Assignment 12, Linear Algebra 1

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Contents

1 Problems

1.1 Problem 1

1. Given the system of linear equations below:

$$\left.\begin{array}{llll} x & +2y & +az = & -3-a \\ x & +(2-a)y-z & =1-a & \\ ax+ay & & =6 & \end{array}\right\} \qquad a \in \mathbb{R}$$

- What assignments to a produce no solutions?
- What assignments to a produce single solution?
- What assignments to a produce infinitely many solutions?

1.1.1 Answer 1

First, express x, y and z through a:

```
programmode: false;
linsystem: [  x + 2*y       - a*z = -3 - a,
              x + (2 - a)*y -   z = 1  - a,
            a*x + a*y             = 6];
print(linsolve(linsystem, [x, y, z]));
```

```
Solution:
                                   3      2
                                  a  - 8 a  + 9 a - 12
                           x = - --------------------
                                      3    2
                                     a  - a  + a
                                     2
                                    a  + 3 a + 2
                               z = ------------
                                      2
                                     a  - a + 1
                                   3      2
                                  a  - 2 a  + 3 a - 6
                             y = -------------------
                                      3    2
                                     a  - a  + a
[%t1, %t2, %t3]
```

It's easy to see now that by solving $a^3 - a^2 + a = 0$ we can find such an assignment to a, which produces no solutions for the system.

```
programmode: false;
print(solve([a^3 - a^2 + a = 0], a));
```

```
solve: solution:
                                         sqrt(3) %i - 1
                                   a = - --------------
                                               2
                                        sqrt(3) %i + 1
                                    a = --------------
                                              2
                                           a = 0
[%t1, %t2, %t3]
```

Ignoring the complex solutions, we are left with $a = 0$ assignement. Let's verify it:

```
programmode: false;
a: 0;
linsystem: [  x + 2*y        - a*z = -3 - a,
              x + (2 - a)*y -    z = 1  - a,
            a*x + a*y              = 6];
print(linsolve(linsystem, [x, y, z]));
```

```
[]
```

Indeed, produces no solutions.

To find out whether there might be infinite solutions, let's first find the reduced echelon form of the system:

```
programmode: false;
linsystem: [  x + 2*y        - a*z = -3 - a,
              x + (2 - a)*y -    z = 1  - a,
            a*x + a*y              = 6];
print(triangularize(coefmatrix(linsystem, [x, y, z])));
```

```
[ 1   2       - a     ]
[                     ]
[              2      ]
[ 0  - a      a       ]
[                     ]
[          3    2     ]
[ 0   0   a  - a  + a ]
```

By examining the reduced echelon form, we can see that infinitely many solutions are only possible if a was equal to 0, but we know that at 0 system has no solutions. Thus any assignment to a will produce a unique solution.

1.2 Problem 2

1. Given the system of linear equations below:

$$\left.\begin{array}{llllll} x & +ay & +bz & +aw & =b \\ x & +(a+1)y & +(a+b)z & +(a+b)w & =a+b \\ ax+a^2y & & +(ab+1)z & +(a+a^2)w & =b+ab \\ 2x+(2a+1)y & & +(a+2b)z & +aw & =2b-2a-ab \end{array}\right\} \qquad a,b\in\mathbb{R}$$

- What assignments to a and b produce no solutions?
- What assignments to a and b produce single solution?
- What assignments to a and b produce infinitely many solutions?

1.2.1 Answer 2

First, express a and b through x, y and z:

```
programmode: false;
linsystem: [  x + a*y         + b*z         + a*w         = b,
              x + (a + 1)*y   + (a + b)*z   + (a + b)*w   = a   + b,
            a*x + a^2*y       + (a*b + 1)*z + (a + a^2)*w = b   + a*b,
            2*x + (2*a + 1)*y + (a + 2*b)*z  + a*w        = 2*b - 2*a - a*b];
print(linsolve(linsystem, [x, y, z, w]));
```

```
Solution:
        3         2             2     4    3      2             4    3      2
       b  + (- 3 a  + a - 1) b  + (a  - a  - 4 a  - a) b + 3 a  + a  + 3 a
  x = - -------------------------------------------------------------------
                                     b + a
                                 2         2         2
                                b  + (a - a ) b - 3 a
                          z = ----------------------
                                    b + a
                             2       3    2                 3    2
                      2 a b  + (- a  + a  + 2 a) b - 3 a  - a
                y = - ---------------------------------------
                                    b + a
                                  a b + 3 a
                              w = ---------
                                    b + a
[%t1, %t2, %t3, %t4]
```

One can see that assignment $a = -b$ will result in solutions (any such combination will be equivalent to divsion by zero).

