1. 考虑一个由电子和正电子组成的体系，粒子之间存在由最小电磁耦合所描写的电磁相互作用。

(1) 写出该体系的拉氏密度。 (2) 写出体系所满足的运动方程。

解：(1) $\mathcal{L}_{Dirac} = :\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi:$

$\mathcal{L}_{e.m} = :-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial\cdot A)^2:$ $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu \partial_\mu$.

$\mathcal{L}_i = -e:\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x):$

$\mathcal{L} = :\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi: + :\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial\cdot A)^2: + -e:\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x):$

2) 对 ψ、$\bar{\psi}$、A_μ

① 对于 ψ：

$\partial_\mu\frac{\partial\mathcal{L}_{Dirac}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}_{Dirac}}{\partial\psi} = \partial_\mu(\bar{\psi}i\gamma^\mu) + m\bar{\psi} = -\bar{\psi}(-\partial_\mu\gamma^\mu i - m) = -\bar{\psi}(\gamma^\mu\overleftarrow{\partial_\mu}i - m)$

$\partial_\mu\frac{\partial\mathcal{L}_i}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}_i}{\partial\psi} = 0 + e:\bar{\psi}(x)\gamma^\mu A_\mu(x):$

$\partial_\mu\frac{\partial\mathcal{L}_{e.m}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}_{e.m}}{\partial\psi} = 0$

$\therefore \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} - \frac{\partial\mathcal{L}}{\partial\psi} = -\bar{\psi}(\gamma^\mu\partial_\mu i - m) + e:\bar{\psi}(x)\gamma^\mu A_\mu(x): = 0$

$\therefore \bar{\psi}(i\gamma^\mu\overleftarrow{\partial_\mu} - m) = e:\bar{\psi}(x)\gamma^\mu A_\mu(x):$

② 对于 $\bar{\psi}$：

$\partial_\mu\frac{\partial\mathcal{L}_{Dirac}}{\partial(\partial_\mu\bar{\psi})} - \frac{\partial\mathcal{L}_{Dirac}}{\partial\bar{\psi}} = 0 - (i\gamma^\mu\partial_\mu - m)\psi$

$\partial_\mu\frac{\partial\mathcal{L}_{e.m.}}{\partial(\partial_\mu\bar{\psi})} - \frac{\partial\mathcal{L}_{e.m.}}{\partial\bar{\psi}} = 0$

$\partial_\mu\frac{\partial\mathcal{L}_i}{\partial(\partial_\mu\bar{\psi})} - \frac{\partial\mathcal{L}_i}{\partial\bar{\psi}} = +e:\gamma^\mu\psi(x)A_\mu(x):$

$\therefore \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} - \frac{\partial\mathcal{L}}{\partial\bar{\psi}} = (i\gamma^\mu\partial_\mu - m)\psi - e:\gamma^\mu\psi(x)A_\mu(x): = 0$

$\therefore (i\gamma^\mu\partial_\mu - m)\psi = e:\gamma^\mu\psi(x)A_\mu(x):$

③ 对于 A_μ :

$$\partial_\mu \frac{\partial \mathcal{L}_{Dirac}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

$$\partial_\mu \frac{\partial \mathcal{L}_{e.m}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}_{e.m.}}{\partial A_\nu} =$$

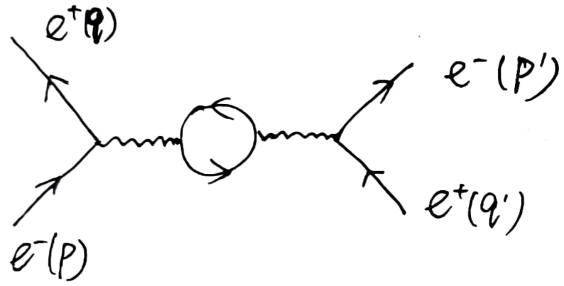
$$= \partial_\mu \partial^\rho A^\mu - \partial_\mu \partial^\mu A_\mu - \partial_\mu \partial_\rho A^\mu$$

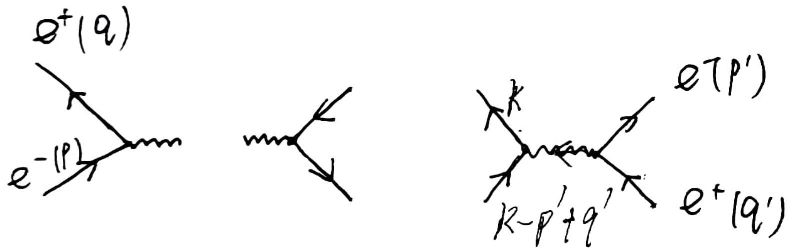
$$= +\Box A^\mu$$

$$\partial_\mu \frac{\partial \mathcal{L}_i}{\partial(\partial_\mu A_\mu)} - \frac{\partial \mathcal{L}_i}{\partial A_\mu} = -e_i \bar\psi \gamma^\mu \psi ;$$

$$\therefore \quad \Box A^\mu = e_i \bar\psi \gamma^\mu \psi ;$$

三. 写出如下 Feynman 图的 S矩阵元.





