3.4 Maxwell场的正则量子化

- 正则量子化的困难

$$\mathcal{L} = -\frac{1}{4}F^2, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

A_ρ的共轭动量为

$$\pi^\rho = \frac{\partial \mathcal{L}}{\partial \dot{A}_\rho} = F^{\rho 0},$$

$$\Rightarrow \qquad \pi^0 = 0.$$

- 克服困难的办法：1) 采用辐射规范；

2) Gupta - Bleuter的不定度规量子化

3.4.1 不定度规

- 运动方程和对易关系

修改电磁场的Lagrangian为

$$\mathcal{L} = -\frac{1}{4}F^2 - \frac{\lambda}{2}(\partial \cdot A)^2,$$

取$\lambda = 1$（Feynman规范），则

$$\mathcal{L} = -\frac{1}{4}F^2 - \frac{1}{2}(\partial \cdot A)^2, \tag{3.133}$$

A_ρ的共轭动量为

$$\pi^\rho = \frac{\partial \mathcal{L}}{\partial \dot{A}_\rho} = F^{\rho 0} - g^{\rho 0}(\partial \cdot A), \tag{3.134}$$

$$\pi^0 = -(\partial \cdot A) \neq 0.$$

Maxwell方程变为：

$$\Box A_\mu = 0, \tag{3.135}$$

$$\Rightarrow \qquad \Box \partial \cdot A = 0. \tag{3.136}$$

为量子化，假定等时正则对易关系：

$$[\pi^\nu(\vec{x}, t), A_\rho(\vec{y}, t)] = -i g^\nu_\rho \delta^3(\vec{x} - \vec{y}), \tag{3.137}$$

$$[A_\rho(\vec{x}, t), A_\nu(\vec{y}, t)] = [\pi_\rho(\vec{x}, t), \pi_\nu(\vec{y}, t)] = 0, \tag{3.138}$$

(3.137)中取$\rho = \nu = 0$，则有

$$[\pi^0(\vec{x}, t), A_0(\vec{y}, t)] = -i\delta^3(\vec{x} - \vec{y}) \neq 0.$$

对易关系(3.137)和(3.138)可化成：

$$[A_\rho(\vec{x},t),A_\nu(\vec{y},t)]=[\dot{A}_\rho(\vec{x},t),\dot{A}_\nu(\vec{y},t)]=0,$$

$$[\dot{A}_\rho(\vec{x},t),A_\nu(\vec{y},t)]=ig_{\rho\nu}\delta^3(\vec{x}-\vec{y}). \tag{3.139}$$

注意： 标量场 $[\dot{\varphi}(\vec{x},t),\varphi(\vec{y},t)]=-i\delta^3(\vec{x}-\vec{y});$

矢量场 $[\dot{A}_i(\vec{x},t),A_j(\vec{y},t)]=-i\delta^3(\vec{x}-\vec{y}),$

$$[\dot{A}_0(\vec{x},t),A_0(\vec{y},t)]=i\delta^3(\vec{x}-\vec{y}).$$

场的平面波展开：

$$A_\mu(x)=\int \tilde{d}k\sum_{\lambda=0}^{3}[a^{(\lambda)}(k)\varepsilon_\mu^{(\lambda)}(k)e^{-ikx}+a^{(\lambda)^+}(k)\varepsilon_\mu^{(\lambda)^*}(k)e^{ikx}],$$

k满足

$$k^2={k^0}^2-\vec{k}^2=0,\qquad k^0=\left|\vec{k}\right|>0, \tag{3.141}$$

即k在正光锥上。光锥条件$k^2=0$表明场量子无质量。

$$\tilde{d}k=\frac{d^3k}{(2\pi)^3 2k^0},$$

对正光锥上的每一个k, $\varepsilon_\mu^{(\lambda)}$是4个线性无关的4矢量，称为极化矢量.

