第五章　S矩阵和微扰论

§5-1 相互作用绘景、U矩阵和S矩阵

➢ 三种绘景

① S绘景： $\dot{O}_{\mathrm{S}}(t)=0,\qquad \mathrm{i}\frac{\partial}{\partial t}|t\rangle_{\mathrm{S}}=H_{\mathrm{S}}|t\rangle_{\mathrm{S}}.$

② H绘景： $\dot{O}_{\mathrm{H}}(t)=\mathrm{i}[H_{\mathrm{H}},O_{\mathrm{H}}(t)],\qquad \frac{\partial}{\partial t}|t\rangle_{\mathrm{H}}=0.$

③ I 绘景： $\dot{O}_{\mathrm{I}}(t)=\mathrm{i}[(H_0)_{\mathrm{I}},O_{\mathrm{I}}(t)],\qquad \mathrm{i}\frac{\partial}{\partial t}|t\rangle_{\mathrm{I}}=(H_i(t))_{\mathrm{I}}|t\rangle_{\mathrm{I}}.$

$$H_{\mathrm{I}}=(H_0)_{\mathrm{I}}+(H_i)_{\mathrm{I}}$$

绘景变换:

S-H:

$$|t\rangle_H = e^{iH_S t}|t\rangle_S, \qquad O_H(t) = e^{iH_S t} O_S(t) e^{-iH_S t},$$

$$H_H(t) = e^{iH_S t} H_S e^{-iH_S t} = H_S = H.$$

S-I:

$$|t\rangle_I = e^{i(H_0)_S t}|t\rangle_S, \qquad O_I(t) = e^{i(H_0)_S t} O_S e^{-i(H_0)_S t},$$

$$(H_0)_I = e^{i(H_0)_S t}(H_0)_S e^{-i(H_0)_S t} = (H_0)_S = H_0,$$

H-I:

$$|t\rangle_I = e^{i(H_0)_S t}|t\rangle_S = e^{i(H_0)_S t} e^{-iHt}|t\rangle_H,$$

$$O_I(t) = e^{i(H_0)_S t} O_S e^{-i(H_0)_S t} = e^{i(H_0)_S t} e^{-iHt} O_H(t) e^{iHt} e^{-i(H_0)_S t}.$$

物理观测量:

$$\left(\langle a|O|b\rangle\right)_S = \left(\langle a|O|b\rangle\right)_H = \left(\langle a|O|b\rangle\right)_I.$$

➢ U矩阵

$$|t\rangle_{\mathrm{I}} = U(t,t_0)|t_0\rangle_{\mathrm{I}}, \tag{5.17}$$

$$\mathrm{i}\frac{\partial}{\partial t}U(t,t_0) = \left(H_i(t)\right)_{\mathrm{I}} U(t,t_0). \tag{5.22}$$

性质： $U(t_0,t_0)=1,$

$$U(t_1,t_2)U(t_2,t_3)=U(t_1,t_3),$$

$$U(t,t_0)=U^{-1}(t_0,t),$$

$$U(t,t_0)=U^{+}(t,t_0).$$

➢ S矩阵

$$H = H_0 + H_i,$$

$$S = \lim_{\substack{t_0 \to -\infty \\ t \to +\infty}} U(t, t_0) = U(\infty, -\infty), \tag{5.25}$$

为使上式的双向极限存在，H_0和H必须有相同的谱。

例1： 一维谐振子

$$H = \omega a^+ a, \tag{5.26}$$

$$H = \omega_0 a^+ a + (\omega - \omega_0) a^+ a = H_0 + H_i, \tag{5.27}$$

由(5.22)式，有

$$\mathrm{i} \frac{\partial}{\partial t} U(t, t_0) = H_i U(t, t_0) = (\omega - \omega_0) a^+ a U(t, t_0), \tag{5.28}$$

$$\Rightarrow \quad U(t, t_0) = \mathrm{e}^{-\mathrm{i}(\omega - \omega_0)(t - t_0) N}, \quad N = a^+ a, \tag{5.29}$$

$N=a^{+}a$的本征方程为：

$$N|n\rangle=n|n\rangle,$$

$$\therefore\quad \langle n|U(t,t_0)|n\rangle=\langle n|\,\mathrm{e}^{-\mathrm{i}(\omega-\omega_0)(t-t_0)N}\,|n\rangle=\mathrm{e}^{-\mathrm{i}(\omega-\omega_0)(t-t_0)n},$$

仅当$\omega=\omega_0$时，极限 $\lim\limits_{\substack{t_0\to-\infty\\ t\to+\infty}}\langle n|U(t,t_0)|n\rangle=\lim\limits_{\substack{t_0\to-\infty\\ t\to+\infty}}\mathrm{e}^{-\mathrm{i}(\omega-\omega_0)(t-t_0)n}$ 存在，

H_0和H有相同的谱。

例2：单个实标量场

$$\mathcal{H} =: \frac{1}{2}[\pi^2 + (\nabla\varphi)^2 + m_0^2\varphi^2] + \lambda_0\varphi^4 :, \qquad (5.31)$$

无法显示该图片。

λ_0 — 裸耦合常数，

$$\mathcal{H}\left|\text{vac}\right\rangle = 0. \qquad (5.32)$$

$\lambda_0\varphi^4 > 0$，排斥力，系统无稳定束缚态，物理态可表为单粒子态的乘积。$\mathcal{H}$具有平移不变性 $\rightarrow$ 存在守恒量P^μ，

$$P^\mu\left|k\right\rangle = k^\mu\left|k\right\rangle, \qquad (5.33)$$

$$k_\mu k^\mu = m^2, \qquad (5.34)$$

m — 物理质量。单粒子能量$k^0 = \sqrt{\vec{k}^2 + m^2}$，系统总能量

$$E=\sum_{n_k} n_k k^0=\sum_{n_{\vec{k}}} n_{\vec{k}}\sqrt{\vec{k}^2+m^2}, \tag{5.35}$$

$n_{\vec{k}}=0,1,2,\cdots$是具有物理质量m，动量$\vec{k}$的0自旋粒子数。

引入

$$\mathcal{H}_0 =:\frac{1}{2}[\pi^2+(\nabla\varphi)^2+m^2\varphi^2]:, \tag{5.36}$$

$$\mathcal{H}_0\left|0\right\rangle=0, \tag{5.37}$$

则

$$\mathcal{H}_i =:\frac{1}{2}(m_0^2-m^2)\varphi^2+\lambda_0\varphi^4:, \tag{5.38}$$

H_0和H有全同的谱。

I绘景中，场算符φ_r及π_r满足自由场的运动方程和等时对易关系，因而场算符可按自由场的c-数本征解展开。

对实标量场，

$$(\Box + m^2)\varphi(x) = 0,$$

$$(\Box + m^2)\pi(x) = 0,$$

对任意时间t，有展开式

$$\varphi(x) = \int \mathrm{d}\tilde{k}[a(k)\mathrm{e}^{-\mathrm{i}kx} + a^+(k)\mathrm{e}^{\mathrm{i}kx}],$$

$$\pi(x) = \int \mathrm{d}\tilde{k}(-\mathrm{i}\,\omega_k)[a(k)\mathrm{e}^{-\mathrm{i}kx} - a^+(k)\mathrm{e}^{\mathrm{i}kx}],$$

$$\omega_k = \sqrt{\vec{k}^2 + m^2},$$

$$[\varphi(x), \pi(x')]_{t=t'} = \mathrm{i}\,\delta^3(\vec{x} - \vec{x}'),$$

$a^+(k)|0\rangle$ — 具有物理质量的自由粒子。

§ 5-2 微扰展开

$$\begin{cases} \mathrm{i}\dfrac{\partial}{\partial t}U(t,t_0)=H_i(t)U(t,t_0), \\ U(t_0,t_0)=1 \end{cases} \tag{5.41}$$

化为积分方程：

$$U(t,t_0)=1-i\int_{t_0}^{t}dt_1H_i(t_1)U(t_1,t_0), \tag{5.42}$$

叠代得到

$$U(t,t_0)=1-i\int_{t_0}^{t}dt_1H_i(t_1)[1-i\int_{t_0}^{t_1}dt_2H_i(t_2)U(t_2,t_0)]$$

$$=1-i\int_{t_0}^{t}dt_1H_i(t_1)+(-i)^2\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2H_i(t_1)H_i(t_2)+\cdots$$

$$+(-i)^n\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{n-1}}dt_nH_i(t_1)H_i(t_2)\cdots H_i(t_n)+\cdots$$

$n=2$的项：

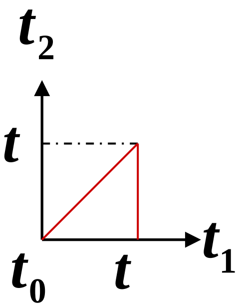
$$U^{(2)}=\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2 H_i(t_1)H_i(t_2)=\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2 TH_i(t_1)H_i(t_2),$$

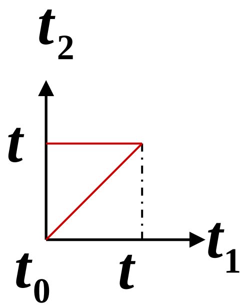
$$TH_i(t_1)H_i(t_2)=TH_i(t_2)H_i(t_1),$$

$$U^{(2)}=\frac{1}{2}\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2 TH_i(t_1)H_i(t_2)+\frac{1}{2}\int_{t_0}^{t}dt_2\int_{t_0}^{t_2}dt_1 TH_i(t_2)H_i(t_1),$$

$$\int_{t_0}^{t}dt_2\int_{t_0}^{t_2}dt_1=\int_{t_0}^{t}dt_2\int_{t_0}^{t_2}dt_1\theta(t_2-t_1)=\int_{t_0}^{t}dt_2\int_{t_0}^{t}dt_1\theta(t_2-t_1)$$

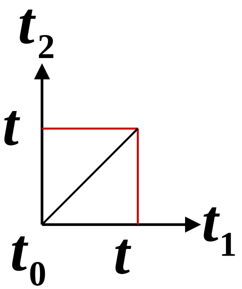
$$=\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\theta(t_2-t_1)=\int_{t_0}^{t}dt_1\int_{t_1}^{t}dt_2,$$





$$U^{(2)} = \frac{1}{2}\left(\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 + \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2\right) T H_i(t_1) H_i(t_2)$$

$$= \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T H_i(t_1) H_i(t_2),$$



$$\Rightarrow \quad \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 = \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 = \frac{1}{2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2, \tag{5.49}$$

n级项：

$$U^{(n)}=\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{n-1}}dt_nH_i(t_1)H_i(t_2)\cdots H_i(t_n),\quad t_1>t_2>\cdots>t_n$$

$$=\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\cdots\int_{t_0}^{t_{n-1}}dt_nTH_i(t_1)H_i(t_2)\cdots H_i(t_n)$$

$$=\frac{1}{n!}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\cdots\int_{t_0}^{t}dt_nTH_i(t_1)H_i(t_2)\cdots H_i(t_n),$$

$U(t,t_0)$的级数展开为：

$$U(t,t_0)=1+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}dt_1\cdots\int_{t_0}^{t}dt_nTH_i(t_1)\cdots H_i(t_n),\tag{5.51}$$

或

$$U(t,t_0)=T\exp\left[-i\int_{t_0}^{t}H_i(t_1)dt_1\right].\tag{5.52}$$

设

$$H_i(t)=\int d^3x\,\mathcal{H}_i(x), \tag{5.53}$$

$U(t,t_0)$的级数展开可写为：

$$U(t,t_0)=T\exp\left[-i\int_{t_0}^{t}d^4x\mathcal{H}_i(x)\right]$$

$$=1+\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\int_{t_0}^{t}d^4x_1\cdots\int_{t_0}^{t}d^4x_nT\mathcal{H}_i(x_1)\cdots\mathcal{H}_i(x_n). \tag{5.54}$$

由于微观因果性：

$$[\mathcal{H}_i(x_1),\mathcal{H}_i(x_2)]=0,\quad \text{对于}(x_1-x_2)^2<0, \tag{5.55}$$

$T\mathcal{H}_i(x_1)\cdots\mathcal{H}_i(x_n)$是LT不变的，

$$T\mathcal{H}_i(x_1)\cdots\mathcal{H}_i(x_n)\longrightarrow T\mathcal{H}_i(x_1')\cdots\mathcal{H}_i(x_n').$$

S矩阵的级数展开：

$$S = U(\infty,-\infty) = T\exp\left[-i\int_{-\infty}^{\infty} d^4x \mathcal{H}_i(x)\right]$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d^4x_1 \cdots \int_{-\infty}^{\infty} d^4x_n T\mathcal{H}_i(x_1)\cdots\mathcal{H}_i(x_n). \qquad (5.56)$$

$\langle f|S|i\rangle$— 从初态$|i\rangle$至终态$|f\rangle$的跃迁振幅，

$$H = H_0 + H_i,$$

$|i\rangle$和$|f\rangle$为物理的自由粒子，可取为H_0的本征态。

以QED为例，

$$H = H_0 + H_i,$$

$$H_0 = \int \mathrm{d}^3 x (\mathcal{H}_{\text{e.m.}} + \mathcal{H}_{\text{Dirac}}),$$

$$\mathcal{H}_{\text{e.m.}} =: -\frac{1}{2} \dot{A}^{\mu} \dot{A}_{\mu} + \nabla A^{\mu} \cdot \nabla A_{\mu} :,$$

$$\mathcal{H}_{\text{Dirac}} =: \overline{\psi} (-i\vec{\gamma} \cdot \nabla + m) \psi :,$$

$$H_i = \int \mathrm{d}^3 x \mathcal{H}_i(x),$$

$$\mathcal{H}_i(x) =: e\overline{\psi}(x) \not{A}(x) \psi(x) - \delta m \overline{\psi}(x) \psi(x) :,$$

$$\delta m = m - m_0 .$$

$|i\rangle$和$|f\rangle$可取为H_0的本征态$|n\rangle$,

$$|n\rangle = \prod a^{(\lambda_i)+}(k_i) b^{+}_{\alpha_i}(p_i) d^{+}_{\alpha_j}(p_j) |0\rangle, \tag{5.60}$$

$|0\rangle$为H_0的真空态，满足

$$H_0|0\rangle = 0.$$

H_0和H有全同的谱。设$|n\rangle$满足

$$H_0|n\rangle = E_n|n\rangle,$$

引入

$$|n^{in}\rangle \equiv U(0,-\infty)|n\rangle,$$

$$|n^{out}\rangle \equiv U(0,\infty)|n\rangle,$$

满足

$$H\left|n^{in}\right\rangle = E_n\left|n^{in}\right\rangle,$$

$$H\left|n^{out}\right\rangle = E_n\left|n^{out}\right\rangle。$$

S矩阵元为

$$\langle n'|S|n\rangle = \langle n'|U(\infty,0)U(0,-\infty)|n\rangle = \left\langle n'^{out}\middle|n^{in}\right\rangle.$$

§5-3 Wick定理

$$e^- \quad + \quad e^+ \quad \to \quad \gamma \quad + \quad \gamma$$

$$(p,\alpha) \qquad (q,\beta) \qquad (k,\lambda) \qquad (k',\lambda')$$

$$|i\rangle = b_\alpha^+(p)d_\beta^+(q)|0\rangle, \qquad |f\rangle = a_\lambda^+(k)a_{\lambda'}^+(k')|0\rangle,$$

$$S_{fi} = \langle f|S|i\rangle = \langle 0|a_\lambda(k)a_{\lambda'}(k')Sb_\alpha^+(p)d_\beta^+(q)|0\rangle,$$

S_{fi}可由S中正规乘积项$a_\lambda^+(k)a_{\lambda'}^+(k')b_\alpha(p)d_\beta(q)$的系数给出。

$$\text{编时乘积} \xrightarrow{\text{wick定理}} \text{正规乘积}$$

设$\varphi(x)$为任意定域场，推广正规乘积的定义：

$$:c\varphi(x_1)\cdots\varphi(x_n):=c:\varphi(x_1)\cdots\varphi(x_n): \qquad (5.68)$$

记$\varphi(x_i)\equiv\varphi_i$.

