第三章　场的正则量子化

3.1　正则形式与粒子力学的量子化

经典力学——→量子力学：

(1)将经典力学纳入正则形式

考虑一个单粒子系统,其广义坐标为$q_i\ (i=1,2,\cdots,N)$,体系的作用量

$$I=\int_{t_1}^{t_2}\mathrm{d}t\,L(q_i,\dot{q}_i),\qquad \left(\dot{q}_i\equiv\frac{\mathrm{d}}{\mathrm{d}t}q_i\right)\tag{3.1}$$

由$\delta I/\delta q_i = 0$，得到质点力学的Lagrange运动方程：

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \quad (i = 1,2,\cdots,N) \tag{2.5}$$

定义q_i的共轭动量（正则动量）：

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \qquad (i = 1,2,\cdots,N) \tag{3.2}$$

通过Legendre变换得到系统的哈密顿量为

$$H(q_i, p_i) = p_i \dot{q}_i - L, \tag{3.3}$$

可导出经典Hamilton正则运动方程：

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}. \quad (i = 1,\cdots N) \tag{3.4}$$

(3.4)的导出：

(3.2)对时间求导，得

$$\dot{p}_i=\frac{\mathrm{d}p_i}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}=\frac{\partial L}{\partial q_i}=-\frac{\partial H}{\partial q_i};$$

(3.3)对p_i求偏导，得

$$\frac{\partial H}{\partial p_i}=\dot{q}_i+p_j\left(\frac{\partial \dot{q}_j}{\partial p_i}\right)-\left(\frac{\partial L}{\partial \dot{q}_j}\right)\left(\frac{\partial \dot{q}_j}{\partial p_i}\right)$$

$$=\dot{q}_i+p_j\left(\frac{\partial \dot{q}_j}{\partial p_i}\right)-p_j\left(\frac{\partial \dot{q}_j}{\partial p_i}\right)$$

$$=\dot{q}_i.$$

(2)从经典力学的正则形式过度到量子力学

q_i和p_i满足如下对易关系：

$$\left.\begin{aligned}&[q_i(t), p_j(t)] = i\delta_{ij},\\ &[q_i(t), q_j(t)] = [p_i(t), p_j(t)] = 0,\end{aligned}\right. \quad (i, j = 1, \cdots, N) \qquad (3.5)$$

可证：

$$[q_i^n, p_i] = inq_i^{n-1} = i\frac{\partial q_i^n}{\partial q_i},$$

$$[p_i^n, q_i] = -inp_i^{n-1} = -i\frac{\partial p_i^n}{\partial p_i},$$

一般地，$A = a_{mn}^{ij} q_i^m p_j^n$，则有

$$[A, p_i] = i\frac{\partial A}{\partial q_i}, \qquad [A, q_i] = -i\frac{\partial A}{\partial p_i},$$

令$A = H$，有

$$[H, p_i] = i\frac{\partial H}{\partial q_i}, \qquad [H, q_i] = -i\frac{\partial H}{\partial p_i}, \tag{*}$$

将经典正则方程中的量看成算符，得量子正则方程，

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}. \quad (i = 1, \cdots N) \tag{3.4}$$

联合(*)和(3.4)，得正则形式的量子运动方程

$$\begin{aligned} \dot{p}_i(t) &= i[H, p_i(t)], \\ \dot{q}_i(t) &= i[H, q_i(t)]. \end{aligned} \qquad \text{(Heisenberg方程)} \tag{3.6}$$

✓ 一维谐振子的正则量子化

$$L = \frac{1}{2}\dot{q}^2 - \frac{\omega}{2}q^2, \tag{3.8}$$

① 纳入正则形式

定义动量：

$$p \equiv \frac{\partial L}{\partial \dot{q}} = \dot{q}, \tag{3.9}$$

哈密顿量为

$$H = p\dot{q} - L = \frac{1}{2}(p^2 + \omega q^2), \tag{3.10}$$

Hamilton 正则方程成为

$$\dot{p}=-\frac{\partial H}{\partial q}=-\omega q,\qquad \dot{q}=\frac{\partial H}{\partial p}=p. \tag{3.11}$$

② 进行量子化

引入正则对易关系

$$[q,p]=i, \tag{3.12}$$

Heisenberg方程为

$$\dot{p}=i[H,p]=-\omega q,\qquad \dot{q}=i[H,q]=p. \tag{3.11}$$

③ 本征值问题

引入线性组合

$$a=\sqrt{\frac{1}{2\omega}}(\omega q+ip),\quad a^{+}=\sqrt{\frac{1}{2\omega}}(\omega q-ip),\qquad (3.14)$$

满足

$$[a,a^{+}]=1,\quad [a,a]=[a^{+},a^{+}]=0,\qquad (3.15)$$

哈密顿量可表为

$$H=\frac{1}{2}\omega(a^{+}a+aa^{+})=\omega(a^{+}a+\frac{1}{2}),\qquad (3.16)$$

令

$$N = a^{+}a, \tag{3.17}$$

则

$$H = \omega(N + \frac{1}{2}). \tag{3.18}$$

对任意态$|\psi\rangle$，有

$$\langle\psi|N|\psi\rangle = \langle\psi|a^{+}a|\psi\rangle = \left|a|\psi\rangle\right|^{2} \geq 0.$$