$\varepsilon^{(1)},\varepsilon^{(2)}$—横向极化矢量；$\varepsilon^{(3)}$—纵向极化矢量；

$\varepsilon^{(0)}$—标量极化矢量。

当$\vec{k}$平行于$\varepsilon^{(3)}(k)$时，极化矢量可写为

$$\varepsilon^{(0)}=n=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \varepsilon^{(1)}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \varepsilon^{(2)}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \varepsilon^{(3)}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \quad (3.143)$$

一般情况下，$\varepsilon^{(\lambda)}(k)$可以是复的，并总有

$$\sum_{\lambda}\frac{\varepsilon_{\mu}^{(\lambda)}(k)\varepsilon_{\nu}^{(\lambda)^*}(k)}{\varepsilon^{(\lambda)}(k)\varepsilon^{(\lambda)^*}(k)}=g_{\mu\nu},\qquad \varepsilon^{(\lambda)}(k)\varepsilon^{(\lambda)^*}(k)=g^{\lambda\lambda'},\quad (3.144)$$

$$\sum_{\lambda,\lambda'}g^{\lambda\lambda'}\varepsilon_{\mu}^{(\lambda)}(k)\varepsilon_{\nu}^{(\lambda')^*}(k)=g_{\mu\nu}.$$

可证明，只要

$$[a^{(\lambda)}(k), a^{(\lambda')+}(k')] = -g^{\lambda\lambda'} 2k^0 (2\pi)^3 \delta^3(\vec{k} - \vec{k}'),$$

$$[a^{(\lambda)}(k), a^{(\lambda')}(k')] = [a^{(\lambda)+}(k), a^{(\lambda')+}(k')] = 0, \tag{3.145}$$

则可得出等时正则对易关系(3.139)，$\therefore$ (3.145) $\leftrightarrow$ (3.139)。

利用(3.145)还可求出非等时对易关系：

$$[A_\rho(x), A_\nu(x)] = -g_{\rho\nu} \int \widetilde{d}k [e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}]$$

$$= -ig_{\rho\nu} \Delta(x - y)\Big|_{m=0}. \tag{3.146}$$

- 构造Fock空间

定义真空态$|0\rangle$

$$a^{(\lambda)}(k)|0\rangle = 0, \quad \text{(对所有的}\lambda\text{，}k\text{)} \tag{3.147}$$

存在的问题：①有非物理自由度；

②可能有负模方态。

考虑单粒子标量极化态

$$|1\rangle = \int \tilde{d}k f(k) a^{(0)+}(k)|0\rangle,$$

则其模方

$$\langle 1|1\rangle = \iint \tilde{d}k \tilde{d}k' f^*(k) f(k') \langle 0|a^{(0)}(k) a^{(0)+}(k')|0\rangle$$

$$=\iint \tilde{d}k\tilde{d}k' f^*(k)\, f(k')\langle 0|[a^{(0)}(k), a^{(0)+}(k')]|0\rangle$$

$$=\iint \tilde{d}k\tilde{d}k' f^*(k)\, f(k')\langle 0|[-g^{00}2k'^0(2\pi)^3\delta^3(\vec{k}-\vec{k}')]|0\rangle$$

$$=-\langle 0|0\rangle\int |f(k)|^2\,\tilde{d}k<0.$$ (负模方态)

横向与纵向极化单粒子态为正模方态。

若引入厄密度规算符η，满足$\eta^2=1$，则此Fock空间中态矢的内积为：

$$\langle a|\eta|b\rangle=\begin{cases}-\delta_{ab} & a\text{为标量极化单粒子态}\\ +\delta_{ab} & a\text{为横、纵极化单粒子态}\end{cases}$$ （不定度规）

回到Maxwell理论：

设法选择物理态$|\psi\rangle$，满足

$$\langle\psi|\partial\cdot A|\psi\rangle=0, \tag{3.148}$$

$|\psi\rangle$—物理Hilbert空间$\mathcal{H}_1$。为保持$\mathcal{H}_1$为一线性空间，要求

$$\partial^{\mu}A_{\mu}^{(+)}|\psi\rangle=0, \tag{3.149}$$

即$\partial\cdot A$的正频(消灭算符)部分消灭$\mathcal{H}_1$。

分析$\mathcal{H}_1$中的态：

考虑基矢态$|\psi\rangle \sim a^+ a^+ \cdots a^+ |0\rangle$,

a^+按极化分类，可将$|\psi\rangle$因子化为

$$|\psi\rangle = |\psi_T\rangle |\phi\rangle, \tag{3.150}$$

其中$|\psi_T\rangle$—横光子态，$|\phi\rangle$—纵、标量光子态。

$\because \quad \varepsilon^{(\lambda)} \cdot k = 0, \quad$ 对$\lambda = 1,2$

$$\therefore \quad i\partial \cdot A^{(+)} = \int \tilde{d}k e^{-ikx} \sum_{\lambda=0,1,2,3} a^{(\lambda)}(k) \varepsilon_\mu^{(\lambda)}(k) k^\mu$$

$$= \int \tilde{d}k e^{-ikx} \sum_{\lambda=0,3} a^{(\lambda)}(k) \varepsilon^{(\lambda)}(k) \cdot k, \tag{3.151}$$