➢ Wick定理：

$$T\varphi_1\cdots\varphi_n$$

$$=:\varphi_1\cdots\varphi_n:$$

$$+:\underbracket{\varphi_1\varphi_2}\varphi_3\cdots\varphi_n:+:\underbracket{\varphi_1\varphi_2\varphi_3}\cdots\varphi_n:+\text{所有含一对算符收缩的项}$$

$$+:\underbracket{\varphi_1\varphi_2}\underbracket{\varphi_3\varphi_4}\varphi_5\cdots\varphi_n:+:\underbracket{\varphi_1\varphi_2\varphi_3}\varphi_4\varphi_5\cdots\varphi_n:+\text{所有含两对算符收缩的项}$$

$$+\cdots$$

$$+\begin{cases}\underbracket{\varphi_1\varphi_2}\cdots\underbracket{\varphi_{n-1}\varphi_n}+\text{所有含}n/2\text{对算符收缩的项}(n\text{偶})\\ \underbracket{\varphi_1\varphi_2}\cdots\underbracket{\varphi_{n-2}\varphi_{n-1}}\varphi_n+\text{所有含}(n-1)/2\text{对算符收缩的项}(n\text{偶})\end{cases}\qquad(5.69)$$

其中

$$\underbracket{\varphi_1\varphi_2}\equiv\langle 0|T\varphi_1\varphi_2|0\rangle\text{—算符的收缩，对应Feynman传播子.}$$

$$:\underbracket{\varphi_1\varphi_2}\varphi_3\cdots\varphi_n := \underbracket{\varphi_1\varphi_2}:\varphi_3\cdots\varphi_n:$$

$$:\underbracket{\varphi_1\varphi_2\varphi_3}\cdots\varphi_n := :\delta p\,\underbracket{\varphi_1\varphi_3}\varphi_2\cdots\varphi_n := \delta p\,\underbracket{\varphi_1\varphi_3}:\varphi_2\cdots\varphi_n:$$

$$\delta p = \begin{cases} -1 & \text{若}\varphi_2\varphi_3\text{同为费米子算符,} \\ +1 & \text{其它情况.} \end{cases}$$

➢ 算符的正规乘积：

$$\varphi = \varphi^{(+)} + \varphi^{(-)},$$

$\varphi^{(+)}$只含消灭算符，$\varphi^{(-)}$只含产生算符，

$$\varphi^{(+)}\left|0\right\rangle = \left\langle 0\right|\varphi^{(-)} = 0,$$

$$:\varphi_1\varphi_2: = :(\varphi_1^{(+)} + \varphi_1^{(-)})(\varphi_2^{(+)} + \varphi_2^{(-)}):$$

$$= :\varphi_1^{(+)}\varphi_2^{(+)} + \varphi_1^{(-)}\varphi_2^{(+)} + \varphi_1^{(+)}\varphi_2^{(-)} + \varphi_1^{(-)}\varphi_2^{(-)}:$$

$$= \varphi_1^{(+)}\varphi_2^{(+)} + \varphi_1^{(-)}\varphi_2^{(+)} \mp \varphi_2^{(-)}\varphi_1^{(+)} + \varphi_1^{(-)}\varphi_2^{(-)}, \tag{5.75}$$

$$:\varphi_1\cdots\varphi_n: = \sum_{A,B}\delta p \prod_{i\subset A}\varphi_i^{(-)} \prod_{j\subset B}\varphi_j^{(+)}, \tag{5.76}$$

对 $:\varphi_1\varphi_2:$，集合A和B可为：

i	1,2	1	2	–
j	–	2	1	1,2

$$:\varphi_1\varphi_2: = \varphi_1^{(-)}\varphi_2^{(-)} + \varphi_1^{(-)}\varphi_2^{(+)} + \delta p\,\varphi_2^{(-)}\varphi_1^{(+)} + \varphi_1^{(+)}\varphi_2^{(+)}$$

对 $:\varphi_1\varphi_2\varphi_3:$，集合A和B可为：

i	1,2,3	1,2	1,3	2,3	1	2	3	–
j	–	3	2	1	2,3	1,3	1,2	1,2,3

$$\begin{aligned}:\varphi_1\varphi_2\varphi_3: &= \varphi_1^{(-)}\varphi_2^{(-)}\varphi_3^{(-)} + \varphi_1^{(-)}\varphi_2^{(-)}\varphi_3^{(+)} + \delta p_3\varphi_1^{(-)}\varphi_3^{(-)}\varphi_2^{(+)} \\ &+ \delta p_4\varphi_2^{(-)}\varphi_3^{(-)}\varphi_1^{(+)} + \varphi_1^{(-)}\varphi_2^{(+)}\varphi_3^{(+)} + \delta p_6\varphi_2^{(-)}\varphi_1^{(+)}\varphi_3^{(+)} \\ &+ \delta p_7\varphi_3^{(-)}\varphi_1^{(+)}\varphi_2^{(+)} + \varphi_1^{(+)}\varphi_2^{(+)}\varphi_3^{(+)}\end{aligned}$$

➢ Wick定理的证明（归纳法）：

$a)$ $n=1$时，$T\varphi_1 =: \varphi_1 := \varphi_1$, (19)式成立；

$b)$ $n=2$时，

$$\varphi_1\varphi_2 = (\varphi_1^{(+)} + \varphi_1^{(-)})(\varphi_2^{(+)} + \varphi_2^{(-)})$$

$$= \varphi_1^{(+)}\varphi_2^{(+)} + \varphi_1^{(-)}\varphi_2^{(+)} + \varphi_1^{(+)}\varphi_2^{(-)} + \varphi_1^{(-)}\varphi_2^{(-)}$$

$$\mp \varphi_2^{(-)}\varphi_1^{(+)} \pm \varphi_2^{(-)}\varphi_1^{(+)}$$

由(20)式，有

$$\varphi_1\varphi_2 = :\varphi_1\varphi_2: + c-\text{数项}. \tag{5.77}$$

$$T\varphi_1\varphi_2 = \theta(x_1^0 - x_2^0)\varphi_1\varphi_2 \mp \theta(x_2^0 - x_1^0)\varphi_2\varphi_1$$

$$= \theta(x_1^0 - x_2^0):\varphi_1\varphi_2: \mp \theta(x_2^0 - x_1^0):\varphi_2\varphi_1: + c-\text{数项}$$

$$T\varphi_1\varphi_2=[\theta(x_1^0-x_2^0)+\theta(x_2^0-x_1^0)]:\varphi_1\varphi_2:+c-\text{数项}$$

$$=\ :\varphi_1\varphi_2:\ +c-\text{数项}.$$

上式求真空平均值,

$$\langle 0|T\varphi_1\varphi_2|0\rangle=\langle 0|:\varphi_1\varphi_2:|0\rangle+c-\text{数项},$$

利用$\langle 0|:\varphi_1\varphi_2:|0\rangle=0$，可得

$$c-\text{数项}=\langle 0|T\varphi_1\varphi_2|0\rangle=\underbracket{\varphi_1\varphi_2},$$

$$\therefore\ T\varphi_1\varphi_2=:\varphi_1\varphi_2:+\underbracket{\varphi_1\varphi_2},$$

即$n=2$时,(5.69)式成立;

c）假设(5.69)式对某个n成立，证明它对$n+1$也成立。

选择t_{n+1}为最早的时间，则

$$T\varphi_1\cdots\varphi_n\varphi_{n+1}=T(\varphi_1\cdots\varphi_n)\varphi_{n+1}$$

$$=\{:\varphi_1\cdots\varphi_n:+:\underbrace{\varphi_1\varphi_2}\varphi_3\cdots\varphi_n:+\cdots\}\varphi_{n+1}$$

$$:\varphi_1\cdots\varphi_n:\varphi_{n+1}=\sum_{\mathrm{A,B}}\delta p\prod_{i\subset A}\varphi_i^{(-)}\prod_{j\subset B}\varphi_j^{(+)}(\varphi_{n+1}^{(+)}+\varphi_{n+1}^{(-)})$$

$$=\sum_{\mathrm{A,B}}\delta p\prod_{i\subset A}\varphi_i^{(-)}\prod_{j\subset B}\varphi_j^{(+)}\varphi_{n+1}^{(+)}+\sum_{\mathrm{A,B}}\delta p\prod_{i\subset A}\varphi_i^{(-)}\prod_{j\subset B}\varphi_j^{(+)}\varphi_{n+1}^{(-)}$$

上式第二项：

$$\sum_{\mathrm{A,B}}\delta p\prod_{i\subset A}\varphi_i^{(-)}\prod_{j\subset B}\varphi_j^{(+)}\varphi_{n+1}^{(-)}$$

$$=\sum_{\mathrm{A,B}}\delta p\prod_{i\subset A}\varphi_i^{(-)}\prod_{\substack{j\subset B\\ j\neq k}}\varphi_j^{(+)}[\varphi_k^{(+)}\varphi_{n+1}^{(-)}\pm\varphi_{n+1}^{(-)}\varphi_k^{(+)}\mp\varphi_{n+1}^{(-)}\varphi_k^{(+)}]$$

因而

$$:\varphi_1\cdots\varphi_n:\varphi_{n+1}$$

$$=\sum_{\mathrm{A,B}}\delta p\prod_{i\subset A}\varphi_i^{(-)}\prod_{j\subset B}\varphi_j^{(+)}\varphi_{n+1}^{(+)}+\sum_{\mathrm{A,B}}\delta p'\prod_{i\subset A}\varphi_i^{(-)}\varphi_{n+1}^{(-)}\prod_{j\subset B}\varphi_j^{(+)}$$

$$+\sum_{\mathrm{A,B}}\prod_{i\subset A}\varphi_i^{(-)}\sum_{k\subset B}\delta p''\prod_{\substack{j\subset B\\ j\neq k}}\varphi_j^{(+)}[\varphi_k^{(+)}\varphi_{n+1}^{(-)}\pm\varphi_{n+1}^{(-)}\varphi_k^{(+)}],\qquad(5.81)$$

其中

$$\delta p' = \delta p \times [\varphi_{n+1}^{(-)}\text{置换到}\prod_{j\subset B}\varphi_j^{(+)}\text{的左侧出现的符号因子}],$$

$$\delta p'' = \delta p \times \delta p_k,$$

$$\delta p_k = \varphi_{n+1}^{(-)}\text{置换到}\varphi_k^{(+)}\text{的右侧出现的符号因子。}$$

(5.81)式可写为

$$:\varphi_1\cdots\varphi_n:\varphi_{n+1}$$

$$=:\varphi_1\cdots\varphi_n\varphi_{n+1}:+\sum_{\mathrm{A,B}}\prod_{i\subset A}\varphi_i^{(-)}\sum_{k\subset B}\delta p''\prod_{\substack{j\subset B\\ j\neq k}}\varphi_j^{(+)}[\varphi_k^{(+)}\varphi_{n+1}^{(-)}\pm\varphi_{n+1}^{(-)}\varphi_k^{(+)}],$$

又

$$\varphi_k \varphi_{n+1} = (\varphi_k^{(+)} + \varphi_k^{(-)})(\varphi_{n+1}^{(+)} + \varphi_{n+1}^{(-)})$$

$$= \varphi_k^{(+)}\varphi_{n+1}^{(+)} + \varphi_k^{(-)}\varphi_{n+1}^{(+)} + \varphi_k^{(+)}\varphi_{n+1}^{(-)} + \varphi_k^{(-)}\varphi_{n+1}^{(-)},$$

$$\underbracket{\varphi_k \varphi_{n+1}} = \langle 0 | T\varphi_k \varphi_{n+1} | 0 \rangle = \langle 0 | \varphi_k \varphi_{n+1} | 0 \rangle = \langle 0 | \varphi_k^{(+)} \varphi_{n+1}^{(-)} | 0 \rangle$$

$$= \underbracket{\varphi_k^{(+)} \varphi_{n+1}^{(-)}} = \langle 0 | \varphi_k^{(+)} \varphi_{n+1}^{(-)} \pm \varphi_{n+1}^{(-)} \varphi_k^{(+)} | 0 \rangle$$

$$= \varphi_k^{(+)} \varphi_{n+1}^{(-)} \pm \varphi_{n+1}^{(-)} \varphi_k^{(+)},$$

因而

$$:\varphi_1 \cdots \varphi_n : \varphi_{n+1}$$

$$=:\varphi_1 \cdots \varphi_n \varphi_{n+1}: + \sum_{\mathrm{A,B}} \prod_{i \subset A} \varphi_i^{(-)} \sum_{k \subset B} \delta p'' \prod_{\substack{j \subset B \\ j \neq k}} \varphi_j^{(+)} \underbracket{\varphi_k^{(+)} \varphi_{n+1}^{(-)}}, \tag{5.83}$$

上式第二部分来自$:\varphi_1\cdots\varphi_n:$中的$\varphi_k^{(+)}(k=1,2,\cdots,n)$与$\varphi_{n+1}^{(-)}$的收缩项，该项的全体可写为

$$\delta p_k:\varphi_1\cdots\underbracket{\varphi_k^{(+)}\varphi_{n+1}^{(-)}}\cdots\varphi_n:=\delta p_k:\varphi_1\cdots\underbracket{\varphi_k\varphi_{n+1}}\cdots\varphi_n:$$

$$=:\varphi_1\cdots\underbracket{\varphi_k\cdots\varphi_n\varphi_{n+1}}:$$

又

$$\sum_B\sum_{k\subset B}\to\sum_{k=1}^{n}$$

(5.83)式成为

$$:\varphi_1\cdots\varphi_n:\varphi_{n+1}=:\varphi_1\cdots\varphi_n\varphi_{n+1}:+\sum_{k=1}^{n}:\varphi_1\cdots\underbracket{\varphi_k\cdots\varphi_n\varphi_{n+1}}: \qquad (5.84)$$

因而

$$T\varphi_1\cdots\varphi_{n+1}$$

$$=:\varphi_1\cdots\varphi_{n+1}:+\sum_{k=1}^{n}:\varphi_1\cdots\underbrace{\varphi_k\cdots\varphi_{n+1}}:$$

$$+\sum_{\substack{x_1,\cdots x_n\\ \text{的所有置换}}}:\underbrace{\varphi_1\varphi_2}\varphi_3\cdots\varphi_{n+1}:+\sum_{\substack{x_1,\cdots x_n\\ \text{的所有置换}}}\sum_{k=3}^{n}:\underbrace{\varphi_1\varphi_2}\varphi_3\cdots\underbrace{\varphi_k\cdots\varphi_n}\varphi_{n+1}:$$

$$+\cdots$$

$$=:\varphi_1\cdots\varphi_{n+1}:+\sum_{\substack{x_1,\cdots x_{n+1}\\ \text{的所有置换}}}:\underbrace{\varphi_1\varphi_2}\varphi_3\cdots\varphi_{n+1}:+\cdots$$

即对于$n+1$, (5.69)式成立，因而 (5.69)式对任意n均成立。

➢ 算符的收缩：

1. 电磁场算符的收缩

$$\underbrace{A_\mu(x_1)A_\nu}(x_2)=\langle 0|TA_\mu(x_1)A_\nu(x_2)|0\rangle=\mathrm{i}\,g_{\mu\nu}G_F(x_1-x_2)\Big|_{m=0}$$

$$=\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}k\cdot(x_1-x_2)}\frac{-\mathrm{i}\,g_{\mu\nu}}{k^2+\mathrm{i}\varepsilon},\tag{26}$$

其中

$$G_F(x_1-x_2)=-\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{e}^{-\mathrm{i}k\cdot(x_1-x_2)}}{k^2-m^2+\mathrm{i}\varepsilon}.\tag{27}$$

2. 费米场算符的收缩

$$\psi(x_1)\psi(x_2) \equiv \langle 0|T\psi_1\psi_2|0\rangle$$

$$= \theta(x_1^0 - x_2^0)\langle 0|\psi_1\psi_2|0\rangle - \theta(x_2^0 - x_1^0)\langle 0|\psi_2\psi_1|0\rangle$$

$$= \theta(x_1^0 - x_2^0)\langle 0|\psi_1^{(+)}\psi_2^{(-)}|0\rangle - (x_1 \leftrightarrow x_2)$$

$$= -\theta(x_1^0 - x_2^0)\langle 0|\psi_2^{(-)}\psi_1^{(+)}|0\rangle - (x_1 \leftrightarrow x_2)$$

$$= 0, \tag{28}$$

同理，

$$\overline{\psi}(x_1)\overline{\psi}(x_2) = 0, \tag{29}$$

$$\psi(x_1)\overline{\psi}(x_2) \equiv \langle 0 | T\psi_1\overline{\psi}_2 | 0 \rangle = \mathrm{i}\,S(x_1 - x_2)$$

$$= \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}k\cdot(x_1 - x_2)} \frac{\mathrm{i}(\not{k} + m)}{k^2 - m^2 + \mathrm{i}\varepsilon}, \tag{30}$$