求证：H的本征态$|n\rangle$满足

$$H|n\rangle=\omega(n+\frac{1}{2})|n\rangle,\quad (n=0,1,2,\cdots) \tag{3.19}$$

$$|n\rangle=\frac{1}{\sqrt{n!}}a^{+n}|0\rangle, \tag{3.20}$$

其中$|0\rangle$为本征值最小的本征态 满足

$$H|0\rangle=\frac{\omega}{2}|0\rangle. \tag{3.21}$$

证：令$|\ \rangle$是N的任一本征态，满足

$$N|\ \rangle=l|\ \rangle, \tag{3.22}$$

由于：

$$Na^+ = a^+(N+1), \qquad Na = a(N-1),$$

$$Na^{+^2} = a^+(N+2), \qquad Na^2 = a^2(N-2),$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$Na^{+^n} = a^{+^n}(N+n), \qquad Na^n = a^n(N-n)$$

固有：

$$Na^+|\ \rangle = (l+1)a^+|\ \rangle, \qquad Na|\ \rangle = (l-1)a|\ \rangle,$$

$$Na^{+^2}|\ \rangle = (l+1)a^{+^2}|\ \rangle, \qquad Na^2|\ \rangle = (l-2)a|\ \rangle,$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$Na^{+^n}|\ \rangle = (l+n)a^{+^n}|\ \rangle, \qquad Na^n|\ \rangle = (l-n)a^n|\ \rangle.$$

$l=$非负整数，对于$n=l$，有

$$|0\rangle \equiv a^{l}|\ \ \rangle,$$

满足

$$N|0\rangle = 0.$$

以$|0\rangle$代替$|\ \ \rangle$，得到

$$Na^{+^n}|0\rangle = na^{+^n}|0\rangle,$$

或

$$N|n\rangle = n|n\rangle, \quad n = 0,1,2,\cdots \tag{3.25}$$

$$Na|0\rangle = (aN - a)|0\rangle = -a|0\rangle,$$

$$\Rightarrow \qquad a|0\rangle = 0. \tag{3.26}$$

在坐标表象中，$p=-i\dfrac{\partial}{\partial q}$，令 $\psi(q)=\langle q|0\rangle$，(3.26)可写成

$$(\omega q+\frac{\partial}{\partial q})\psi(q)=0,$$

其解为

$$\psi(q)\propto e^{-\frac{\omega^2}{2}q^2},$$

因此 $|0\rangle$ 不仅存在且是非简并的。由此得证。

$a^+|n\rangle=\sqrt{n+1}|n+1\rangle$，($a^+$ — 产生算符)

$a|n\rangle=\sqrt{n}|n-1\rangle$，($a$ — 消灭算符)

$N|n\rangle=n|n\rangle$.　(N — 粒子数算符)

3.2　场的正则量子化——一般表述

- 单个定域场情况

	粒子力学	场论
广义坐标	$q_i(t), i=1,\cdots,N$	$\varphi(\vec{x},t), \vec{x}$连续变化
	$\dot{q}_i(t)$	$\dfrac{\partial}{\partial t}\varphi(\vec{x},t)$

➢ 纳入正则形式

将3维空间分成无穷多个体积元ΔV_i，定义第i个广义坐标

$$\varphi_i(t)\equiv\frac{1}{\Delta V_i}\int_{(\Delta V_i)}\mathrm{d}^3x\,\varphi(\vec{x},t),\qquad(3.30)$$

其时间微商为

$$\dot{\varphi}_i(t)\equiv\frac{1}{\Delta V_i}\int_{(\Delta V_i)}d^3x\frac{\partial}{\partial t}\varphi_i(\vec{x},t),\qquad(3.31)$$

则拉氏量可重写为：

$$L=\int d^3x\mathcal{L}(\varphi(x),\partial_\mu\varphi(x))$$

$$\to\sum_i\Delta V_i\overline{\mathcal{L}}_i(\varphi_i(t),\dot{\varphi}_i(t),\varphi_{i\pm s}(t),\cdots),\qquad(3.32)$$

与$\varphi_i(t)$相联系的共轭动量（正则动量）为

$$p_i(t)=\frac{\partial \mathrm{L}}{\partial \dot{\varphi}_i(t)}$$

$$=\sum_i \Delta V_i \frac{\partial \overline{L}_i}{\partial \dot{\varphi}_i(t)} \equiv \Delta V_i \pi_i(t), \ (i\text{不求和}) \qquad (3.33)$$

通过Legendre变换，得到系统的哈密顿量为

$$H=\sum_{\mathrm{i}} p_i \dot{\varphi}_i-\mathrm{L}=\sum_i \Delta V_i(\pi_i \dot{\varphi}_i-\overline{\mathcal{L}}_i). \qquad (3.34)$$

➢ 量子化

① 引入正则对易关系

$$[\varphi_i(t),\varphi_j(t)]=[p_i(t),p_j(t)]=0,$$

$$[p_i(t),\varphi_j(t)]=-i\delta_{ij},\ \text{或}\ [\pi_i(t),\varphi_j(t)]=-i\frac{\delta_{ij}}{\Delta V_i},\tag{3.35}$$

则φ_i和p_i满足Heisenberg方程

$$\dot{\varphi}_i=i[H,\varphi_i],\qquad \dot{p}_i=i[H,p_i].\tag{3.36}$$

② 回到连续极限

作代换：

$$\sum_i \Delta V_i \to \int d^3x, \qquad \bar{\mathcal{L}}_i \to \mathcal{L},$$

$$\varphi_i(t) \to \varphi(\vec{x},t), \qquad \dot{\varphi}_i(t) \to \dot{\varphi}(\vec{x},t),$$

$$\pi_i(t) \to \pi(\vec{x},t),$$

其中$\pi(\vec{x},t)$**是**$\varphi(\vec{x},t)$**的共轭动量，定义为**

$$\pi(\vec{x},t) \equiv \frac{\partial \mathcal{L}(\varphi, \partial_\mu \varphi)}{\partial \dot{\varphi}(\vec{x},t)}, \tag{3.37}$$

