2.4 场作用量、最小作用量原理和Noether定理

1. 对场作用量的一般要求

① 只限于定域场；

$$I = \int_{\tau_1}^{\tau_2} \mathrm{d}^4 x \mathcal{L}, \qquad [\mathcal{L}] = \mathrm{L}^{-4}, \tag{2.125}$$

$$\mathrm{d}^4 x = \mathrm{d}x^0 \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3, \tag{2.126}$$

$\mathcal{L}$为Lagrange密度(Lagrangian),

$$\mathcal{L} = \mathcal{L}(\phi(x), \partial_\mu \phi(x)). \tag{2.127}$$

②作用量应为实的；

③从作用量导出的经典运动方程不含高于2阶的微商；

④应具有Poincaré不变性；

⑤存在对作用量的其它不变性要求。

2. 经典场论中的最小作用量原理

$$I = \int_{\tau_1}^{\tau_2} \mathrm{d}^4 x \mathcal{L}(\phi, \partial_\mu \phi),$$

当$\phi \to \phi + \delta\phi$时，

$$\delta I = 0$$

$$\Rightarrow \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \qquad (2.134)$$

(Euler-Lagrange eq.)

若

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \partial_\mu \Lambda^\mu, \qquad \text{（正则变换）}$$

$$\Lambda^\mu = \Lambda^\mu(\phi, \partial_\mu \phi)$$

则$\mathcal{L}$和$\mathcal{L}'$导致相同的运动方程（平坦时空中）

3. Noether 定理

在某种对称变换下，

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \delta x^{\mu}, \quad \phi(x) \to \phi'(x') = \phi(x) + \delta\phi,$$

$$\delta I = \int_{\tau_1'}^{\tau_2'} \mathrm{d}^4 x' \, \mathcal{L}'(x') - \int_{\tau_1}^{\tau_2} \mathrm{d}^4 x \, \mathcal{L}(x)$$

$$= \int_{\tau_1}^{\tau_2} \{[\mathrm{d}^4 x + \delta(\mathrm{d}^4 x)][\mathcal{L}(x) + \delta\mathcal{L}] - \mathrm{d}^4 x \mathcal{L}(x)\}$$

$$= \int_{\tau_1}^{\tau_2} [\delta(\mathrm{d}^4 x)\mathcal{L} + \mathrm{d}^4 x \delta\mathcal{L}], \tag{2.136}$$

$$\delta(\mathrm{d}^4 x) = \mathrm{d}^4 x' - \mathrm{d}^4 x = \mathrm{d}^4 x \left| \frac{\partial x'^{\mu}}{\partial x^{\nu}} \right| - \mathrm{d}^4 x$$

$$= \mathrm{d}^4 x (\partial_{\mu} \delta x^{\mu}), \tag{2.137}$$

$$\delta = \delta_0 + \delta x^{\mu} \partial_{\mu},$$

$$\delta \mathcal{L} = \delta x^{\mu} \partial_{\mu} \mathcal{L} + \delta_0 \mathcal{L}(\phi, \partial_{\mu} \phi)$$

$$= \delta x^{\mu} \partial_{\mu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \delta_0 \phi + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} \phi)} \delta_0 (\partial_{\mu} \phi)$$

$$= \delta x^{\mu} \partial_{\mu} \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \delta_0 \phi + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu} \phi)} \partial_{\mu} (\delta_0 \phi)$$

$$\delta\mathcal{L} = \delta x^{\mu}\partial_{\mu}\mathcal{L} + \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\right]\delta_0\phi + \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta_0\phi\right)$$

$$= \delta x^{\mu}\partial_{\mu}\mathcal{L} + \partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta_0\phi\right), \qquad (2.139)$$

将(2.137)和(2.139)代入(2.136)，得

$$\delta I=\int_{\tau_1}^{\tau_2}\mathrm{d}^4x\left[(\partial_\mu\delta x^\mu)\mathcal{L}+\delta x^\mu\partial_\mu\mathcal{L}+\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0\phi\right)\right]$$

$$=\int_{\tau_1}^{\tau_2}\mathrm{d}^4x\partial_\mu\left[\delta x^\mu\mathcal{L}+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_0\phi\right]$$

$$=\int_{\tau_1}^{\tau_2}\mathrm{d}^4x\partial_\mu\left[\left(\mathcal{L}g^\mu_\rho-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\rho\phi\right)\delta x^\rho+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right],(2.140)$$

引入整体变换参数$\delta\omega^a$，

$$\delta x^\rho = \frac{\delta x^\rho}{\delta\omega^a}\delta\omega^a, \quad \delta\phi = \frac{\delta\phi}{\delta\omega^a}\delta\omega^a,$$

$$\delta I = \int_{\tau_1}^{\tau_2} \mathrm{d}^4 x \partial_\mu \left[\underbrace{\left(\mathcal{L} g^\mu_\rho - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\rho\phi \right)\frac{\delta x^\rho}{\delta\omega^a} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\delta\omega^a}}_{\equiv -j^\mu_a,\text{（流密度）}} \right]\delta\omega^a,$$

若对所有$\delta\omega^a$，$\delta I = 0$，则得

$$\partial_\mu j^\mu_a = 0. \quad \text{（流守恒方程）} \tag{2.145}$$

Noether定理：作用量I的某种不变性将导致流守恒方程。

将(2.145)式展开成

$$\partial_0 j_a^0 + \partial_i j_a^i = 0,$$

对上式积分$\left(\int_{T_1}^{T_2} \mathrm{d}\, x^0 \int_{-\infty}^{\infty} \mathrm{d}^3\, x\right)$，则有

$$\int_{T_1}^{T_2} \mathrm{d}\, x^0 \int_{-\infty}^{\infty} \mathrm{d}^3\, x \partial_0 j_a^0 + \int_{T_1}^{T_2} \mathrm{d}\, x^0 \int_{-\infty}^{\infty} \mathrm{d}^3\, x \partial_i j_a^i = 0,$$

$$\Rightarrow \quad \int_{T_1}^{T_2} \mathrm{d}\, x^0 \frac{\mathrm{d}}{\mathrm{d}\, x^0} \int_{-\infty}^{\infty} \mathrm{d}^3\, x j_a^0(t, \vec{x}) = 0,$$

定义荷Q_a：

$$Q_a(T) = \int_{-\infty}^{\infty} \mathrm{d}^3 x j_a^0(t, \vec{x}), \tag{2.148}$$

$$\Rightarrow \quad \frac{dQ_a}{dt} = 0. \tag{2.149}$$

（Q_a守恒）

在某种整体对称变换下 $\delta I = 0$导致了守恒荷的存在

4. 守恒流不唯一

(1) $j_a^\mu \to j_a'^\mu = j_a^\mu + A^\mu,$

若j_a^μ为守恒流，且$\partial_\mu A^\mu = 0$，则$j_a'^\mu$也是守恒流；

(2) $j_a^\mu \to j_a^\mu + \partial_\nu t_a^{\nu\mu},$

$t_a^{\nu\mu}$为任意反对称张量。这种不唯一性是由于

$$\mathcal{L} \to \mathcal{L} + \partial_\nu \Lambda^\nu$$

的贡献。

2.5 0自旋场

1. 单个标量场

① Lagrangian

$$\mathcal{L}=\frac{1}{2}\partial_\mu\varphi(x)\partial^\mu\varphi(x)-V(\varphi(x)). \tag{2.153}$$

在变换$\varphi\to\varphi+a$下不变.