其中，

$$S(x_1 - x_2) = -(\mathrm{i}\not{\partial}_{x_1} + m)G_F(x_1 - x_2). \tag{31}$$

3. 两个不同场算符的收缩为零

设A, B为不同场，则

$$\underbrace{A(x_1)B(x_2)} \equiv \langle 0|TA(x_1)B(x_2)|0\rangle$$

$$= \theta(x_1^0 - x_2^0)\langle 0|A_1 B_2|0\rangle \mp \theta(x_2^0 - x_1^0)\langle 0|B_2 A_1|0\rangle,$$

$$\because \quad \langle 0|A_1 B_2|0\rangle = \langle 0|A_1^{(+)} B_2^{(-)}|0\rangle = \mp\langle 0|B_2^{(-)} A_1^{(+)}|0\rangle = 0,$$

$$\langle 0|B_2 A_1|0\rangle = \langle 0|B_2^{(+)} A_1^{(-)}|0\rangle = \mp\langle 0|A_1^{(-)} B_2^{(+)}|0\rangle = 0,$$

$$\therefore \quad \underbrace{A(x_1)B(x_2)} = 0. \tag{5.87}$$

4. 正规乘积中，同一时刻的场算符收缩为零

$$
\begin{aligned}
:\underbracket{A(x)B(x)}: &\equiv \langle 0 | T : A(x)B(x) : | 0 \rangle = \langle 0 | : A(x)B(x) : | 0 \rangle \\
&= 0. \qquad\qquad (5.86)
\end{aligned}
$$

$\therefore$ 属于同一$\mathcal{H}_i(x)$的场算符间的收缩为零。

§5-4 Feynman图-(QED)

§5-4-1 QED中S矩阵的正规乘积分解

$$H = H_0 + H_i,$$

$$H_0 = \int \mathrm{d}^3 x (\mathcal{H}_{\mathrm{e.m.}} + \mathcal{H}_{\mathrm{Dirac}}),$$

$$\mathcal{H}_{\mathrm{e.m.}} =: \frac{1}{2} \dot{A}^\mu \dot{A}_\mu + \nabla A^\mu \cdot \nabla A_\mu :,$$

$$\mathcal{H}_{\mathrm{Dirac}} =: \bar{\psi}(-i\vec{\gamma} \cdot \nabla + m)\psi :$$

$$H_i = \int \mathrm{d}^3 x \mathcal{H}_i(x),$$

$$\mathcal{H}_i(x) =: e\bar{\psi}(x) \not{A}(x) \psi(x) - \delta m \bar{\psi}(x) \psi(x) :,$$

$$\delta m = m - m_0 .$$

$$S = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} \mathrm{d}^4\, x_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}^4\, x_n T \mathcal{H}_i(x_1) \cdots \mathcal{H}_i(x_n). \tag{5.88}$$

$$\begin{aligned}
S^{(2)} &= 1 - \mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}^4\, x T \mathcal{H}_i(x) + \frac{(-\mathrm{i})^2}{2!} \int_{-\infty}^{\infty} \mathrm{d}^4\, x_1\, \mathrm{d}^4\, x_2 T \mathcal{H}_i(x_1) \mathcal{H}_i(x_2) \\
&= 1 - \mathrm{i}\, e \int_{-\infty}^{\infty} \mathrm{d}^4\, x T : \overline{\psi}(x) \not{A}(x) \psi(x) : + \mathrm{i}\, \delta m \int_{-\infty}^{\infty} \mathrm{d}^4\, x T : \overline{\psi}(x) \psi(x) : \\
&\quad + \frac{(-\mathrm{i}\, e)^2}{2!} \int_{-\infty}^{\infty} \mathrm{d}^4\, x_1\, \mathrm{d}^4\, x_2 T : \overline{\psi}(x_1) \not{A}(x_1) \psi(x_1) :: \overline{\psi}(x_2) \not{A}(x_2) \psi(x_2) : \\
&\quad + \frac{(-\mathrm{i})^2}{2!} e \delta m \int_{-\infty}^{\infty} \mathrm{d}^4\, x_1\, \mathrm{d}^4\, x_2 T : \overline{\psi}(x_1) \not{A}(x_1) \psi(x_1) :: \overline{\psi}(x_2) \psi(x_2) : \\
&\quad + \frac{(-\mathrm{i})^2}{2!} e \delta m \int_{-\infty}^{\infty} \mathrm{d}^4\, x_1\, \mathrm{d}^4\, x_2 T : \overline{\psi}(x_1) \psi(x_1) :: \overline{\psi}(x_2) \not{A}(x_2) \psi(x_2) : \\
&\quad + \frac{(-\mathrm{i}\, \delta m)^2}{2!} \int_{-\infty}^{\infty} \mathrm{d}^4\, x_1\, \mathrm{d}^4\, x_2 T : \overline{\psi}(x_1) \psi(x_1) :: \overline{\psi}(x_2) \psi(x_2) : \\
&= 1 + S^{(1)}_{\mathrm{e.m.}} + S^{(1)}_{\delta m} + S^{(2)}_{\mathrm{e.m.}} + \cdots
\end{aligned} \tag{33}$$

利用Wick定理，有

$$S_{\mathrm{e.m.}}^{(1)}=-\mathrm{i}\,e\int_{-\infty}^{\infty}\mathrm{d}^4x:\bar{\psi}(x)\not{A}(x)\psi(x):$$

$$S_{\delta m}^{(1)}=\mathrm{i}\,\delta m\int_{-\infty}^{\infty}\mathrm{d}^4x:\bar{\psi}(x)\psi(x):$$

$$S_{\mathrm{e.m.}}^{(2)}=\frac{(-\mathrm{i}\,e)^2}{2!}\int_{-\infty}^{\infty}\mathrm{d}^4x_1\,\mathrm{d}^4x_2$$

$$\times\{:(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}:$$

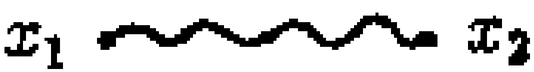
$$+:(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}:+:(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}:+:(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}:$$

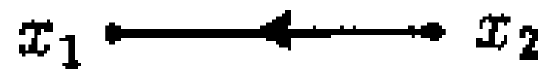
$$+:(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}:+:(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}:+:(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}:$$

$$+:(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}:\}$$

传播子

$A_\mu(x_1)A_\nu(x_2)$ $\psi(x_1)\bar{\psi}(x_2)$





外线

$A_\mu(x)$ $\psi(x)$ $\bar{\psi}(x)$

坐标表象中传播子与外线的图示

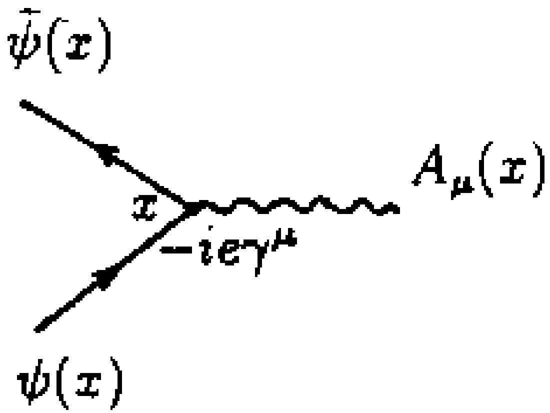
§5-4-2 S矩阵元

➢ 一级正规乘积：

$$S^{(2)}=1+S^{(1)}_{\mathrm{e.m.}}+S^{(1)}_{\delta m}+S^{(2)}_{\mathrm{e.m.}}+\cdots \tag{5.88}$$

$$S^{(1)}_{\mathrm{e.m.}}=-\mathrm{i}e\int_{-\infty}^{\infty}\mathrm{d}^4x:\overline{\psi}(x)A(x)\psi(x):$$

$$S^{(1)}_{\delta m}=\mathrm{i}\,\delta m\int_{-\infty}^{\infty}\mathrm{d}^4x:\overline{\psi}(x)\psi(x):$$



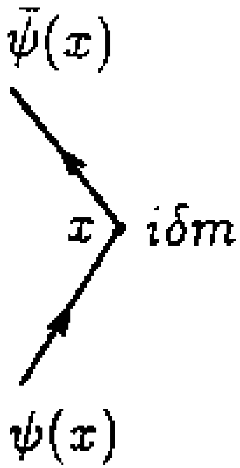


图5.2　最低级相互作用顶角

$$\psi = \psi^{(+)} + \psi^{(-)} = \text{电子消灭} + \text{正电子产生}$$

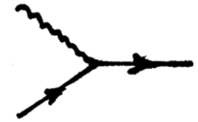
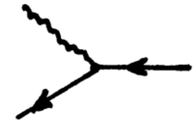
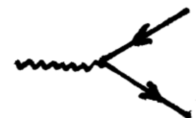
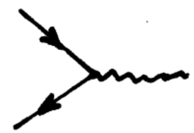
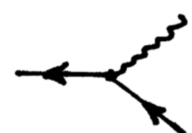
$$\overline{\psi} = \overline{\psi}^{(-)} + \overline{\psi}^{(+)} = \text{电子产生} + \text{正电子消灭}$$

$$A = A^{(-)} + A^{(+)} = \text{光子产生} + \text{光子消灭}$$

$$S^{(1)}_{\mathrm{e.m.}} = -\mathrm{i}\,e\int_{-\infty}^{\infty}\mathrm{d}^4x : (\overline{\psi}^{(-)} + \overline{\psi}^{(+)})\gamma^{\mu}(\psi^{(+)} + \psi^{(-)})\cdot(A^{(+)}_{\mu} + A^{(-)}_{\mu}) :$$

$$\begin{aligned} = -\mathrm{i}\,e\int_{-\infty}^{\infty}\mathrm{d}^4x : \{ & \overline{\psi}^{(-)}\gamma^{\mu}\psi^{(+)}A^{(+)}_{\mu} + \overline{\psi}^{(-)}\gamma^{\mu}\psi^{(+)}A^{(-)}_{\mu} \\ & + \overline{\psi}^{(-)}\gamma^{\mu}\psi^{(-)}A^{(+)}_{\mu} + \overline{\psi}^{(-)}\gamma^{\mu}\psi^{(-)}A^{(-)}_{\mu} \\ & + \overline{\psi}^{(+)}\gamma^{\mu}\psi^{(+)}A^{(+)}_{\mu} + \overline{\psi}^{(+)}\gamma^{\mu}\psi^{(+)}A^{(-)}_{\mu} \\ & + \overline{\psi}^{(+)}\gamma^{\mu}\psi^{(-)}A^{(+)}_{\mu} + \overline{\psi}^{(+)}\gamma^{\mu}\psi^{(-)}A^{(-)}_{\mu} :\} \end{aligned}$$

$$S^{(1)}_{\delta m} = \mathrm{i}\,\delta m\int_{-\infty}^{\infty}\mathrm{d}^4x : (\overline{\psi}^{(-)} + \overline{\psi}^{(+)})(\psi^{(+)} + \psi^{(-)}) :$$

	正规乘积	$\vert i\rangle$	$\vert f\rangle$	图
a)	$\bar{\psi}^{(-)}\gamma^{\mu}\psi^{(+)}A_{\mu}^{(+)}$	γe^{-}	e^{-}	
b)	$\bar{\psi}^{(+)}\gamma^{\mu}\psi^{(-)}A_{\mu}^{(+)}$	γe^{+}	e^{+}	
c)	$\bar{\psi}^{(-)}\gamma^{\mu}\psi^{(-)}A_{\mu}^{(+)}$	γ	$e^{+}e^{-}$	
d)	$\bar{\psi}^{(+)}\gamma^{\mu}\psi^{(+)}A_{\mu}^{(-)}$	$e^{+}e^{-}$	γ	
e)	$\bar{\psi}^{(-)}\gamma^{\mu}\psi^{(+)}A_{\mu}^{(-)}$	e^{-}	γe^{-}	
f)	$\bar{\psi}^{(+)}\gamma^{\mu}\psi^{(-)}A_{\mu}^{(-)}$	e^{+}	γe^{+}	
g)	$\bar{\psi}^{(-)}\gamma^{\mu}\psi^{(-)}A_{\mu}^{(-)}$		$\gamma e^{+}e^{-}$	
h)	$\bar{\psi}^{(+)}\gamma^{\mu}\psi^{(+)}A_{\mu}^{(+)}$	$\gamma e^{+}e^{-}$		

- 几个有用的结果:

$$A_\mu^{(+)}=\int \tilde{\mathrm{d}}k\, a^{(\lambda)}(k)\varepsilon_\mu^{(\lambda)}(k)\mathrm{e}^{-\mathrm{i}kx},$$

$$\psi^{(+)}=\int \tilde{\mathrm{d}}k\, b_\alpha(k)\mathrm{e}^{-\mathrm{i}kx}\,u^{(\alpha)}(k),\qquad \overline{\psi}^{(-)}=\int \tilde{\mathrm{d}}k\, b_\alpha^+(k)\mathrm{e}^{\mathrm{i}kx}\,\overline{u}^{(\alpha)}(k),$$

$$\psi^{(-)}=\int \tilde{\mathrm{d}}k\, d_\alpha^+(k)\mathrm{e}^{\mathrm{i}kx}\,v^{(\alpha)}(k),\qquad \overline{\psi}^{(+)}=\int \tilde{\mathrm{d}}k\, d_\alpha(k)\mathrm{e}^{-\mathrm{i}kx}\,\overline{v}^{(\alpha)}(k),$$

$$[a^{(\lambda)}(k),a^{(\lambda')+}(k')]=(2\pi)^3 2k^0\delta^3(\vec{k}-\vec{k}')\delta^{\lambda\lambda'},\tag{34}$$

$$\{b_\alpha(k),b_\beta^+(k')\}=\{d_\alpha(k),d_\beta^+(k')\}=(2\pi)^3\frac{k^0}{m}\delta^3(\vec{k}-\vec{k}')\delta_{\alpha\beta},\tag{35}$$

$$\begin{aligned}A_\mu^{(+)}(x)a^{(\lambda)+}(k)|0\rangle&=\int \tilde{\mathrm{d}}k'\, a^{(\lambda')}(k')\varepsilon_\mu^{(\lambda')}(k')\mathrm{e}^{-\mathrm{i}k'x}\,a^{(\lambda)+}(k)|0\rangle,\\&=\int \tilde{\mathrm{d}}k'\,\varepsilon_\mu^{(\lambda')}(k')\mathrm{e}^{-\mathrm{i}k'x}(2\pi)^3 2k^{0'}\delta^{\lambda\lambda'}\delta(\vec{k}'-\vec{k})|0\rangle\\&=\varepsilon_\mu^{(\lambda)}(k)\mathrm{e}^{-\mathrm{i}kx}|0\rangle,\end{aligned}\tag{36}$$

$$\psi^{(+)}(x) b_\alpha^+(p)\left|0\right\rangle = \int \mathrm{d}\tilde{q}\, b_\beta(q) u^{(\beta)}(q)\,\mathrm{e}^{-\mathrm{i}qx}\, b_\alpha^+(p)\left|0\right\rangle$$

$$= u^{(\alpha)}(p)\,\mathrm{e}^{-\mathrm{i}\,px}\left|0\right\rangle, \tag{37}$$

$$\overline{\psi}^{(+)}(x) d_\alpha^+(p)\left|0\right\rangle = \overline{v}^{(\alpha)}(p)\,\mathrm{e}^{-\mathrm{i}\,px}\left|0\right\rangle, \tag{38}$$

$$\left\langle 0\right| a^{(\lambda')}(k') A_\mu^{(-)}(x) = \left\langle 0\right| \varepsilon_\mu^{(\lambda')}(k')\,\mathrm{e}^{-\mathrm{i}k'x}, \tag{39}$$

$$\left\langle 0\right| b_{\alpha'}(p')\overline{\psi}^{(-)}(x) = \left\langle 0\right| \overline{u}^{(\alpha')}(p')\,\mathrm{e}^{\mathrm{i}\,p'x}, \tag{40}$$

$$\left\langle 0\right| d_{\alpha'}(p')\psi^{(-)}(x) = \left\langle 0\right| v^{(\alpha')}(p')\,\mathrm{e}^{\mathrm{i}\,p'x}\,. \tag{41}$$

- 含三线性耦合纯正规乘积因子的项对S矩阵元的贡献为0.