$p'-q'$

$$\langle f|S|i\rangle = (2\pi)^4 \delta^4(p'+q'-p-q) \cdot \bar u(p')(-ie\gamma^\mu) v(q') \frac{-ig_{\mu\nu}}{(p'-q')^2+i\varepsilon}$$

$$\int dk' \frac{i}{\not k' - m + i\varepsilon} (-ie\gamma^\mu) \frac{i}{\not k - m + i\varepsilon} (-ie\gamma^\mu) \frac{-ig_{\mu\nu}}{(p'-q')^2+i\varepsilon}$$

$$\bar v(q) (-ie\gamma^\mu) u(p)$$

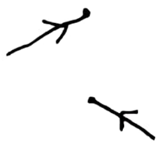
四. 电子质量为 m_e，考虑 Møller 散射过程 $e^-(p)+e^-(p')\to e^-(k)+e^-(k')$
其中括号内的量为各粒子的四动量.

(1) 写出相应的 Feynman 规则.

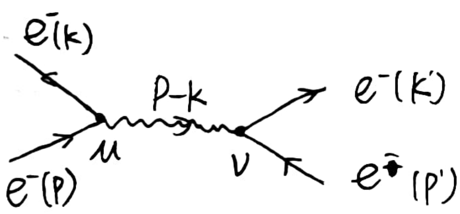
(2) 画出该散射过程相应的 Feynman 图(至二级微扰展开).

(3) 利用 Feynman 规则，求该散射过程的非极化散射截面(至二阶微扰展开).

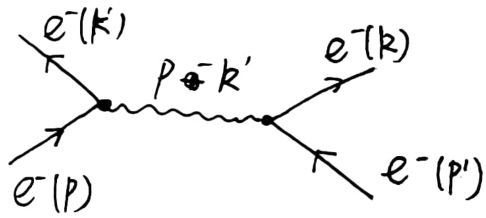
解: 1) Feynman 规则.

	图形	S矩阵中的因子	
Fermion子外线		$\frac{i}{\not p - m + i\varepsilon}$	$u(p)$ $\bar{u}(p')$
光子内线		$\frac{-i g_{\mu\nu}}{k^2+i\varepsilon}$	
电磁作用顶角		$-ie\gamma^\mu$	
质量抵消项顶角		$i\delta m$	

2)



(a)



(b)

3)
$$T(a) = \bar{u}(k')(-ie\gamma^\nu)u(p')\frac{-ig_{\mu\nu}}{(p-k)^2+i\varepsilon}\bar{u}(k)(-ie\gamma^\mu)u(p)$$

$$T(b) = \bar{u}(k)(-ie\gamma^\nu)u(p')\frac{-ig_{\mu\nu}}{(p-k')+i\varepsilon}\bar{u}(k')(-ie\gamma^\mu)u(p)$$

总振幅 $A = W_aT_a + W_bT_b = T_a + T_b$

$$= (-ie)^2\left[\bar{u}(k')\gamma^\mu u(p')\frac{-ig_{\mu\nu}}{(p-k)^2+i\varepsilon}\bar{u}(k)\gamma^\mu u(p) + \bar{u}(k)\gamma^\mu u(p')\frac{-ig_{\mu\nu}}{(p-k')^2+i\varepsilon}\bar{u}(k')\gamma^\mu u(p)\right]$$

	外线因子	S矩阵元	内线	S矩阵元传播子
Fermion子	p 入射	$u^{(\alpha)}(p)$	p	$\frac{i}{\not p - m + i\epsilon}$
	p 出射	$\bar{u}^{(\alpha)}(p)$		
反Fermion子	p	$v^{(\alpha)}(p)$		
	p	$\bar{v}^{(\alpha)}(p)$		
光子外线	k 入射	$\epsilon^{(\lambda)}(k)$	k, μ ν	$\frac{-ig_{\mu\nu}}{k^2+i\epsilon}$
	出射			
实标量$\lambda\varphi^4$耦合		1		$\frac{i}{k^2-\mu^2+i\epsilon}$
电磁作用顶角因子		$-ie\gamma^\mu$		π^0-N耦合
质量抵消项顶角因子		$i\delta m$		π-N同位旋SU(2)不变耦合

4. 每个顶角处的4动量因子：入射为负，出射为正，守恒 $=0$

5. 积分因子：对每一个内线圈有一积分 $\int d^4p/(2\pi)^4$

6. 求迹因子：对每一个Fermion子内线圈，有一求迹因子：$-\mathrm{Tr}$

7. 符号因子：全同Fermion外线交换产生负号.

