第二章　经典场

量子场

↑ 二次量子化

经典场

↑

符合物理要求的作用量

经典物理中：作用量的极值 ⟹ 经典运动方程；

作用量的对称性 ⟹ 物理的守恒律。

2.1 作用量

1. 定义

考虑一个经典的点粒子系统，其广义坐标为$q_i(t)$ $(i=1,2,\cdots,N)$，拉氏量为$L(q_i,\dot{q}_i)$，则体系的作用量定义为：

$$I=\int_{t_1}^{t_2}\mathrm{d}t L(q_i,\dot{q}_i). \qquad \left(\dot{q}_i\equiv\frac{\mathrm{d}}{\mathrm{d}t}q_i\right) \qquad (2.1)$$

N—体系的自由度

2. 作用量的极值与经典运动方程

设

$$q_i(t) \to q_i(t) + \delta q_i(t), \quad \dot{q}_i(t) \to \dot{q}_i(t) + \delta\dot{q}_i(t), \quad (2.2)$$

则

$$\delta I = \int_{t_1}^{t_2} dt\, \delta L = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

$$= \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right\}$$

$$\delta I=\int_{t_1}^{t_2}dt\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}\right)\delta q_i+\boxed{\int_{t_1}^{t_2}dt\,\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\delta q_i\right)}. \qquad (2.3)$$

"表面项"

若变分时限定 $\delta q_i(t_1)=\delta q_i(t_2)=0$，则表面项为0.

由最小作用量原理，

$$\delta I=\int_{t_1}^{t_2}dt\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}\right)\delta q_i=0,$$

得到Euler—Lagrange（拉氏）方程：

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_i}=0. \quad (i=1,2,\cdots,N) \qquad (2.5)$$

（经典运动方程）

3. 作用量的对称性与守恒量

假设对于某种坐标变换

$$q_i(t) \to q_i(t) + \delta q_i(t), \qquad (2.8)$$

有 $\delta I = 0$，即

$$\delta I = \int_{t_1}^{t_2} dt\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\delta q_i + \int_{t_1}^{t_2} dt \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) = 0,$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) = 0. \quad (\text{守恒律}) \qquad (2.10)$$

Noether定理：作用量对于某种数学变换的不变性将导致物理的守恒定律。

物理上的对称性要求能够对作用量加以某种限制，减少作用量的任意性。

目的：建立一套能描述高速运动的微观系统的理论。

高速系统必须满足狭义相对性要求：

任何物理规律在**Lorentz**变换下是不变的。

物理规律具有时空平移不变性。

$$\left.\begin{array}{l}\textbf{Lorentz}\text{变换}\\ \text{时空平移变换}\end{array}\right\}\textbf{Poincare}\text{变换}$$

描述高速运动的作用量必须是**Poincare**不变的。

2.2 Lorentz变换和Poincaré变换

2.2.1 Lorentz变换（LT）

1. LT的定义

联系事件x^μ和x'^ν并保持其度规不变性的线性变换,

$$S^2 = g_{\mu\nu}x^\mu x^\nu = g_{\mu\nu}x'^\mu x'^\nu, \tag{2.16}$$

$$g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1). \tag{2.17}$$

设$\Lambda^{\mu}{}_{\nu}$为LT算符，即

$$x^{\mu} \xrightarrow{\text{LT}} x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} = \Lambda^{\mu}{}_{0} x^{0} + \Lambda^{\mu}{}_{i} x^{i}, \qquad (2.18)$$

要求$\Lambda^{\mu}{}_{\nu}$满足

$$g_{\rho\sigma} \boxed{x^{\rho} x^{\sigma}} \equiv g_{\mu\nu} x'^{\mu} x'^{\nu} = g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} \boxed{x^{\rho} x^{\sigma}},$$

$$\Rightarrow \quad g_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = g_{\rho\sigma}. \text{ (LT矩阵满足的条件)} \qquad (2.19)$$

用矩阵形式表示，

$$g_{\mu\nu} \to \mathrm{diag}(g_{00}, g_{11}, g_{22}, g_{33}) \equiv g,$$

$$x^{\mu} \to \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \equiv X, \qquad \Lambda^{\mu}{}_{\nu} \to \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \cdots \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \cdots \\ \vdots & \vdots & \cdots \end{pmatrix} \equiv L,$$

则度规不变性、LT、LT矩阵满足的条件可写为

$$S^2 = X^T g X, \tag{2.20}$$

$$X' = LX, \tag{2.21}$$

$$g = L^T g L. \tag{2.22}$$

2. LT的分类

① 依据LT矩阵的行列式符号分类

$$\det \boldsymbol{g} = \det \boldsymbol{L}^T \det \boldsymbol{g} \det \boldsymbol{L}, \quad \Rightarrow \quad (\det \boldsymbol{L})^2 = \mathbf{1},$$

$$\det \boldsymbol{L} = \pm \mathbf{1}. \tag{2.23}$$

$$\det \boldsymbol{L} = \begin{cases} +1 & \textbf{proper LT (正LT)} \\ -1 & \textbf{improper LT (负LT)} \end{cases}$$

例子： $L = g$ **为 improper LT,**

对应空间反射($x^0 \to x^0, x^i \to -x^i$).