得到运动方程：

$$\partial_\mu\partial^\mu\varphi=-V'(\varphi). \tag{2.158}$$

***a*) Klein-Gordon Lagrangian :**

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2; \qquad (2.154)$$

$$\Rightarrow \quad (\Box + m)\varphi(x) = 0. \quad \text{(K-G方程)}$$

***b*) $\lambda\varphi^4$自作用理论：**

$$\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\varphi^4; \qquad (2.156)$$

***c*) Sine-Gordon Lagrangian :**

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{m^4}{\lambda}\left(\cos\frac{\sqrt{\lambda}\varphi}{m} - 1\right). \qquad (2.157)$$

② 守恒量

a) 无穷小时空平移

$$\delta x^{\mu} = \varepsilon^{\mu}, \quad \delta\varphi = 0,$$

由(2.144)，流密度可写为

$$j_{\mu a} = -\left[\mathcal{L} g_{\mu\rho} - \frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\varphi)}\partial_{\rho}\varphi\right]\frac{\delta x^{\rho}}{\delta\omega^{a}} - \frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\varphi)}\frac{\delta\varphi}{\delta\omega^{a}},$$

取$a = \nu$，$\delta\omega^{a} = \delta\omega^{\nu} = \varepsilon^{\nu}$，可得

$$\frac{\delta x^{\rho}}{\delta\omega^{\nu}} = g^{\rho}{}_{\nu},$$

$$
\begin{aligned}
j_{\mu\nu} &= -\left[\mathcal{L} g_{\mu\rho} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\rho \varphi\right] g^{\rho}{}_{\nu} \\
&= -\left[\mathcal{L} g_{\mu\rho} - \partial_\mu \varphi \partial_\rho \varphi\right] g^{\rho}{}_{\nu} \\
&= -g_{\mu\nu} \mathcal{L} + \partial_\mu \varphi \partial_\nu \varphi, \qquad (2.162)
\end{aligned}
$$

$j_{\mu\nu}$为能–动量张量密度，

$$
\partial^\mu j_{\mu\nu} = 0, \qquad (2.163)
$$

相应的守恒荷是场的4动量：

$$
P_\nu = \int d^3 x j_{0\nu} = \int d^3 x (-g_{0\nu} \mathcal{L} + \partial_0 \varphi \partial_\nu \varphi), \qquad (2.164)
$$

P_0是场的能量，能量密度为

$$j_{00} = -\mathcal{L} + \partial_0\varphi\partial_0\varphi$$

$$= \frac{1}{2}\partial_0\varphi\partial_0\varphi + \frac{1}{2}\nabla\varphi\cdot\nabla\varphi + V(\varphi), \qquad (2.165)$$

$V > 0$时，j_{00}正定。j_{00}的最小值出现于静场φ_0情况，

$$\partial_0\varphi_0 = \partial_i\varphi_0 = 0.$$

***b*) 无穷小LT**

$$\delta x^{\mu} = \varepsilon^{\mu\nu} x_{\nu}, \qquad \delta\varphi = 0,$$

取$a = \nu\rho$, $\delta\omega^{a} \equiv \delta\omega^{\nu\rho} = \varepsilon^{\nu\rho}$, 可得

$$\frac{\delta x^{\mu}}{\delta\omega^{\nu\rho}} = \frac{1}{2}(-g^{\mu}{}_{\rho} x_{\nu} + g^{\mu}{}_{\nu} x_{\rho}),$$

守恒流密度为：

$$\begin{aligned} j_{\mu\nu\rho} &= -[-\mathcal{L} g_{\mu\lambda} + \partial_{\mu}\varphi \partial_{\lambda}\varphi](g^{\lambda}_{\nu} x_{\rho} - g^{\lambda}_{\rho} x_{\nu}) \\ &= x_{\nu} j_{\mu\rho} - x_{\rho} j_{\mu\nu}, \end{aligned} \tag{2.166}$$

$j_{\mu\nu\rho}$称为广义的角动量张量密度，相应的守恒荷

$$M_{\nu\rho}=\int \mathrm{d}^3\, x j_{0\nu\rho}=\int \mathrm{d}^3\, x(x_\nu j_{0\rho}-x_\rho j_{0\nu}) \qquad (2.167)$$

称为广义的角动量。

2. 多个标量场

考虑N个实标量场$\varphi_a\,(a=1,\cdots,N)$,

$$\mathcal{L}=\frac{1}{2}\partial_\mu\varphi_a\partial^\mu\varphi_a-V(\varphi_a\varphi_a), \tag{2.168}$$

整体转动下，

$$\delta\varphi_a=\varepsilon_{ab}\varphi_b,\quad \varepsilon_{ab}=-\varepsilon_{ba},$$

$$\delta x^\rho=0,$$

转动参数ε_{ab}有$N(N-1)/2$个独立分量。

由(2.144)，Noether流密度为

$$j_f^\mu = -\left[\mathcal{L} g_\rho^\mu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_c)} \partial_\rho \varphi_c\right] \frac{\delta x^\rho}{\delta \omega^f} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_c)} \frac{\delta \varphi_c}{\delta \omega^f},$$

取$f = ab$，$\delta\omega^f = \delta\omega^{ab} = \varepsilon^{ab}$，可得

$$\frac{\delta \varphi_c}{\delta \omega^{ab}} = g^c{}_a \varphi_b - g^c{}_b \varphi_a,$$

$$j_{ab}^\mu = \partial^\mu \varphi_c (g^c{}_a \varphi_b - g^c{}_b \varphi_a)$$

$$= \varphi_a \partial^\mu \varphi_b - \varphi_b \partial^\mu \varphi_a. \tag{2.170}$$

2.6 自旋1/2场

2.6.1 作用量、运动方程和守恒量

1. Lagrangian

自由场情况下，

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi, \quad \bar{\psi} = \psi^{+}\gamma^{0}, \qquad (2.171)$$

或

$$\mathcal{L}' = i\bar{\psi}\gamma^{\mu}\overleftrightarrow{\partial}_{\mu}\psi - m\bar{\psi}\psi, \qquad (2.172)$$

其中$\overleftrightarrow{\partial}_{\mu}$定义为

$$b\overleftrightarrow{\partial}_{\mu}a \equiv b\partial_{\mu}a - (\partial_{\mu}b)a. \qquad (2.173)$$

2. Dirac 方程

Euler – Lagrange Eq.

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \qquad (2.134)$$

由(2.171)，对$\overline{\psi}$和ψ独立进行变分，得

$$(i\not{\partial} - m)\psi = 0, \quad (\psi\text{满足的Dirac Eq.}) \qquad (2.174a)$$

$$\overline{\psi}(i\overleftarrow{\not{\partial}} - m) = 0, \quad (\overline{\psi}\text{满足的Dirac Eq.}) \qquad (2.174b)$$

其中

$$\not{a} \equiv \gamma^\mu a_\mu = \gamma_\mu a^\mu, \qquad a\overleftarrow{\partial}_\mu \equiv -\partial_\mu a.$$

3. 守恒量

a) 无穷小时空平移

$$\delta x^{\mu}=\varepsilon^{\mu}, \quad \delta\psi=0, \quad \delta\overline{\psi}=0,$$

由(2.144)，流密度可写为

$$j_{\mu a}=-\left[\mathcal{L}g_{\mu\rho}-\frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)}\partial_{\rho}\phi\right]\frac{\delta x^{\rho}}{\delta\omega^{a}}-\frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\phi)}\frac{\delta\phi}{\delta\omega^{a}},$$