8. 外线4动量守恒因子：$(2\pi)^4\delta^4\{\sum P\}$

9. 拓扑权因子：$W=$(拓扑权)×(相互作用顶角置换因子)×(微扰级数展开因子)

$\lambda\varphi^4$自作用顶角		质量抵消项顶角			
	$-i\lambda_0$		$i\delta\mu^2$		
π^0-N 顶角		核子质量抵消项顶角		介子质量抵消项顶角	
	$g_0\gamma_5$				$i\delta\mu^2$
π-N顶角		介子内线			
i	$g_0\gamma_5\tau_i$	k, i j	$\frac{i\delta_{ij}}{k^2-\mu^2+i\epsilon}$		

1. 质量为 m 的自由标量场 $\varphi(x)$ 由如下拉氏量描写， 30-33

$$\mathcal{L}(x) = -\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$$ 体系在 Lorentz 变换下保持不变，导出 Lorentz 不变性所导致的 Noether 守恒流.

解： 流密度. $$j^\mu_\alpha = -\left[\mathcal{L} g^\mu_\rho - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\rho\varphi\right]\frac{\delta x^\rho}{\delta\omega^\alpha} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\frac{\delta_0\varphi}{\delta\omega^\alpha} \quad (2.144)$$

$$j^\mu_{\nu\rho} = -\left[\mathcal{L} g^\mu_{\rho'} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_{\rho'}\varphi\right]\frac{\delta x^{\rho'}}{\delta\omega^{\nu\rho}} - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\frac{\delta_0\varphi}{\delta\omega^{\nu\rho}}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{\partial}{\partial(\partial_\mu\varphi)}\left(-\frac{1}{2}\partial_\nu\varphi\,\partial^\nu\varphi\right)$$

$$= \frac{\partial}{\partial(\partial_\mu\varphi)}\left(-\frac{1}{2}\partial_\nu\varphi\, g^{\nu\rho}\,\partial_\rho\varphi\right)$$

$$= -\frac{1}{2}\left(\delta_{\mu\nu} g^{\nu\rho}\partial_\rho\varphi + \partial_\nu\varphi\, g^{\nu\rho}\delta_{\mu\rho}\varphi\right)$$

$$= -\frac{1}{2}\left(\partial^\mu\varphi + \partial^\mu\varphi\right) = -\partial^\mu\varphi$$

$(\rho' = \lambda,\ \rho = \nu,\ \lambda = \mu)$

$$\delta x^{\rho'} = \varepsilon^{\rho'\rho} x_\rho \quad (2.149) \quad = g^{\rho'}{}_\lambda \varepsilon^{\lambda\rho} x_\rho$$

$$= \frac{1}{2} g^{\rho'}{}_\lambda \varepsilon^{\lambda\rho} x_\rho + \frac{1}{2} g^{\rho'}{}_\lambda \varepsilon^{\lambda\rho} x_\rho$$

$$= \frac{1}{2}\varepsilon^{\lambda\rho} g^{\rho'}{}_\lambda x_\rho - \frac{1}{2}\varepsilon^{\rho\lambda} g^{\rho'}{}_\lambda x_\rho \quad (\varepsilon^{\lambda\rho} = -\varepsilon^{\rho\lambda})$$

$$= \frac{1}{2}\varepsilon^{\lambda\rho} g^{\rho'}{}_\lambda x_\rho - \underline{\frac{1}{2}\varepsilon^{\lambda\rho} g^{\rho'}{}_\rho x_\lambda} \quad (\text{互换 } \lambda\rho \text{ 符号})$$

$$= \frac{1}{2}\varepsilon^{\lambda\rho}\left(g^{\rho'}{}_\lambda x_\rho - g^{\rho'}{}_\rho x_\lambda\right)$$

$$\frac{\delta x^{\rho'}}{\delta\omega^{\nu\rho}} = \frac{1}{2}\left(g^{\rho'}{}_\nu x_\rho - g^{\rho'}{}_\rho x_\nu\right) \quad (2.151)$$

