 0$ we can find such an assignment to a, which produces no solutions for the system.

```
programmode: false;
print(solve([a^3 - a^2 + a = 0], a));
```

```
solve: solution:
                                      sqrt(3) %i - 1
                                a = - --------------
                                            2
                                     sqrt(3) %i + 1
                                 a = --------------
                                           2
                                        a = 0
[%t1, %t2, %t3]
```

Ignoring the complex solutions, we are left with $a = 0$ assignement. Let's verify it:

```
programmode: false;
a: 0;
linsystem: [  x + 2*y        - a*z = -3 - a,
              x + (2 - a)*y -    z = 1  - a,
            a*x + a*y              = 6];
print(linsolve(linsystem, [x, y, z]));
```

```
[]
```

Indeed, produces no solutions.

To find out whether there might be infinite solutions, let's first find the reduced echelon form of the system:

```
programmode: false;
linsystem: [  x + 2*y        - a*z = -3 - a,
              x + (2 - a)*y -    z = 1  - a,
            a*x + a*y              = 6];
print(triangularize(coefmatrix(linsystem, [x, y, z])));
```

```
[ 1   2       - a     ]
[                     ]
[              2      ]
[ 0  - a      a       ]
[                     ]
[          3    2     ]
[ 0   0   a  - a  + a ]
```

By examining the reduced echelon form, we can see that infinitely many solutions are only possible if a was equal to 0, but we know that at 0 system has no solutions. Thus any assignment to a will produce a unique solution.

1.2 Problem 2

1. Given the system of linear equations below:

$$\left.\begin{array}{lllll} x & + ay & + bz & + aw & = b \\ x & + (a+1)y & + (a+b)z & + (a+b)w & = a+b \\ ax + a^2y & & + (ab+1)z & + (a+a^2)w & = b+ab \\ 2x & + (2a+1)y & + (a+2b)z & + aw & = 2b-2a-ab \end{array}\right\} \quad a, b \in \mathbb{R}$$

- What assignments to a and b produce no solutions?
- What assignments to a and b produce single solution?
- What assignments to a and b produce infinitely many solutions?

1.2.1 Answer 2

First, express a and b through x, y and z:

```
programmode: false;
linsystem: [  x + a*y          + b*z          + a*w          = b,
              x + (a + 1)*y    + (a + b)*z    + (a + b)*w    = a    + b,
            a*x + a^2*y        + (a*b + 1)*z + (a + a^2)*w = b    + a*b,
            2*x + (2*a + 1)*y + (a + 2*b)*z  + a*w          = 2*b - 2*a - a*b];
print(linsolve(linsystem, [x, y, z, w]));
```

```
Solution:
        3          2            2     4    3      2             4    3     2
       b  + (- 3 a  + a - 1) b  + (a  - a  - 4 a  - a) b + 3 a  + a  + 3 a
  x = - --------------------------------------------------------------------
                                   b + a
                             2         2         2
                            b  + (a - a ) b - 3 a
                        z = ----------------------
                                   b + a
                         2       3    2                 3    2
                   2 a b  + (- a  + a  + 2 a) b - 3 a  - a
              y = - ---------------------------------------
                                   b + a
                                 a b + 3 a
                             w = ---------
                                   b + a
[%t1, %t2, %t3, %t4]
```

One can see that assignment $a = -b$, the system has no solutions.

Since reduced echelon form of this system is:

```
programmode: false;
linsystem: [  x + a*y         + b*z         + a*w         = b,
              x + (a + 1)*y   + (a + b)*z   + (a + b)*w   = a   + b,
            a*x + a^2*y       + (a*b + 1)*z + (a + a^2)*w = b   + a*b,
            2*x + (2*a + 1)*y + (a + 2*b)*z + a*w         = 2*b - 2*a - a*b];
print(triangularize(coefmatrix(linsystem, [x, y, z, w])));
```

```
[ 1  a  b     a    ]
[                  ]
[ 0  1  a     b    ]
[                  ]
[ 0  0  1     a    ]
[                  ]
[ 0  0  0  - b - a ]
```

In order to find an assignment, which would eliminate one pivot from reduced echelon form, we would need to solve $-b - a = 0$, but this is exactly the assignment which gives single solution. So, as before, there appear to be no assignment that produces infinitely many solutions.