$\pi(\vec{x},t)$在体积元ΔV_i中的平均值为

$$\frac{1}{\Delta V_i}\int_{\Delta V_i}\mathrm{d}^3 x\,\pi(\vec{x},t)=\frac{1}{\Delta V_i}\int_{\Delta V_i}\mathrm{d}^3 x\,\frac{\partial\mathcal{L}(\varphi,\partial_\mu\varphi)}{\partial\dot{\varphi}(\vec{x},t)}$$

$$=\frac{1}{\Delta V_i}\cdot\Delta V_i\,\frac{\partial\overline{\mathcal{L}}_i}{\partial\dot{\varphi}(t)}=\pi_i(t).$$

在连续极限下，哈密顿量(3.34)成为

$$H=\int d^3x\mathcal{H}(\pi(\vec{x},t),\varphi(\vec{x},t)),$$
$$\mathcal{H}=\pi\dot{\varphi}-\mathcal{L}.\quad \text{(Hamiltonian)} \tag{3.38}$$

对易关系(3.35)成为

$$[\varphi(\vec{x},t),\varphi(\vec{x}',t)]=[\pi(\vec{x},t),\pi(\vec{x}',t)]=0,$$
$$[\pi(\vec{x},t),\varphi(\vec{x}',t)]=-i\delta^3(\vec{x}-\vec{x}'). \tag{3.39}$$

Heisenberg方程(3.36)成为

$$\dot{\varphi}_i(\vec{x},t)=i[H,\varphi_i(\vec{x},t)],$$
$$\dot{\pi}_i(\vec{x},t)=i[H,\pi_i(\vec{x},t)]. \tag{3.40}$$

▪ 多个定域场情况

➢ 纳入正则形式

设有n个独立场$\varphi_r(\vec{x},t)$, $(r=1,\cdots,n)$, 对每个场$\varphi_r(\vec{x},t)$

引入共轭动量

$$\pi_r(\vec{x},t)=\frac{\partial L}{\partial\dot{\varphi}_r(\vec{x},t)}, \tag{3.41}$$

通过Legendre变换，定义Hamiltonian和哈密顿量

$$\mathcal{H}(\pi_r,\cdots,\varphi_r,\cdots)=\sum_{r=1}^{n}\pi_r\dot{\varphi}_r-\mathcal{L},$$
$$H=\int \mathrm{d}^3x\mathcal{H}. \tag{3.42}$$

➢ 量子化

算符φ_r和π_r满足对易关系：

$$
\begin{aligned}
&[\varphi_r(\vec{x},t),\varphi_s(\vec{x}',t)]=[\pi_r(\vec{x},t),\pi_s(\vec{x}',t)]=0, \\
&[\pi_r(\vec{x},t),\varphi_s(\vec{x}',t)]=-i\delta_{rs}\delta^3(\vec{x}-\vec{x}'),
\end{aligned}
\tag{3.43}
$$

运动方程为

$$
\begin{aligned}
&\dot{\varphi}_r(\vec{x}',t)=i[H,\varphi_r(\vec{x},t)], \\
&\dot{\pi}_r(\vec{x},t)=i[H,\pi_r(\vec{x},t)].
\end{aligned}
\tag{3.44}
$$

3.3 Klein-Gordon场的正则量子化

3.3.1 实标量场的量子化与粒子解释

- 实标量场的量子化

$$\mathcal{L}=\frac{1}{2}\partial^{\mu}\varphi\partial_{\mu}\varphi-\frac{1}{2}m^{2}\varphi^{2}, \qquad (3.55)$$

运动方程

$$(\Box+m^{2})\varphi(x)=0,\quad \text{（K-G方程）} \qquad (3.56)$$

➢ 纳入正则形式

共轭动量

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} = \dot{\varphi}(x), \qquad (3.57)$$

哈密顿量为

$$H = \int d^3x \mathcal{H}(\pi, \varphi)$$

$$\mathcal{H}(\pi, \varphi) = \pi\dot{\varphi} - \mathcal{L}$$

$$= \frac{1}{2}[\pi(\vec{x}, t)^2 + |\nabla\varphi(\vec{x}, t)|^2 + m^2\varphi(\vec{x}, t)^2]. \quad (3.58)$$

➢ 正则量子化

$\pi, \varphi \to$ 厄密算符，满足等时对易子：

$$[\varphi(\vec{x},t),\varphi(\vec{x}',t)] = [\pi(\vec{x},t),\pi(\vec{x}',t)] = 0,$$
$$[\pi(\vec{x},t),\varphi(\vec{x}',t)] = -i\delta^3(\vec{x}-\vec{x}'), \tag{3.59}$$

运动方程(3.57)和(3.56)等同于Heisenberg方程：

$$\dot{\varphi}(\vec{x},t) = i[H,\varphi(\vec{x},t)],$$
$$\dot{\pi}(\vec{x},t) = i[H,\pi(\vec{x},t)]. \tag{3.60}$$

