Generating function method

Here we present the details of solving the PDEs in the paper with generating functions.

All-cell growth model:

In[1]:=
```
ClearAll[G, z, kb, kd, pde]
```

In[2]:=
$$pde = (kd\, z - kb - kd)\, z\, G'[z] - kb \frac{z}{1-z} G[z] + kb \frac{z}{1-z} == 0;$$

```
DSolve[pde, G[z], z] // FullSimplify
```

Out[3]=
$$\left\{\left\{G[z] \to \frac{kb + (-1+z)\, c_1}{kb + kd - kd\, z}\right\}\right\}$$

One-end growth model:

In[4]:=
```
ClearAll[G, z, pde, γ, Gsum, GsumPrime]
```

In[5]:=
$$pde = -\gamma\, z\, G'[z] - \left((1-z) + \gamma \frac{z}{1-z}\right) G[z] + \gamma \frac{z}{1-z} == 0;$$

```
DSolve[pde, G[z], z] // FullSimplify
```

Out[6]=
$$\left\{\left\{G[z] \to -e^{z/\gamma} (-1+z)\, z^{-1/\gamma} \left(c_1 + \int_1^z \frac{e^{-\frac{K[1]}{\gamma}} K[1]^{\frac{1}{\gamma}}}{(-1+K[1])^2} dK[1]\right)\right\}\right\}$$

In[7]:=
$$G[z\_] := e^{\frac{z}{\gamma}} (1-z)\, z^{-1/\gamma} \int_0^z \frac{e^{-\frac{x}{\gamma}} x^{1/\gamma}}{(1-x)^2} dx;$$

```
Highlighted[G[z]]
```

Out[8]=
$$e^{z/\gamma} (1-z)\, z^{-1/\gamma} \int_0^z \frac{e^{-\frac{x}{\gamma}} x^{\frac{1}{\gamma}}}{(1-x)^2} dx$$

In[9]:=
```
Series[G[z], {z, 0, 5}, Assumptions → γ > 0]
```

Out[9]=
$$\frac{\gamma z}{1+\gamma} + \frac{2\gamma z^2}{(1+\gamma)(1+2\gamma)} + \frac{3\gamma z^3}{(1+\gamma)(1+2\gamma)(1+3\gamma)} + \frac{4\gamma z^4}{(1+\gamma)(1+2\gamma)(1+3\gamma)(1+4\gamma)} + \frac{5\gamma z^5}{(1+\gamma)(1+2\gamma)(1+3\gamma)(1+4\gamma)(1+5\gamma)} + O[z]^6$$

```
(* Check to see that this sum form is the solution to the original PDE *)
```

In[10]:=
$$Gsum = Sum\left[\frac{n\,\gamma^{1-n}}{Pochhammer[(1+\gamma)/\gamma, n]} * z^n, \{n, 1, \infty\}\right] // FullSimplify;$$

```
GsumPrime = D[Gsum, z] // FullSimplify;
```

In[12]:= $-\gamma\, z\, \text{GsumPrime} - \left((1-z) + \gamma \frac{z}{1-z}\right) \text{Gsum} + \gamma \frac{z}{1-z}$ // FullSimplify

Out[12]=

0

Two-end growth model:

In[13]:= ClearAll[G, z, pde, γ, Gsum, GsumPrime]

In[14]:= $\text{pde} = z\, G'[z] == -\left(\frac{2}{\gamma}(1-z) + \frac{z}{1-z}\right) G[z] + \frac{z}{1-z} + \frac{1}{1+\gamma} z\,(1-z);$

DSolve[pde, G[z], z] // FullSimplify

Out[15]=

$$\left\{\left\{G[z] \to -e^{\frac{2z}{\gamma}} (-1+z)\, z^{-2/\gamma} \left(c_1 + \int_1^z \frac{e^{-\frac{2K[1]}{\gamma}} K[1]^{2/\gamma} (2+\gamma+(-2+K[1])K[1])}{(1+\gamma)(-1+K[1])^2} \, dK[1]\right)\right\}\right\}$$