Since reduced echelon form of this system is:

```
programmode: false;
linsystem: [  x + a*y         + b*z         + a*w         = b,
              x + (a + 1)*y   + (a + b)*z   + (a + b)*w   = a   + b,
            a*x + a^2*y       + (a*b + 1)*z + (a + a^2)*w = b   + a*b,
            2*x + (2*a + 1)*y + (a + 2*b)*z + a*w         = 2*b - 2*a - a*b];
print(triangularize(coefmatrix(linsystem, [x, y, z, w])));
```

```
[ 1  a  b     a    ]
[                  ]
[ 0  1  a     b    ]
[                  ]
[ 0  0  1     a    ]
[                  ]
[ 0  0  0  - b - a ]
```

In order to find an assignment, which would eliminate one pivot from reduced echelon form, we would need to solve $-b - a = 0$, but this is exactly the assignment which gives single solution. So, as before, there appear to be no assignment that produces infinitely many solutions.

1.3 Problem 3

1. Solve the system of linear equations:

$$\left.\begin{aligned} \frac{1}{x} + \frac{2}{y} - \frac{4}{z} &= 1 \\ \frac{2}{x} + \frac{3}{y} + \frac{8}{z} &= 0 \\ \frac{1}{x} + \frac{9}{y} - \frac{10}{z} &= 5 \end{aligned}\right\} \qquad x, y, z \in \mathbb{R}$$

1.3.1 Answer 3

```
programmode: false;
linsystem: [ 1/x + 2/y -  4/z = 1,
             2/x + 3/y +  8/z = 0,
             1/x + 9/y + 10/z = 5];
linsolve(linsystem, [x, y, z]);
```

```
Solution:
                                  x = 0
                                  y = 0
                                  z = 0
```

The reduced echelon form of the matrix of this system has a pivot element in every column, which means that this system can have at most one solution. Unfortunately, this solution encounters division by zero, which renders this system as having no solutions.

```
programmode: false;
solution: triangularize(coefmatrix(
[ -x*y*z + 2*x*z - 4*x*y = 0,
  2*y*z + 3*x*z + 8*x*y = 0,
  y*z + 9*x*z + 10*x*y - 5*x*y*z = 0],
  [x, y, z]));
print(solution);
```

The system above would be equivalent to the given system under assumption that $x \neq 0$, $y \neq 0$ and $z \neq 0$.

```
        [ 3 z + 8 y ]         [              2 z + 8 x              ]
        [           ]         [                                     ]
Col 1 = [     0     ] Col 2 = [                  2                  ]
        [           ]         [ (2 y - 3 x - 4) z  + (8 y - 28 x) z ]
        [     0     ]         [                                     ]
                              [                  0                  ]
                               [                2 y + 3 x                ]
                               [                                         ]
                               [      2                        2         ]
                       Col 3 = [   (2 y  - 4 y) z + (8 - 8 x) y  + 28 x y   ]
                               [                                         ]
                               [      2                     2           2 ]
                               [ ((98 x  + 168 x) y - 52 x y ) z - 5 x y z  ]
```

1.4 Problem 4

Given $U = \{\vec{u_1}, \vec{u_2}, \vec{u_3}, \vec{u_4}\}$ is a linearly independant set of vectors in $\mathbb{R}^5$ and vectors:

$$\begin{aligned}
v_1 &= 8au_1 + \quad 2u_2 + u_3 \\
v_2 &= \qquad\quad 16au_2 + u_4 \\
v_3 &= u_1 - \quad \frac{1}{2}u_3 + au_4 \\
a &\in \mathbb{R}
\end{aligned}$$

1. Find all a such that $V = \{v_1, v_2, v_3\}$ is linearly dependent.
2. For every a found in (1), write v_2 as linear combination of v_1 and v_3.
3. Is it possible to adjoin the vectors v_i to U such that $U \cup \{v_i\}$ would become a basis in $\mathbb{R}$?

1.4.1 Answer 4

First we will arrange all coefficients describing vectors v_i as rows of the matrix:

```
programmode: false;
solution: triangularize(transpose(matrix(
    [8*a, 2,    1,    0],
    [0,   16*a, 0,    1],
    [1,   0,    -1/2, a])));
print(solution);
```

```
[ 2   0        - 1    ]
[                     ]
[ 0  32 a       2     ]
[                     ]
[               2     ]
[ 0   0     32 a  - 2 ]
[                     ]
[ 0   0         0     ]
```

In order for this matrix to represent linearly dependent combination, it should be the case that $32^a - 2 = 0$. Otherwise, this system has no solutions (but it has to, becuase it is given that v_i is linearly dependant, which requires that linear combinations of all dependent vectors be equal to zero vector).

```
solution: solve([32*a^2 - 2], a);
tex(solution);
```

$$\left[a = -\frac{1}{4}, a = \frac{1}{4}\right]$$

Now we can write v_2 as linear combination of v_1 and v_3 for $\frac{1}{4}$:

$$\begin{aligned}
(0, \frac{32}{4}, 0, 0) &= x(2, 0, 0, 0) &&+ y(-1, 2, \frac{32}{4^2} - 2) \\
(0, 8, 0, 0) &= x(2, 0, 0, 0) &&+ y(-1, 2, 0, 0) \\
(0, 8, 0, 0) &= 4(-1, 2, 0, 0) &&+ 2(2, 0, 0, 0) \\
v_2 &= 4v_3 &&+ 2v_1
\end{aligned}$$

and similarly for $-\frac{1}{4}$:

$$
\begin{aligned}
(0,-\frac{32}{4},0,0) &= x(2,0,0,0) &&+ y(-1,2,\frac{32}{-4^2}-2)\\
(0,-8,0,0) &= x(2,0,0,0) &&+ y(-1,2,0,0)\\
(0,-8,0,0) &= -4(-1,2,0,0) &&+ -2(2,0,0,0)\\
v_2 &= -4v_3 &&+ -2v_1
\end{aligned}
$$