考虑$\langle f | S^{(1)}_{\text{e.m.}} | i \rangle_a$：

$$|i\rangle = a^{(\lambda)+}(k) b^+_\alpha(p) |0\rangle, \quad \langle f | = \langle 0 | b_{\alpha'}(p'),$$

$$\langle f | S^{(1)}_{\text{e.m.}} | i \rangle_a$$

$$= -\mathrm{i}e \int \mathrm{d}^4 x \langle 0 | b_{\alpha'}(p') \overline{\psi}^{(-)}(x) \gamma^\mu \psi^{(+)}(x) A^{(+)}_\mu(x) a^{(\lambda)+}(k) b^+_\alpha(p) | 0 \rangle$$

$$= -\mathrm{i}e \int \mathrm{d}^4 x \overline{u}^{(\alpha')}(p') \mathrm{e}^{\mathrm{i}p'x} \gamma^\mu u^{(\alpha)}(p) \mathrm{e}^{-\mathrm{i}px} \varepsilon^{(\lambda)}_\mu(k) \mathrm{e}^{-\mathrm{i}kx} \langle 0 | 0 \rangle$$

$$= -\mathrm{i}e(2\pi)^4 \delta^4(p' - p - k) \overline{u}^{(\alpha')}(p') \gamma^\mu u^{(\alpha)}(p) \varepsilon^{(\lambda)}_\mu(k), \tag{42}$$

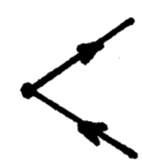
4动量守恒条件：$p' - p - k = 0,$

质壳条件：$p^2 = p'^2 = m^2$，$k^2 = 0,$

以上两个条件相矛盾，因而$\langle f | S^{(1)}_{\text{e.m.}} | i \rangle_a = 0.$

▪ 含双线性耦合纯正规乘积因子的项对S矩阵元的贡献

$$S_{\delta m}^{(1)} = \mathrm{i}\,\delta m \int_{-\infty}^{\infty} \mathrm{d}^4 x : (\overline{\psi}^{(-)} + \overline{\psi}^{(+)})(\psi^{(+)} + \psi^{(-)}) :$$

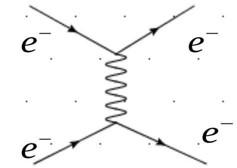
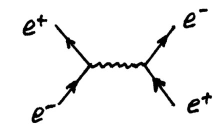
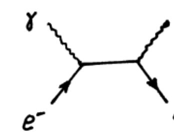
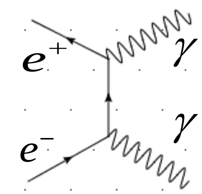
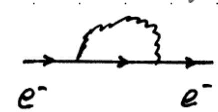
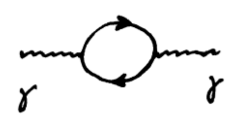
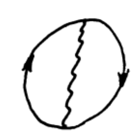
	正规乘积	$\vert i\rangle$	$\vert f\rangle$	图
a)	$\bar{\psi}^{(-)}\psi^{(+)}$	e^-	e^-	
b)	$\bar{\psi}^{(+)}\psi^{(-)}$	e^+	e^+	
c)	$\bar{\psi}^{(-)}\psi^{(-)}$		e^+e^-	
d)	$\bar{\psi}^{(+)}\psi^{(+)}$	e^+e^-		

考虑$\left\langle f \middle| S_{\delta m}^{(1)} \middle| i \right\rangle_a$：

$$\left| i \right\rangle = b_\alpha^+(p)\left| 0 \right\rangle, \quad \left\langle f \right| = \left\langle 0 \right| b_{\alpha'}(p'),$$

$$\begin{aligned}
&\left\langle f \middle| S_{\delta m}^{(1)} \middle| i \right\rangle_a \\
&= \mathrm{i}\,\delta m \int \mathrm{d}^4 x \left\langle 0 \middle| b_{\alpha'}(p')\overline{\psi}^{(-)}(x)\psi^{(+)}(x) b_\alpha^+(p) \middle| 0 \right\rangle \\
&= \mathrm{i}\,\delta m \int \mathrm{d}^4 x\, \overline{u}^{(\alpha')}(p')\,\mathrm{e}^{\mathrm{i}p'x}\, u^{(\alpha)}(p)\,\mathrm{e}^{-\mathrm{i}px} \left\langle 0 \middle| 0 \right\rangle \\
&= \mathrm{i}\,\delta m (2\pi)^4 \delta^4(p'-p)\overline{u}^{(\alpha')}(p')u^{(\alpha)}(p).
\end{aligned} \tag{43}$$

➢ 二级正规乘积：

正规乘积	$\vert i\rangle$	$\vert f\rangle$	过程	
$:\bar{\psi}^{(-)}\gamma^{\mu}\psi^{(+)}\bar{\psi}^{(-)}\gamma^{\nu}\psi^{(+)}A_{\mu}A_{\nu}:$	e^-e^-	e^-e^-	$e^-e^- \to e^-e^-$	
$:\bar{\psi}^{(-)}\gamma^{\mu}\psi^{(+)}\bar{\psi}^{(+)}\gamma^{\nu}\psi^{(-)}A_{\mu}A_{\nu}:$	e^+e^-	e^+e^-	$e^+e^- \to e^+e^-$	
$\bar{\psi}^{(-)}\gamma^{\mu}\psi\bar{\psi}\gamma^{\nu}\psi^{(+)}A_{\mu}^{(-)}A_{\nu}^{(+)}$	γe^-	γe^-	$\gamma e^- \to \gamma e^-$	
$\bar{\psi}^{(+)}\gamma^{\mu}\psi\bar{\psi}\gamma^{\nu}\psi^{(+)}A_{\mu}^{(-)}A_{\nu}^{(-)}$	e^+e^-	2γ	$e^+e^- \to 2\gamma$	
$\bar{\psi}^{(-)}\gamma^{\mu}\psi\bar{\psi}\gamma^{\nu}\psi^{(+)}A_{\mu}A_{\nu}$	e^-	e^-	电子自能跃迁	
$\bar{\psi}\gamma^{\mu}\psi\bar{\psi}\gamma^{\nu}\psi A_{\mu}^{(-)}A_{\nu}^{(+)}$	γ	γ	光子自能跃迁	
$\bar{\psi}\gamma^{\mu}\psi\bar{\psi}\gamma^{\nu}\psi A_{\mu}A_{\nu}$	$\vert 0\rangle$	$\vert 0\rangle$	真空－真空跃迁	

§ 5-4-3　物理过程的二级S矩阵元举例

1.　Bhabba散射（ $e^+e^- \to e^+e^-$ ）

$$|i\rangle = b_\alpha^+(p)d_\beta^+(q)|0\rangle, \quad \langle f| = \langle 0|d_{\beta'}(q')b_{\alpha'}(p'),$$

$$\langle f|S_{\text{e.m.}}^{(2)}|i\rangle = \frac{(-\mathrm{i}e)^2}{2}\int_{-\infty}^{\infty}\mathrm{d}^4x_1\,\mathrm{d}^4x_2\langle f|:(\bar{\psi}\gamma^\mu A_\mu\psi)_{x_1}(\bar{\psi}\gamma^\nu A_\nu\psi)_{x_2}:|i\rangle$$

$$= \frac{(-\mathrm{i}e)^2}{2}\int_{-\infty}^{\infty}\mathrm{d}^4x_1\,\mathrm{d}^4x_2\langle 0|d_{\beta'}(q')b_{\alpha'}(p')$$

$$\times:\{\bar{\psi}^{(-)}(x_1)\gamma^\mu\psi^{(+)}(x_1)\bar{\psi}^{(+)}(x_2)\gamma^\nu\psi^{(-)}(x_2)+(x_1\leftrightarrow x_2)$$

$$+\bar{\psi}^{(+)}(x_1)\gamma^\mu\psi^{(+)}(x_1)\bar{\psi}^{(-)}(x_2)\gamma^\nu\psi^{(-)}(x_2)+(x_1\leftrightarrow x_2)\}:$$

$$\times A_\mu(x_1)A_\nu(x_2)b_\alpha^+(p)d_\beta^+(q)|0\rangle,$$

利用(26)、(37)－(41)式，得到

$$\left\langle f \left| S^{(2)}_{\mathrm{e.m.}} \right| i \right\rangle\Big|_{e^+e^-\to e^+e^-}$$

$$= (-\mathrm{i}\, e)^2 \int \mathrm{d}^4 x_1\, \mathrm{d}^4 x_2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-\mathrm{i}\, g_{\mu\nu}}{k^2 + \mathrm{i}\,\varepsilon} \mathrm{e}^{-\mathrm{i}\, k\cdot(x_1 - x_2)}$$

$$\times \Big\{ - e^{ip'x_1} \overline{u}^{(\alpha')}(p')\gamma^\mu u^{(\alpha)}(p)\, \mathrm{e}^{-\mathrm{i}\,px_1} \mathrm{e}^{-\mathrm{i}\,qx_2}\, \overline{v}^{(\beta)}(q)\gamma^\nu v^{(\beta')}(q')\, \mathrm{e}^{\mathrm{i}\,q'x_2}$$

$$+ \mathrm{e}^{-\mathrm{i}\,qx_1}\, \overline{v}^{(\beta)}(q)\gamma^\mu u^{(\alpha)}(p)\, \mathrm{e}^{-\mathrm{i}\,px_1}\, e^{ip'x_2} \overline{u}^{(\alpha')}(p')\gamma^\nu v^{(\beta')}(q')\, \mathrm{e}^{\mathrm{i}\,q'x_2} \Big\}$$

$$= (-\mathrm{i}\, e)^2 \int \mathrm{d}^4 x_1\, \mathrm{d}^4 x_2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-\mathrm{i}\, g_{\mu\nu}}{k^2 + \mathrm{i}\,\varepsilon}$$

$$\times \Big\{ - \overline{u}^{(\alpha')}(p')\gamma^\mu u^{(\alpha)}(p) \overline{v}^{(\beta)}(q)\gamma^\nu v^{(\beta')}(q')\, \mathrm{e}^{\mathrm{i}(-k+p'-p)x_1} \mathrm{e}^{\mathrm{i}(k-q+q')x_2}$$

$$+ \overline{v}^{(\beta)}(q)\gamma^\mu u^{(\alpha)}(p) \overline{u}^{(\alpha')}(p')\gamma^\nu v^{(\beta')}(q')\, \mathrm{e}^{-\mathrm{i}(k+p+q)x_1} \mathrm{e}^{\mathrm{i}(k+p'+q')x_2} \Big\}$$

$$
\begin{aligned}
&\left\langle f \left| S^{(2)}_{\mathrm{e.m.}} \right| i \right\rangle \Big|_{e^+e^- \to e^+e^-} \\
&= (-\mathrm{i}\,e)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{-\mathrm{i}\, g_{\mu\nu}}{k^2 + \mathrm{i}\,\varepsilon} \\
&\quad \times \Big\{ -(2\pi)^4 \delta^4(-k + p' - p)(2\pi)^4 \delta^4(k - q + q') \\
&\qquad\qquad \cdot \overline{u}^{(\alpha')}(p')\gamma^\mu u^{(\alpha)}(p)\overline{v}^{(\beta)}(q)\gamma^\nu v^{(\beta')}(q') \\
&\qquad + (2\pi)^4 \delta^4(k + p + q)(2\pi)^4 \delta^4(k + p' + q') \\
&\qquad\qquad \cdot \overline{v}^{(\beta)}(q)\gamma^\mu u^{(\alpha)}(p)\overline{u}^{(\alpha')}(p')\gamma^\nu v^{(\beta')}(q') \Big\}
\end{aligned}
$$

$$\langle f | S^{(2)}_{\text{e.m.}} | i \rangle \Big|_{e^+e^- \to e^+e^-}$$

$$= (-\mathrm{i}e)^2 (2\pi)^4 \delta^4(p' + q' - p - q)$$

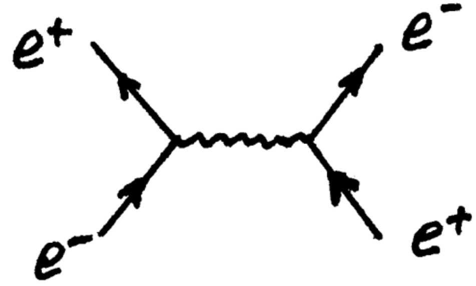
$$\times \{ -\bar{u}^{(\alpha')}(p')\gamma^\mu u^{(\alpha)}(p) \frac{-\mathrm{i}g_{\mu\nu}}{(p'-p)^2 + \mathrm{i}\varepsilon} \bar{v}^{(\beta)}(q)\gamma^\nu v^{(\beta')}(q')$$

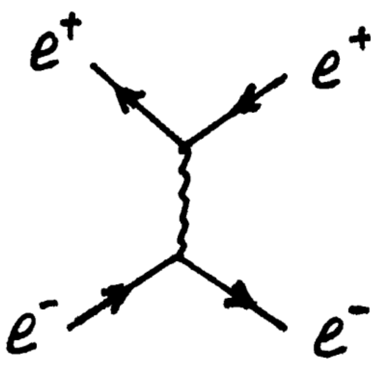
$$+ \bar{v}^{(\beta)}(q)\gamma^\mu u^{(\alpha)}(p) \frac{-\mathrm{i}g_{\mu\nu}}{(p+q)^2 + \mathrm{i}\varepsilon} \bar{u}^{(\alpha')}(p')\gamma^\nu v^{(\beta')}(q') \} \qquad (5.109)$$

$$= A + B,$$

$$B = -A \Big|_{\bar{u}^{(\alpha')}(p') \leftrightarrow \bar{v}^{(\beta)}(q),\, p' \leftrightarrow -q} .$$

Bhabba散射的二级图:





2. 光子自能跃迁

$$|i\rangle = a^{(\lambda)+}(k)|0\rangle, \quad \langle f| = \langle 0|a^{(\lambda')}(k'),$$

$$\langle f|S^{(2)}_{\text{e.m.}}|i\rangle = \frac{(-\mathrm{i}e)^2}{2}\int_{-\infty}^{\infty}\mathrm{d}^4x_1\,\mathrm{d}^4x_2\langle f|:(\bar{\psi}\gamma^\mu A_\mu\psi)_{x_1}(\bar{\psi}\gamma^\nu A_\nu\psi)_{x_2}:|i\rangle$$

$$= \frac{(-\mathrm{i}e)^2}{2}\int_{-\infty}^{\infty}\mathrm{d}^4x_1\,\mathrm{d}^4x_2\langle 0|a^{(\lambda')}(k'):\bar{\psi}(x_1)\gamma^\mu\psi(x_1)\bar{\psi}(x_2)\gamma^\nu\psi(x_2)$$

$$\times[A^{(-)}_\mu(x_1)A^{(+)}_\nu(x_2) + A^{(+)}_\mu(x_1)A^{(-)}_\nu(x_2)]:a^{(\lambda)+}(k)|0\rangle$$

$$= (-\mathrm{i}e)^2\int_{-\infty}^{\infty}\mathrm{d}^4x_1\,\mathrm{d}^4x_2\,\mathrm{e}^{\mathrm{i}k'x_1}\,\varepsilon^{(\lambda')}_\mu(k')$$

$$\times:\bar{\psi}(x_1)\gamma^\mu\psi(x_1)\bar{\psi}(x_2)\gamma^\nu\psi(x_2):\varepsilon^{(\lambda)}_\nu(k)\mathrm{e}^{-\mathrm{i}kx_2}$$

令$O\equiv\gamma^{\mu}\psi(x_1)\overline{\psi}(x_2)\gamma^{\nu}$,则

$$I\equiv\ :\overline{\psi}(x_1)\gamma^{\mu}\psi(x_1)\overline{\psi}(x_2)\gamma^{\nu}\psi(x_2):\ =\ :\overline{\psi}(x_1)O\psi(x_2):$$

$$=\ :\overline{\psi}_{\alpha}(x_1)O_{\alpha\beta}\psi_{\beta}(x_2):\ =-:O_{\alpha\beta}\psi_{\beta}(x_2)\overline{\psi}_{\alpha}(x_1):$$

$$=-\mathrm{Tr}:O\psi(x_2)\overline{\psi}(x_1):$$

$$:\overline{\psi}(x_1)\gamma^{\mu}\psi(x_1)\overline{\psi}(x_2)\gamma^{\nu}\psi(x_2):$$

$$=-\mathrm{Tr}:\gamma^{\mu}\psi(x_1)\overline{\psi}(x_2)\gamma^{\nu}\psi(x_2)\overline{\psi}(x_1):$$