In[16]:= $G[z\_] := e^{\frac{2z}{\gamma}} (1-z)\, z^{-2/\gamma} \int_0^z \frac{e^{-\frac{2x}{\gamma}} x^{2/\gamma} \left(x^2 - 2x + 2 + \gamma\right)}{(1+\gamma)(1-x)^2} \, dx;$

Highlighted[G[z]]

Out[17]=

$$e^{\frac{2z}{\gamma}} (1-z)\, z^{-2/\gamma} \int_0^z \frac{e^{-\frac{2x}{\gamma}} x^{2/\gamma} \left(2 - 2x + x^2 + \gamma\right)}{(1-x)^2 (1+\gamma)} \, dx$$

In[18]:= Series[G[z], {z, 0, 6}, Assumptions → γ > 0]

Out[18]=

$$\frac{\gamma z}{1+\gamma} + \frac{\gamma z^2}{(1+\gamma)^2} + \frac{3\gamma z^3}{(1+\gamma)^2 (2+3\gamma)} + \frac{4\gamma z^4}{(1+\gamma)^2 (1+2\gamma)(2+3\gamma)} + \frac{10\gamma z^5}{(1+\gamma)^2 (1+2\gamma)(2+3\gamma)(2+5\gamma)} + \frac{12\gamma z^6}{(1+\gamma)^2 (1+2\gamma)(1+3\gamma)(2+3\gamma)(2+5\gamma)} + O[z]^7$$

(* Check to see that this sum form is the solution to the original PDE *)

In[19]:= $\text{Gsum} = \frac{\gamma}{1+\gamma} z + \text{Sum}\left[\frac{2+\gamma}{2(1+\gamma)} \frac{n\,(\gamma/2)^{1-n}}{\text{Pochhammer}[(2+\gamma)/\gamma, n]} * z^n, \{n, 2, \infty\}\right]$ // FullSimplify;

GsumPrime = D[Gsum, z] // FullSimplify;

In[21]:= $z\, \text{GsumPrime} + \left(\frac{2}{\gamma}(1-z) + \frac{z}{1-z}\right) \text{Gsum} - \frac{z}{1-z} - \frac{1}{1+\gamma} z\,(1-z)$ // FullSimplify

Out[21]=

0

Conditional survival probability:

In[22]:= ClearAll[G, z, t, kb, kd, pde, initCond]

In[23]:=
```
pde = D[G[z, t], t] == -(kb + kd) z D[G[z, t], z] + kd D[G[z, t], z] - (kd / z) G[z, t];
```
$$\texttt{initCond = G[z, 0] == } \sum_{n=1}^{\infty} z^n;$$

In[25]:=
```
sol = DSolve[{pde, initCond}, G, {z, t}]
```

Out[25]=

$$\left\{\left\{G \to \text{Function}\left[\{z, t\}, \frac{(kb + kd)\ z}{\left(-1 + e^{kb\left(t - \frac{\text{Log}[kd\ (-1+z)+kb\ z]}{kb+kd}\right) + kd\left(t - \frac{\text{Log}[kd\ (-1+z)+kb\ z]}{kb+kd}\right)}\ kb\right)\ (-kd + kb\ z + kd\ z)}\right]\right\}\right\}$$

In[26]:=
```
G = sol[[1, 1, 2, 2]] // FullSimplify;
Highlighted[G]
```

Out[27]=

$$\frac{(kb + kd)\ z}{e^{(kb+kd)\ t}\ kb + kd - (kb + kd)\ z}$$

In[28]:=
```
SeriesCoefficient[G, {z, 0, n}]
```

Out[28]=

$$\begin{cases} \left(\frac{kb+kd}{e^{(kb+kd)\ t}\ kb+kd}\right)^n & n \geq 1 \\ 0 & \text{True} \end{cases}$$