② 依据LT矩阵的00分量符号分类

取(2.19)式的00分量，

$$1 = g_{\mu\nu}\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0} = g_{00}\Lambda^{0}{}_{0}\Lambda^{0}{}_{0} + g_{ii}\Lambda^{i}{}_{0}\Lambda^{i}{}_{0}$$

$$= (\Lambda^{0}{}_{0})^{2} - (\Lambda^{i}{}_{0})^{2},$$

$$\Rightarrow \qquad |\Lambda^{0}{}_{0}| \geq 1, \tag{2.24}$$

$$\begin{cases} \Lambda^{0}{}_{0} \geq 1 & \text{orthochronous LT（顺时LT）} \\ \Lambda^{0}{}_{0} \leq -1 & \text{non-orthochronous LT（逆时LT）} \end{cases}$$

$\det L$和$\Lambda^{0}{}_{0}$的符号具有LT不变性。

根据$\det L$和$\Lambda^0{}_0$的符号，LT可分为4类：

1) proper orthochronous LT (顺时正LT) $L_+^{\uparrow}$

2) proper non-orthochronous LT (逆时正LT) $L_+^{\downarrow}$

3) improper orthochronous LT (顺时负LT) $L_-^{\uparrow}$

4) improper non-orthochronous LT (逆时负LT) $L_-^{\downarrow}$

3. LT的几个例子

a) 空间转动：

$$x'^0 = x^0, x'^i = a^i{}_j x^j, a\text{是正交矩阵},$$

则

$$L = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \qquad (2.25)$$

3×3矩阵

$$\det L = \det a = \pm 1$$

$$\det L = \det a = 1 \Rightarrow L_+^\uparrow,$$

$$\det L = \det a = -1 \Rightarrow L_-^\uparrow$$

***b*) 平动(boosts)（属$L_+^\uparrow$）**

$$\begin{cases} x'^0 = x^0 \operatorname{ch}\eta - x^1 \operatorname{sh}\eta \\ x'^1 = -x^0 \operatorname{sh}\eta + x^1 \operatorname{ch}\eta, \\ x'^2 = x^2, \quad x'^3 = x^3 \end{cases} \qquad \begin{cases} \operatorname{ch}\eta = \dfrac{1}{\sqrt{(1-v^2)}} \\ \operatorname{sh}\eta = \dfrac{v}{\sqrt{(1-v^2)}} \end{cases},$$

$$L = \begin{pmatrix} \operatorname{ch}\eta & -\operatorname{sh}\eta & \multicolumn{2}{c}{\mathbf{0}} \\ -\operatorname{sh}\eta & \operatorname{ch}\eta & & \\ \multicolumn{2}{c}{\mathbf{0}} & 1 & 0 \\ & & 0 & 1 \end{pmatrix}, \tag{2.26}$$

$$\det L = \operatorname{ch}^2\eta - \operatorname{sh}^2\eta = 1, \quad \Lambda^0{}_0 = \operatorname{ch}\eta \geq 1.$$

c) 时间反演（属$L_-^{\downarrow}$）

$$x'^0 = -x^0, x'^i = x^i; \qquad L = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

有$\det L = -1$且$\Lambda^0{}_0 = -1$

故时间反演属$L_-^{\downarrow}$.

$d)$ 空间反射（属$L_{-}^{\uparrow}$）

$$x'^0 = x^0, x'^i = -x^i; \qquad L = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

有$\det L = -1$且$\Lambda^0{}_0 = 1$,

故空间反射属$L_{-}^{\uparrow}$.

$e)$ 全反演（属$L_+^{\downarrow}$）

$$x'^0 = -x^0, x'^i = -x^i; \qquad L = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}.$$

有$\det L = 1$, $\Lambda^0{}_0 = -1$

故全反演属$L_+^{\downarrow}$.

所有的LT都可分解为以上几种典型变换的乘积

$$L_+^{\uparrow} \times (\text{空间反射}, L_-^{\uparrow}) \rightarrow L_-^{\uparrow};$$

$$L_+^{\uparrow} \times (\text{时间反演}, L_-^{\downarrow}) \rightarrow L_-^{\downarrow};$$

$$L_+^{\uparrow} \times (\text{全反演}, L_+^{\downarrow}) \rightarrow L_+^{\downarrow}.$$

$L_+^{\uparrow}$类称为正LT，转动和平动代表了所有可能的$L_+^{\uparrow}$类变换。转动是3个空间坐标之间的线性变换；平动是空间坐标与时间坐标之间的线性变换。

4. 正LT的参数和生成元

① 参数

3个空间分量×(1个转动+1个平动)= 6.

