Assignment 15, Linear Algebra 1

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1 Problems

1.1 Problem 1

Check whether the following transformations are linear:

1. $T_1 : \mathbb{R}^2 \to \mathbb{R}^2, T_1(x, y) = (\sin x, y)$.
2. $T_2 : \mathbb{R} \to \mathbb{R}, T_2(x) = |x|$.
3. $T_3 : \mathbb{R}[x] \to \mathbb{R}[x], T_3(p(x)) = (x + 1)p'(x) - p(x)$.

1.1.1 Answers 1, 2, 3

1. T_1 is not a linear transformation because it is not a bijection: it will send any multiple of π to the same value in the image.
2. T_2 is not a linear transformation by the same reasoning: it fails to be a bijection because it will send every value in the domain and its multiple with -1 to the same value in the image.
3. T_3, if I understood it correctly (there was a typo in the assignment description—a missing parenthesis), is indeed a linear transformation: the derivative (I believe that p' is a first order derivative, but it could be anything else really as long as it's some kind of polynomial of a finite degree) will be dominated by the polynomial of which it is a derivative, else it will be some finite order polynomial, in which case it is still possible to find a bijectin and a total function.

1.2 Problem 2

Does there exist a linear non-zero transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(1, 0, 1) = T(1, 2, 1) = T(0, 1, 1) = T(2, 3, 3)$? Prove an example if exists, else explain in detail why it doesn't.

1.2.1 Answer 4

No, such transformation doesn't exist. The easiest way to see why is by attempting to construct one. In order to do so, let's give names to the elements of the first row of the matrix representing the transformation T. *(We will only need the first row because the same equation will be obtained by multiplying any row of the matrix with a vector).* Hence, to reduce the verbosity of the proof, let a, b, c be the elements of the first row of T. Then $T(1, 0, 1)_1 = a + c$ *(I will use subscripts to denote the elements within vectors).* Other elements of the vector $T(1, 0, 1)$ will be, similarly, the sums of the first and the last elements of the consequent rows of T.

Similarly, we find that $T(1, 2, 1)_1 = a + 2b + c$, $T(0, 1, 1)_1 = b + c$, $T(2, 3, 3)_1 = 2a + 3b + 3c$. Now, since we know that the vectors are equal (from the given), we may equate them. Thus obtains:

$$
\begin{aligned}
a + c &= a + 2b + c \\
a + c &= b + c \\
2a + 3b + 3c &= a + 2b + c
\end{aligned}
$$

solving these two equations gives us that $a = b$ and $2b = 0$, hence $b = 0$, then, $a + b + 2c = 0$, which then gives $2c = 0$, hence $c = 0$. In other words the only transformation which satisfies the given condition is the transformation, where all elements of the matrix representing it are zeros.

1.3 Problem 3

Given $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent group in linear vector space V. Provided $T : V \to V$ prove or disprove:

1. If $\{T(v_1), T(v_2), \ldots, T(v_3)\}$ is linearly independent, then $\dim(\operatorname{Im}(T)) = k$.
2. If $\{T(v_1), T(v_2), \ldots, T(v_3)\}$ spans V, then $\dim V = k$.

1.3.1 Answer 5

This is not necessarily true. There could be some other vector, let's call it $v_{k+1} \in V$, linearly independent from $\{T(v_1), T(v_2), \ldots, T(v_3)\}$ and yet $T(v_{k+1}) \in V$. Let, for the sake of simplicity, choose $k = 2$ and $V = \operatorname{Sp}\{e_1, e_2, e_3\}$, where e_i are the vectors from the standard basis, T would be the identity transformation.

It is easy to see that T will send all $\{e_1, e_2, e_3\}$ back to themselves, thus $\dim(\operatorname{Im}(T)) = 3$, yet was assumed to be 2. This completes the proof.

1.3.2 Answer 6

First, observe that k cannot be smaller than $\dim V$, because the span generated by T will be at least the basis of V, and basis needs to have the same dimension as the space of which it is a basis. The domain of T is also given as linearly independant. In other words, the dimension of its span cannot be larger than that of V. Since T is a function, it cannot assign more than one element in its image to an element in its domain. Thus, the only possible way this transformation could have the given properties is if it was an injection, and this would mean that its domain can have at most k linearly independent vectors in it, hence $k = \dim V$.

1.4 Problem 4

Given

$$A = \begin{bmatrix} 1 & -2 & -2 \\ 2 & -1 & 5 \\ -2 & 3 & 1 \end{bmatrix}$$

And linear transformation $T : \mathbf{M}^{\mathbb{R}}_{3\times 3} \to \mathbf{M}^{\mathbb{R}}_{3\times 3}$ defined as $T(X) = AX$ for all $X \in \mathbf{M}^{\mathbb{R}}_{3\times 3}$. Let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the matrix defined using A, in other words $\forall x \in \mathbb{R}^3 : T_A(x) = Ax$.

1. Find basis for $\operatorname{Ker} T_A$ and $\operatorname{Im} T_A$.
2. Prove that T is not invertible.
3. Find basis for $\operatorname{Ker} T$ and $\operatorname{Im} T$.
4. Prove that if $Y \in \operatorname{Im} T$, then $\rho(Y) \leq \dim(\operatorname{Im}(T_A))$.
5. Prove that if $Y \in \operatorname{Ker} T$, then $\rho(Y) \leq \dim(\operatorname{Ker}(T_A))$.

1.5 Problem 5

Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation such that $\dim(\operatorname{Ker}(T)) > \dim(\operatorname{Im}(T))$ and the matrix representing the transformation T with the basis $B = ((1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0))$ is given by:

$$[T]_B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & a_1 & b_1 & c_1 \\ 1 & a_2 & b_2 & c_2 \\ 1 & a_3 & b_3 & c_3 \end{bmatrix}$$

1. Find a_i, b_i, c_i for $1 \leq i \leq 3$.
2. Find basis of $\operatorname{Im} T$ and $\operatorname{Ker} T$.

1.6 Problem 6

1. Let V be a finitely generated vector space and $T : V \to V$ a linear transformation. Prove that if T is not an isomorphism, then there exists a baisis B in V such that $[T]_B$ is a matrix with a column of zeros.
2. Prove that if A is singular, then A is similar to a matrix with a column of zeros.