于是，条件(3.149)化为

$$\sum_{\lambda=0,3} k\cdot\varepsilon^{(\lambda)}(k)a^{(\lambda)}(k)\left|\phi\right\rangle = 0. \tag{3.152}$$

下面证实，由于限制条件(3.152)，在$\mathcal{H}_1$子空间中的态矢量将有正的模方：

若取$\vec{k}$平行于第3轴，则

$$k^{\mu} = (k^0,0,0,k^0), \quad \varepsilon^{(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad \varepsilon^{(3)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

$$\sum_{\lambda=0,3} k\cdot\varepsilon^{(\lambda)}(k)a^{(\lambda)}(k)$$

$$=k\cdot\varepsilon^{(0)}(k)a^{(0)}(k)+k\cdot\varepsilon^{(3)}(k)a^{(3)}(k)$$

$$=k^0a^{(0)}(k)-k^0a^{(3)}(k)=k^0[a^{(0)}(k)-a^{(3)}(k)],$$

因而条件(3.152)等价于

$$[a^{(0)}(k)-a^{(3)}(k)]|\phi\rangle=0, \tag{3.153}$$

设$|\phi_n\rangle$是n个标量或纵光子态，则可以写

$$|\phi\rangle=c_0|\phi_0\rangle+c_1|\phi_1\rangle+\cdots+c_n|\phi_n\rangle, \tag{3.154}$$

其中$|\phi\rangle\equiv|0\rangle$是真空态。

纵光子与标量光子的粒子数算符为：

$$N' = \int \tilde{d}k [a^{(3)^+}(k) a^{(3)}(k) - a^{(0)^+}(k) a^{(0)}(k)]. \qquad (3.155)$$

$|\phi_n\rangle$必须满足(3.153)，即

$$[a^{(0)}(k) - a^{(3)}(k)]|\phi_n\rangle = 0,$$

$$\langle\phi_n|[a^{(3)^+}(k) - a^{(0)^+}(k)] = 0,$$

由此得

$$\langle\phi_n|N'|\phi_n\rangle = n\langle\phi_n|\phi_n\rangle$$

$$= \langle\phi_n|\int \tilde{d}k [a^{(3)^+}(k) a^{(3)}(k) - a^{(0)^+}(k) a^{(0)}(k)]|\phi_n\rangle = 0,$$

$$\therefore \quad n\langle\phi_n|\phi_n\rangle = 0,$$

$\Rightarrow \quad \langle \phi_n | \phi_n \rangle = \delta_{n0}.$ (当$n \neq 0$时，$|\phi_n\rangle$有0模)

于是对$\mathcal{H}_1$中的任意态$|\phi\rangle$，有

$$\langle \phi | \phi \rangle = |c_0|^2 \geq 0,$$

对$\mathcal{H}_1$中的任意态矢量$|\psi\rangle = |\psi_T\rangle |\phi\rangle$，也有模方

$$\langle \psi | \psi \rangle \geq 0.$$

还可证明$|\phi\rangle$的任意性不影响物理观测量。

结论：如果只限于物理的Hilbert空间$\mathcal{H}_1$，则不仅负几率消失而且非物理的纵光子和标量光子对平均值（对物理观测）没有贡献，物理上将只表现横光子的作用。

3.4.2 传播子

编时乘积：

$$TA_\mu(x)A_\nu(y)=\theta(x^0-y^0)A_\mu(x)A_\nu(y)$$
$$+\theta(y^0-x^0)A_\nu(y)A_\mu(x)$$

光子传播子为：

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle$$
$$=\langle 0|\theta(x^0-y^0)A_\mu(x)A_\nu(y)|0\rangle$$
$$+\langle 0|\theta(y^0-x^0)A_\nu(y)A_\mu(x)|0\rangle$$

利用A_μ的平面波展开式(3.140)，以及$a^{(\lambda)}(k)$、$a^{(\lambda')+}(k')$间的对易关系(3.145)，可得到

$$\langle 0|TA_\mu(x)A_\nu(y)|0\rangle = ig_{\mu\nu}G_F(x-y)\Big|_{m=0}$$

$$= -ig_{\mu\nu}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2+i\varepsilon}.$$