取$a=\nu$,

$$\frac{\delta x^{\rho}}{\delta\omega^{\nu}}=g^{\rho}{}_{\nu},$$

守恒流密度为

$$\Theta_{\mu\nu}=-\left[\mathcal{L}g_{\mu\rho}-\frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\psi)}\partial_{\rho}\psi\right]g^{\rho}{}_{\nu}=\frac{\partial\mathcal{L}}{\partial(\partial^{\mu}\psi)}\partial_{\nu}\psi-g_{\mu\nu}\mathcal{L}$$

$$=\frac{\partial}{\partial(\partial^{\mu}\psi)}[i\overline{\psi}\gamma_{\mu}\partial^{\mu}\psi-m\overline{\psi}\psi]\partial_{\nu}\psi-g_{\mu\nu}\mathcal{L}$$

$$=i\overline{\psi}\gamma_{\mu}\partial_{\nu}\psi-g_{\mu\nu}\overline{\psi}(i\gamma^{\sigma}\partial_{\sigma}-m)\psi$$

$$=i\overline{\psi}\gamma_{\mu}\partial_{\nu}\psi, \tag{2.177}$$

$\Theta_{\mu\nu}$为能–动量张量密度，相应的守恒荷为

$$P_{\nu}=\int d^{3}x\Theta_{0\nu}=i\int d^{3}x\psi^{+}\partial_{\nu}\psi. \tag{2.178}$$

b) 无穷小LT

$$\delta x^{\mu} = \varepsilon^{\mu\nu} x_{\nu},$$

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x) = [I - \frac{i}{4}\varepsilon^{\alpha\beta}\sigma_{\alpha\beta}],$$

$$\delta\psi = \psi'(x') - \psi(x) = -\frac{i}{4}\varepsilon^{\alpha\beta}\sigma_{\alpha\beta}.$$

LT不变性导致的守恒流密度

$$J_{\mu\nu\rho} = x_{\nu}\Theta_{\mu\rho} - x_{\rho}\Theta_{\mu\nu} + \bar{\psi}\gamma_{\mu}\frac{\sigma_{\nu\rho}}{2}\psi, \qquad (2.179)$$

$J_{\mu\nu\rho}$称为广义的角动量张量密度，

相应的守恒荷是广义角动量：

$$M_{\nu\rho} = \int d^3 x J_{0\nu\rho}$$

$$= \int d^3 x [x_\nu \Theta_{0\rho} - x_\rho \Theta_{0\nu} + \bar{\psi}\gamma_0 \frac{\sigma_{\nu\rho}}{2}\psi]. \quad (2.180)$$

c) 整体相位变换

$$\psi \to e^{i\alpha}\psi, \qquad \bar{\psi} \to \bar{\psi} e^{-i\alpha},$$

(2.168)所示的**Lagrangian**在整体相位变换下不变

Noether流密度为

$$j_a^{\mu} = -\left[\mathcal{L} g_{\rho}^{\mu} - \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\psi)}\partial_{\rho}\psi\right]\frac{\delta x^{\rho}}{\delta\omega^a} - \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\psi)}\frac{\delta\psi}{\delta\omega^a},$$

$$\delta\omega^a = \alpha, \quad \frac{\delta x}{\delta\alpha} = 0, \quad \frac{\delta\psi}{\delta\alpha} = i\psi,$$

$$J^{\mu} = -\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\psi)}\frac{\delta\psi}{\delta\omega^a} = -i\bar{\psi}\gamma^{\mu}(i\psi) = \bar{\psi}\gamma^{\mu}\psi. \qquad (2.182)$$

守恒荷：

$$Q=\int \mathrm{d}^3 x\,\bar{\psi}\gamma^0\psi=\int \mathrm{d}^3 x\,\psi^+\psi,$$

$$\psi=\psi_L+\psi_R,$$

$$\psi_L=\frac{1}{2}(1-\gamma_5)\psi,\quad \psi_R=\frac{1}{2}(1+\gamma_5)\psi,$$

$$Q=\int \mathrm{d}^3 x(\psi_L^++\psi_R^+)(\psi_L+\psi_R),$$

由于$(1-\gamma_5)(1+\gamma_5)=0$，则

$$\psi_L^+\psi_R=\psi_R^+\psi_L=0,$$

$$\therefore\qquad Q=\int d^3x(\psi_L^+\psi_L+\psi_R^+\psi_R).\qquad(2.184)$$

2.6.2 平面波解、投影算符

1. Dirac 方程的平面波解

$$(i\not\partial - m)\psi = 0, \tag{2.174}$$

其平面波解可写为

$$\begin{aligned} \psi^{(+)}(x) &= e^{-ikx}u(k), \quad \text{（正能）} \\ \psi^{(-)}(x) &= e^{ikx}v(k), \quad \text{（负能）} \end{aligned} \tag{2.185}$$

$kx = k_\mu x^\mu$, $k^2 = m^2$. u和v满足方程：

$$\begin{aligned} (\not k - m)u(k) &= 0, \\ (\not k + m)v(k) &= 0. \end{aligned} \qquad (\not k = \gamma^\mu k_\mu) \tag{2.186}$$

由

$$(\not k - m)(\not k + m) = k^2 - m^2 = 0,$$

方程(2.186)的解为

$$\begin{aligned} u^{(\alpha)}(k) &= \frac{\not k + m}{\sqrt{2m(k^0+m)}} u_0^{(\alpha)} = \frac{\not k + m}{\sqrt{2m(k^0+m)}} \begin{pmatrix} \varphi^{(\alpha)} \\ 0 \end{pmatrix}, \\ v^{(\alpha)}(k) &= \frac{-\not k + m}{\sqrt{2m(k^0+m)}} v_0^{(\alpha)} = \frac{-\not k + m}{\sqrt{2m(k^0+m)}} \begin{pmatrix} 0 \\ \chi^{(\alpha)} \end{pmatrix}, \end{aligned} \tag{2.187}$$

其中$\varphi^{(\alpha)}$和$\chi^{(\alpha)}$$(\alpha = 1,2)$均为2分量旋量。

$\varphi^{(\alpha)}$和$\chi^{(\alpha)}$的两种形式：

① 取为静止系中Pauli矩阵σ^3的本征态

$$\varphi^{(\alpha)}(m,0)=\chi^{(\alpha)}(m,0)=\begin{cases}\begin{pmatrix}1\\0\end{pmatrix} & \alpha=1\\ \begin{pmatrix}0\\1\end{pmatrix} & \alpha=2\end{cases} \tag{2.188}$$

$$\sigma^3\begin{Bmatrix}\varphi^{(\alpha)}(m,0)\\ \chi^{(\alpha)}(m,0)\end{Bmatrix}=\varepsilon_\alpha\begin{Bmatrix}\varphi^{(\alpha)}(m,0)\\ \chi^{(\alpha)}(m,0)\end{Bmatrix} \qquad \varepsilon_1=-\varepsilon_2=1$$

② 取为$\vec{\sigma}\cdot\hat{k}(\hat{k}=\vec{k}/|\vec{k}|)$的本征态

$$\varphi^{(\alpha)}(\hat{k})=\chi^{(\alpha)}(\hat{k})=\begin{cases}\begin{pmatrix}\cos\theta/2\\ \sin\dfrac{\theta}{2}e^{i\varphi}\end{pmatrix} & \alpha=1\\ \begin{pmatrix}-\sin\dfrac{\theta}{2}e^{-i\varphi}\\ \cos\theta/2\end{pmatrix} & \alpha=2\end{cases} \tag{2.190}$$