$\because \delta\varphi = 0$

$$\therefore j^\mu_{\nu\rho} = -\left[\mathcal{L} g^\mu{}_{\rho'} + \partial^\mu\varphi\,\partial_{\rho'}\varphi\right]\left(\frac{1}{2}g^{\rho'}{}_\nu x_\rho - \frac{1}{2}g^{\rho'}{}_\rho x_\nu\right)$$

$$= -\frac{1}{2}\left[x_\rho\left(\mathcal{L} g^\mu{}_{\rho'} + \partial^\mu\varphi\,\partial_{\rho'}\varphi\right)g^{\rho'}{}_\nu - x_\nu\left(\mathcal{L} g^\mu{}_{\rho'} + \partial^\mu\varphi\,\partial_{\rho'}\varphi\right)g^{\rho'}{}_\rho\right.$$

二. 质量为 μ 的中性 π^0 介子由赝标量场 $\varphi(x)$ 描写，质量为 m 的中子 n 由 Dirac 场 $\psi(x)$ 描写，π^0 和 n 之间的强相互作用 Lagrangian 可写为 $L_I(x) = ig\bar{\psi}\gamma_5\psi\varphi$ 其中 g 为实耦合常数.

(1) 写出该体系的 Lagrangian $L(x)$； (2) 推导出体系所满足的运动方程.

解：(1) $\mathcal{L} = \mathcal{L}_{KG} + \mathcal{L}_{Dirac} + L_I$ (4.55)

$$= \frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - \mu^2\varphi^2) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + ig\bar{\psi}\gamma_5\psi\varphi$$

(2) 将 $\psi, \bar{\psi}, \varphi$ 代入 Euler-Lagrange 方程. $\partial_\mu \frac{\partial L}{\partial(\partial_\mu\varphi)} - \frac{\partial L}{\partial\varphi} = 0$.

① 对 φ：$\partial_\mu \frac{\partial L}{\partial(\partial_\mu\varphi)} - \frac{\partial L}{\partial\varphi} = 0$

$$\frac{\partial L}{\partial(\partial_\mu\varphi)} = \frac{\partial}{\partial(\partial_\mu\varphi)}\Big[\frac{1}{2}(\partial_\mu\varphi\partial^\mu\varphi - \underbrace{\mu^2\varphi^2) + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + ig\bar{\psi}\gamma_5\psi\varphi)}_{=0}$$

$$= \frac{1}{2}\frac{\partial}{\partial(\partial_\mu\varphi)}(\partial_\mu\varphi\partial^\mu\varphi)$$

$$= \partial^\mu\varphi$$

$$\frac{\partial L}{\partial\varphi} = \frac{\partial}{\partial\varphi}\left(-\frac{1}{2}\mu^2\varphi^2 + ig\bar{\psi}\gamma_5\psi\varphi\right)$$

$$= -\mu^2\varphi + ig\bar{\psi}\gamma_5\psi$$

∴ 由欧拉方程. 可知 $\partial_\mu \frac{\partial L}{\partial(\partial_\mu\varphi)} = \frac{\partial L}{\partial\varphi}$

$$\therefore \partial_\mu\partial^\mu\varphi = -\mu^2\varphi + ig\bar{\psi}\gamma_5\psi$$

$$\underbrace{\partial_\mu\partial^\mu}_{=\Box}\varphi + \mu^2\varphi = ig\bar{\psi}\gamma_5\psi$$

$$\therefore (\Box + \mu^2)\varphi = ig\bar{\psi}\gamma_5\psi.$$

π - N

Yukawa 耦合:

$$\mathcal{L}_I = -ig :\bar{N}(x)\gamma_5\vec{\tau}N(x)\cdot\vec{\varphi}(x):$$

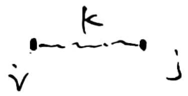
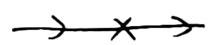
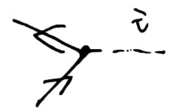
$$H_I = -ig :\bar{N}(x)\gamma_5\vec{\tau}N(x)\cdot\vec{\varphi}(x): - \delta m :\bar{N}(x)N(x): - \frac{1}{2}\delta\mu^2 :\vec{\varphi}(x)^2:$$