3.5 Dirac 场的正则量子化

3.5.1 反对易子

$$\mathcal{L} = i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi,$$

ψ的共轭动量为

$$\pi_{\psi} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^{+}. \qquad (3.165)$$

不同的量子化程序

标量场、矢量场：引入场算符的正则对易关系

$\rightarrow$ 证明它们与Poincare不变性自恰

旋量场：由Poincare不变性决定场算符的对易关系。

一、场算符的平面波展开

ψ满足Dirac方程

$$(i\partial\!\!\!/ - m)\psi = 0, \tag{2.174}$$

其平面波解的一般形式可写为

$$\psi^{(+)}(x) = e^{-ikx}u(k), \quad （正能）$$
$$\psi^{(-)}(x) = e^{ikx}v(k). \quad （负能） \tag{2.185}$$

将场算符$\psi(x),\overline{\psi}(x)$按Dirac方程的平面波解展开为

$$\psi(x) = \int \widetilde{d}k \sum_{\alpha=1,2} [b_\alpha(k)u^{(\alpha)}(k)e^{-ikx} + d_\alpha^+(k)v^{(\alpha)}(k)e^{ikx}],$$
$$\overline{\psi}(x) = \int \widetilde{d}k \sum_{\alpha=1,2} [b_\alpha^+(k)\overline{u}^{(\alpha)}(k)e^{ikx} + d_\alpha(k)\overline{v}^{(\alpha)}(k)e^{-ikx}], \tag{3.166}$$

其中费米子场的积分测度

$$\tilde{d}k = \frac{d^3k}{(2\pi)^3}\frac{m}{k^0}.$$（对$m \neq 0$的费米子）

二、反对易关系

由Poincare不变性要求，算符b, d, b^+, d^+所满足的反对易关系为：

$$\{b_\alpha(q), b_\beta^+(q)\} = \{d_\alpha(q), d_\beta^+(q)\} = (2\pi)^3\frac{k^0}{m}\delta^3(\vec{k}-\vec{q})\delta_{\alpha\beta}, \quad (3.172)$$

所有其它对易子为0.

利用(3.172)，可证明ψ, ψ^+满足等时反对易关系：

$$\begin{aligned} &\{\psi_\xi(\vec{x},t), \psi_\eta^+(\vec{y},t)\} = \delta^3(\vec{k}-\vec{q})\delta_{\xi\eta}, \\ &\{\psi_\xi(\vec{x},t), \psi_\eta(\vec{y},t)\} = \{\psi_\xi^+(\vec{x},t), \psi_\eta^+(\vec{y},t)\} = 0. \end{aligned} \quad (3.173)$$

三、场的物理量

LT不变性：

利用场算符的反对易子(3.173)，还可证明广义角动量算符与场算符$\psi(x)$满足对易关系

$$[M^{\mu\nu},\psi(x)]=-[i(x^{\mu}\partial^{\nu}-x^{\nu}\partial^{\mu})+\frac{1}{2}\sigma^{\mu\nu}]\psi(x), \qquad (3.174)$$

∴理论将保持LT不变。

时空平移不变性：

$$P_{\mu}=\int d^{3}x i\overline{\psi}\gamma^{0}\partial_{\mu}\psi$$

$$=\int\tilde{d}k\sum_{\alpha}k_{\mu}[b_{\alpha}^{+}(k)b_{\alpha}(k)-d_{\alpha}(k)d_{\alpha}^{+}(k)].$$

$U(1)$整体相位变换不变性

$$J^{\mu}=\bar{\psi}\gamma^{\mu}\psi,$$

$$Q=\int d^{3}xJ^{0}(x)=\int d^{3}x\psi^{+}(x)\psi(x)$$

$$=\int \tilde{d}k\sum_{\alpha}[b_{\alpha}^{+}(k)b_{\alpha}(k)+d_{\alpha}(k)d_{\alpha}^{+}(k)],$$