$(\theta,\varphi)=\hat{k}$,

$$\vec{\sigma}\cdot\hat{k}\begin{cases}\varphi^{(\alpha)}(\hat{k})\\ \chi^{(\alpha)}(\hat{k})\end{cases}=\varepsilon_\alpha\begin{cases}\varphi^{(\alpha)}(\hat{k})\\ \chi^{(\alpha)}(\hat{k})\end{cases},\qquad \varepsilon_1=-\varepsilon_2=1$$

(2.187)的共轭旋量为:

$$\bar{u}^{(\alpha)}(k) \equiv u^{(\alpha)+}(k)\gamma^0 = \bar{u}_0^{(\alpha)} \frac{\not{k}+m}{\sqrt{2m(k^0+m)}},$$

$$\bar{v}^{(\alpha)}(k) \equiv v^{(\alpha)+}(k)\gamma^0 = \bar{v}_0^{(\alpha)} \frac{-\not{k}+m}{\sqrt{2m(k^0+m)}}. \tag{2.192}$$

***u*, *v*的正交归一关系**

$$\bar{u}^{(\alpha)}(k)u^{(\beta)}(k) = \delta_{\alpha\beta}, \quad \bar{u}^{(\alpha)}(k)v^{(\beta)}(k) = 0,$$

$$\bar{v}^{(\alpha)}(k)v^{(\beta)}(k) = -\delta_{\alpha\beta}, \quad \bar{v}^{(\alpha)}(k)u^{(\beta)}(k) = 0. \tag{2.193}$$

2. 投影算符

① 定义

定义矩阵

$$\Lambda_+(k) \equiv \sum_{\alpha=1,2} u^{(\alpha)}(k)\overline{u}^{(\alpha)}(k)$$

$$= \sum_{\alpha=1,2} \frac{(\not{k}+m)}{\sqrt{2m(k^0+m)}} u_0^{(\alpha)}\overline{u}_0^{(\alpha)} \frac{(\not{k}+m)}{\sqrt{2m(k^0+m)}}$$

$$= \frac{1}{2m(k^0+m)}(\not{k}+m)\sum_{\alpha=1,2} u_0^{(\alpha)}u_0^{(\alpha)+}\gamma^0(\not{k}+m)$$

$$\Lambda_{+}(k) = C(\not{k}+m)\sum_{\alpha=1,2}\begin{pmatrix}\varphi^{(\alpha)} \\ 0\end{pmatrix}\begin{pmatrix}\varphi^{(\alpha)+} & 0\end{pmatrix}\begin{pmatrix}I & 0 \\ 0 & -I\end{pmatrix}(\not{k}+m)$$

$$= C(\not{k}+m)\sum_{\alpha=1,2}\begin{pmatrix}\varphi^{(\alpha)}\varphi^{(\alpha)+} & 0 \\ 0 & 0\end{pmatrix}(\not{k}+m)$$

$$= C(\not{k}+m)\begin{pmatrix}I & 0 \\ 0 & 0\end{pmatrix}(\not{k}+m)$$

$$= C(\not{k}+m)\frac{I+\gamma^{0}}{2}(\not{k}+m)$$

$$= \frac{C}{2}[(\not{k}+m)(\not{k}+m)+(\not{k}+m)\gamma^{0}(\not{k}+m)]$$

利用等式

$$(\not{k}+m)\gamma^0(\not{k}+m)=2k^0(\not{k}+m),\quad (\text{对}k^2=m^2)$$

则

$$\begin{aligned}\Lambda_+(k)&=\frac{C}{2}[2m\not{k}+2m^2+2k^0(\not{k}+m)]\\&=\frac{1}{2m(k^0+m)}(k^0+m)(\not{k}+m)\\&=\frac{(\not{k}+m)}{2m}.\end{aligned}\tag{2.194}$$

$\Lambda_+(k)$ — 正能投影算符。

类似地，定义矩阵

$$\Lambda_-(k) \equiv -\sum_{\alpha=1,2} v^{(\alpha)}(k)\bar{v}^{(\alpha)}(k)$$

$$= \frac{-\not{k}+m}{2m}, \qquad (2.195)$$

$\Lambda_-(k)$—负能投影算符。

证明：$(\not{k}+m)\gamma^0(\not{k}+m)=2k^0(\not{k}+m)$.

$$(\not{k}+m)\gamma^0(\not{k}+m)$$

$$=\not{k}\gamma^0(\not{k}+m)+m\gamma^0\not{k}+\gamma^0 m^2$$

$$=\not{k}\gamma^0(\not{k}+m)+m\gamma^0\not{k}+\gamma^0\not{k}\not{k}$$

$$=\not{k}\gamma^0(\not{k}+m)+\gamma^0\not{k}(\not{k}+m)$$

$$=(\not{k}\gamma^0+\gamma^0\not{k})(\not{k}+m)$$

$\because \not{k}\gamma^0+\gamma^0\not{k}=\gamma^0 k^0\gamma^0+\gamma^i k_i\gamma^0+\gamma^0\gamma^0 k^0+\gamma^0\gamma^i k_i=2k^0$,

$\therefore (\not{k}+m)\gamma^0(\not{k}+m)=2k^0(\not{k}+m)$.

② 性质

1) $\Lambda_{\pm}^{2}(k)=\Lambda_{\pm}(k)$;

证明：$$\Lambda_{+}^{2}(k)=\left[\frac{(\not{k}+m)}{2m}\right]^{2}=\frac{k^{2}+2m\not{k}+m^{2}}{4m^{2}}$$

$$=\frac{2m^{2}+2m\not{k}}{4m^{2}}=\frac{(\not{k}+m)}{2m}=\Lambda_{+}(k).$$

2) $\mathrm{Tr}\,\Lambda_{\pm}(k)=2$;

证明:$$\mathrm{Tr}\,\Lambda_{\pm}(k)=\mathrm{Tr}\left(\frac{\pm\not{k}}{2m}+\frac{1}{2}I\right)=0+\frac{1}{2}\times 4=2.$$

3) $\Lambda_{+}(k)+\Lambda_{-}(k)=I.$

3. u和v 的极化自由度

$$S^{\rho\sigma}=\frac{1}{2}\sigma^{\rho\sigma}, \qquad \sigma^{\rho\sigma}=\frac{i}{2}[\gamma^{\rho},\gamma^{\sigma}],$$

Pauli – Lubanski 矢量：

$$W_{\mu}=-\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}S^{\rho\sigma}\partial^{\nu}=-\frac{i}{4}\varepsilon_{\mu\nu\rho\sigma}\sigma^{\rho\sigma}\partial^{\nu}$$

$$=-\frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}\sigma^{\rho\sigma}k^{\nu}, \tag{2.202}$$