利用

$$\psi(x_1)\overline{\psi}(x_2)=\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}p\cdot(x_1-x_2)}\frac{\mathrm{i}(p+m)}{p^2-m^2+\mathrm{i}\varepsilon},$$

$$\frac{p+m}{p^2-m^2+\mathrm{i}\varepsilon}=(p+m)\frac{1}{(p+m-\mathrm{i}\varepsilon)(p-m+\mathrm{i}\varepsilon)}=\frac{1}{(p-m+\mathrm{i}\varepsilon)},$$

得到

$$\left\langle f\left|S_{\mathrm{e.m.}}^{(2)}\right|i\right\rangle=(-\mathrm{i}e)^2\delta^4(k'-k)\varepsilon_\mu^{(\lambda')}(k')\Pi^{\mu\nu}\varepsilon_\nu^{(\lambda)}(k), \tag{45}$$

其中$\Pi^{\mu\nu}$为真空极化张量：

$$\Pi^{\mu\nu}=-(-\mathrm{i}e)^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{Tr}[\gamma^\mu\frac{\mathrm{i}}{(\not{p}-m+\mathrm{i}\varepsilon)}\gamma^\nu\frac{\mathrm{i}}{(\not{p}-\not{k}-m+\mathrm{i}\varepsilon)}], \tag{46}$$

（平方紫外发散）

当$p\to\infty$时，

$$\Pi^{\mu\nu}\sim\int\frac{\mathrm{d}^4p}{|p|^2}.$$

§5-4-4 QED Feynman规则

正规乘积项 → 物理过程的S矩阵元 → Feynman规则

物理过程的Feynman图 $\xrightarrow{\text{Feynman规则}}$ S矩阵元

Feynman规则：

1. 外线因子

Feimion子外线

$u^{(\alpha)}(p)$

$\bar{u}^{(\alpha)}(p)$

$v^{(\alpha)}(p)$

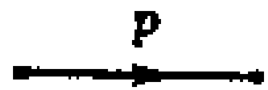
$\bar{v}^{(\alpha)}(p)$

入射光子外线 

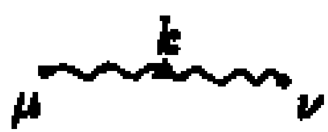
出射光子外线 

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \varepsilon^{(\lambda)}(k)$$

2. 内线因子（动量表象中的传播子）

Feimion子内线 

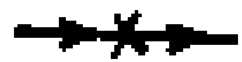
$$\frac{\mathrm{i}}{p-m+\mathrm{i}\,\varepsilon}$$

光子内线 

$$\frac{-\mathrm{i}}{k^{2}+\mathrm{i}\,\varepsilon}$$

3. 顶角因子

电磁作用顶角  $-\mathrm{i}e\gamma^{\mu}$

质量抵消项顶角  $\mathrm{i}\,\delta m$

4. 每一顶角处的4动量因子：

入射为负，出射为正，$\sum_i p_i = 0.$

5. 积分因子：

对每一内线圈，有一积分 $\int \mathrm{d}^4 p \big/ (2\pi)^4$.

6. 求迹因子：

对每一个Feimion子内线圈，有一求迹因子：−Tr .

7. 符号因子

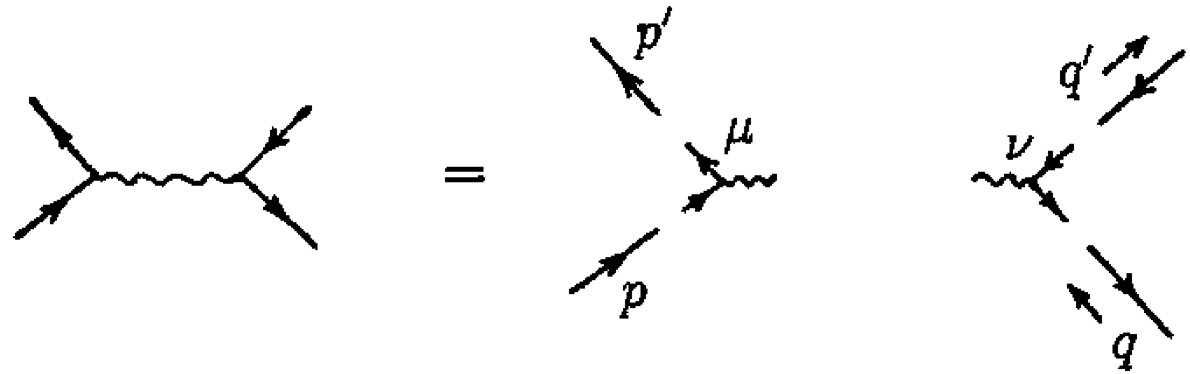
全同Fermion外线交换产生负号。

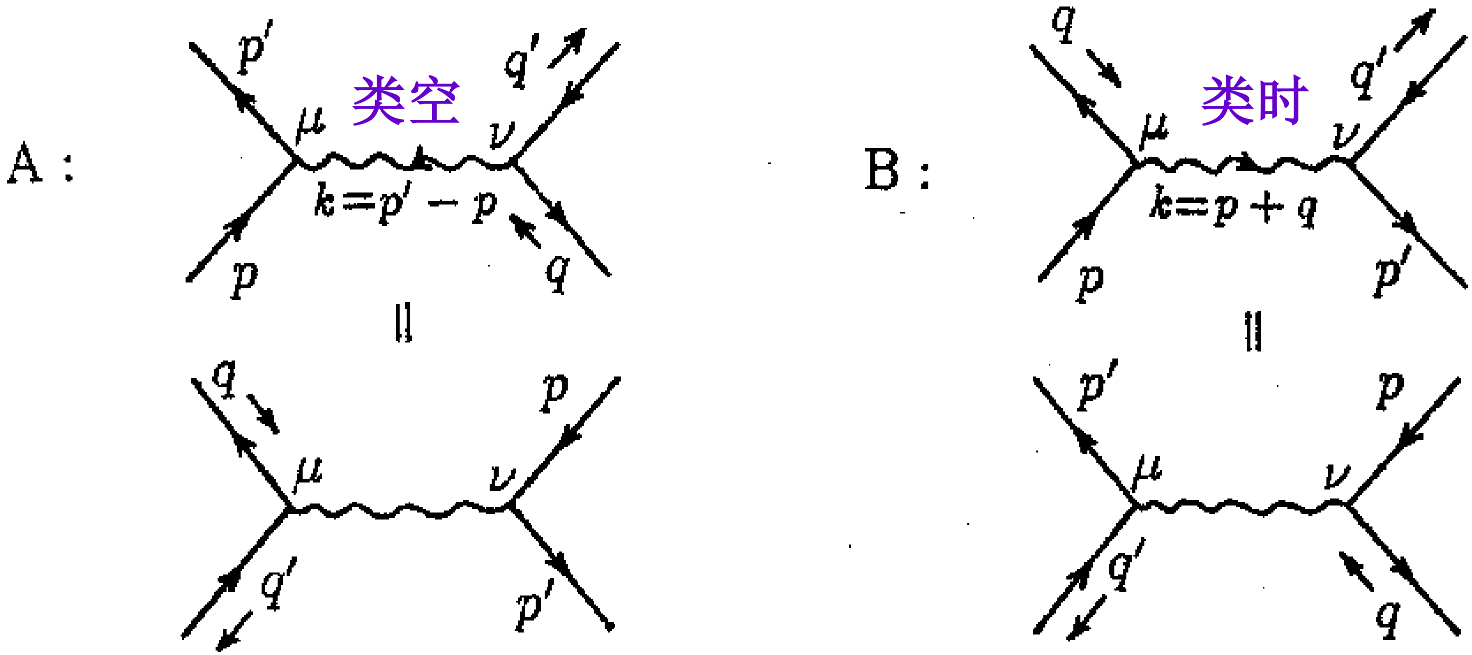
8. 外线4动量守恒因子：$(2\pi)^4\delta^4\{\sum p\}$

9. 拓扑权因子：

$$W=(\text{拓扑权})\times(\text{相互作用顶角置换因子})\times(\text{微扰级数展开因子})$$

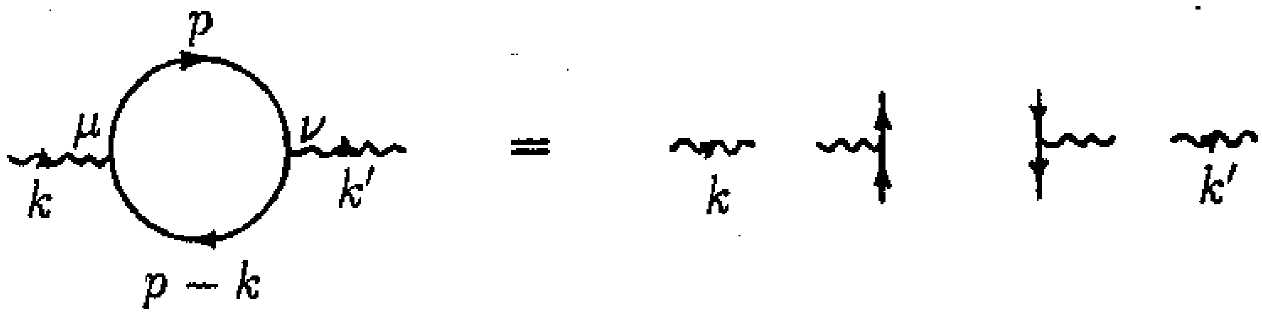
例一：$e^- + e^+ \to e^- + e^+$





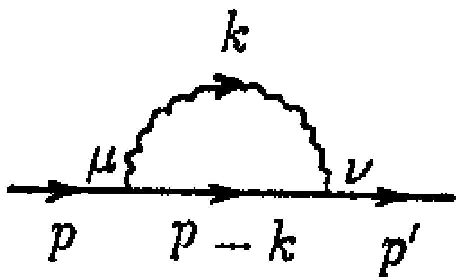
Bhabba散射图的拓扑权

例二：光子自能跃迁



§5-4-5　电子和光子的自能跃迁

1. 电子自能跃迁



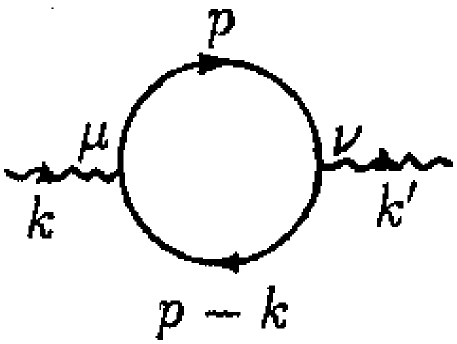
$$\left\langle f \left| S^{(2)}_{\text{e.m.}} \right| i \right\rangle = (2\pi)^4 \delta^4 (p' - p) \bar{u}^{(\alpha')}(p')[-\mathrm{i}\Sigma(p)] u^{(\alpha)}(p), \tag{47}$$

其中，

$$-\mathrm{i}\Sigma(p) = (-\mathrm{i}e)^2 \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \gamma^\mu \frac{\mathrm{i}}{\not{p} - \not{k} - m + \mathrm{i}\varepsilon} \gamma^\nu \frac{-\mathrm{i} g_{\mu\nu}}{k^2 + \mathrm{i}\varepsilon}. \tag{48}$$

（线性紫外发散）

2. 光子自能跃迁



$$\left\langle f\left|S_{\text{e.m.}}^{(2)}\right|i\right\rangle=(-\mathrm{i}e)^2\delta^4(k'-k)\varepsilon_\mu^{(\lambda')}(k')\Pi^{\mu\nu}\varepsilon_\nu^{(\lambda)}(k), \tag{45}$$

其中

$$\Pi^{\mu\nu}=-(-\mathrm{i}e)^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{Tr}\left[\gamma^\mu\frac{\mathrm{i}}{(\not p-m+\mathrm{i}\varepsilon)}\gamma^\nu\frac{\mathrm{i}}{(\not p-\not k-m+\mathrm{i}\varepsilon)}\right]. \tag{46}$$

（平方紫外发散）

3. 质量重整化

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_i,$$

$$\mathcal{H}_i(x) =: e\overline{\psi}(x)\not{A}(x)\psi(x) - \delta m\overline{\psi}(x)\psi(x):,$$

$$\delta m = m - m_0.$$

$$\langle f|S_{\delta m}^{(1)}|i\rangle = \langle 0|b_{\alpha'}(p')S_{\delta m}^{(1)}b_{\alpha}^{+}(p)|0\rangle$$

$$= \mathrm{i}\,\delta m(2\pi)^4\delta^4(p'-p)\overline{u}^{(\alpha')}(p')u^{(\alpha)}(p), \tag{43}$$

想法：使$\langle f|S_{\delta m}^{(1)}|i\rangle$与 $\langle f|S_{\mathrm{e.m.}}^{(2)}|i\rangle$相抵消。

做法：1）将发散分离出来（正规化）；

2）选择适当的m_0以抵消$[-\mathrm{i}\Sigma(p)]$中的发散（减除）。

➢ 常用的正规化方法：

a. 动量截断法

$$\int \mathrm{d}^4 k \to \int \mathrm{d} k^0 \int_{|\vec{k}|<\Lambda} \mathrm{d}^3 k, \quad \Lambda \text{有限}$$

b. Pauli-Villars 正规化

将光子传播子作代换：

$$\frac{1}{k^2 + i\varepsilon} \to \frac{1}{k^2 + i\varepsilon} - \frac{1}{k^2 - \Lambda^2 + i\varepsilon}$$

$$= \frac{-\Lambda^2}{(k^2 + i\varepsilon)(k^2 - \Lambda^2 + i\varepsilon)} \xrightarrow{k\to\infty} \frac{1}{k^4},$$

则 $$-\mathrm{i}\Sigma(p) \xrightarrow{k\to\infty} \int \frac{\mathrm{d}^4 k}{k^5}.$$

c. Hooft-Veltman 维数正规化

将时空维数作解析延拓：

$$\mathrm{d}^4 k \ \rightarrow\ \mathrm{d}^D k,\quad D\text{为复变量}$$

$$D = 4 - 2\varepsilon,\quad \varepsilon \rightarrow 0^+.$$

§ 5-5 含标量粒子的Feynman规则

§ 5-5-1 实标量场$\lambda\varphi^4$耦合

$$\mathcal{L} =: \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}\mu_0^2\varphi^2 - \frac{\lambda_0}{4!}\varphi^4 :$$

$$\mathcal{H}_i(x) = \frac{\lambda_0}{4!} : \varphi^4 : - \frac{1}{2}\delta\mu^2 : \varphi^2(x) :, \qquad \delta\mu^2 = \mu^2 - \mu_0^2 .$$

$$\varphi(x_1)\varphi(x_2) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\frac{\mathrm{i}}{k^2 - \mu^2 + \mathrm{i}\varepsilon} e^{-\mathrm{i}k\cdot(x_1 - x_2)},$$

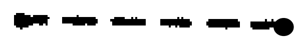
$\rightarrow$ **标量场传播子。**

S矩阵的一级正规乘积：

$$S^{(1)} = -\mathrm{i}\int \mathrm{d}^4 x\,\mathcal{H}_i(x) = -\mathrm{i}\int \mathrm{d}^4 x\left[\frac{\lambda_0}{4!} : \varphi^4 : - \frac{1}{2}\delta\mu^2 : \varphi^2(x) :\right],$$

$\rightarrow$ **相互作用顶角。**

表5.4　实标量场$\lambda\varphi^4$耦合的部分Feynman规则

	图形	S矩阵中的因子
标量粒子外线		1
标量粒子内线		$\dfrac{i}{k^2-\mu^2+i\varepsilon}$
$\lambda\varphi^4$自作用顶角		$-i\lambda_0$
质量抵消项顶角		$i\delta\mu^2$
拓扑权因子		W