为减除真空的无穷大贡献，必须把算符排列成

正规乘积。

费米子场的正规乘积

$$\begin{aligned}
&:d_{\alpha}d_{\beta}^{+}:=-d_{\beta}^{+}d_{\alpha},\\
&:d_{\alpha}^{+}d_{\beta}:=d_{\alpha}^{+}d_{\beta},\\
&:d_{\alpha}^{+}d_{\beta}^{+}:=d_{\alpha}^{+}d_{\beta}^{+}=-d_{\beta}^{+}d_{\alpha}^{+},\\
&:d_{\alpha}d_{\beta}:=d_{\alpha}d_{\beta}=-d_{\beta}d_{\alpha}.
\end{aligned}\tag{3.175}$$

$$P^{\mu}=\int\frac{d^3k}{(2\pi)^3}\frac{m}{k^0}k^{\mu}\sum_{\alpha=1,2}:b_{\alpha}^{+}(k)b_{\alpha}(k)-d_{\alpha}(k)d_{\alpha}^{+}(k):$$

$$=\int\frac{d^3k}{(2\pi)^3}\frac{m}{k^0}k^{\mu}\sum_{\alpha=1,2}[b_{\alpha}^{+}(k)b_{\alpha}(k)+d_{\alpha}^{+}(k)d_{\alpha}(k)],\qquad(3.176)$$

或 $$\mathcal{L}=:i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi-m\overline{\psi}\psi:\qquad(3.177)$$

$$Q=\int\tilde{d}k\sum_{\alpha}:[b_{\alpha}^{+}(k)b_{\alpha}(k)+d_{\alpha}(k)d_{\alpha}^{+}(k)]:$$

$$=\int\tilde{d}k\sum_{\alpha}[b_{\alpha}^{+}(k)b_{\alpha}(k)-d_{\alpha}^{+}(k)d_{\alpha}(k)].$$

引进粒子数算符

$$N_k^{(+)}=b_{\alpha}^{+}(k)b_{\alpha}(k),\qquad N_k^{(-)}=d_{\alpha}^{+}(k)d_{\alpha}(k),$$

$$N_k^{(+)}\left|0\right\rangle=0,\qquad N_k^{(-)}\left|0\right\rangle=0.$$

则场的物理量可写为

$$P^{\mu}=\int \tilde{d}k k^{\mu}\sum_{\alpha=1,2}[N_k^{(+)}+N_k^{(-)}],$$

$$Q=\int \tilde{d}k\sum_{\alpha=1,2}[N_k^{(+)}-N_k^{(-)}].$$

b型粒子—费米子； d型粒子—反费米子

b型粒子和d型粒子的静质量为m，自旋为$1/2$，但具有相反的Q荷。

b^{+},b—费米子的产生、消灭算符；

d^{+},d—反费米子的产生、消灭算符；

$N^{(+)}$—费米子的粒子数算符；

$N^{(-)}$—反费米子的粒子数算符

3.5.2 Fermions 的 Fock 空间

- **考虑k给定的单粒子态**

对于一个给定的4动量k，有4个简并态$|1\rangle_a \ (a=1,2,3,4)$:

$$|1\rangle_1 = b_1^+(k)|0\rangle, \qquad |1\rangle_2 = b_2^+(k)|0\rangle,$$

$$|1\rangle_3 = d_1^+(k)|0\rangle, \qquad |1\rangle_4 = d_2^+(k)|0\rangle,$$

满足

$$P_\mu|1\rangle_a = k_\mu|1\rangle_a. \tag{3.178}$$

区分这4个态的力学量：Q, 自旋

① 守恒荷

$$Q=\int \tilde{d}k\sum_{\alpha}[b_{\alpha}^{+}(k)b_{\alpha}(k)-d_{\alpha}^{+}(k)d_{\alpha}(k)], \qquad (3.179)$$

$$[Q,b_{\alpha}^{+}(k)]=b_{\alpha}^{+}(k), \qquad [Q,d_{\alpha}^{+}(k)]=-d_{\alpha}^{+}(k).$$

由于$Q|0\rangle=0$，所以

$$\begin{aligned} Qb_{\alpha}^{+}(k)|0\rangle &= [Q,b_{\alpha}^{+}(k)]|0\rangle = b_{\alpha}^{+}(k)|0\rangle, \\ Qd_{\alpha}^{+}(k)|0\rangle &= [Q,d_{\alpha}^{+}(k)]|0\rangle = -d_{\alpha}^{+}(k)|0\rangle, \end{aligned} \qquad (3.181)$$

即

$$Q|1\rangle_a=\begin{cases} +|1\rangle_a & a=1,2 \quad b\text{型粒子，}Q\text{荷}+1; \\ -|1\rangle_a & a=3,4 \quad d\text{型粒子，}Q\text{荷}-1; \end{cases} \qquad (3.182)$$

可证明

$$[Q, P_\mu] = 0, \tag{3.183}$$

$$[Q, \psi(x)] = -\psi(x), \qquad [Q, \overline{\psi}(x)] = \overline{\psi}(x), \tag{3.184}$$