利用γ矩阵的性质： $\gamma_5\sigma_{\mu\nu}=\frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}\sigma^{\rho\sigma}$, **则**

$$W_{\mu}=\frac{i}{2}\gamma_5\sigma_{\mu\nu}k^{\nu}. \tag{2.203}$$

引入类空单位矢量,

$$n^2 = n_\mu n^\mu = -1, \qquad n_\mu k^\mu = 0, \tag{2.205}$$

则有

$$-\frac{W \cdot n}{m} = -\frac{W_\mu n^\mu}{m}$$

$$= -\frac{1}{m}\frac{i}{2}\gamma_5 \sigma_{\mu\nu} k^\nu n^\mu$$

$$= -\frac{i}{2m}\gamma_5 \cdot \frac{i}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu) k^\nu n^\mu$$

$$= \frac{1}{4m}\gamma_5 (2\gamma_\mu \gamma_\nu - 2g_{\mu\nu}) k^\nu n^\mu$$

$$-\frac{W\cdot n}{m}=\frac{1}{2m}\gamma^5(\not{n}\not{k}-n\cdot k)$$

$$=\frac{1}{2m}\gamma^5\not{n}\not{k}. \qquad (2.206)$$

设$t\equiv(1,0,0,0)$是k矢量空间中的时间轴，

则(t,k)平面内的4矢量

$$n=\left(k\frac{k\cdot t}{m^2}-t\right)\frac{m}{|\vec{k}|}$$

满足(2.205)。

$$n^2 = \left(k^\mu \frac{k\cdot t}{m^2} - t^\mu\right)\left(k_\mu \frac{k\cdot t}{m^2} - t_\mu\right)\left(\frac{m}{|\vec{k}\,|}\right)^2$$

$$= \left[\frac{k^2(k\cdot t)^2}{m^4} - \frac{2(k\cdot t)^2}{m^2} + t^2\right]\frac{m^2}{|\vec{k}\,|^2}$$

$$= [\frac{k_0{}^2}{m^2} - \frac{2k_0{}^2}{m^2} + t^2]\frac{m^2}{\left|\vec{k}\right|^2} = \frac{m^2 - k_0{}^2}{m^2}\cdot\frac{m^2}{|\vec{k}\,|^2}$$

$$= \frac{k_0{}^2 - |\vec{k}\,|^2 - k_0{}^2}{|\vec{k}\,|^2} = -1;$$

$$n\cdot k = \left(k_\mu \frac{k\cdot t}{m^2} - t_\mu\right)\frac{m}{|\vec{k}\,|}k^\mu = \left(\frac{k^2(k\cdot t)}{m^2} - k\cdot t\right)\frac{m}{|\vec{k}\,|} = 0.$$

将(2.207)代入(2.206)，得到

$$-\frac{W\cdot n}{m}=\frac{1}{2m}\gamma^{5}\left(\not{k}\frac{k\cdot t}{m^{2}}-\not{t}\right)\frac{m}{|\vec{k}|}\not{k}$$

$$=\frac{1}{2}\gamma^{5}\left[\frac{k^{2}(k\cdot t)}{m^{2}}-\not{t}\not{k}\right]\frac{1}{|\vec{k}|}$$

$$=\frac{1}{2}\gamma^{5}\left(k\cdot t-\not{t}\not{k}\right)\frac{1}{|\vec{k}|}$$

$$=\frac{1}{2}\gamma^{5}\left(k_{0}-\gamma^{0}\not{k}\right)\frac{1}{|\vec{k}|}$$

$$=\frac{1}{2}\gamma^{5}\left[k_{0}-\gamma^{0}\left(\gamma^{0}k_{0}-\gamma^{i}k_{i}\right)\right]\frac{1}{|\vec{k}|}$$

$$-\frac{W\cdot n}{m}=\frac{1}{2}\gamma^5\gamma^0\gamma^i\ \frac{k_i}{|\vec{k}\,|}=\frac{1}{2}\gamma^5\gamma^0\vec{\gamma}\cdot\vec{k}\ \frac{1}{|\vec{k}\,|}$$

$$=\frac{1}{2}\vec{\Sigma}\cdot\hat{k}, \tag{2.208}$$

其中

$$\vec{\Sigma}=\gamma_5\gamma^0\vec{\gamma}=\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix},$$

$-\dfrac{W\cdot n}{m}$是$+\vec{k}$方向的自旋投影算符。

$$[\vec{\Sigma}\cdot\hat{k}, \not{k}] = \left[\begin{pmatrix} \vec{\sigma}\cdot\vec{k} & 0 \\ 0 & \vec{\sigma}\cdot\vec{k} \end{pmatrix}, \begin{pmatrix} k_0 & -\vec{\sigma}\cdot\vec{k} \\ \vec{\sigma}\cdot\vec{k} & -k_0 \end{pmatrix}\right] = 0,$$

$$\frac{1}{2}\vec{\Sigma}\cdot\hat{k}u^{(\alpha)}(k)$$

$$= \frac{1}{2}\vec{\Sigma}\cdot\hat{k}\,\frac{\not{k}+m}{\sqrt{2m(k^0+m)}}\begin{pmatrix} \varphi^{(\alpha)}(\hat{\mathbf{k}}) \\ 0 \end{pmatrix}$$

$$= \frac{\not{k}+m}{\sqrt{2m(k^0+m)}}\,\frac{1}{2}\begin{pmatrix} \vec{\sigma}\cdot\hat{k} & 0 \\ 0 & \vec{\sigma}\cdot\hat{k} \end{pmatrix}\begin{pmatrix} \varphi^{(\alpha)}(\hat{k}) \\ 0 \end{pmatrix}$$

$$\frac{1}{2}\vec{\Sigma}\cdot\hat{k}u^{(\alpha)}(k)=\frac{1}{2}\frac{\not{k}+m}{\sqrt{2m(k^0+m)}}\begin{pmatrix}(\vec{\sigma}\cdot\hat{k})\varphi^{(\alpha)}(\hat{k})\\ 0\end{pmatrix}$$

$$=\frac{1}{2}\frac{\not{k}+m}{\sqrt{2m(k^0+m)}}\begin{pmatrix}\varepsilon_\alpha\varphi^{(\alpha)}(\hat{k})\\ 0\end{pmatrix}$$

$$=\frac{\varepsilon_\alpha}{2}u^{(\alpha)}(k),$$

$$\frac{1}{2}\vec{\Sigma}\cdot\hat{k}v^{(\alpha)}(k)=\frac{\varepsilon_\alpha}{2}v^{(\alpha)}(k),$$

$$\varepsilon_1=-\varepsilon_2=1.$$

2.6.3 零质量粒子

1. Lagrangian与守恒量

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi, \tag{2.214}$$

在整体手征变换下，

$$\psi \to \psi' = e^{i\beta\gamma_5}\psi,$$

$$\bar{\psi} \to \bar{\psi}' = \psi'^{+}\gamma^{0} = \psi^{+}e^{-i\beta\gamma_5}\gamma^{0} = \bar{\psi}e^{i\beta\gamma_5},$$

$$\mathcal{L}' = i\bar{\psi}'\gamma^{\mu}\partial_{\mu}\psi' = i\bar{\psi}e^{i\beta\gamma_5}\gamma^{\mu}\partial_{\mu}e^{i\beta\gamma_5}\psi$$

$$= i\bar{\psi}e^{i\beta\gamma_5}\gamma^{\mu}e^{i\beta\gamma_5}\partial_{\mu}\psi = i\bar{\psi}e^{i\beta\gamma_5}e^{-i\beta\gamma_5}\gamma^{\mu}\partial_{\mu}\psi$$

$$= i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi = \mathcal{L},$$

相应的守恒流为

$$J_5^{\mu} = \overline{\psi}\gamma^{\mu}\gamma_5\psi, \tag{2.216}$$

$$\partial_{\mu}J_5^{\mu} = 0, \tag{2.217}$$

守恒荷为

$$Q_5 = \int d^3x\overline{\psi}\gamma^0\gamma_5\psi = \int d^3x\psi^{+}\gamma_5\psi,$$

利用 :$\psi = \psi_L + \psi_R$, $\psi_L = \frac{1}{2}(1-\gamma_5)\psi$, $\psi_R = \frac{1}{2}(1+\gamma_5)\psi$,

$$\gamma_5\psi_L = -\psi_L, \qquad \gamma_5\psi_R = +\psi_R.$$

$$\Rightarrow \quad Q_5 = \int \mathrm{d}^3x(\psi_R^{+}\psi_R - \psi_L^{+}\psi_L). \tag{2.218}$$