守恒的4动量算符和广义角动量算符为

$$P^\mu = \int d^3x j^{0\mu} = \int d^3x(-g^{0\mu}\mathcal{L} + \partial^0\varphi\partial^\mu\varphi)$$
$$= \int d^3x(-g^{0\mu}\mathcal{L} + \pi\partial^\mu\varphi), \tag{3.61}$$

$$
\begin{aligned}
P^0 &= \int d^3x\,(-g^{00}\mathcal{L} + \pi\dot{\varphi}) = H, \\
M^{\nu\rho} &= \int d^3x\,(x^\nu j^{0\rho} - x^\rho j^{0\nu}).
\end{aligned}
\tag{3.62}
$$

利用π,φ的正则对易关系，可证实

$$
\begin{aligned}
i[P^\mu,\varphi(x)] &= \partial^\mu\varphi(x), \\
i[M^{\nu\rho},\varphi(x)] &= (x^\nu\partial^\rho - x^\rho\partial^\nu)\varphi(x).
\end{aligned}
\tag{3.63}
$$

- 粒子解释

➢ 平面波展开

$$\varphi(\vec{x},t)=\int \tilde{d}k[a(k)e^{-ikx}+a^{+}(k)e^{ikx}],$$

$$\nabla\varphi(\vec{x},t)=\int \tilde{d}k(i\vec{k})[a(k)e^{-ikx}-a^{+}(k)e^{ikx}], \qquad (3.64)$$

$$\pi(\vec{x},t)=\dot{\varphi}(\vec{x},t)$$

$$=\int \tilde{d}k(-i\omega_k)[a(k)e^{-ikx}-a^{+}(k)e^{ikx}], \qquad (3.67)$$

其中积分测度

$$\tilde{d}k=\frac{d^3k}{(2\pi)^3 2\omega_k}=\frac{d^4k}{(2\pi)^4}\delta(k^2-m^2)\theta(k^0)2\pi,$$

$$\omega_k = \sqrt{\vec{k}^2 + m^2}, \qquad \theta(k^0) = \frac{1}{2}[1 + \varepsilon(k^0)],$$

$$\varepsilon(k^0) = \begin{cases} 1 & k^0 > 0 \\ -1 & k^0 < 0 \end{cases}$$

d^4k和$\varepsilon(k^0)$均是LT不变的，因而积分测度$\widetilde{d}k$是LT不变的。

平面波e^{-ikx}满足正交条件：

$$
\begin{aligned}
& i\int d^3x e^{ikx}\overleftrightarrow{\partial}_0 e^{-ik'x} = (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}'), \\
& i\int d^3x e^{ikx}\overleftrightarrow{\partial}_0 e^{ik'x} = 0,
\end{aligned}
\tag{3.68}
$$

则$a(k)$和$a^+(k)$可通过$\varphi(\vec{x},t)$来表达：

$$
\begin{aligned}
& a(k) = i\int d^3x e^{ikx}\overleftrightarrow{\partial}_0 \varphi(\vec{x},t), \\
& a^+(k) = -i\int d^3x e^{-ikx}\overleftrightarrow{\partial}_0 \varphi(\vec{x},t),
\end{aligned}
\tag{3.69}
$$

$a(k)$与t无关，

$$
\dot{a}(k) = \frac{\partial}{\partial t}a(k) = i\int d^3x \frac{\partial}{\partial t}[e^{ikx}\overleftrightarrow{\partial}_0 \varphi(\vec{x},t)]
$$

$$\dot{a}(k)=i\int d^3x\frac{\partial}{\partial t}[e^{ikx}\frac{\partial}{\partial t}\varphi(\vec{x},t)-(\frac{\partial}{\partial t}e^{ikx})\varphi(\vec{x},t)]$$

$$=i\int d^3x\left[e^{ikx}\frac{\partial^2\varphi}{\partial t^2}-(\frac{\partial^2}{\partial t^2}e^{ikx})\varphi\right]$$

$$=i\int d^3x\left[e^{ikx}(\nabla^2-m^2)\varphi-(\frac{\partial^2}{\partial t^2}e^{ikx})\varphi\right]$$

$$=i\int d^3x\varphi\left[\nabla^2-m^2-\frac{\partial^2}{\partial t^2}\right]e^{ikx}=0,$$

其中两次利用了$\mathrm{K-G}$方程：

$$\frac{\partial^2}{\partial t^2}\varphi=(\nabla^2-m^2)\varphi.$$

$a(k), a^{+}(k)$满足的代数：

$$
\begin{aligned}
&[a(k), a^{+}(k')] = (2\pi)^{3} 2\omega_{k} \delta^{3}(\vec{k} - \vec{k}'), \\
&[a(k), a(k')] = [a^{+}(k), a^{+}(k')] = 0.
\end{aligned}
\tag{3.70}
$$

利用$a(k), a^{+}(k)$来表达场的总能量和总动量算符：

$$
\begin{aligned}
H &= \frac{1}{2}\int d^{3}x[\pi(\vec{x},t)^{2} + \left|\nabla\varphi(\vec{x},t)\right|^{2} + m^{2}\varphi(\vec{x},t)^{2}] \\
&= \frac{1}{2}\int \tilde{d}k\omega_{k}[a(k)a^{+}(k) + a^{+}(k)a(k)],
\end{aligned}
\tag{3.71}
$$

$$
\begin{aligned}
\vec{P} &= -\int d^{3}x\pi(\vec{x},t)\nabla\varphi(\vec{x},t) \\
&= -\frac{1}{2}\int \tilde{d}k\vec{k}[a(k)a^{+}(k) + a^{+}(k)a(k)].
\end{aligned}
\tag{3.72}
$$