② 自旋投影算符 $(\vec{S}\cdot\hat{k})$

$$\vec{S}\cdot\hat{k}|1\rangle_a = \begin{cases} +\dfrac{1}{2}|1\rangle_a, & a=1 \\ -\dfrac{1}{2}|1\rangle_a, & a=2 \end{cases}$$

$$\vec{S}\cdot\hat{k}|1\rangle_a = \begin{cases} -\dfrac{1}{2}|1\rangle_a, & a=3 \\ +\dfrac{1}{2}|1\rangle_a, & a=4 \end{cases}$$

单粒子态的完全标记

态	$\vert 1\rangle_1$	$\vert 1\rangle_2$	$\vert 1\rangle_3$	$\vert 1\rangle_4$
	$b_1^+(k)\vert 0\rangle$	$b_2^+(k)\vert 0\rangle$	$d_1^+(k)\vert 0\rangle$	$d_2^+(k)\vert 0\rangle$
动量	k	k	k	k
粒子数	1	1	-1	-1
极化	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$

$$\psi(x)=\int\tilde{d}k\sum_{\alpha=1,2}[b_\alpha(k)u^{(\alpha)}(k)e^{-ikx}+d_\alpha^+(k)v^{(\alpha)}(k)e^{ikx}].$$

▪ 多粒子态

Fock空间的基矢一般表为

$$\sim \int f(1,\cdots,n) a^{+}(1)\cdots a^{+}(n)\left|0\right\rangle,$$

$\because a^{+}$彼此反对易，$\therefore f(1,\cdots,n)$对所有宗量全反对称

若有两个粒子处于同一个量子数的态，

则$f \equiv 0$（Pauli不相容原理）。

3.5.3 传播子

两个Dirac场的编时乘积定义为：

$$
\begin{aligned}
& T\psi_{\xi}(x)\overline{\psi}_{\xi'}(y) \\
&= \theta(x^0 - y^0)\psi_{\xi}(x)\overline{\psi}_{\xi'}(y) - \theta(y^0 - x^0)\overline{\psi}_{\xi'}(y)\psi_{\xi}(x) \\
&= -T\overline{\psi}_{\xi'}(y)\psi_{\xi}(x),
\end{aligned}
\tag{3.201}
$$

Dirac场的传播子为

$$
\langle 0|T\psi_{\xi}(x)\overline{\psi}_{\xi'}(y)|0\rangle = iS(x-y)_{\xi\xi'}, \tag{3.202}
$$

$$
S(x-y) = -(i\partial\!\!\!/_x + m)G_F(x-y)
$$

$$
= \int \frac{d^4k}{(2\pi)^4}\frac{k\!\!\!/ + m}{k^2 - m^2 + i\varepsilon}e^{-ik\cdot(x-y)}.
$$

3.6　分立对称性

$$\begin{cases}\text{空间反射或宇称变换} & \mathcal{P} \\ \text{电荷共轭变换} & \mathcal{C} \\ \text{时间反演} & \mathcal{T}\end{cases}$$

3.6.1 宇称变换($\mathcal{P}$)

- K-G 场的宇称变换

$$\varphi(x) \xrightarrow{\mathcal{P}} \begin{cases} +\varphi(\tilde{x}) \to \text{标量} \\ -\varphi(\tilde{x}) \to \text{赝标量} \end{cases}, \quad (\tilde{x}^{\mu} = x_{\mu}) \qquad (3.204)$$

±号对应φ的内禀宇称（intrinsicparity）.

在此变换下，Lagrangian

$$\mathcal{L}(x) = \partial_{\mu}\varphi^{+}(x)\partial^{\mu}\varphi(x) - m^{2}\varphi^{+}(x)\varphi(x) \xrightarrow{\mathcal{P}} \mathcal{L}(\tilde{x}),$$

因而，作用量

$$I = \int d^{4}x\mathcal{L}(x) \xrightarrow{\mathcal{P}} \int d^{4}x\mathcal{L}(\tilde{x}) = \int d^{4}x\mathcal{L}(x) = I,$$