➢ 拓扑权因子的计算

$$W=(\text{拓扑权})\times(\text{相互作用顶角置换因子})\times(\text{微扰级数展开因子})$$

例一：

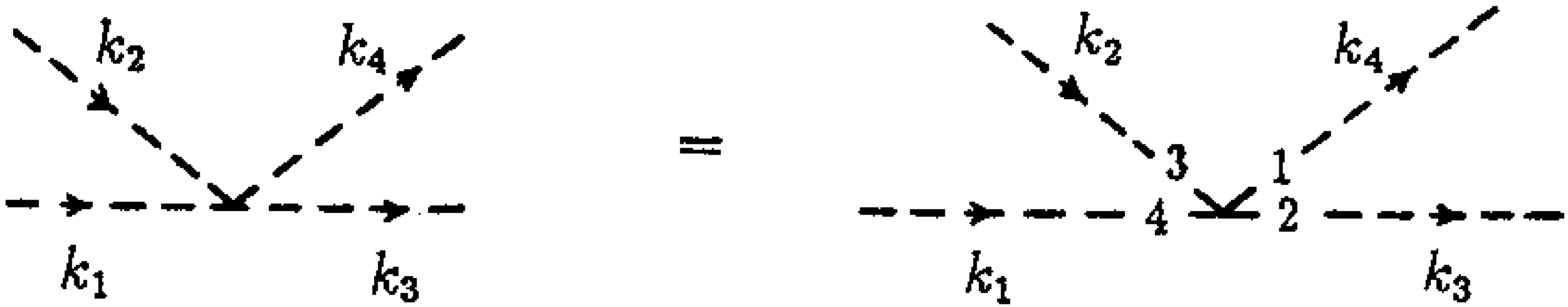


图5.13　两标量粒子散射的最低级图

$$\text{拓扑权}=4\times3\times2\times1=4!,\qquad \text{顶角置换因子}=\frac{1}{4!}$$

$$W=\frac{4!}{4!}=1.$$

例二：

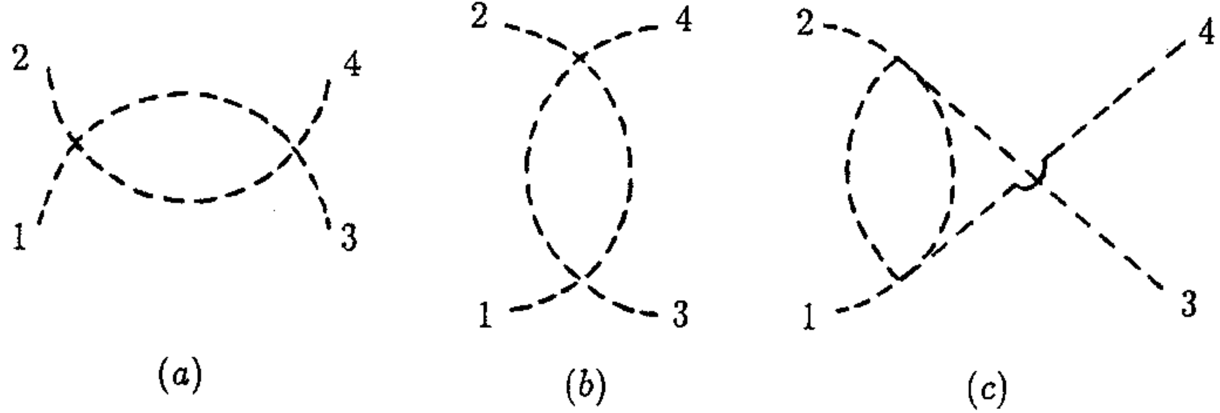
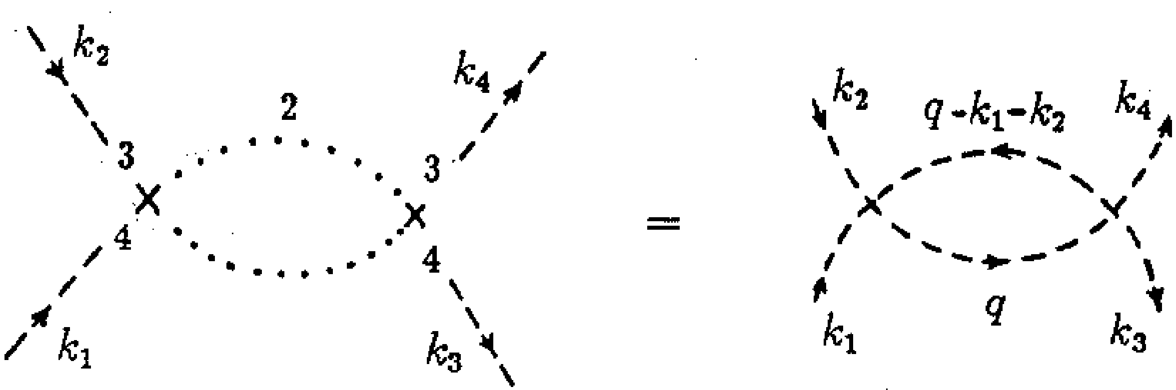


图5. 14　两标量粒子散射的二级图



拓扑权 $= 2\times(4\times3\times4\times3\times2) = (4!)^2$，**顶角置换因子** $= \dfrac{1}{(4!)^2}$,

$$W = \frac{(4!)^2}{2!(4!)^2} = \frac{1}{2}.$$

§ 5-5-2　π-N Yukawa 耦合

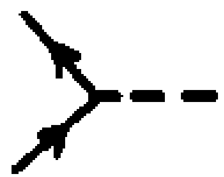
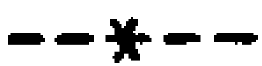
1.　π^0-N 耦合

$$\mathcal{H}_i(x) = \mathrm{i}\, g_0 :\bar{\psi}(x)\gamma_5\psi(x)\varphi(x):$$

$$-\delta m :\bar{\psi}(x)\psi(x): -\frac{1}{2!}\delta\mu^2 :\varphi^2(x):,$$

$$\delta m = m - m_0, \qquad \delta\mu^2 = \mu^2 - \mu_0^2.$$

表5.5 π^0-N 耦合的部分Feynman规则

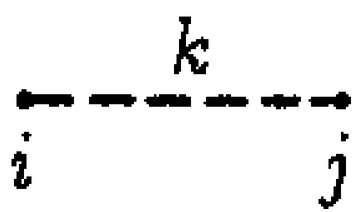
	图形	S矩阵中的因子
π^0-N 顶角		$g_0\gamma_5$
核子质量抵消项顶角		$\mathrm{i}\,\delta m$
介子质量抵消项顶角		$\mathrm{i}\,\delta\mu^2$

2. π−N 同位旋SU(2)不变耦合

$$\mathcal{H}_i(x) = \mathrm{i}\, g_0 : \overline{N}(x)\gamma_5 \vec{\tau} N(x) \cdot \vec{\varphi}(x) :$$

$$- \delta m : \overline{N}(x) N(x) : - \frac{1}{2!}\delta\mu^2 : \vec{\varphi}(x) \cdot \vec{\varphi}(x) : .$$

π−N 同位旋SU(2)不变耦合的部分Feynman规则

	图形	S矩阵中的因子
π−N 顶角		$g_0\gamma_5\tau_i$
介子内线		$\dfrac{\mathrm{i}\,\delta_{ij}}{k^2-\mu^2+\mathrm{i}\varepsilon}$

§ 5-6 截面与寿命

➢ 跃迁几率

$$S_{fi} = I_{fi} + \mathrm{i}(2\pi)^4 \delta^4 (P_f - P_i) T_{fi}, \tag{51}$$

跃迁几率：

$$\omega_{fi} = [(2\pi)^4 \delta^4 (P_f - P_i)]^2 \left| T_{fi} \right|^2, \tag{52}$$

a）单位时空内跃迁至某一终态的几率

$$\because \qquad (2\pi)^4 \delta^4 (0) = \int \mathrm{d}^4 x e^{-\mathrm{i}(P-P)\cdot x} = VT,$$

可得

$$\Omega_{fi} = (2\pi)^4 \delta^4 (P_f - P_i) \left| T_{fi} \right|^2, \tag{53}$$

b）单位时空内跃迁至一群连续末态的几率

$$\Omega_{fi} \times \text{终态粒子相空间的态数}$$

对于确定的极化，相空间 $1 \cdot \mathrm{d}^3 p$ 中的态数：

$$\tilde{d}p = \begin{cases} \dfrac{\mathrm{d}^3 p}{(2\pi)^3 2p^0} & \text{对玻色子和0质量旋量粒子} \\[2ex] \dfrac{\mathrm{d}^3 p}{(2\pi)^3} \dfrac{m}{p^0} & \text{对}m \neq 0\text{的旋量粒子} \end{cases}$$

考虑末态相空间因子后，单位时空的跃迁几率为

$$\Omega_{fi} \prod_{j=1}^{n} \tilde{\mathrm{d}}\, p_j' \frac{1}{S}, \tag{54}$$

统计因子 $S = \prod_i m_i!.$

➢ 散射截面

$$1+2 \to 1'+2'+\cdots+n'$$

$$\mathrm{d}\sigma = \frac{\text{单位时空的跃迁几率}}{\text{入射粒子流}\times\text{靶粒子密度}}$$

$$= \frac{1}{v_{12}\rho_1\rho_2}\Omega(1'\cdots n'|12)\prod_{j=1}^{n}\tilde{\mathrm{d}}\, p'_j \,\frac{1}{S}. \qquad (55)$$

a）初态为两个玻色子或两个0质量旋量粒子

$$\langle p | p \rangle = (2\pi)^3\delta^3(0)2p^0 = V\cdot 2p^0,$$

$$\Rightarrow \quad \rho = 2p^0,$$

$$\mathrm{d}\sigma(1+2\to 1'+\cdots+n')$$

$$=\frac{1}{4[(p_1\cdot p_1)^2-m_1^2m_2^2]^{1/2}}$$

$$\times\left|\langle 1'\cdots n'|T|12\rangle\right|^2\prod_{j=1}^{n}\tilde{\mathrm{d}}\,p_j'(2\pi)^4\delta^4\left(\sum_{j=1}^{n}p_j'-p_1-p_2\right)\frac{1}{S}. \qquad (56)$$

b）初态为一个玻色子（或0质量旋量粒子）和一个非零质量旋量粒子

$$\langle p|p\rangle=V\cdot 2p^0 \quad\to\quad \langle p,\alpha|p,\alpha\rangle=V\frac{p^0}{m},$$

$$\mathrm{d}\sigma(1+2\to 1'+\cdots+n')=(56)\text{式}\Big|_{\frac{1}{2}\to m\text{（代换后的旋量粒子的质量）}}. \qquad (57)$$

➢ 衰变几率

单位时间的衰变几率：

$$\Gamma = \frac{\text{单位时空的跃迁几率}}{\text{初态粒子密度}}.$$

a）$|i\rangle$为玻色子态或0质量旋量粒子态（ $\rho_i = 2p_i^0$ ）

$$\Gamma = \frac{1}{2p_i^0}\int\prod_j \tilde{\mathrm{d}}\, p_j'\Omega_{fi}\frac{1}{S}$$

$$= \frac{1}{2p_i^0}\int\prod_{j=1}^{n}\tilde{\mathrm{d}}\, p_j'\left|\langle 1'\cdots n'|T|i\rangle\right|^2(2\pi)^4\delta^4\left(\sum_{j=1}^{n}p_j' - p_i\right)\frac{1}{S}; \quad (58)$$

b） $|i\rangle$为非零质量旋量粒子态

$$\Gamma = \frac{m}{p_i^0}\int\prod_{j=1}^{n}\tilde{\mathrm{d}}\,p_j'\left|\langle 1'\cdots n'|T|i\rangle\right|^2(2\pi)^4\delta^4\left(\sum_{j=1}^{n}p_j'-p_i\right)\frac{1}{S}; \quad (59)$$

衰变粒子的寿命：

$$\tau = \frac{1}{\Gamma}. \quad (60)$$

$$\sigma_{\text{不极化}} = \frac{1}{4}\sum_{\text{初态自旋}}\ \sum_{\text{终态自旋}}\sigma_{\text{极化}},$$

$$\Gamma_{\text{不极化}} = \frac{1}{2}\sum_{\text{初态自旋}}\ \sum_{\text{终态自旋}}\Gamma_{\text{极化}}.$$

§5-7 应用举例

§5-7-1　一些有用的公式

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}\{1,-1,-1,-1\};$$

$$\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu};$$

$$\not{a} = a_{\mu}\gamma^{\mu},$$

$$\gamma_5 = \mathrm{i}\,\gamma^0\gamma^1\gamma 2\gamma^3,$$

$$\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}).$$

➢ γ矩阵的乘积

$$\gamma^\mu \gamma_\mu = 4,$$

$$\gamma_5 \gamma_5 = 1,$$

$$\{\gamma^\mu, \gamma_5\} = 0,$$

$$\not{a}\not{a} = a \cdot a,$$

$$\not{a}\not{b} + \not{b}\not{a} = 2(a \cdot b),$$

$$\gamma^\mu \not{a} \gamma_\mu = -2\not{a},$$

$$\gamma^\mu \not{a}\not{b} \gamma_\mu = 4(a \cdot b),$$

$$\gamma^\mu \not{a}\not{b}\not{c} \gamma_\mu = -2\not{c}\not{b}\not{a}.$$

$$\gamma^0 \gamma^{\mu +} \gamma^0 = \gamma^\mu,$$

$$\gamma^0 \sigma^{\mu\nu +} \gamma^0 = \sigma^{\mu\nu},$$

$$\gamma_5^+ = \gamma_5.$$

➢ γ矩阵乘积的迹

1) $\mathrm{Tr}\, I = 4;$

2) $\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n+1}}) = 0;$

3) $\mathrm{Tr}(\gamma^{\mu_1}\cdots\gamma^{\mu_{2n}}) = \mathrm{Tr}(\gamma^{\mu_{2n}}\gamma^{\mu_{2n-1}}\cdots\gamma^{\mu_1});$

4) $\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}) = 4g^{\mu\nu},\qquad \mathrm{Tr}(\not{a}\not{b}) = 4a\cdot b,\qquad \mathrm{Tr}\,\sigma^{\mu\nu} = 0;$

5) $\mathrm{Tr}(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}),$

$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4[(a\cdot b)(c\cdot d) - (a\cdot c)(b\cdot d) + (a\cdot d)(b\cdot c)];$

6) $\mathrm{Tr}\,\gamma_5 = \mathrm{Tr}(\gamma_5\gamma^{\mu}) = \mathrm{Tr}(\gamma_5\gamma^{\mu}\gamma^{\nu}) = \mathrm{Tr}(\gamma_5\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}) = 0,$

$\mathrm{Tr}(\gamma_5\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}) = -i4\varepsilon^{\mu\nu\rho\sigma}.$

➢ 旋量自旋求和

对于$m \neq 0$的旋量，有

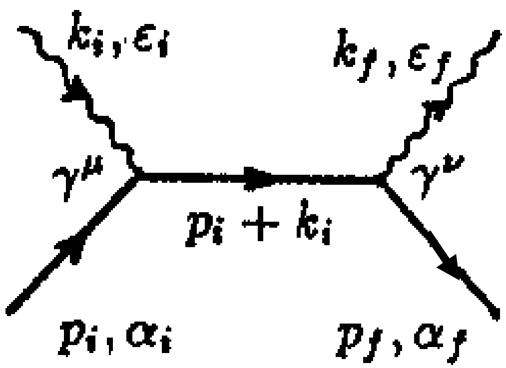
$$\sum_{\alpha=1,2} u^{(\alpha)}(p)\bar{u}^{(\alpha)}(p) = \frac{\not{p}+m}{2m},$$

$$\sum_{\alpha=1,2} v^{(\alpha)}(p)\bar{v}^{(\alpha)}(p) = \frac{\not{p}-m}{2m}.$$

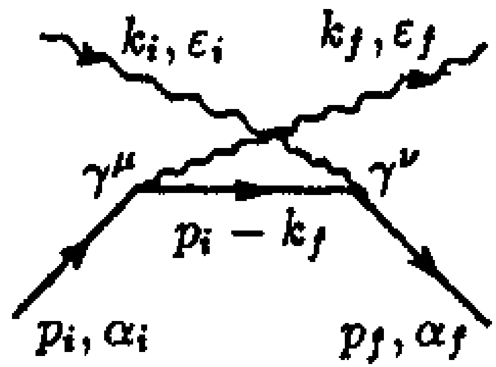
对于$m = 0$的旋量，有

$$\sum_{\alpha=1,2} u^{(\alpha)}(p)\bar{u}^{(\alpha)}(p) = \sum_{\alpha=1,2} v^{(\alpha)}(p)\bar{v}^{(\alpha)}(p) = \not{p}.$$

§ 5-7-2　Compton 散射



(a)



(b)

$$T_a = \bar{u}(p_f, \alpha_f)(-\mathrm{i}e)\gamma^\nu \varepsilon_\nu^{(\lambda_f)}(k_f)\frac{1}{\not{p}_i + \not{k}_i - m + \mathrm{i}\varepsilon}(-\mathrm{i}e)\gamma^\mu \varepsilon_\mu^{(\lambda_i)}(k_i)u(p_i, \alpha_i)$$

$$= (-\mathrm{i}e)^2 \bar{u}(p_f, \alpha_f)\not{\varepsilon}_f \frac{1}{\not{p}_i + \not{k}_i - m}\not{\varepsilon}_i u(p_i, \alpha_i),$$

$$T_b = T_a\Big|_{(\varepsilon_i \leftrightarrow \varepsilon_f, k_i \leftrightarrow -k_f)}.$$

总振幅： $A = T_a + T_b$.