考虑无穷小LT,

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \varepsilon^{\mu}{}_{\nu}, \tag{2.28}$$

代入$g_{\rho\sigma} = g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}$中，并只保留$\varepsilon^{\mu}{}_{\nu}$的一次项，得

$$0 = g_{\nu\rho}\varepsilon^{\rho}{}_{\mu} + g_{\mu\rho}\varepsilon^{\rho}{}_{\nu} = \varepsilon_{\nu\mu} + \varepsilon_{\mu\nu}, \tag{2.32}$$

$\Rightarrow \quad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}.$

（二阶反对称张量，有6个独立参数）

② 生成元

无穷小LT下，

$$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} = (\delta^{\mu}{}_{\nu} + \varepsilon^{\mu}{}_{\nu}) x^{\nu} = x^{\mu} + \varepsilon^{\mu}{}_{\nu} x^{\nu},$$

$$\Rightarrow \delta x^{\mu} = x'^{\mu} - x^{\mu} = \varepsilon^{\mu}{}_{\nu} x^{\nu} = \varepsilon^{\mu\rho} x_{\rho}. \tag{2.33}$$

引入厄米算符：

$$L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}),\text{（角动量算符）} \tag{2.34}$$

$$L_{\mu\nu} = -L_{\nu\mu},\quad\text{（反对称）}$$

$$\frac{i}{2}\varepsilon^{\rho\sigma} L_{\rho\sigma} x^{\mu} = \frac{i}{2}\varepsilon^{\rho\sigma} \cdot i(x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho}) x^{\mu},$$

$$\frac{i}{2}\varepsilon^{\rho\sigma}L_{\rho\sigma}x^{\mu}=-\frac{1}{2}\varepsilon^{\rho\sigma}(x_{\rho}\delta^{\mu}{}_{\sigma}-x_{\sigma}\delta^{\mu}{}_{\rho})$$

$$=-\frac{1}{2}\varepsilon^{\rho\mu}x_{\rho}+\frac{1}{2}\varepsilon^{\mu\sigma}x_{\sigma}$$

$$=\varepsilon^{\mu\sigma}x_{\sigma}=\delta x^{\mu}, \qquad (2.36)$$

$\therefore L_{\mu\nu}$为LT生成元，满足对易关系

$$[L_{\mu\nu},L_{\rho\sigma}]=ig_{\mu\sigma}L_{\nu\rho}+ig_{\nu\rho}L_{\mu\sigma}-ig_{\mu\rho}L_{\nu\sigma}-ig_{\nu\sigma}L_{\mu\rho}. \qquad (2.37)$$

[SO(3,1)代数]

SO(3,1)群生成元的最一般表示为：

$$M_{\mu\nu} = \underbrace{L_{\mu\nu}}_{\text{轨道空间}} + \underbrace{S_{\mu\nu}}_{\text{自旋空间}}, \tag{2.38}$$

其中$S_{\mu\nu}$厄米，

$$S_{\mu\nu} = -S_{\nu\mu}, \quad M_{\mu\nu} = -M_{\nu\mu}, \quad [L, S] = 0.$$

$M_{\mu\nu}$（或$S_{\mu\nu}$）有6个独立的分量，

满足和$L_{\mu\nu}$同样的李代数，

$$[S_{\mu\nu}, S_{\rho\sigma}] = ig_{\mu\sigma}S_{\nu\rho} + ig_{\nu\rho}S_{\mu\sigma} - ig_{\mu\rho}S_{\nu\sigma} - ig_{\nu\sigma}S_{\mu\rho},$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = ig_{\mu\sigma}M_{\nu\rho} + ig_{\nu\rho}M_{\mu\sigma} - ig_{\mu\rho}M_{\nu\sigma} - ig_{\nu\sigma}M_{\mu\rho}.$$

1) 考虑M_{ij}：

$$J_i \equiv \frac{1}{2}\varepsilon_{ijk}M_{jk}, \tag{2.39}$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{若}ijk\text{是123的偶排列} \\ -1 & \text{若}ijk\text{是123的奇排列} \\ 0 & \text{其他} \end{cases}$$

$$\varepsilon_{123} = \varepsilon_{312} = \cdots = 1, \quad \varepsilon_{132} = \varepsilon_{321} = \cdots = -1, \quad \varepsilon_{112} = \varepsilon_{122} = \cdots = 0,$$

$$\varepsilon_{ilk}\varepsilon_{jmk} = \delta_{ij}\delta_{lm} - \delta_{im}\delta_{lj}.$$

可以证明

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \quad [\mathrm{SU}(2)\text{或}\mathrm{SO}(3)\text{代数}] \tag{2.41}$$