$\mathcal{L}$中的质量项将破坏手征不变性：

$$-m\bar{\psi}\psi \to -m\bar{\psi}'\psi' = -m\bar{\psi}e^{i\beta\gamma_5}e^{i\beta\gamma_5}\psi = -m\bar{\psi}e^{2i\beta\gamma_5}\psi.$$

由(2.214)得到的运动方程是：

$$\not{P}\psi = 0, \tag{2.220}$$

$$P_\mu = i\partial_\mu, \qquad P_\mu = (P^0, -\vec{P}),$$

$$P^2 = m^2 = 0, \qquad |P^0| = |\vec{P}|.$$

2. 手征性与 helicity 的关系

以$\gamma_5\gamma^0$左乘(2.220)，得

$$0=\gamma_5\gamma^0\not{P}\psi=\gamma_5\gamma^0(\gamma^0P^0-\vec{\gamma}\cdot\vec{P})\psi$$

$$=(\gamma_5P^0-\gamma_5\gamma^0\vec{\gamma}\cdot\vec{P})\psi=(\gamma_5P^0-\vec{\Sigma}\cdot\vec{P})\psi,$$

$$\Rightarrow\quad \gamma_5P^0=\vec{\Sigma}\cdot\vec{P},\qquad \text{或}\qquad \frac{\vec{\Sigma}\cdot\vec{P}}{|\vec{P}|}=\gamma_5\varepsilon(P^0),$$

$$\therefore\quad \gamma_5=\frac{\vec{\Sigma}\cdot\vec{P}}{|\vec{P}|}\begin{cases}+ & \text{对}P^0>0\ \text{（同）}\\ - & \text{对}P^0<0\ \text{（反）}\end{cases}\tag{2.223}$$

3. 平面波解的显示形式

$$\psi(x)=\begin{cases} e^{-ikx}u^{(\alpha)}(k), \\ e^{ikx}v^{(\alpha)}(k), \end{cases} \qquad k^2=0, \quad k^0=|\vec{k}|>0, \tag{2.224}$$

$$\vec{\Sigma}\cdot\hat{k}\begin{cases} u^{(\alpha)}(k) \\ v^{(\alpha)}(k) \end{cases}=\varepsilon_\alpha\begin{cases} u^{(\alpha)}(k), \\ v^{(\alpha)}(k), \end{cases} \qquad (\varepsilon_1=-\varepsilon_2=1) \tag{2.225}$$

$$\gamma_5\begin{cases} u^{(\alpha)}(k) \\ v^{(\alpha)}(k) \end{cases}=\varepsilon_\alpha\begin{cases} u^{(\alpha)}(k), \\ v^{(\alpha)}(k). \end{cases} \qquad (\varepsilon_1=-\varepsilon_2=1) \tag{2.226}$$

Dirac表象中，

$$\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$$u^{(\alpha)}(k) = \sqrt{k^0}\begin{pmatrix} \varphi^{(\alpha)}(\hat{k}) \\ \varepsilon_\alpha \varphi^{(\alpha)}(\hat{k}) \end{pmatrix}, \quad (\varepsilon_1 = -\varepsilon_2 = 1) \qquad (2.227)$$

其中$\varphi^{(\alpha)}(\hat{k})$取为$\vec{\sigma}\cdot\hat{k}$的本征态，由(2.190)给出。

定义：

$$\left.\begin{aligned} v^{(1)}(k) &\equiv C\overline{u}^{T(2)}(k) = -u^{(1)}(k), \\ v^{(2)}(k) &\equiv C\overline{u}^{T(1)}(k) = -u^{(2)}(k). \end{aligned}\right. \quad (C \equiv i\gamma^2\gamma^0) \quad (2.228)$$

4. 归一化条件与自旋求和公式

归一化条件：

$$u^{(\alpha)+}(k)u^{(\beta)}(k) = v^{(\alpha)+}(k)v^{(\beta)}(k) = 2k^0\delta^{\alpha\beta},$$

$$v^{(\alpha)+}(k)u^{(\beta)}(\tilde{k}) = u^{(\alpha)+}(k)v^{(\beta)}(\tilde{k}) = 0. \qquad (2.229)$$

$$\tilde{k} = (k^0, -\vec{k})$$

自旋求和公式：

$$\sum_{\alpha=1,2} u^{(\alpha)}(k)\bar{u}^{(\alpha)}(k) = \sum_{\alpha=1,2} v^{(\alpha)}(k)\bar{v}^{(\alpha)}(k) = \not{k}. \qquad (2.230)$$

证明(2.229)式：

$$v^{(\alpha)+}(k)v^{(\beta)}(k)=u^{(\alpha)+}(k)u^{(\beta)}(k)$$

$$=k^0\left(\varphi^{(\alpha)+}(\hat{k}) \quad \varepsilon_\alpha\varphi^{(\alpha)+}(\hat{k})\right)\begin{pmatrix}\varphi^{(\beta)}(\hat{k})\\ \varepsilon_\beta\varphi^{(\beta)}(\hat{k})\end{pmatrix}$$

$$=k^0(1+\varepsilon_\alpha\varepsilon_\beta)\varphi^{(\alpha)+}(\hat{k})\varphi^{(\beta)}(\hat{k}),$$

由$\varphi^{(\alpha)+}(\hat{k})\varphi^{(\beta)}(\hat{k})=\delta^{\alpha\beta}$及$1+\varepsilon_\alpha\varepsilon_\beta=2\delta^{\alpha\beta}$，则

$$u^{(\alpha)+}(k)u^{(\beta)}(k)=v^{(\alpha)+}(k)v^{(\beta)}(k)=2k^0\delta^{\alpha\beta};$$

$$\hat{k}=(\theta,\varphi),\qquad \hat{\tilde{k}}=-\hat{k}=(\pi-\theta,\pi+\varphi),$$

$$v^{(\alpha)+}(k)u^{(\beta)}(\tilde{k})=u^{(\alpha)+}(k)v^{(\beta)}(\tilde{k})$$

$$=-u^{(\alpha)+}(k)u^{(\beta)}(\tilde{k})$$

$$=-k^{0}\begin{pmatrix}\varphi^{(\alpha)+}(\hat{k}) & \varepsilon_{\alpha}\varphi^{(\alpha)+}(\hat{k})\end{pmatrix}\begin{pmatrix}\varphi^{(\beta)}(\hat{\tilde{k}})\\ \varepsilon_{\beta}\varphi^{(\beta)}(\hat{\tilde{k}})\end{pmatrix}$$

$$=-k^{0}(1+\varepsilon_{\alpha}\varepsilon_{\beta})\varphi^{(\alpha)+}(\hat{k})\varphi^{(\beta)}(\hat{\tilde{k}}),$$

1)当$\alpha\neq\beta$时，$1+\varepsilon_{\alpha}\varepsilon_{\beta}=0$，上式$=0$;

2) 当 $\alpha=\beta=1$ 时，

$$\varphi^{(1)+}(\hat{k})\varphi^{(1)}(\hat{\tilde{k}})=\begin{pmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi}\end{pmatrix}\begin{pmatrix}\cos\frac{\pi-\theta}{2}\\ \sin\frac{\pi-\theta}{2}\mathrm{e}^{-\mathrm{i}(\pi+\varphi)}\end{pmatrix}$$

$$=\cos\frac{\theta}{2}\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\sin\frac{\theta}{2}=0;$$

3) 当 $\alpha=\beta=2$ 时，

$$\varphi^{(2)+}(\hat{k})\varphi^{(2)}(\hat{\tilde{k}})=0.$$

$$\therefore\quad v^{(\alpha)+}(k)u^{(\beta)}(\tilde{k})=u^{(\alpha)+}(k)v^{(\beta)}(\tilde{k})=0.$$