等时对易关系：

$$[\varphi(x),\pi(x')]_{t=t'}=i\delta^3(\vec{x}-\vec{x}'),$$

$$\downarrow \mathcal{P}$$

$$[\varphi'(x),\pi'(x')]_{t=t'}=[\pm\varphi(\tilde{x}),\pm\pi(\tilde{x}')]_{t=t'}$$

$$=[\varphi(-\vec{x},t),\pi(-\vec{x}',t')]_{t=t'}$$

$$=i\delta^3(\vec{x}-\vec{x}').$$

寻求Hilbert空间中的幺正算符$\mathcal{P}_s$和$\mathcal{P}_{ps}$，使得

$$\mathcal{P}_s\varphi(x)\mathcal{P}_s^{-1}=+\varphi(\tilde{x}),\qquad (\tilde{x}^\mu=x_\mu) \tag{3.205}$$

$$\mathcal{P}_{ps}\varphi(x)\mathcal{P}_{ps}^{-1}=-\varphi(\tilde{x}). \tag{3.206}$$

① 标量场的宇称算符

利用$\varphi(x)$的展开式

$$\varphi(x)=\int \tilde{d}k[a(k)e^{-ikx}+b^{+}(k)e^{ikx}], \tag{3.113}$$

(3.205)的左边和右边分别为

$$左边=\int \tilde{d}k[\mathcal{P}_s a(k)\mathcal{P}_s^{-1}\mathcal{P}_s e^{-ikx}\mathcal{P}_s^{-1}+\mathcal{P}_s b^{+}(k)\mathcal{P}_s^{-1}\mathcal{P}_s e^{ikx}\mathcal{P}_s^{-1}]$$

$$=\int \tilde{d}k[\mathcal{P}_s a(k)\mathcal{P}_s^{-1}e^{-ikx}+\mathcal{P}_s b^{+}(k)\mathcal{P}_s^{-1}e^{ikx}], \tag{1}$$

$$右边=\int \tilde{d}k[a(k)e^{-ik\tilde{x}}+b^{+}(k)e^{ik\tilde{x}}]$$

$$=\int \tilde{d}k[a(k)e^{-i\tilde{k}x}+b^{+}(k)e^{i\tilde{k}x}], \quad (\tilde{k}^{\mu}=k_{\mu})$$

$$=\int \tilde{d}k[a(\tilde{k})e^{-ikx}+b^{+}(\tilde{k})e^{ikx}], \tag{2}$$

比较(1)和(2)，得到

$$\mathcal{P}_s a(k)\mathcal{P}_s^{-1} = a(\tilde{k}), \qquad \mathcal{P}_s a^+(k)\mathcal{P}_s^{-1} = a^+(\tilde{k}), \qquad (3.207a)$$

$$\mathcal{P}_s b(k)\mathcal{P}_s^{-1} = b(\tilde{k}), \qquad \mathcal{P}_s b^+(k)\mathcal{P}_s^{-1} = b^+(\tilde{k}). \qquad (3.207b)$$

令

$$\mathcal{P}_s = e^{iP_s}, \qquad (P_s = P_s^+) \qquad (3.208)$$

上式代入(3.207a)，并利用公式

$$e^A B e^{-A} = \sum_{j=0}^{\infty}\frac{1}{j!}[A^{(j)}, B],$$

$$[A^{(j)}, B] = [A,[A^{(j-1)}, B]], \qquad [A^{(0)}, B] = B,$$

得

$$\mathcal{P}_s a(k) \mathcal{P}_s^{-1} = e^{iP_s} a(k) e^{-iP_s}$$

$$= a(k) + i[P_s, a(k)] + \frac{i^2}{2}[P_s^{(2)}, a(k)] + \cdots + \frac{i^n}{n!}[P_s^{(n)}, a(k)] + \cdots$$

$$= a(\tilde{k}), \tag{3.209}$$

为满足上式，选择

$$[P_s, a(k)] = \frac{\lambda}{2}[a(k) + \delta_P a(\tilde{k})], \tag{3.210}$$

其中$\delta_P = +$或-1，λ待定。由此

$$[P_s, a(\tilde{k})] = \frac{\lambda}{2}[a(\tilde{k}) + \delta_P a(k)],$$

$$[P_s^{(2)}, a(k)] = [P_s, [P_s, a(k)]] = \frac{\lambda^2}{2}[a(k) + \delta_P a(\tilde{k})] = \lambda[P_s, a(k)],$$

$$[P_s^{(3)}, a(k)] = \frac{\lambda^3}{2}[a(k) + \delta_P a(\tilde{k})] = \lambda^2 [P_s, a(k)],$$

$$\cdots\cdots\cdots$$

$$[P_s^{(n)}, a(k)] = \frac{\lambda^n}{2}[a(k) + \delta_P a(\tilde{k})] = \lambda^{n-1} [P_s, a(k)],$$