$J_i\,(i = 1,2,3)$为角动量算符，是LT的3个转动生成元。

2) 考虑M_{0i}：

$$K_i \equiv M_{0i},\text{（平动生成元）} \tag{2.44}$$

满足对易关系：

$$[K_i, K_j] = -i\varepsilon_{ijk} J_k, \tag{2.45}$$

$$[J_i, K_j] = i\varepsilon_{ijk} K_k. \tag{2.46}$$

2.2.2 Poincaré变换

（Lorentz变换+时空平移变换）

1. 时空平移变换

均匀的时空平移变换为:

$$x^{\mu} \rightarrow x'^{\mu} = x^{\mu} + a^{\mu}, \tag{2.47}$$

其中a^{μ}为任意常数4矢量。

考虑无穷小平移

$$\delta x^{\mu} = \varepsilon^{\mu} = \boxed{-\mathrm{i}\,\varepsilon^{\rho} P_{\rho}}\, x^{\mu}, \tag{2.49}$$

坐标*x*的无穷小平移算符

其中P_{ρ}为4动量算符

$$P_{\rho} = \mathrm{i}\,\partial_{\rho} = \mathrm{i}\frac{\partial}{\partial x^{\rho}}. \tag{2.50}$$

(时空平移变换的生成元)

2. Poincaré变换

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu. \qquad (2.51)$$

Poincaré群的生成元：　6个$M_{\mu\nu}$ + 4个P_μ

Poincaré代数：

$$[P_\mu, P_\nu] = 0, \qquad (2.52)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = ig_{\mu\sigma}M_{\nu\rho} + ig_{\nu\rho}M_{\mu\sigma} - ig_{\mu\rho}M_{\nu\sigma} - ig_{\nu\sigma}M_{\mu\rho}, \qquad (2.53)$$

$$[M_{\mu\nu}, P_\rho] = -ig_{\mu\rho}P_\nu + ig_{\nu\rho}P_\mu. \qquad (2.54)$$

3. Casimir算符

若算符A与某个群的所有生成元都对易，则A称为该群的Casimir算符。群表示可由该群的Casimir算符的本征值进行分类。

SU(2)群的Casimir算符： $J^2 = J^i J_i$.

Poincaré群的Casimir算符：

$i)$ $P_\mu P^\mu$

$ii)$ $W_\mu W^\mu$

W^{μ}为Pauli－Lubanski矢量：

$$W^{\mu}=-\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}P_{\nu}M_{\rho\sigma}, \tag{2.55}$$

$$\varepsilon^{\mu\nu\rho\sigma}=\begin{cases}1 & \text{当}\mu\nu\rho\sigma\text{是0123的偶排列}\\ -1 & \text{当}\mu\nu\rho\sigma\text{是0123的奇排列}\\ 0 & \text{其它}\end{cases}$$

$\varepsilon^{\mu\nu\rho\sigma}$是4维Levi－Civita符号，是4阶全反对称张量。

利用$\varepsilon^{\mu\nu\rho\sigma}P_{\mu}P_{\nu}=0$可证明:

$$[W^{\mu},\ P^{\rho}]=0, \tag{2.56}$$

$$[M_{\mu\nu},W_{\rho}]=-ig_{\mu\rho}W_{\nu}+ig_{\nu\rho}W_{\mu}. \tag{2.57}$$

2.3　定域场的Poincaré变换

1. 无穷小坐标变换引起的场函数改变

在时空中进行任意无穷小变换，

$$x \to x' = x + \delta x,$$

则场函数的变化为

$$\delta f \equiv f'(x') - f(x) = f'(x+\delta x) - f(x),$$

$$f'(x+\delta x) = f'(x) + \delta x^{\mu}\partial_{\mu} f'(x) + O(\delta x^{2}),$$

$$\delta f = f'(x) - f(x) + \delta x^{\mu}\partial_{\mu} f'(x) + O(\delta x^{2}),$$

$$f'(x) = f(x) + O(\delta x),$$

$$\delta f \approx f'(x) - f(x) + \delta x^{\mu}\partial_{\mu} f(x) + O(\delta x^{2}),$$

只保留$O(\delta x)$的项，则

$$\delta f = \delta_0 f + \delta x^{\mu}\partial_{\mu} f. \tag{2.74}$$

$\delta_0 f \equiv f'(x) - f(x),$

坐标x相同时，f的改变。

输运项，

由于坐标变化而引起的f的变化。

$$\delta = \delta_0 + \delta x^{\mu}\partial_{\mu}.$$ （算符方程）

2. 定域场的时空平移变换

在时空平移变换下，定域场f不变，

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \varepsilon^{\mu}, \qquad \delta x^{\mu} = \varepsilon^{\mu},$$

$$f(x) \to f'(x') = f(x), \quad \delta f = 0,$$

$$0 = \delta f = \delta_0 f + \delta x^{\mu} \partial_{\mu} f = \delta_0 f + \varepsilon^{\mu} \partial_{\mu} f,$$

$$\Rightarrow \qquad \delta_0 f = -\varepsilon^{\mu} \partial_{\mu} f = +\boxed{i\varepsilon^{\mu} P_{\mu} f}. \qquad (2.79)$$