将3维动量空间分成体积元ΔV_k，则

$$\int d^3k \to \sum_k \Delta V_k, \qquad \delta^3(\vec{k}-\vec{k}') \to \frac{\delta_{kk'}}{\Delta V_k},$$

令

$$a_k = \sqrt{\frac{\Delta V_k}{(2\pi)^3 2\omega_k}}\, a(k), \tag{3.73}$$

则

$$[a_k, a_{k'}^+] = \delta_{kk'}, \quad [a_k, a_{k'}] = [a_k^+, a_{k'}^+] = 0. \tag{3.74}$$

$$H = \sum_k \omega_k (a_k^+ a_k + \frac{1}{2}), \tag{3.75}$$

$$\vec{P} = \sum_k \vec{k} a_k^+ a_k \,. \tag{3.76}$$

➢ 本征值与本征态

对每一本征振动，能量与动量算符各为

$$H_k = \omega_k (a_k^+ a_k + \frac{1}{2}) = \omega_k (N_k + \frac{1}{2}),$$

$$\vec{P}_k = \vec{k} a_k^+ a_k = \vec{k} N_k, \qquad (3.77)$$

其中$N_k = a_k^+ a_k$,

$$N_k |n_k\rangle_k = n_k |n_k\rangle_k, \quad (n_k = 0,1,2,\cdots)$$

$$|n_k\rangle_k = \frac{1}{\sqrt{n_k!}} a_k^{+\,n_k} |0\rangle_k, \qquad (3.78)$$

基态$|0\rangle_k$**定义为：**

$$a_k |0\rangle_k = 0. \qquad (3.79)$$

$$H_k|n_k\rangle_k = \omega_k(N_k + \frac{1}{2})|n_k\rangle_k = \omega_k(n_k + \frac{1}{2})|n_k\rangle_k, \tag{3.80}$$

$$\vec{P}_k|n_k\rangle_k = \vec{k}N_k|n_k\rangle_k = \vec{k}n_k|n_k\rangle_k,$$

总哈密顿量和总动量为

$$H = \sum_k H_k, \qquad \vec{P} = \sum_k \vec{P}_k, \tag{3.81}$$

$$|n\rangle = \prod_k |n_k\rangle_k,$$

$$H|n\rangle = \sum_k \omega_k(n_k + \frac{1}{2})|n\rangle,$$

$$\vec{P}|n\rangle = \sum_k \vec{k}n_k|n\rangle.$$

系统的基态为

$$\left|0\right\rangle = \prod_k \left|0\right\rangle_k, \text{（真空态）}$$

满足

$$\begin{aligned} &a_k\left|0\right\rangle = 0 \qquad \text{(对所有}k\text{)}, \\ &\left\langle 0 \middle| 0\right\rangle = 1. \end{aligned} \tag{3.82}$$

真空能量

$$E = \left\langle 0\right|H\left|0\right\rangle = \left\langle 0\right|\sum_k \omega_k (a_k^+ a_k + \frac{1}{2})\left|0\right\rangle$$

$$= \sum_k \frac{1}{2}\omega_k = \infty, \tag{3.83}$$

为消除此发散，重新定义哈密顿量为：

$$H=\sum_k\omega_k(a_k^+a_k+\frac{1}{2})-\langle 0|\sum_k\omega_k(a_k^+a_k+\frac{1}{2})|0\rangle$$

$$=\sum_k\omega_k a_k^+a_k, \tag{3.84}$$

正规乘积:

$$:\frac{1}{2}(a_k^+a_k+a_ka_k^+):\equiv a_k^+a_k. \tag{3.85}$$

➢ 粒子解释

态$|n_k\rangle_k$满足方程组：

$$N_k|n_k\rangle_k = n_k|n_k\rangle_k,$$

$$H_k|n_k\rangle_k = n_k\omega_k|n_k\rangle_k,$$

$$\vec{P}_k|n_k\rangle_k = n_k\vec{k}|n_k\rangle_k.$$

$N_k = a_k^+ a_k$ —粒子数算符，在态$|n_k\rangle_k$上，有n_k个能量为ω_k、动量为$\vec{k}$的粒子。可证明，

$$\begin{aligned} N_k a_k^+|n_k\rangle_k &= (n_k+1)a_k^+|n_k\rangle_k, \\ N_k a_k|n_k\rangle_k &= (n_k-1)a_k|n_k\rangle_k, \end{aligned} \tag{3.86}$$

$a_k^+(a_k)$—量子数为k的粒子的产生（消灭)算符。

$$[a_k^+, a_{k'}^+] = 0,$$

$$\Rightarrow \qquad a_k^+ a_{k'}^+ \left|0\right\rangle = a_{k'}^+ a_k^+ \left|0\right\rangle, \text{（满足Bose统计）} \qquad (3.87)$$