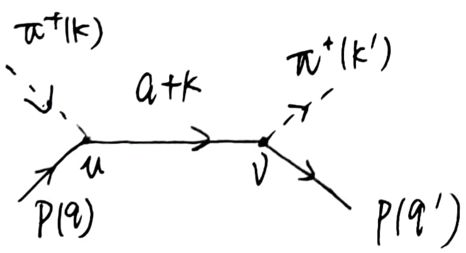
$\delta m = m - m_0$　　　　核子场ψ 的质量和裸质量

$\delta\mu^2 = \mu^2 - \mu_0^2$　　　　π介子场φ

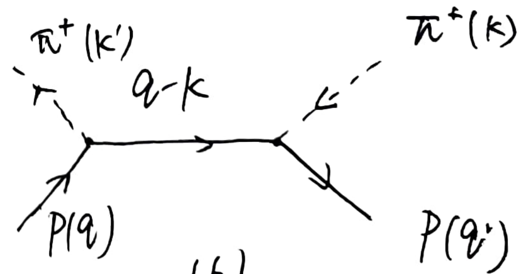
π介子为赝标量介子. 核子为 Fermion.

相应 Feynman 规则:

	图	S
介子外线		1
介子内线		$\dfrac{i\delta_{ij}}{k^2-\mu^2+i\varepsilon}$
介子质量附件		$i\delta\mu^2$
核质		$i\delta m$
Fermion子外线		$u(p)$
		$\bar{u}(p)$
内线		$\dfrac{i}{\not{p}-m+i\varepsilon}$
π-N 顶角		$-g_0\gamma_5\tau_i$



(a)



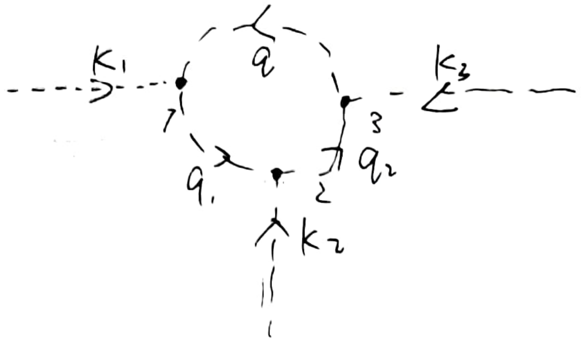
(b)

$$T(a) = \bar{u}(q')(-g_0\gamma_5\tau_i)\frac{i}{\not{q}+\not{k}-m_n+i\varepsilon}(-g_0\gamma_5\tau_i)\,u(q)$$

$$T(b) = \bar{u}(q')(-g_0\gamma_5\tau_i)\frac{i}{\not{q}-\not{k}-m_n+i\varepsilon}(-g_0\gamma_5\tau_i)\,u(q)$$

$$d\sigma = \frac{1}{2}\sum_{q,k}\frac{m_\pi m_p}{[(q\cdot k)^2-m_\pi^2 m_p^2]^{\frac{1}{2}}}|A|^2\frac{m_\pi d^3(k')}{(2\pi)^3 k'^0}\frac{m_p d^3(q')}{(2\pi)^3 q'^0}(2\pi)^4\delta^4(k'+q'-k-q)$$

写下如下 Feynman 图的S矩阵元.



$\mathcal{L}_I(x) = -\frac{\lambda}{3!}\phi^3(x)$

因为 $\mathcal{L}_I(x) = -\frac{\lambda}{3!}\phi^3(x)$

$H_I = \frac{\lambda}{3!}\phi^3(x)$

∴ 顶角为: $-i\lambda$

1. $-k_1 - q + q_1 = 0 \Rightarrow q_1 = k_1 + q$

2. $-q_1 - k_2 + q_2 = 0 \Rightarrow q_2 = q_1 + k_2 = k_1 + k_2 + q$

拓扑数: $3! \times 3 \times 3 \times 3 \times 4 \times 2$

拓扑数因子 = 拓扑数 × 相互作用顶角因子 × 微扰级数展开.

$$= \frac{3! \times 3 \times 3 \times 3 \times 4 \times 2}{(3!)^3 \times 3!} = 1$$

$$\langle f | S_{e.m} | i \rangle = (2\pi)^4 \delta(-k_1 - k_2 - k_3)\, (-i\lambda) \int \frac{d^4 q}{(2\pi)^4} \frac{i}{(q+k_1)^2 - \mu^2 + i\varepsilon} (-i\lambda)$$
$$\times \frac{i}{(q+k_1+k_2)^2 - \mu^2 + i\varepsilon} (-i\lambda) \frac{i}{q^2 - \mu^2 + i\varepsilon}$$

$$= (2\pi)^4 \delta(-k_1 - k_2 - k_3)\, (-i\lambda)^3 \int \frac{d^4 q}{(2\pi)^4} \sim\sim$$