函数f的无穷小平移算符

3. 定域场的Lorentz变换

a) 标量场、矢量场和张量场

① 标量场 (scalar field)

两个惯性系的场函数分别为$\varphi(x)$和$\varphi'(x')$，若

$$\varphi(x) \xrightarrow{\text{LT}} \varphi'(x') = \varphi(x), \tag{2.80}$$

则$\varphi(x)$称为LT下的标量。

无穷小LT下，坐标的变换为

$$\delta x^{\mu} = \varepsilon^{\mu}{}_{\nu} x^{\nu} = \varepsilon^{\mu\nu} x_{\nu}. \tag{2.33}$$

$$0 = \delta\varphi = \delta_0 \varphi + \delta x^{\mu} \partial_{\mu} \varphi,$$

$\delta_0\varphi$定义了*Lorentz*生成元在标量场φ上的表示，可设

$$\delta_0\varphi = \boxed{-\frac{i}{2}\varepsilon^{\mu\nu}\boxed{M_{\mu\nu}}}\varphi, \tag{2.82}$$

标量场φ的LT生成元

标量场φ的无穷小LT算符

又 $$\delta_0\varphi = -\delta x^{\mu}\partial_{\mu}\varphi = -\varepsilon^{\mu\nu}x_{\nu}\partial_{\mu}\varphi$$

$$= \frac{1}{2}\varepsilon^{\nu\mu}(x_{\nu}\partial_{\mu} - x_{\mu}\partial_{\nu})\varphi,$$

$$\Rightarrow \quad M_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}), \text{（对标量场}\varphi\text{）} \tag{2.83}$$

即，对于标量场，$S_{\mu\nu} = 0$（自旋为0）.

② 矢量场和张量场

考虑$\partial_\mu \varphi$的无穷小LT，由(2.74)式，有

$$\delta_0(\partial_\mu \varphi) = \delta(\partial_\mu \varphi) - \delta x^\rho \partial_\rho (\partial_\mu \varphi),$$

$$\delta\partial_\mu \varphi = \delta\partial_\mu \varphi - \partial_\mu \delta\varphi + \partial_\mu \delta\varphi = [\delta, \partial_\mu]\varphi + \partial_\mu \delta\varphi$$

$$= [\delta, \partial_\mu]\varphi, \qquad (2.84)$$

$$[\delta, \partial_\mu] = [\delta_0, \partial_\mu] + [\delta x^\nu \partial_\nu, \partial_\mu] = [\delta x^\nu, \partial_\mu]\partial_\nu$$

$$= [\varepsilon^{\rho\nu} x_\rho, \partial_\mu]\partial_\nu = -\varepsilon^\nu{}_\mu \partial_\nu = \varepsilon_\mu{}^\nu \partial_\nu. \qquad (2.86)$$

$$\therefore \quad \delta\partial_\mu \varphi = \varepsilon_\mu{}^\nu \partial_\nu \varphi. \qquad (2.87)$$

$$\delta_0(\partial_\mu \varphi) = \varepsilon_\mu{}^\nu \partial_\nu \varphi - \delta x^\rho \partial_\rho \partial_\mu \varphi,$$

$$\delta_0(\partial_\mu\varphi) = \varepsilon_\mu{}^\nu\partial_\nu\varphi - \varepsilon^\rho{}_\nu x^\nu\partial_\rho\partial_\mu\varphi,$$

$$= -\frac{i}{2}\varepsilon^{\rho\sigma}L_{\rho\sigma},$$

$$\delta_0\partial_\mu\varphi = \varepsilon_\mu{}^\nu\partial_\nu\varphi - \frac{i}{2}\varepsilon^{\rho\sigma}L_{\rho\sigma}\partial_\mu\varphi$$

$$= -\frac{i}{2}\varepsilon^{\rho\sigma}(S_{\rho\sigma})_\mu{}^\nu\partial_\nu\varphi - \frac{i}{2}\varepsilon^{\rho\sigma}L_{\rho\sigma}\partial_\mu\varphi, \tag{2.88}$$

$$= -\frac{i}{2}\varepsilon^{\rho\sigma}M_{\rho\sigma}(\partial_\mu\varphi),$$

其中

$$(S_{\rho\sigma})_\mu{}^\nu = i(g_{\rho\mu}g_\sigma^\nu - g_{\sigma\mu}g_\rho^\nu). \tag{2.89}$$

矢量场：在LT下，变换规律遵循(2.88)式的场；(A^{μ})