2 Exercises

Given O is a homogeneous system of linear equations, and M is not homogeneous system of linear equations, which share the coefficients of the row vectors of their respective matrices sans the last one. Both O and M have m equations and n unknowns.

- **a** if only the first statement is correct.
- **b** if only the second statement is correct.
- **c** if both statements are correct.
- **d** if neither statement is correct.

2.1 Exercise 1

1. There are infinitely many solutions (to the system of linear equations given below).
2. The homogeneous matrix created using the given system of linear equations has infinitely many solutions.

```
programmode: false;
linsystem: [  a + 2*b -   c +   d = 2,
            2*a + 3*b - 3*c + 2*d = 3,
             -a -   b + 2*c -   d = -1,
            2*a + 4*b - 2*c + 3*d = 3,
            2*a + 2*b - 4*c + 2*d = 2];
linsolve(linsystem, [a, b, c, d]);
```

```
solve: dependent equations eliminated: (2 5)
Solution:
                              a = 4 - 3 %r1
                                 d = - 1
                               c = 1 - %r1
                                 b = %r1
```

Answer: **c**

2.2 Exercise 2

1. The system given below has no solutions.
2. The system given below taken without its first equantion has no solutions.

```
programmode: false;
linsystem: [  a +   b +         d -   e = -1,
                          c +   d + 2*e = 2,
            3*a + 3*b + 2*c + 5*d + 2*e = 2,
            3*a + 3*b + 4*c + 7*d + 6*e = 2];
linsolve(linsystem, [a, b, c, d, e]);
```

```
programmode: false;
linsystem: [              c +   d + 2*e = 2,
            3*a + 3*b + 2*c + 5*d + 2*e = 2,
            3*a + 3*b + 4*c + 7*d + 6*e = 2];
linsolve(linsystem, [a, b, c, d, e]);
```

Answer: **c**

2.3 Exercise 3

```
programmode: false;
linsystem: [  x + 2*y -  3*z = a,
            2*x + 6*y - 11*z = b,
              x - 2*y +  7*z = c];
linsolve(linsystem, [a, b, c]);
```

```
Solution:
                                a = - 3 z + 2 y + x
                              b = - 11 z + 6 y + 2 x
                                 c = 7 z - 2 y + x
```

1. There exist such a, b and c, which are the unique solution to the system.
2. There are such a, b and c, which are not a solution of the system.

Assignment $a = 1$, $b = 1$ and $c = 1$ gives no solutions.

```
programmode: false;
solution: triangularize(coefmatrix(
[ -3*x - 11*y + 7*z = a,
   2*x +  6*y - 2*z = b,
     x +  2*y +   z = c],
     [x, y, z]));
print(solution);
```

```
[ - 3   - 11   7  ]
[                 ]
[  0     4    - 8 ]
[                 ]
[  0     0     0  ]
```

Triangulated matrix of the above solution doesn't have pivot in the third column, thus it doesn't have a unique solution.

Answer: **b**

2.4 Exercise 4

1. If O has infinitely many solutions, then $n \geq m$.
2. If $n > m$, then M has infinitely many solutions.

(1) Not necessarily so because it is possible to have dependent equations. We could simply repeat the same euqation $n + 1$ times to find a counterexample.

(2) Not necessarily so because it is possible to have such matrices, which don't have solutions at all.

Answer: **d**

2.5 Exercise 5

1. If $\vec{c}$, $\vec{d}$ are solutions of M, and $\mu\vec{d}$, $\lambda\vec{c}$ are solutions of M then $\lambda + \mu = 1$.
2. If $\vec{c}$ is a solution of M and $\vec{d}$ is a solution of O, then $\vec{c} - 3\vec{d}$ is a solution of M.

(1) Would be true, if $\vec{d}$ and $\vec{c}$ were the same vector and M had only one solution thus. But if $\vec{c}$ and $\vec{d}$ are distinct, this warrants infinitely many solutions, thus there is no requirement that a scalar multiplier of the elementary operations performed on the solution be any particular value.

(2) If $\vec{c}$ is a unique solution of M, then O has a unique solution too. Since a solution of homogenous matrix is a zero vector, then adding any multiple of it won't change the value of $\vec{c}$. But if M has infinitely many solutions, then it is possible to see $\vec{c} - 3\vec{d}$ as bein an elementary operation, which we can accomodate in place of at least one free unknown, which has to be present in this case.

Answer: **b**