K-G场量子化后描述的是静质量为m，服从Bose统计的0自旋（标量）粒子。

➢ Fock 空间与连续动量下的归一化

在连续的k下，

$$H=\int \tilde{d}k\frac{\omega_k}{2}:[a(k)a^+(k)+a^+(k)a(k)]:$$

$$=\int \tilde{d}k\omega_k a^+(k)a(k), \tag{3.88}$$

$$\vec{P}=\int \tilde{d}k\frac{\vec{k}}{2}[a(k)a^+(k)+a^+(k)a(k)]$$

$$=\int \tilde{d}k\vec{k}a^+(k)a(k), \tag{3.89}$$

以上两式合写为

$$P^\mu=\int \tilde{d}kk^\mu a^+(k)a(k), \tag{3.90}$$

总粒子数算符

$$N=\int \tilde{d}k\, a^{+}(k)a(k). \tag{3.91}$$

Fock 空间—

在连续k下，Fock空间的基矢为

$$|r\rangle = \left(r!\int \tilde{d}k_1 \cdots \tilde{d}k_r \left|F(k_1,\cdots k_r)\right|^2\right)^{-\frac{1}{2}}$$

$$\cdot \int \tilde{d}k_1 \cdots \tilde{d}k_r F(k_1,\cdots k_r) a^+(k_1)\cdots a^+(k_r)|0\rangle, \qquad (3.92)$$

其中真空态满足

$$a_k|0\rangle = 0 \qquad (\text{对所有}k),$$

$$\langle 0|0\rangle = 1, \qquad (3.93)$$

$F(k_1,\cdots k_r)a^+(k_1)$是动量空间波函数。$|r\rangle$满足

$$\langle r|r\rangle = 1,$$

$$N|r\rangle = r|r\rangle.$$

3.3.2 场的可观测性与微观因果性

在量子理论中，仅当

$$[\varphi(x),\ \varphi(y)]=0$$

时，才有可能精确测量x, y点的场强。

在固定时刻t，有

$$[\varphi(\vec{x},t),\ \varphi(\vec{y},t)]=0,$$

一般情况下，

$$[\varphi(x),\ \varphi(y)]$$

$$=\int \tilde{d}k\tilde{d}k'\{[a(k),a^{+}(k')]e^{-ikx+ik'y}+[a^{+}(k),a(k')]e^{ikx-ik'y}\}$$

$$=\int \tilde{d}k[e^{-ik(x-y)}-e^{ik(x-y)}]=i\Delta(x-y), \tag{3.97}$$

其中

$$\Delta(x)=\frac{1}{i}\int\tilde{d}k(e^{-ikx}-e^{ikx})$$

$$=\frac{1}{i}\int\frac{d^3k}{(2\pi)^3 2\omega_k}e^{i\vec{k}\cdot\vec{x}}(e^{-i\omega_k t}-e^{i\omega_k t})$$

$$=\frac{1}{i}\int\frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\varepsilon(k^0)e^{-ik\cdot x}, \tag{3.98}$$

$\Delta(x)$是LT不变的，并有如下性质

1) $\Delta(x)^*=\Delta(x)$;

2) $\Delta(-x)=-\Delta(x)$;

3) $(\Box+m)\Delta(x)=0$;

4) $\Delta(\vec{x},0)=0;$

5) $\left.\frac{\partial}{\partial t}\Delta(\vec{x},t)\right|_{t=0}=-\delta^3(\vec{x});$

6) $\left.\partial_{x^j}\Delta(x)\right|_{t=0}=0.$

由性质4)，有

$$\Delta(\vec{x}-\vec{y},x^0-y^0=0)=0,$$

由于类空间隔

$$(x-y)^2=(x^0-y^0)^2-(\vec{x}-\vec{y})^2<0$$

总是可以通过LT成为等时间隔，且$\Delta(x-y)$是LT不变的，

则可得到

$$\Delta(x-y)=0, \qquad [\text{对}(x-y)^2<0]$$

$$[\varphi(x),\varphi(y)]=0, \quad [\text{对}(x-y)^2<0]$$—微观因果性条件

3.3.3 复标量场与正反粒子

- 复标量场的 Lagrangian 和守恒量

为了描述正反粒子，引入两个实厄密标量场φ_1, φ_2，

并组合成复场φ(非厄密)—荷电标量场

$$\varphi = \frac{\varphi_1 + i\varphi_2}{\sqrt{2}}, \qquad \varphi^+ = \frac{\varphi_1 - i\varphi_2}{\sqrt{2}} \tag{3.102}$$

系统的Lagrangian为

$$\mathcal{L} = \mathcal{L}(\varphi_1) + \mathcal{L}(\varphi_2)$$

$$= \frac{1}{2}(\partial_\mu \varphi_1 \partial^\mu \varphi_1 + \partial_\mu \varphi_2 \partial^\mu \varphi_2) - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2)$$

$$= (\partial_\mu \varphi^+) \partial^\mu \varphi - m^2 \varphi^+ \varphi, \tag{3.103}$$

Euler-Lagrange方程为

$$(\Box+m^2)\varphi(x)=0, \qquad (\Box+m^2)\varphi^+(x)=0, \tag{3.104}$$

在整体相变换下，

$$\varphi\to e^{i\alpha}\varphi, \qquad \varphi^+\to e^{-i\alpha}\varphi^+, \tag{3.105a}$$

或

$$\delta\varphi=i\delta\alpha\,\varphi, \qquad \delta\varphi^+=-i\delta\alpha\,\varphi^+, \tag{3.105b}$$

$\mathcal{L}$保持不变，相应的守恒流密度为

$$j^\mu=-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\frac{\delta\varphi}{\delta\alpha}-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^+)}\frac{\delta\varphi^+}{\delta\alpha}$$

$$=-(\partial^\mu\varphi^+)i\varphi-(-i\varphi^+)\partial^\mu\varphi$$

$$=i\varphi^+\partial^\mu\varphi-i(\partial^\mu\varphi^+)\varphi$$

$$j^{\mu} = i\varphi^{+}\overleftrightarrow{\partial}^{\mu}\varphi, \tag{3.106}$$