张量场：带有多个Lorentz指标，在LT下，每个指标都如矢量变换的场。($B_{\mu\nu}$,$C_{\alpha\beta\gamma}$)

对于正LT下的标量场：

对应空间反射不变号的标量称为标量(scalar);

对应空间反射变号的标量称为赝标量(pseudoscalar)。

对于正LT下的矢量场，在空间反射下，

A^μ中的A^0不变号，A^i变号的矢量称为矢量(vector);

A^μ中的A^0变号，A^i不变号的矢量称为赝矢量或轴矢量。

(pseudovector, Axial vector)

③ Poincaré不变量的构造

标量场： $\varphi(x),\ \cos\varphi(x),$

$$\partial_\mu\partial^\mu\varphi(x),\ \partial_\mu\varphi(x)\partial^\mu\varphi(x),\ \cdots$$

矢量场： $A_\mu(x)A^\mu(x),\ \ \partial_\mu A_\nu(x)\partial^\mu A^\nu(x),$

$$\partial^\mu A^\nu(x)\partial_\mu A_\nu(x),\ \ \partial^\mu A^\mu(x),\ \cdots$$

2阶张量场： $B_{\mu\nu}(x)B^{\mu\nu}(x),\ \ \partial_\rho B_{\mu\nu}(x)\partial^\rho B^{\mu\nu}(x),$

$$\partial_\rho B_{\mu\nu}(x)\partial^\mu B^{\rho\nu}(x),\ \cdots$$

b) 旋量场

① γ代数

Dirac方程：

$$i\frac{\partial}{\partial t}\psi=(-i\vec{\alpha}\cdot\nabla+\beta m)\psi. \tag{1.6}$$

旋量场—符合Dirac eq.的场。

引入γ矩阵：

$$\gamma^0=\beta, \qquad \gamma^i=\beta\alpha^i, \tag{2.92}$$

Dirac方程可写为：

$$(i\gamma^\mu\frac{\partial}{\partial x^\mu}-m)\psi(x)=0. \tag{2.94}$$

定义矩阵

$$\gamma^5 \equiv \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{2.114}$$

$$= -\frac{i}{4}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. \tag{2.118}$$

γ矩阵满足反对易关系

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{2.93}$$

$$\{\gamma^5, \gamma^\mu\} = 0.$$

$$(\gamma^0)^2 = I, \qquad (\gamma^i)^2 = -I, \qquad (\gamma^5)^2 = I.$$

$$\gamma^{0+} = \gamma^0, \qquad \gamma^{i+} = -\gamma^i, \qquad \gamma^{5+} = \gamma^5,$$

$$\gamma^0\gamma^{\mu+}\gamma^0 = \gamma^\mu.$$

在Dirac表象中：

$$\gamma^0=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},\quad \gamma^i=\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},\quad \gamma^5=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

在Weyl表象中：

$$\gamma^0=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},\quad \gamma^i=\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},\quad \gamma^5=\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

Weyl表象与Dirac表象之间的变换关系是

$$\gamma^\mu_{\mathrm{Weyl}}=U\gamma^\mu_{\mathrm{Dirac}}U^+,\qquad U=\frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}.$$

② LT算符$S(\Lambda)$

考虑2个以LT相联系的惯性系，

$$x^{\mu} \xrightarrow{\text{LT}} x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu},$$

$$\psi(x) \xrightarrow{\text{LT}} \psi'(x') = S(\Lambda)\psi(x), \qquad (2.96)$$

其中$S(\Lambda)$是一个非奇异的4×4矩阵。

$\psi(x)$和$\psi'(x')$均满足Dirac方程：

$$(i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - m)\psi(x) = 0, \qquad (2.94)$$

$$(i\gamma^{\mu}\frac{\partial}{\partial x'^{\mu}} - m)\psi'(x') = 0, \qquad (2.97)$$

$$\Rightarrow \quad i\gamma^{\mu}\frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial}{\partial x^{\nu}}S(\Lambda)\psi(x)-mS(\Lambda)\psi(x)=0,$$

$$\Rightarrow \quad iS^{-1}(\Lambda)\gamma^{\nu}S(\Lambda)\frac{\partial x^{\mu}}{\partial x'^{\nu}}\frac{\partial}{\partial x^{\mu}}\psi(x)-m\psi(x)=0,$$

比较上式和(2.94)式，有

$$S^{-1}(\Lambda)\gamma^{\nu}S(\Lambda)\frac{\partial x^{\mu}}{\partial x'^{\nu}}=\gamma^{\mu},$$

利用$\frac{\partial x^{\mu}}{\partial x'^{\nu}}=(\Lambda^{-1})^{\mu}{}_{\nu}$，得

$$S(\Lambda)\gamma^{\mu}S^{-1}(\Lambda)=(\Lambda^{-1})^{\mu}{}_{\nu}\gamma^{\nu}=\Lambda_{\nu}{}^{\mu}\gamma^{\nu}, \tag{2.98}$$