5. Weyl 旋量与 Majorana 旋量

1) Weyl 旋量场

在Weyl表象中，Dirac旋量可写成

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$\mathcal{L} = L_L + L_R$$

$$\mathcal{L}_L = i\psi_L^+ \bar{\sigma}^\mu \partial_\mu \psi_L, \qquad \bar{\sigma}^\mu = (I, -\vec{\sigma}) \qquad (2.231)$$

$$\mathcal{L}_R = i\psi_R^+ \sigma^\mu \partial_\mu \psi_R, \qquad \sigma^\mu = (I, \vec{\sigma})$$

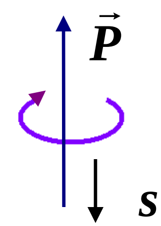
① 以ψ_L描述0质量旋量场

$$\mathcal{L}_L = i\psi_L^+ \bar{\sigma}^\mu \partial_\mu \psi_L,$$

则运动方程为

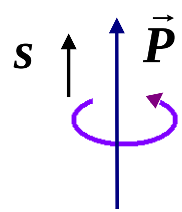
$$i\bar{\sigma}^\mu \partial_\mu \psi_L = 0 \quad \text{或} \quad (P^0 + \vec{\sigma}\cdot\vec{P})\psi_L = 0, \qquad (2.232)$$

粒子 （$P^0 > 0$） helicity负



自旋-动量反平行（左旋

反粒子（$P^0 < 0$） helicity正



自旋-动量平行 （右旋）

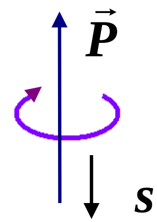
② 以ψ_R描述0质量旋量场

$$\mathcal{L}_R = i\psi_L^{+}\sigma^{\mu}\partial_{\mu}\psi_L ,$$

则运动方程为

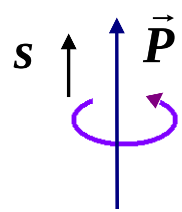
$$i\sigma^{\mu}\partial_{\mu}\psi_R = 0 \quad 或 \quad (P^0 - \vec{\sigma}\cdot\vec{P})\psi_R = 0, \qquad (2.233)$$

反粒子($P^0 < 0$)　helicity负



自旋-动量反平行（左旋

粒子（$P^0 > 0$）　helicity正



自旋-动量平行　（右旋）

2) Majorana 旋量场

动能项为

$$\mathcal{L}_M = \frac{i}{2}\overline{\psi}_M \gamma^\mu \partial_\mu \psi_M, \quad (\psi_M)^C = \psi_M, \qquad (2.234)$$

质量项为：

$$\mathcal{L}_M^m = \frac{-1}{2} m \overline{\psi}_M \psi_M. \qquad (2.235)$$

Weyl表象中，

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}, \tag{2.236}$$

$$\psi_M = \begin{pmatrix} \psi_L \\ -i\sigma^2\psi_L^* \end{pmatrix} = (\psi_M)^C \tag{2.237}$$

$$\mathcal{L}_M^m = \frac{1}{2} im(\psi_L^+\sigma^2\psi_L^* - \psi_L^T\sigma^2\psi_L). \tag{2.238}$$

2.7 Maxwell 场

vector bosons :

γ **(the electromagnetic interaction)**

$W^{\pm}, Z^{0}$ **(the weak interaction)**

g**luons (the strong interaction)**

vector mesons$(\rho, \omega, J/\psi, \psi' \cdots)$

1. Lagrangian和运动方程

$$\mathcal{L}(x) = -\frac{1}{2}[a\partial_\mu A^\nu \partial^\mu A_\nu + b\partial_\mu A^\nu \partial_\nu A^\mu + c(\partial_\mu A^\mu) + dA_\mu A^\mu], \tag{2.239}$$

Euler – Lagrange方程：

$$a\partial^\nu \partial_\nu A^\mu + (b+c)\partial^\mu \partial^\nu A_\nu - dA^\mu = 0. \tag{2.240}$$

真空中的Maxwell方程：

$$\nabla \cdot \vec{E} = 0, \quad \nabla \times \vec{B} = \frac{\partial}{\partial t}\vec{E}, \tag{2.241}$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}. \tag{2.242}$$

引入电磁场强张量

$$F^{\mu\nu}=\begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix},(F^{0i}=-E^i,\quad F^{ij}=-\varepsilon^{ijk}B^k),$$

其对偶张量为

$$\tilde{F}^{\mu\nu}=\frac{i}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}=F^{\mu\nu}(E^i\leftrightarrow \mathrm{i}\,B^i)$$

$$=i\begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix},(\tilde{F}^{0i}=-iB^i,\quad \tilde{F}^{ij}=i\varepsilon^{ijk}E^k)$$

$$F^{\mu\nu} = -F^{\nu\mu}, \qquad \tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu},$$

$$F_{\mu\nu}F^{\mu\nu} = -\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = -2(\vec{E}^2 - \vec{B}^2),$$

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\vec{E}\cdot\vec{B}.$$

真空中的Maxwell方程可写为

$$\partial_\mu F^{\mu\nu} = 0, \tag{2.241$'$}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{2.242$'$}$$

引入4维矢量势$A^\mu = (A^0, \vec{A})$,

$$\vec{B} = \nabla \times \vec{A}, \qquad [B^i = \partial_j (\partial^j A^i - \partial^i A^j)]$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla A^0, \qquad [E^i = \partial^i A^0 - \partial^0 A^i]$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

则

$$\partial_\mu \tilde{F}^{\mu\nu} = \partial_\mu \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$= \frac{i}{2} \varepsilon^{\mu\nu\rho\sigma} (\partial_\mu \partial_\rho A_\sigma - \partial_\mu \partial_\sigma A_\rho) = 0,$$

$$\partial_\nu F^{\nu\mu} = 0,$$

$$\Rightarrow \qquad \partial_\nu \partial^\nu A^\mu - \partial^\mu \partial_\nu A^\nu = 0, \tag{2.246}$$

上式与Euler – Lagrange方程

$$a\partial^\nu \partial_\nu A^\mu + (b+c)\partial^\mu \partial^\nu A_\nu - dA^\mu = 0 \tag{2.240}$$

比较，得到

$$a = 1, \qquad b + c = -1, \qquad d = 0.$$

Lagrangian为

$$\mathcal{L}(x) = -\frac{1}{2}[a\partial_\mu A^\nu \partial^\mu A_\nu + b\partial_\mu A^\nu \partial_\nu A^\mu + c(\partial_\mu A^\mu) + dA_\mu A^\mu]$$

$$= -\frac{1}{2}(\partial_\mu A^\nu \partial^\mu A_\nu - \partial_\mu A^\nu \partial_\nu A^\mu) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.248}$$

2. 规范不变性与规范条件

$$A^{\mu}(x) \to A'^{\mu}(x) = A^{\mu}(x) + \partial^{\mu}\alpha(x), \text{（规范变换）} \tag{2.249}$$

$$F^{\mu\nu} \to F'^{\mu\nu} = F^{\mu\nu}, \qquad \mathcal{L} \to \mathcal{L}' = \mathcal{L}.$$

1) 辐射规范（横规范）

$$\nabla \cdot \vec{A} = 0, \qquad A^{0} = 0. \tag{2.250}$$

优点：没有非物理自由度；

缺点：失去明显的Lorentz协变性。

2) Lorentz 规范

$$\partial_\nu A^\nu(x) = 0, \tag{2.251}$$

作规范变换：

$$A'^\nu = A^\nu - \partial^\nu \alpha,$$

$$\partial_\nu A'^\nu = \partial_\nu (A^\nu - \partial^\nu \alpha) = \partial_\nu A^\nu - \Box \alpha,$$

选择α使$\Box \alpha = \partial_\nu A^\nu$，则$\partial_\nu A'^\nu = 0$。

在Lorentz规范下，Maxwell方程(2.246)成为

$$\Box A^\mu(x) = 0, \tag{2.252}$$

此时A^μ仍有规范任意性，$A^\mu \to A'^\mu = A^\mu + \partial^\mu \alpha$,

当$\Box \alpha = 0$时，$\Box A'^\mu = 0$.