守恒荷为

$$Q = \int d^3x j^0(x) = i\int d^3x \varphi^{+}\overleftrightarrow{\partial}^{0}\varphi. \tag{3.107}$$

▪ 复标量场的量子化

定义共轭动量：

$$\pi = \frac{\partial L}{\partial \dot{\varphi}} = \dot{\varphi}^{+}, \qquad \pi^{+} = \frac{\partial L}{\partial \dot{\varphi}^{+}} = \dot{\varphi}, \tag{3.108}$$

$$\mathcal{L} = (\partial_{\mu}\varphi^{+})\partial^{\mu}\varphi - m^2\varphi^{+}\varphi$$

$$= (\partial_{0}\varphi^{+})\partial^{0}\varphi - (\partial_{i}\varphi^{+})\partial^{i}\varphi - m^2\varphi^{+}\varphi$$

$$= \dot{\varphi}^{+}\dot{\varphi} - \nabla\varphi^{+}\cdot\nabla\varphi - m^2\varphi^{+}\varphi = \pi\pi^{+} - \nabla\varphi^{+}\cdot\nabla\varphi - m^2\varphi^{+}\varphi,$$

$$H=\int d^3x:(\pi\dot{\varphi}+\pi^+\dot{\varphi}^+-\mathcal{L}):=\int d^3x:(\pi\pi^++\pi^+\pi-\mathcal{L}):$$

$$=\int d^3x:(\pi^+\pi+\nabla\varphi^+\cdot\nabla\varphi+m^2\varphi^+\varphi):. \tag{3.109}$$

利用正则对易关系

$$[\varphi_i(x),\varphi_j(y)]=i\delta_{ij}\Delta(x-y),$$

可得到复场形式下的对易子：

$$\begin{aligned}&[\varphi(x),\varphi(y)]=[\varphi^+(x),\varphi^+(y)]=0,\\&[\varphi(x),\varphi^+(y)]=i\Delta(x-y),\end{aligned} \tag{3.111}$$

(3.111)最后一式两边依次对x^0和y^0微商，并取

$x^0=y^0=t$，则得到等时对易子

$$[\varphi(\vec{x},t),\pi(\vec{y},t)]=[\varphi^+(\vec{x},t),\pi^+(\vec{y},t)]=i\delta^3(\vec{x}-\vec{y}). \tag{3.112}$$

▪ 场的粒子性

将场$\varphi(x),\varphi^+(x)$作平面波展开：

$$\varphi(x)=\int \tilde{d}k[a(k)e^{-ikx}+b^+(k)e^{ikx}],$$

$$\varphi^+(x)=\int \tilde{d}k[a^+(k)e^{ikx}+b(k)e^{-ikx}], \tag{3.113}$$

相应的实场

$$\varphi_i(x)=\int \tilde{d}k[a_i(k)e^{-ikx}+a_i^+(k)e^{ikx}],\ (i=1,2) \tag{3.114}$$

由(3.102)可得

$$a(k)=\frac{1}{\sqrt{2}}[a_1(k)+ia_2(k)],\quad a^+(k)=\frac{1}{\sqrt{2}}[a_1^+(k)-ia_2^+(k)],$$

$$b(k)=\frac{1}{\sqrt{2}}[a_1(k)-ia_2(k)],\quad b^+(k)=\frac{1}{\sqrt{2}}[a_1^+(k)+ia_2^+(k)],$$

满足对易关系

$$[a(k),a^+(k')]=[b(k),b^+(k')]=(2\pi)^3 2\omega_k\delta^3(\vec{k}-\vec{k}'),\tag{3.116}$$

其他对易子为0,

φ — 消灭a型粒子，产生b型粒子；

φ^+ — 产生a型粒子，消灭b型粒子。

粒子数算符分别为：

$$N_a=\int\tilde{d}k a^+(k)a(k),\qquad N_b=\int\tilde{d}k b^+(k)b(k),\tag{3.117}$$

守恒的4动量算符为：

$$P^\mu=\int\tilde{d}k k^\mu[a^+(k)a(k)+b^+(k)b(k)].$$

真空态$|0\rangle$满足

$$a(k)|0\rangle=b(k)|0\rangle=0.\qquad (\text{对所有}k)$$

量子化后，与$U(1)$内部对称性相应的Noether流算符写成

$$j^\mu = i:\varphi^+\overleftrightarrow{\partial}^\mu\varphi:,$$

守恒荷为

$$Q = i\int d^3x :\varphi^+\overleftrightarrow{\partial}^0\varphi:$$

$$= \int \widetilde{d}k :[a^+(k)a(k) - b^+(k)b(k)]:$$

$$= N_a - N_b,$$

$$\dot{Q} = i[H,Q] = i[H, N_a - N_b] = 0,$$

$$\partial_\mu j^\mu = i\partial_\mu :\varphi^+\partial^\mu\varphi - (\partial^\mu\varphi^+)\varphi:$$

$$= i:\varphi^+\partial_\mu\partial^\mu\varphi - (\partial_\mu\partial^\mu\varphi^+)\varphi:$$

$$= i:-m^2\varphi^+ + m^2\varphi^+\varphi:$$

$$= 0.$$

3.3.4 编时乘积与 Feynman 传播子

考虑自由荷电复标量场 $\varphi(x'),\varphi^+(x)$:

$$\varphi(x')=\int\tilde{d}k[a(k)e^{-ikx'}+b^+(k)e^{ikx'}],$$

$$\varphi^+(x)=\int\tilde{d}k[a^+(k)e^{ikx}+b(k)e^{-ikx}],$$

$\varphi^+(x)$作用在态$|\ \rangle$上，

$$\left.\begin{array}{l}\text{将产生}Q=+1\text{的粒子}\\ \text{或消灭}Q=-1\text{的粒子}\end{array}\right\}\text{在}x\text{点，}Q\text{荷增加}1;$$