上式是Dirac方程的Lorentz不变性对$S(\Lambda)$所加的条件。

③ 无穷小LT下$S(\Lambda)$的形式

对于无穷小LT：

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \varepsilon^{\mu}{}_{\nu}, \qquad (\Lambda^{-1})^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} - \varepsilon^{\mu}{}_{\nu}, \tag{2.99}$$

$$\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu},$$

相应地，设

$$S(\Lambda) = I - \frac{i}{4}\varepsilon^{\rho\sigma}\sigma_{\rho\sigma} + \cdots, \tag{2.100}$$

$$S^{-1}(\Lambda) = I + \frac{i}{4}\varepsilon^{\rho\sigma}\sigma_{\rho\sigma} + \cdots,$$

其中$\sigma_{\rho\sigma} + \sigma_{\sigma\rho} = 0.$

将(2.99)和(2.100)代入(2.98)，只保留$O(\varepsilon)$的项，得

$$\varepsilon^{\rho\sigma}[\gamma^{\mu},\sigma_{\rho\sigma}] = 4i\varepsilon^{\mu\rho}\gamma_{\rho}$$

$$= 4ig^{\mu}{}_{\sigma}\varepsilon^{\sigma\rho}\gamma_{\rho} = 2i\varepsilon^{\rho\sigma}(g^{\mu}{}_{\rho}\gamma_{\sigma} - g^{\mu}{}_{\sigma}\gamma_{\rho}),$$

$$\Rightarrow \qquad [\gamma^{\mu},\sigma_{\rho\sigma}] = 2i(g^{\mu}{}_{\rho}\gamma_{\sigma} - g^{\mu}{}_{\sigma}\gamma_{\rho}), \qquad (2.101)$$

$$\sigma_{\rho\sigma} = \frac{i}{2}[\gamma_{\rho},\gamma_{\sigma}]. \qquad (2.102)$$

无穷小LT下$S(\Lambda)$的形式如下：

$$S(\Lambda) = I - \frac{i}{4}\varepsilon^{\rho\sigma}\sigma_{\rho\sigma} + \cdots$$

$$= I + \frac{1}{8}\varepsilon^{\rho\sigma}[\gamma_{\rho},\gamma_{\sigma}] + \cdots$$

④ 有限LT下$S(\Lambda)$的形式

对于有限的LT，将有限变换参数$\omega^{\rho\sigma}$分成n等分并令$n\to\infty$，则$\omega^{\rho\sigma}/n$为无穷小，

$$S(\Lambda)=\left(I-\frac{i}{4}\frac{\omega^{\rho\sigma}}{n}\sigma_{\rho\sigma}\right)^{n}\Big|_{n\to\infty}$$

$$=\mathrm{e}^{-\frac{i}{4}\frac{\omega^{\rho\sigma}}{n}\sigma_{\rho\sigma}}. \qquad (2.103)$$

⑤ LT生成元对旋量场作用的一般形式

对于无穷小LT：

$$\psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x)$$

$$= (I - \frac{i}{4}\varepsilon^{\rho\sigma}\sigma_{\rho\sigma})\psi(x^{\mu} - \varepsilon^{\mu\nu}x_{\nu})$$

$$= (I - \frac{i}{4}\varepsilon^{\rho\sigma}\sigma_{\rho\sigma} - \varepsilon^{\rho\sigma}x_{\sigma}\partial_{\rho})\psi(x)$$

$$= \{I - \frac{i}{2}\varepsilon^{\rho\sigma}[\underbrace{\frac{1}{2}\sigma_{\rho\sigma} + i(x_{\rho}\partial_{\sigma} - x_{\sigma}\partial_{\rho})}_{=M_{\rho\sigma}}]\}\psi(x),$$

$$\delta_0\psi = \psi'(x) - \psi(x) = -\frac{i}{2}\varepsilon^{\rho\sigma}M_{\rho\sigma}\psi(x),$$

其中$M_{\rho\sigma}$为Lorentz生成元，

$$M_{\rho\sigma} = S_{\rho\sigma} + L_{\rho\sigma} = \frac{1}{2}\sigma_{\rho\sigma} + i(x_\rho\partial_\sigma - x_\sigma\partial_\rho), \qquad (2.104)$$

$$\gamma^0\sigma^+_{\rho\sigma}\gamma^0 = \sigma_{\rho\sigma}, \qquad \sigma^+_{ij} = \sigma_{ij}, \qquad \sigma^+_{i0} = -\sigma_{i0}.$$

$$S(\Lambda) = e^{-\frac{i}{4}\frac{\omega^{\rho\sigma}}{n}\sigma_{\rho\sigma}}$$