微分散射截面：

$$\mathrm{d}\sigma = \frac{m}{2(p_i \cdot k_i)} |A|^2 \frac{\mathrm{d}^3 k_f}{(2\pi)^3 2k_f^0} \frac{m\,\mathrm{d}^3 p_f}{(2\pi)^3 p_f^0} (2\pi)^4 \delta^4(k_f + p_f - k_i - p_i)$$

若靶电子不极化，且对终态电子不测量极化，则

$$\mathrm{d}\sigma = \frac{1}{2} \sum_{\alpha_i, \alpha_f} \frac{m}{2(p_i \cdot k_i)} |A|^2 \frac{\mathrm{d}^3 k_f}{(2\pi)^3 2k_f^0} \frac{m\,\mathrm{d}^3 p_f}{(2\pi)^3 p_f^0} (2\pi)^4 \delta^4(k_f + p_f - k_i - p_i).$$

$$A = (-\mathrm{i}e)^2 \bar{u}(p_f, \alpha_f)[\not{\varepsilon}_f \frac{1}{\not{p}_i + \not{k}_i - m} \not{\varepsilon}_i + \not{\varepsilon}_i \frac{1}{\not{p}_i - \not{k}_f - m} \not{\varepsilon}_f] u(p_i, \alpha_i),$$

利用 $\dfrac{1}{\not{p}_i + \not{k}_i - m} = \dfrac{\not{p}_i + \not{k}_i + m}{(p_i + k_i)^2 - m^2}$,

$$A = (-\mathrm{i}e)^2 \bar{u}(p_f)[\not{\varepsilon}_f \frac{\not{p}_i + \not{k}_i + m}{(p_i + k_i)^2 - m^2} \not{\varepsilon}_i + \not{\varepsilon}_i \frac{\not{p}_i - \not{k}_f + m}{(p_i - k_f)^2 - m^2} \not{\varepsilon}_f] u(p_i)$$

$$A=(-\mathrm{i}e)^2\bar{u}(p_f)[\not{\varepsilon}_f\frac{\not{p}_i+\not{k}_i+m}{2(p_i\cdot k_i)}\not{\varepsilon}_i+\not{\varepsilon}_i\frac{\not{p}_i-\not{k}_f+m}{-2(p_i\cdot k_f)}\not{\varepsilon}_f]u(p_i),$$

由$\not{a}\not{b}=2a\cdot b-\not{b}\not{a}$，有

$$A=(-\mathrm{i}e)^2\bar{u}(p_f)[\not{\varepsilon}_f\frac{2p_i\cdot\varepsilon_i-\not{\varepsilon}_i\not{p}_i+2k_i\cdot\varepsilon_i-\not{\varepsilon}_i\not{k}_i+\not{\varepsilon}_i m}{2(p_i\cdot k_i)}$$

$$+\not{\varepsilon}_i\frac{2\varepsilon_f\cdot p_i-\not{\varepsilon}_f\not{p}_i-2k_f\cdot\varepsilon_f+\not{\varepsilon}_f\not{k}_f+\not{\varepsilon}_f m}{-2(p_i\cdot k_f)}]u(p_i)$$

$$=(-\mathrm{i}e)^2\bar{u}(p_f)[\not{\varepsilon}_f\frac{2(p_i+k_i)\cdot\varepsilon_i+\not{\varepsilon}_i(-\not{p}_i-\not{k}_i+m)}{2(p_i\cdot k_i)}$$

$$+\not{\varepsilon}_i\frac{2(p_i-k_f)\cdot\varepsilon_f+\not{\varepsilon}_f(-\not{p}_i+\not{k}_f+m)}{-2(p_i\cdot k_f)}]u(p_i)$$

$$=(-\mathrm{i}e)^2\bar{u}(p_f)[\not{\varepsilon}_f\frac{2(p_i+k_i)\cdot\varepsilon_i-\not{\varepsilon}_i\not{k}_i}{2(p_i\cdot k_i)}+\not{\varepsilon}_i\frac{2(p_i-k_f)\cdot\varepsilon_f+\not{\varepsilon}_f\not{k}_f}{-2(p_i\cdot k_f)}]u(p_i)$$

取靶电子静止系（实验室系），有

$$p_i=(m,0,0,0).$$

取横光子的极化$\varepsilon_i,\varepsilon_f$使之与时间轴$p_i=(m,0,0,0)$正交，即有

$$\varepsilon_i\cdot p_i=\varepsilon_f\cdot p_i=0,$$

$$\varepsilon_i\cdot k_i=\varepsilon_f\cdot k_i=0.$$

$$A=-(-\mathrm{i}e)^2\bar{u}(p_f,\alpha_f)\left[\frac{\not\varepsilon_f\not\varepsilon_i\not k_i}{2(p_i\cdot k_i)}+\frac{\not\varepsilon_i\not\varepsilon_f\not k_f}{2(p_i\cdot k_f)}\right]u(p_i,\alpha_i)$$

$$=-(-\mathrm{i}e)^2\bar{u}(p_f,\alpha_f)Ou(p_i,\alpha_i),$$

其中

$$O\equiv\frac{\not\varepsilon_f\not\varepsilon_i\not k_i}{2(p_i\cdot k_i)}+\frac{\not\varepsilon_i\not\varepsilon_f\not k_f}{2(p_i\cdot k_f)}.$$

$$X = \sum_{\alpha_i,\alpha_f} |A|^2 = (-\mathrm{i}e)^4 \sum_{\alpha_i,\alpha_f} \left| \bar{u}(p_f,\alpha_f) O u(p_i,\alpha_i) \right|^2$$

$$= (-\mathrm{i}e)^4 \sum_{\alpha_i,\alpha_f} [\bar{u}(p_f,\alpha_f) O u(p_i,\alpha_i)][\bar{u}(p_f,\alpha_f) O u(p_i,\alpha_i)]^+$$

$$= (-\mathrm{i}e)^4 \sum_{\alpha_i,\alpha_f} \bar{u}(p_f,\alpha_f) O u(p_i,\alpha_i) u^+(p_i,\alpha_i) O^+ \bar{u}^+(p_f,\alpha_f)$$

$$= (-\mathrm{i}e)^4 \sum_{\alpha_i,\alpha_f} \bar{u}(p_f,\alpha_f) O u(p_i,\alpha_i) \bar{u}(p_i,\alpha_i) \gamma^0 O^+ \gamma^0 u(p_f,\alpha_f)$$

$$= (-\mathrm{i}e)^4 \,\mathrm{Tr} \sum_{\alpha_i,\alpha_f} O u(p_i,\alpha_i) \bar{u}(p_i,\alpha_i) \gamma^0 O^+ \gamma^0 u(p_f,\alpha_f) \bar{u}(p_f,\alpha_f)$$

$$= (-\mathrm{i}e)^4 \,\mathrm{Tr}[O \frac{\not{p}_i + m}{2m} \gamma^0 O^+ \gamma^0 \frac{\not{p}_f + m}{2m}],$$

$$\gamma^0 O^+ \gamma^0 = \gamma^0 \left[\frac{\not{\varepsilon}_f \not{\varepsilon}_i \not{k}_i}{2(p_i \cdot k_i)} + \frac{\not{\varepsilon}_i \not{\varepsilon}_f \not{k}_f}{2(p_i \cdot k_f)} \right]^+ \gamma^0 = \gamma^0 \frac{\not{k}_i^{\,+} \not{\varepsilon}_i^{\,+} \not{\varepsilon}_f^{\,+}}{2(p_i \cdot k_i)} + \frac{\not{k}_f^{\,+} \not{\varepsilon}_f^{\,+} \not{\varepsilon}_i^{\,+}}{2(p_i \cdot k_f)} \gamma^0,$$

利用$\gamma^0 \gamma^{\mu+} \gamma^0 = \gamma^\mu$以及$\gamma^0 \gamma^0 = I$，得到

$$\gamma^0 O^+ \gamma^0 = \frac{\not{k}_i \not{\varepsilon}_i \not{\varepsilon}_f}{2(p_i \cdot k_i)} + \frac{\not{k}_f \not{\varepsilon}_f \not{\varepsilon}_i}{2(p_i \cdot k_f)},$$

$$X = (-\mathrm{i}\, e)^4 \,\mathrm{Tr} \left[\left(\frac{\not{\varepsilon}_f \not{\varepsilon}_i \not{k}_i}{2(p_i \cdot k_i)} + \frac{\not{\varepsilon}_i \not{\varepsilon}_f \not{k}_f}{2(p_i \cdot k_f)} \right) \frac{\not{p}_i + m}{2m} \right.$$

$$\left. \times \left(\frac{\not{k}_i \not{\varepsilon}_i \not{\varepsilon}_f}{2(p_i \cdot k_i)} + \frac{\not{k}_f \not{\varepsilon}_f \not{\varepsilon}_i}{2(p_i \cdot k_f)} \right) \frac{\not{p}_f + m}{2m} \right]$$

$$X=(-\mathrm{i}e)^4\frac{1}{16m^2}\mathrm{Tr}[\frac{1}{(p_i\cdot k_i)^2}\not{\epsilon}_f\not{\epsilon}_i\not{k}_i(\not{p}_i+m)\not{k}_i\not{\epsilon}_i\not{\epsilon}_f(\not{p}_f+m) \qquad ①$$

$$+\frac{1}{(p_i\cdot k_i)(p_i\cdot k_f)}\not{\epsilon}_f\not{\epsilon}_i\not{k}_i(\not{p}_i+m)\not{k}_f\not{\epsilon}_f\not{\epsilon}_i(\not{p}_f+m) \qquad ②$$

$$+\frac{1}{(p_i\cdot k_i)(p_i\cdot k_f)}\not{\epsilon}_i\not{\epsilon}_f\not{k}_f(\not{p}_i+m)\not{k}_i\not{\epsilon}_i\not{\epsilon}_f(\not{p}_f+m) \qquad ③$$

$$+\frac{1}{(p_i\cdot k_f)^2}\not{\epsilon}_i\not{\epsilon}_f\not{k}_f(\not{p}_i+m)\not{k}_f\not{\epsilon}_f\not{\epsilon}_i(\not{p}_f+m)] \qquad ④$$

$$① =\mathrm{Tr}[\not{\epsilon}_f\not{\epsilon}_i\not{k}_i(\not{p}_i+m)\not{k}_i\not{\epsilon}_i\not{\epsilon}_f(\not{p}_f+m)]$$

$$=\mathrm{Tr}[\not{\epsilon}_f\not{\epsilon}_i\not{k}_i\not{p}_i\not{k}_i\not{\epsilon}_i\not{\epsilon}_f\not{p}_f+m^2\not{\epsilon}_f\not{\epsilon}_i\not{k}_i\not{k}_i\not{\epsilon}_i\not{\epsilon}_f]$$

$$=\mathrm{Tr}[\not{\epsilon}_f\not{\epsilon}_i\not{k}_i\not{p}_i\not{k}_i\not{\epsilon}_i\not{\epsilon}_f\not{p}_f]=\mathrm{Tr}[\not{\epsilon}_f\not{\epsilon}_i(2k_i\cdot p_i-\not{p}_i\not{k}_i)\not{k}_i\not{\epsilon}_i\not{\epsilon}_f\not{p}_f]$$

$$
\begin{aligned}
① &= 2(k_i \cdot p_i)\,\mathrm{Tr}[\not{\varepsilon}_f \not{\varepsilon}_i \not{k}_i \not{\varepsilon}_i \not{\varepsilon}_f \not{p}_f] \\
&= 2(k_i \cdot p_i)\,\mathrm{Tr}[\not{\varepsilon}_f \not{\varepsilon}_i (2k_i \cdot \varepsilon_i - \not{\varepsilon}_i \not{k}_i) \not{\varepsilon}_f \not{p}_f] \\
&= 2(k_i \cdot p_i)\,\mathrm{Tr}[\not{\varepsilon}_f \not{\varepsilon}_i \not{\varepsilon}_f \not{p}_f] \\
&= 2(k_i \cdot p_i) \cdot 4[(\varepsilon_f \cdot k_i)(\varepsilon_f \cdot p_f) - (\varepsilon_f \cdot \varepsilon_f)(k_i \cdot p_f) + (\varepsilon_f \cdot p_f)(k_i \cdot \varepsilon_f)] \\
&= 8(k_i \cdot p_i)[2(\varepsilon_f \cdot k_i)(\varepsilon_f \cdot p_f) + (k_i \cdot p_f)] \\
&= 8(k_i \cdot p_i)[2(\varepsilon_f \cdot k_i)\varepsilon_f \cdot (k_i + p_i - k_f) + (k_i \cdot p_f)] \\
&= 8(k_i \cdot p_i)[2(\varepsilon_f \cdot k_i)^2 + (k_f \cdot p_i)],
\end{aligned}
$$

$$② = \mathrm{Tr}[\not\varepsilon_f \not\varepsilon_i \not k_i (\not p_i + m) \not k_f \not\varepsilon_f \not\varepsilon_i (\not p_f + m)]$$

$$= \mathrm{Tr}[\not\varepsilon_f \not\varepsilon_i \not k_i (\not p_i + m) \not k_f \not\varepsilon_f \not\varepsilon_i (\not p_i + \not k_i - \not k_f + m)]$$

$$= \mathrm{Tr}[\not\varepsilon_f \not\varepsilon_i \not k_i (\not p_i + m) \not k_f \not\varepsilon_f \not\varepsilon_i (\not p_i + m)]$$

$$+ \mathrm{Tr}[\not\varepsilon_f \not\varepsilon_i \not k_i \not p_i \not k_f \not\varepsilon_f \not\varepsilon_i \not k_i] - \mathrm{Tr}[\not\varepsilon_f \not\varepsilon_i \not k_i \not p_i \not k_f \not\varepsilon_f \not\varepsilon_i \not k_f]$$

$$= 2(p_i \cdot k_i)\,\mathrm{Tr}[\not k_f \not\varepsilon_f \not\varepsilon_i \not\varepsilon_f \not\varepsilon_i \not p_i]$$

$$+ 2(k_i \cdot \varepsilon_f)\,\mathrm{Tr}[\not\varepsilon_i \not k_i \not p_i \not k_f \not\varepsilon_f \not\varepsilon_i] - 2(\varepsilon_i \cdot k_f)\,\mathrm{Tr}[\not\varepsilon_f \not\varepsilon_i \not k_i \not p_i \not k_f \not\varepsilon_f]$$

$$= 2(p_i \cdot k_i)\{2(\varepsilon_i \cdot \varepsilon_f)\,\mathrm{Tr}[\not k_f \not\varepsilon_f \not\varepsilon_i \not p_i] - \mathrm{Tr}[\not k_f \not p_i]\}$$

$$- 2(k_i \cdot \varepsilon_f)\,\mathrm{Tr}[\not k_i \not p_i \not k_f \not\varepsilon_f] + 2(\varepsilon_i \cdot k_f)\,\mathrm{Tr}[\not\varepsilon_i \not k_i \not p_i \not k_f]$$

$$= 8(k_i \cdot p_i)(k_f \cdot p_i)[2(\varepsilon_i \cdot \varepsilon_f)^2 - 1]$$

$$- 8(k_i \cdot \varepsilon_f)^2 (k_f \cdot p_i) + 8(\varepsilon_i \cdot k_f)^2 (k_i \cdot p_i),$$

$$③ = ②\Big|_{(\varepsilon_i \leftrightarrow \varepsilon_f, k_i \leftrightarrow -k_f)} = ②,$$

$$④ = ①\Big|_{(\varepsilon_i \leftrightarrow \varepsilon_f, k_i \leftrightarrow -k_f)}$$

$$= -8(k_f \cdot p_i)[2(\varepsilon_i \cdot k_f)^2 - (k_i \cdot p_i)].$$

$$
\begin{aligned}
X = (-\mathrm{i}\,e)^4 \frac{1}{16m^2}\{ &\frac{1}{(p_i \cdot k_i)^2} \cdot 8(k_i \cdot p_i)[2(\varepsilon_f \cdot k_i)^2 + (k_f \cdot p_i)] \\
&+ \frac{2}{(p_i \cdot k_i)(p_i \cdot k_f)} \cdot \Big[8(k_i \cdot p_i)(k_f \cdot p_i)[2(\varepsilon_i \cdot \varepsilon_f)^2 - 1] \\
&\qquad - 8(k_i \cdot \varepsilon_f)^2 (k_f \cdot p_i) + 8(\varepsilon_i \cdot k_f)^2 (k_i \cdot p_i)\Big] \\
&+ \frac{-1}{(p_i \cdot k_f)^2} \cdot 8(k_f \cdot p_i)[2(\varepsilon_i \cdot k_f)^2 - (k_i \cdot p_i)]\} \\
= (-\mathrm{i}\,e)^4 \frac{1}{2m^2}\{ &\frac{k_f \cdot p_i}{k_i \cdot p_i} + 4(\varepsilon_i \cdot \varepsilon_f)^2 - 2 + \frac{k_i \cdot p_i}{k_f \cdot p_i}\} \\
= (-\mathrm{i}\,e)^4 \frac{1}{2m^2}\{ &\frac{mk_f^0}{mk_i^0} + \frac{mk_i^0}{mk_f^0} + 4(\varepsilon_i \cdot \varepsilon_f)^2 - 2\} \\
= (-\mathrm{i}\,e)^4 \frac{1}{2m^2}\{ &\frac{k_f^0}{k_i^0} + \frac{k_i^0}{k_f^0} + 4(\varepsilon_i \cdot \varepsilon_f)^2 - 2\}.
\end{aligned}
$$