2 Exercises

Given O is a homogeneous system of linear equations, and M is not homogeneous system of linear equations, which share the coefficients of the row vectors of their respective matrices sans the last one. Both O and M have m equations and n unknowns.

- **a** if only the first statement is correct.
- **b** if only the second statement is correct.
- **c** if both statements are correct.
- **d** if neither statement is correct.

2.1 Exercise 1

1. There are infinitely many solutions (to the system of linear equations given below).
2. The homogeneous matrix created using the given system of linear equations has infinitely many solutions.

```
programmode: false;
linsystem: [  a + 2*b -   c +   d = 2,
            2*a + 3*b - 3*c + 2*d = 3,
             -a -   b + 2*c -   d = -1,
            2*a + 4*b - 2*c + 3*d = 3,
            2*a + 2*b - 4*c + 2*d = 2];
linsolve(linsystem, [a, b, c, d]);
```

```
solve: dependent equations eliminated: (2 5)
Solution:
                                  a = 4 - 3 %r1
                                     d = - 1
                                   c = 1 - %r1
                                     b = %r1
```

Answer: **c**

2.2 Exercise 2

1. The system given below has no solutions.
2. The system given below taken without its first equantion has no solutions.

```
programmode: false;
linsystem: [  a +   b +         d -   e = -1,
                          c +   d + 2*e = 2,
            3*a + 3*b + 2*c + 5*d + 2*e = 2,
            3*a + 3*b + 4*c + 7*d + 6*e = 2];
linsolve(linsystem, [a, b, c, d, e]);
```

```
programmode: false;
linsystem: [              c +   d + 2*e = 2,
            3*a + 3*b + 2*c + 5*d + 2*e = 2,
            3*a + 3*b + 4*c + 7*d + 6*e = 2];
linsolve(linsystem, [a, b, c, d, e]);
```

Answer: **c**

2.3 Exercise 3

```
programmode: false;
linsystem: [  x + 2*y -  3*z = a,
            2*x + 6*y - 11*z = b,
              x - 2*y +  7*z = c];
linsolve(linsystem, [a, b, c]);
```

```
Solution:
                    a = - 3 z + 2 y + x
                  b = - 11 z + 6 y + 2 x
                     c = 7 z - 2 y + x
```

1. There exist such a, b and c, which are the unique solution to the system.
2. There are such a, b and c, which are not a solution of the system.

Assignment $a = 1$, $b = 1$ and $c = 1$ gives no solutions.

```
programmode: false;
solution: triangularize(coefmatrix(
[ -3*x - 11*y + 7*z = a,
   2*x +  6*y - 2*z = b,
     x +  2*y +   z = c],
     [x, y, z]));
print(solution);
```

```
[ - 3   - 11   7  ]
[                 ]
[  0     4    - 8 ]
[                 ]
[  0     0     0  ]
```

Triangulated matrix of the above solution doesn't have pivot in the third column, thus it doesn't have a unique solution.

Answer: **b**

2.4 Exercise 4

1. If O has infinitely many solutions, then $n \geq m$.
2. If $n > m$, then M has infinitely many solutions.

(1) Not necessarily so because it is possible to have dependent equations. We could simply repeat the same euqation $n + 1$ times to find a counterexample.

(2) Not necessarily so because it is possible to have such matrices, which don't have solutions at all.

Answer: **d**

2.5 Exercise 5

1. If $\vec{c}$, $\vec{d}$ are solutions of M, and $\mu\vec{d}$, $\lambda\vec{c}$ are solutions of M then $\lambda + \mu = 1$.
2. If $\vec{c}$ is a solution of M and $\vec{d}$ is a solution of O, then $\vec{c} - 3\vec{d}$ is a solution of M.

(1) Would be true, if $\vec{d}$ and $\vec{c}$ were the same vector and M had only one solution thus. But if $\vec{c}$ and $\vec{d}$ are distinct, this warrants infinitely many solutions, thus there is no requirement that a scalar multiplier of the elementary operations performed on the solution be any particular value.

(2) If $\vec{c}$ is a unique solution of M, then O has a unique solution too. Since a solution of homogenous matrix is a zero vector, then adding any multiple of it won't change the value of $\vec{c}$. But if M has infinitely many solutions, then it is possible to see $\vec{c}-3\vec{d}$ as bein an elementary operation, which we can accomodate in place of at least one free unknown, which has to be present in this case.

Answer: **b**