$\varphi(x')$作用在态$|\ \rangle$上，

$$\left.\begin{array}{l}\text{将产生}Q=-1\text{的粒子}\\ \text{或消灭}Q=+1\text{的粒子}\end{array}\right\}\text{在}x'\text{点，}Q\text{荷减少}1;$$

$\varphi^+(x)$与$\varphi(x')$的联合作用：

1) $t < t'$,对应$\varphi(x')\varphi^+(x)$,

$$\text{振幅} \sim \left\langle \left|\theta(t'-t)\varphi(\vec{x}',t')\varphi^+(\vec{x},t)\right| \right\rangle;$$

2) $t' < t$,对应$\varphi^+(x)\varphi(x')$,

$$\text{振幅} \sim \left\langle \left|\theta(t-t')\varphi^+(\vec{x},t)\varphi(\vec{x}',t')\right| \right\rangle,$$

其中$\theta(y) = \begin{cases} 1, & y > 0 \\ 0, & y > 0 \end{cases}.$

定义Dyson编时乘积（T乘积）：

$$T\varphi(x')\varphi^+(x)$$

$$= \theta(t'-t)\varphi(\vec{x}')\varphi^+(\vec{x}) + \theta(t-t')\varphi^+(\vec{x})\varphi(\vec{x}'), \qquad (3.125)$$

Boson算符在T运算下是可对易的，即

$$T\varphi(x')\varphi^{+}(x)=T\varphi^{+}(x)\varphi(x'). \tag{3.126}$$

将算符$(\Box_{x'}+m^{2})=(\frac{\partial^{2}}{\partial t'^{2}}-\nabla_{x'}^{2}+m^{2})$作用于(3.125)的两边,得

$$(\Box_{x'}+m^{2})T\varphi(x')\varphi^{+}(x)$$

$$=(\frac{\partial^{2}}{\partial t'^{2}}-\nabla_{x'}^{2}+m^{2})T\varphi(x')\varphi^{+}(x)$$

$$=-i\delta^{4}(x'-x)+T(\frac{\partial^{2}}{\partial t'^{2}}-\nabla_{x'}^{2}+m^{2})\varphi(x')\varphi^{+}(x)$$

$$=-i\delta^{4}(x'-x),$$

$$\Rightarrow\quad(\Box_{x'}+m^{2})iT\varphi(x')\varphi^{+}(x)=\delta^{4}(x'-x), \tag{3.127}$$

上式两边对真空态求平均值，有

$$(\Box_{x'}+m^2)i\langle 0|T\varphi(x')\varphi^+(x)|0\rangle=\delta^4(x'-x), \tag{3.128}$$

定义：

$$G_F(x'-x)=i\langle 0|T\varphi(x')\varphi^+(x)|0\rangle, \tag{3.129}$$

它是K-G算子$(\Box_{x'}+m^2)$的一个Green函数，(3.125)代入上式，得到

$$G_F(x'-x)=i\theta(t'-t)\langle 0|\varphi(x')\varphi^+(x)|0\rangle$$

$$+i\theta(t-t')\langle 0|\varphi^+(x)\varphi(x')|0\rangle, \tag{3.130}$$

利用φ和φ^+的平面波展开式(3.113)，有

$$\langle 0|\varphi(x')\varphi^+(x)|0\rangle=\int \tilde{d}k\tilde{d}k' e^{-ik'x'+ikx}\langle 0|a(k')a^+(k)|0\rangle$$

$$\langle 0|\varphi(x')\varphi^+(x)|0\rangle$$

$$= \int \tilde{d}k\tilde{d}k' e^{-ik'x'+ikx}\langle 0|[a(k'),a^+(k)]|0\rangle$$

$$= \int \tilde{d}k \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} e^{-ik'x'+ikx}(2\pi)^3 2\omega_{k'}\delta^3(\vec{k}'-\vec{k})$$

$$= \int \tilde{d}k e^{-ik\cdot(x'-x)},$$

$$\langle 0|\varphi^+(x)\varphi(x')|0\rangle = \int \tilde{d}k e^{ik\cdot(x'-x)},$$

$$\therefore \quad G_F(x'-x) = i\int \tilde{d}k[\theta(t'-t)e^{-ik\cdot(x'-x)} + \theta(t-t')e^{ik\cdot(x'-x)}]$$

$$= -\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x'-x)}}{k^2-m^2+i\varepsilon}, \tag{3.131}$$

$G_F(x'-x)$称为Feynman传播子，是Poincare不变的，且

$$G_F(x'-x) = G_F(x-x'). \tag{3.132}$$

$G_F(x'-x)$的物理意义：

当$t<t'$时，$G_F(x'-x)$描述一个粒子由$\vec{x}\to\vec{x}'$的传播；

当$t'<t$时，$G_F(x'-x)$描述一个反粒子由$\vec{x}'\to\vec{x}$的传播。