不极化电子的Compton散射微分截面为：

$$\mathrm{d}\sigma = \frac{1}{2}\sum_{\alpha_i,\alpha_f} \frac{m}{2(p_i \cdot k_i)} |A|^2 \frac{\mathrm{d}^3 k_f}{(2\pi)^3 2k_f^0} \frac{m\,\mathrm{d}^3 p_f}{(2\pi)^3 p_f^0} (2\pi)^4 \delta^4(k_f + p_f - k_i - p_i)$$

$$= \frac{1}{8}\frac{1}{(2\pi)^2}$$

$$\times \sum_{\alpha_i,\alpha_f} \frac{m^2}{(p_i \cdot k_i)} |A|^2 \frac{1}{k_f^0 p_f^0} \mathrm{d}^3 k_f\, \mathrm{d}^3 p_f \delta^4(k_f + p_f - k_i - p_i), \qquad (61)$$

$$\int \mathrm{d}^3 k_f\, \mathrm{d}^3 p_f \delta^4(k_f + p_f - k_i - p_i)$$

$$= \int \mathrm{d}\vec{p}_f\, \mathrm{d}\vec{k}_f \delta^3(\vec{k}_f + \vec{p}_f - \vec{k}_i - \vec{p}_i)\delta(k_f^0 + p_f^0 - k_i^0 - p_i^0)$$

$$= \int \mathrm{d}\vec{p}_f\, |\vec{k}_f|^2\, \mathrm{d}|\vec{k}_f|\, \mathrm{d}\Omega\, \delta^3(\vec{k}_f + \vec{p}_f - \vec{k}_i - \vec{p}_i)\delta(k_f^0 + p_f^0 - k_i^0 - p_i^0),$$

$$\because k_f^2 = 0, \quad \Rightarrow k_f^0 = |\vec{k}_f|, \quad \Rightarrow \mathrm{d}|\vec{k}_f| = \mathrm{d}k_f^0,$$

$$\therefore \int \mathrm{d}^3 k_f \, \mathrm{d}^3 p_f \delta^4(k_f + p_f - k_i - p_i)$$

$$= \int \mathrm{d}\vec{p}_f \, \mathrm{d}k_f^0 \, \mathrm{d}\Omega (k_f^0)^2 \delta^3(\vec{k}_f + \vec{p}_f - \vec{k}_i - \vec{p}_i) \delta(k_f^0 + p_f^0 - k_i^0 - p_i^0)$$

$$= \int \mathrm{d}k_f^0 \, \mathrm{d}\Omega (k_f^0)^2 \delta(k_f^0 + p_f^0 - k_i^0 - p_i^0) = \frac{{k_f^0}^2}{\left| \dfrac{\mathrm{d}(k_f^0 + p_f^0)}{\mathrm{d}k_f^0} \right|} \mathrm{d}\Omega,$$

$$\because \frac{\mathrm{d}(k_f^0 + p_f^0)}{\mathrm{d}k_f^0} = 1 + \frac{1}{2p_f^0} \frac{\mathrm{d}{p_f^0}^2}{\mathrm{d}k_f^0} = 1 + \frac{1}{2p_f^0} \frac{\mathrm{d}}{\mathrm{d}k_f^0} [m^2 + (\vec{k}_i + \vec{p}_i - \vec{k}_f)^2]$$

$$= 1 + \frac{1}{2p_f^0} 2(\vec{k}_i + \vec{p}_i - \vec{k}_f) \cdot \frac{\mathrm{d}(-\vec{k}_f)}{\mathrm{d}|\vec{k}_f|} = 1 - \frac{1}{p_f^0} \vec{p}_f \cdot \frac{\vec{k}_f}{k_f^0} = \frac{p_f \cdot k_f}{p_f^0 k_f^0},$$

$$\therefore \int \mathrm{d}^3 k_f \, \mathrm{d}^3 p_f \delta^4(k_f + p_f - k_i - p_i) = \frac{p_f^0 {k_f^0}^3}{p_f \cdot k_f} \mathrm{d}\Omega,$$

代入(61)式，得到

$$\mathrm{d}\sigma = \frac{1}{8}\frac{1}{(2\pi)^2}\sum_{\alpha_i,\alpha_f}|A|^2 \frac{(mk_f^0)^2}{(p_i\cdot k_i)(p_f\cdot k_f)}\mathrm{d}\Omega$$

$$= \frac{1}{8}\frac{1}{(2\pi)^2}\sum_{\alpha_i,\alpha_f}|A|^2\left(\frac{k_f^0}{k_i^0}\right)^2 \mathrm{d}\Omega$$

$$= \frac{1}{8}\frac{1}{(2\pi)^2}e^4\frac{1}{2m^2}\left[\frac{k_f^0}{k_i^0}+\frac{k_i^0}{k_f^0}+4(\varepsilon_i\cdot\varepsilon_f)^2-2\right]\left(\frac{k_f^0}{k_i^0}\right)^2 \mathrm{d}\Omega,$$

极化光子对自由电子Compton散射的微分截面为：

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4m^2}\left[\frac{k_f^0}{k_i^0}+\frac{k_i^0}{k_f^0}+4(\varepsilon_i\cdot\varepsilon_f)^2-2\right]\left(\frac{k_f^0}{k_i^0}\right)^2, \qquad (62)$$

不极化的微分散射截面为：$\frac{d\bar{\sigma}}{d\Omega}=\frac{1}{2}\sum_{\varepsilon_i,\varepsilon_f}\frac{d\sigma}{d\Omega}$.

利用

$$\sum_{\varepsilon_i,\varepsilon_f}(\varepsilon_i\cdot\varepsilon_f)^2\equiv\sum_{\lambda_i,\lambda_f=1,2}[\varepsilon_i^{(\lambda_i)}(k_i)\cdot\varepsilon_f^{(\lambda_f)}(k_f)]^2=1+\cos^2\theta=2-\sin^2\theta,$$

可得

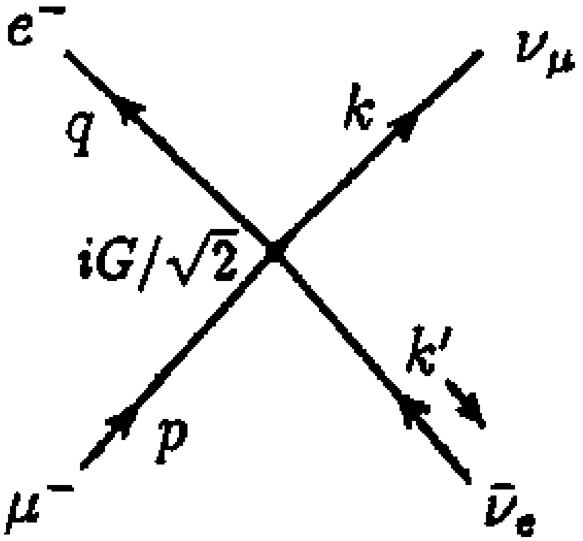
$$\frac{d\bar{\sigma}}{d\Omega}=\frac{1}{2}\sum_{\varepsilon_i,\varepsilon_f}\frac{\alpha^2}{4m^2}\left[\frac{k_f^0}{k_i^0}+\frac{k_i^0}{k_f^0}+4(\varepsilon_i\cdot\varepsilon_f)^2-2\right]\left(\frac{k_f^0}{k_i^0}\right)^2$$

$$=\frac{\alpha^2}{2m^2}\left[\frac{k_f^0}{k_i^0}+\frac{k_i^0}{k_f^0}-\sin^2\theta\right]\left(\frac{k_f^0}{k_i^0}\right)^2, \tag{63}$$

其中

$$\frac{\alpha}{m}=\frac{e^2}{4\pi m}=r_0=2.8\times10^{-13}\,\text{cm}.$$（电子经典半径）

§ 5-7-3 μ衰变寿命



$$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$$

$$\mathcal{H}_i = -\frac{G}{\sqrt{2}} :\bar{\nu}_\mu \gamma_\lambda (1-\gamma_5)\mu \bar{e}\gamma^\lambda (1-\gamma_5)\nu_e : + h.c.$$

4 Fermion顶角对应*S*矩阵中因子：

$$\mathrm{i}\frac{G}{\sqrt{2}}\gamma_\lambda (1-\gamma_5)\cdots\gamma^\lambda (1-\gamma_5),$$

$$\left\langle q,k,k'\left|S\right|p\right\rangle = (2\pi)^4\delta^4(q+k+k'-p)\left\langle q,k,k'\left|T\right|p\right\rangle,$$

$$\left\langle q,k,k'\left|T\right|p\right\rangle = \mathrm{i}\frac{G}{\sqrt{2}}\overline{u}(k)\gamma_\lambda(1-\gamma_5)u(p)\overline{u}(q)\gamma^\lambda(1-\gamma_5)v(k'),$$

$$\tau^{-1} = \frac{m_\mu}{p^0}\frac{1}{2}\sum_{\text{自旋}}\int\left|\left\langle q,k,k'\left|T\right|p\right\rangle\right|^2\frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{m_e}{q^0}$$

$$\times\frac{\mathrm{d}^3 k}{(2\pi)^3 2k^0}\frac{\mathrm{d}^3 k'}{(2\pi)^3 2k'^0}(2\pi)^4\delta^4(q+k+k'-p)$$

$$= \frac{1}{(2\pi)^5}\frac{G^2}{2}\frac{m_\mu m_e}{4p^0}\int\frac{\mathrm{d}^3 q\,\mathrm{d}^3 k\,\mathrm{d}^3 k'}{q^0k^0k'^0}\delta^4(q+k+k'-p)Y,$$

$$Y = \frac{1}{2}\sum_{\text{自旋}}\left|\overline{u}(k)\gamma_\lambda(1-\gamma_5)u(p)\overline{u}(q)\gamma^\lambda(1-\gamma_5)v(k')\right|^2,$$

记$O_\lambda \equiv \gamma_\lambda(1-\gamma_5)$,

$$Y = \frac{1}{2}\sum_{\text{自旋}} [\overline{u}(k)O_\lambda u(p)\overline{u}(q)O^\lambda v(k')][\overline{u}(k)O_\rho u(p)\overline{u}(q)O^\rho v(k')]^*$$

$$= \frac{1}{2}\sum_{\text{自旋}} \overline{u}(k)O_\lambda u(p)[\overline{u}(k)O_\rho u(p)]^+ \overline{u}(q)O^\lambda v(k')[\overline{u}(q)O^\rho v(k')]^+$$

$$= \frac{1}{2}\sum_{\text{自旋}} \overline{u}(k)O_\lambda u(p)\overline{u}(p)\gamma^0 O_\rho^+ \gamma^0 u(k)$$

$$\times \overline{u}(q)O^\lambda v(k')\overline{v}(k')\gamma^0 O^{\rho+}\gamma^0 u(q)$$

$$= \frac{1}{2}\sum_{\text{自旋}} \mathrm{Tr}[O_\lambda u(p)\overline{u}(p)\gamma^0 O_\rho^+ \gamma^0 u(k)\overline{u}(k)]$$

$$\times \mathrm{Tr}[O^\lambda v(k')\overline{v}(k')\gamma^0 O^{\rho+}\gamma^0 u(q)\overline{u}(q)]$$

$$Y=\frac{1}{2}\mathrm{Tr}[O_\lambda\frac{\not p+m_\mu}{2m_\mu}\gamma^0 O_\rho^+\gamma^0\not k]\mathrm{Tr}[O^\lambda\not k'\gamma^0 O^{\rho+}\gamma^0\frac{\not q+m_e}{2m_e}],$$

$$\because\ \gamma^0 O_\rho^+\gamma^0=O_\rho=\gamma_\rho(1-\gamma_5),$$

$$\therefore Y=\frac{1}{2}\mathrm{Tr}[\gamma_\lambda(1-\gamma_5)\frac{\not p+m_\mu}{2m_\mu}\gamma_\rho(1-\gamma_5)\not k]$$

$$\times\mathrm{Tr}[\gamma^\lambda(1-\gamma_5)\not k'\gamma^\rho(1-\gamma_5)\frac{\not q+m_e}{2m_e}],$$

$$=\frac{1}{8m_\mu m_e}\mathrm{Tr}[\gamma_\lambda(1-\gamma_5)\not p\gamma_\rho(1-\gamma_5)\not k]\mathrm{Tr}[\gamma^\lambda(1-\gamma_5)\not k'\gamma^\rho(1-\gamma_5)\not q],$$

利用$(1\pm\gamma_5)^2=2(1\pm\gamma_5)$，上式成为

$$Y=\frac{1}{2m_\mu m_e}\mathrm{Tr}[(1+\gamma_5)\gamma_\lambda\not p\gamma_\rho\not k]\mathrm{Tr}[(1+\gamma_5)\gamma^\lambda\not k'\gamma^\rho\not q],$$

$$\mathrm{Tr}[(1+\gamma_5)\gamma_\lambda \not{p}\gamma_\rho \not{k}] = \mathrm{Tr}[(1+\gamma_5)\gamma_\lambda\gamma_\mu\gamma_\rho\gamma_\nu]p^\mu k^\nu$$

$$= [g_{\lambda\mu}g_{\rho\nu} - g_{\lambda\rho}g_{\mu\nu} + g_{\lambda\nu}g_{\mu\rho} - \mathrm{i}\,\varepsilon_{\lambda\mu\rho\nu}]p^\mu k^\nu,$$

$$\mathrm{Tr}[(1+\gamma_5)\gamma^\lambda \not{k}'\gamma^\rho \not{q}] = \mathrm{Tr}[(1+\gamma_5)\gamma^\lambda\gamma^\sigma\gamma^\rho\gamma^\tau]k'_\sigma q_\tau$$

$$= [g^{\lambda\sigma}g^{\rho\tau} - g^{\lambda\rho}g^{\sigma\tau} + g^{\lambda\tau}g^{\sigma\rho} - \mathrm{i}\,\varepsilon^{\lambda\sigma\rho\tau}]k'_\sigma q_\tau,$$

$$Y = \frac{1}{2m_\mu m_e}[g_{\lambda\mu}g_{\rho\nu} - g_{\lambda\rho}g_{\mu\nu} + g_{\lambda\nu}g_{\mu\rho}]p^\mu k^\nu$$

$$\times[g^{\lambda\sigma}g^{\rho\tau} - g^{\lambda\rho}g^{\sigma\tau} + g^{\lambda\tau}g^{\sigma\rho}]k'_\sigma q_\tau$$

$$= \frac{32}{m_\mu m_e}(p\cdot k')(k\cdot q),$$

$$\tau^{-1} = \frac{1}{(2\pi)^5}\frac{G^2}{2}\frac{m_\mu m_e}{4p^0}\int \frac{\mathrm{d}^3 q\, \mathrm{d}^3 k\, \mathrm{d}^3 k'}{q^0 k^0 k'^0}\delta^4(q+k+k'-p)Y$$

$$= \frac{4G^2}{(2\pi)^5 p^0}\int \frac{\mathrm{d}^3 q\, \mathrm{d}^3 k\, \mathrm{d}^3 k'}{q^0 k^0 k'^0}\delta^4(q+k+k'-p)(p\cdot k')(k\cdot q).$$