在Lagrangian中引入Lagrange乘子λ,

$$\mathcal{L}(x) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial\cdot A)^2, \qquad (2.254)$$

运动方程为

$$\Box A^{\mu} + (\lambda - 1)\partial^{\mu}(\partial\cdot A) = 0. \qquad (2.255)$$

两边取4散度，得

$$\Box\partial_{\mu}A^{\mu} + (\lambda - 1)\Box(\partial\cdot A) = \lambda\Box(\partial\cdot A) = 0,$$

$$\Rightarrow \quad \Box\partial_{\mu}A^{\mu} = 0 \quad (\text{对}\lambda \neq 0), \qquad (2.256)$$

由此可证：若$t=0$时刻，$\chi\equiv\partial_\mu A^\mu=0$， $\frac{\partial}{\partial t}\chi=0$，

则在任意时刻t有$\chi=0$。

证：将χ对t作级数展开，

$$\chi=\chi\big|_{t=0}+\frac{\partial\chi}{\partial t}\bigg|_{t=0}t+\frac{1}{2}\frac{\partial^2\chi}{\partial t^2}\bigg|_{t=0}t^2+\cdots$$

由(2.256)，有

$$\Box\chi=(\frac{\partial^2}{\partial t^2}-\nabla^2)\chi=0,$$

可得

$$\left.\frac{\partial^2 \chi}{\partial t^2}\right|_{t=0} = \nabla^2 \chi\big|_{t=0} = 0,$$

$$\left.\frac{\partial^3 \chi}{\partial t^3}\right|_{t=0} = \nabla^2 \left.\frac{\partial^2 \chi}{\partial t^2}\right|_{t=0} = 0,$$

....

因而χ展开式中所有项为0，

$\therefore$ 在任意时刻t，$\chi \equiv \partial_\mu A^\mu = 0$。

方程(2.255) $\xleftrightarrow{\text{等价于}}$ Lorentz规范下的Maxwell方程。

3. 守恒流与守恒荷

$$\mathcal{L}(x) = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{\lambda}{2}(\partial \cdot A)^2, \tag{2.254}$$

守恒流密度：

$$j^{\mu,a} = -\left[\mathcal{L}g^{\mu\lambda} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)}\partial^\lambda A_\sigma\right]\frac{\delta x_\lambda}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)}\frac{\delta A_\lambda}{\delta \omega_a},$$

$$\frac{\partial}{\partial(\partial_\mu A_\rho)}(F_{\alpha\beta}F^{\alpha\beta}) = 4F^{\mu\rho},$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} = -F^{\mu\sigma} - \lambda g^{\mu\sigma}(\partial \cdot A),$$

① 时空平移变换

$$\delta x_\lambda = \varepsilon_\lambda, \qquad \delta A_\lambda = 0,$$

取$a=\nu$, $\delta\omega_a = \delta\omega_\nu = \varepsilon_\nu$, $\dfrac{\delta x_\lambda}{\delta\omega_a} = \dfrac{\delta x_\lambda}{\delta\omega_\nu} = g^\nu_\lambda$,

$$\begin{aligned}\therefore\ j^{\mu\nu} &= -\left[\mathcal{L} g^{\mu\lambda} + \left(F^{\mu\sigma} + \lambda g^{\mu\sigma}(\partial\cdot A)\right)\partial^\lambda A_\sigma\right] g^\nu_\lambda \\ &= -\left[\mathcal{L} g^{\mu\nu} + \left(F^{\mu\sigma} + \lambda g^{\mu\sigma}(\partial\cdot A)\right)\partial^\nu A_\sigma\right] \\ &= [\frac{1}{4}F^2 + \frac{\lambda}{2}(\partial\cdot A)^2] g^{\mu\nu} \\ &\quad -[F^{\mu\sigma} + \lambda g^{\mu\sigma}(\partial\cdot A)]\partial^\nu A_\sigma,\end{aligned} \tag{2.258}$$

$j^{\mu\nu}$为能–动量张量密度，守恒的动量为

$$P^\nu = \int \mathrm{d}^3 x j^{0\nu}. \tag{2.259}$$

② Lorentz 变换

$$\delta x_\lambda = \varepsilon_{\lambda\sigma} x^\sigma, \qquad \delta A_\lambda = \varepsilon_{\lambda\sigma} A^\sigma,$$

取$a = \nu\rho,\ \delta\omega_a = \delta\omega_{\nu\rho} = \varepsilon_{\nu\rho},$

$$\frac{\delta x_\lambda}{\delta\omega_a} = \frac{\delta x_\lambda}{\delta\omega_{\nu\rho}} = \frac{1}{2}(-g_\lambda^\rho x^\nu + g_\lambda^\nu x^\rho),$$

$$\frac{\delta A_\lambda}{\delta\omega_a} = \frac{\delta A_\lambda}{\delta\omega_{\nu\rho}} = \frac{1}{2}(-g_\lambda^\rho A^\nu + g_\lambda^\nu A^\rho),$$

$$\therefore\ j^{\mu\nu\rho} = -\left[\mathcal{L} g^{\mu\lambda} + F^{\mu\sigma} + \lambda g^{\mu\sigma}(\partial\cdot A)\partial^\lambda A_\sigma\right]$$

$$\times\frac{1}{2}(-g_\lambda^\rho x^\nu + g_\lambda^\nu x^\rho)$$

$$-[-F^{\mu\lambda} - \lambda g^{\mu\lambda}(\partial\cdot A)]\cdot\frac{1}{2}(-g_\lambda^\rho A^\nu + g_\lambda^\nu A^\rho)$$

$$
\begin{aligned}
j^{\mu\nu\rho} = -\frac{1}{2}\{ & -x^{\nu}[\mathcal{L}g^{\mu\rho} + F^{\mu\sigma} + \lambda g^{\mu\sigma}(\partial\cdot A)\partial^{\lambda}A_{\sigma}] \\
& + x^{\rho}[\mathcal{L}g^{\mu\nu} + F^{\mu\sigma} + \lambda g^{\mu\sigma}(\partial\cdot A)\partial^{\nu}A_{\sigma}]\} \\
& -\frac{1}{2}\{A^{\nu}[F^{\mu\rho} + \lambda g^{\mu\rho}(\partial\cdot A)] \\
& - A^{\rho}[F^{\mu\nu} + \lambda g^{\mu\nu}(\partial\cdot A)]\} \\
= -\frac{1}{2}\{ & x^{\nu}j^{\mu\rho} - A^{\rho}j^{\mu\nu} + A^{\nu}[F^{\mu\rho} + \lambda g^{\mu\rho}(\partial\cdot A)] \\
& - A^{\rho}[F^{\mu\nu} + \lambda g^{\mu\nu}(\partial\cdot A)]\},
\end{aligned}
\tag{2.260}
$$

守恒的广义角动量为

$$
M^{\nu\rho} = \int \mathrm{d}^3 x j^{0\nu\rho}. \tag{2.261}
$$