在空间转动下是幺正的[$S^+(\Lambda) = S^-(\Lambda)$]，

在Lorentz平动下是厄米的[$S^+(\Lambda) = S(\Lambda)$]。

⑥ 旋量场在空间反射下的变换性质

$$\Lambda^{\mu}{}_{\nu}=\begin{pmatrix}1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1\end{pmatrix},$$

$$S(\Lambda)\gamma^{\mu}S^{-1}(\Lambda)=(\Lambda^{-1})^{\mu}{}_{\nu}\gamma^{\nu}=\Lambda^{\mu}{}_{\nu}\gamma^{\nu},$$

$$\Rightarrow \quad S(\Lambda)=\eta_P\gamma^{0}, \tag{2.106}$$

$$\therefore \quad \psi'(x')=\eta_P\gamma^{0}\psi(x), \tag{2.107}$$

其中η_P是一个任意的相因子。

⑦ 双线性型$\overline{\psi}\Gamma^a\psi$

定义

$$\overline{\psi} \equiv \psi^{+}\gamma^{0}, \tag{2.108}$$

$$\overline{\psi}(x) \xrightarrow{\text{LT}} \overline{\psi}'(x') = \overline{\psi}(x)S^{-1}(\Lambda), \tag{2.109}$$

$$\overline{\psi}(x) \xrightarrow{\mathcal{P}(\text{空间反射})} \overline{\psi}'(x') = \eta_P^{*}\overline{\psi}(x)\gamma^{0}. \tag{2.110}$$

$$\overline{\psi}(x)\psi(x) \xrightarrow{\text{LT}} \overline{\psi}'(x')\psi'(x') = \overline{\psi}(x)S^{-1}(\Lambda)S(\Lambda)\psi(x)$$

$$= \overline{\psi}(x)\psi(x),\text{（标量）}$$

$$\overline{\psi}(x)\psi(x) \xrightarrow{\mathcal{P}} \overline{\psi}'(x')\psi'(x') = \eta_P^{*}\overline{\psi}(x)\gamma^{0}\cdot\eta_P\gamma^{0}\psi(x)$$

$$= \overline{\psi}(x)\psi(x).\text{（标量）}$$

在$\bar{\psi}$和ψ之间插入任意4×4矩阵，则此类矩阵必满足双线性型。其中独立的分量有16个，如下：

$S:$	$\Gamma^{S}=I$	$\bar{\psi}\psi$	标量 (Scalar)	1
$V:$	$\Gamma^{V}_{\mu}=\gamma_{\mu}$	$\bar{\psi}\gamma_{\mu}\psi$	矢量 (Vector)	4
$T:$	$\Gamma^{T}_{\mu\nu}=\sigma_{\mu\nu}$	$\bar{\psi}\sigma_{\mu\nu}\psi$	张量 (Tensor)	6
$A:$	$\Gamma^{A}_{\mu}=\gamma_{5}\gamma_{\mu}$	$\bar{\psi}\gamma_{5}\gamma_{\mu}\psi$	轴矢量 (Axial vector)	4
$P:$	$\Gamma^{P}=i\gamma_{5}$	$\bar{\psi}\gamma_{5}\psi$	赝标量 (Pseudoscalar)	1

由Dirac旋量构造Poincaré不变量：

$$\bar{\psi}\psi,\quad \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi,\quad \partial_{\mu}\bar{\psi}\gamma^{\mu}\psi,\quad \cdots$$

Γ^a**的性质：**

1) $(\Gamma^a)^2 \equiv \pm I$;

2) $\forall \Gamma^a (\Gamma^a \neq \Gamma^S = I)$, $\exists \Gamma^b$, 使$\{\Gamma^a, \Gamma^b\} = 0$;

3) $\mathrm{Tr}\,\Gamma^a = 0$, $(a \neq S)$;

4) $\Gamma^a \Gamma^b = \Gamma^c$, $(a \neq b, c \neq S)$,

可相差一相因子± 1或$\pm i$;

5) 若$\sum_a \lambda_a \Gamma^a = 0$, 则$\lambda_a = 0$.

⑧ 手征旋量（Weyl 旋量）和Majorana 旋量

定义手征旋量：

$$\begin{cases} \psi_L = \dfrac{1}{2}(1-\gamma_5)\psi, \\ \psi_R = \dfrac{1}{2}(1+\gamma_5)\psi, \end{cases} \quad \text{(2分量复旋量)} \qquad (2.119)$$

$$\psi = \psi_L + \psi_R.$$

引入电荷共轭变换矩阵

$$C = i\gamma^2\gamma^0, \tag{2.122}$$

定义电荷共轭旋量：

$$\psi^C = C\overline{\psi}^T. \tag{2.123}$$

若4分量旋量ψ_M满足自电荷共轭条件

$$(\psi_M)^C = \psi_M, \tag{2.124}$$

则ψ_M称为Majorana旋量。