$$\cdots\cdots\cdots$$

$$\therefore \quad \mathcal{P}_s a(k) \mathcal{P}_s^{-1} = a(k) + \{i + \frac{i^2}{2}\lambda + \frac{i^3}{3!}\lambda^2 + \cdots + \frac{i^n}{n!}\lambda^{n-1} + \cdots\}[P_s, a(k)]$$

$$= a(k) + \frac{1}{\lambda}(e^{i\lambda} - 1) \cdot \frac{\lambda}{2}[a(k) + \delta_P a(\tilde{k})]$$

$$= \frac{1}{2}(e^{i\lambda} + 1) a(k) + \frac{1}{2}\delta_P (e^{i\lambda} - 1) a(\tilde{k})$$

$$= a(\tilde{k}),$$

$$\Rightarrow \begin{cases} \frac{1}{2}(e^{i\lambda}+1)=0, \\ \\ \frac{1}{2}\delta_P(e^{i\lambda}-1)=1, \end{cases} \Rightarrow \quad \lambda=\pm\pi,\ \ \delta_P=-1,$$

$$\therefore \quad [P_s, a(k)] = \frac{\pi}{2}[a(k)-a(\tilde{k})], \tag{3.213a}$$

类似地，将(3.208)代入(3.207b)，可得

$$\therefore \quad [P_s, b(k)] = \frac{\pi}{2}[b(k)-b(\tilde{k})]. \tag{3.213b}$$

满足(3.213a)和(3.213b)的P_s可写为

$$P_s = -\frac{\pi}{2}\int \tilde{d}k\{a^+(k)[a(k)-a(\tilde{k})]+b^+(k)[b(k)-b(\tilde{k})]\},$$

标量场的宇称算符为

$$\mathcal{P}_s = \exp\{-\frac{i\pi}{2}\int \tilde{d}k[a^+(k)a(k) - a^+(k)a(\tilde{k})$$

$$+ b^+(k)b(k) - b^+(k)b(\tilde{k})]\}. \qquad (3.215)$$

② 赝标场的宇称算符

$$\mathcal{P}_{ps}\varphi(x)\mathcal{P}_{ps}^{-1} = -\varphi(\tilde{x}). \qquad (3.206)$$

利用$\varphi(x)$的展开式，得

$$\mathcal{P}_{ps}a(k)\mathcal{P}_{ps}^{-1} = -a(\tilde{k}), \qquad (3.216a)$$

$$\mathcal{P}_{ps}b(k)\mathcal{P}_{ps}^{-1} = -b(\tilde{k}). \qquad (3.216b)$$

写 $\mathcal{P}_{ps} = e^{iP_{ps}}$， $(P_{ps} = P_{ps}^+)$ (3.217)

有

$$[P_{ps}, a(k)] = \frac{\pi}{2}[a(k) + a(\tilde{k})], \tag{3.218a}$$

$$[P_{ps}, b(k)] = \frac{\pi}{2}[b(k) + b(\tilde{k})], \tag{3.218b}$$

于是，赝标场的宇称算符为

$$\mathcal{P}_{ps} = \exp\{\frac{-i\pi}{2}\int \tilde{d}k[a^{+}(k)a(k) + a^{+}(k)a(\tilde{k})$$

$$+ b^{+}(k)b(k) + b^{+}(k)b(\tilde{k})]\}. \tag{3.219}$$

$\mathcal{P}_s$和$\mathcal{P}_{ps}$都是幺正的，且满足：

$$\mathcal{P}_s\left|0\right\rangle = \mathcal{P}_{ps}\left|0\right\rangle = \left|0\right\rangle. \tag{3.220}$$

可证实，4动量算符

$$P^{\mu} = \int \tilde{d}kk^{\mu}[a^{+}(k)a(k) + b^{+}(k)b(k)]$$

在$\mathcal{P}$变换下，为

$$\mathcal{P}_s P^{\mu} \mathcal{P}_s^{-1} = \mathcal{P}_{ps} P^{\mu} \mathcal{P}_{ps}^{-1} = P_{\mu}, \tag{3.221}$$

上式的0分量为

$$\mathcal{P}_s H \mathcal{P}_s^{-1} = H, \Rightarrow [\mathcal{P}_s, H] = 0.$$

▪ Dirac 场的宇称变换

$$\psi(x) \to \gamma^0 \psi(\tilde{x}), \qquad \overline{\psi}(x) \to \psi^+(\tilde{x}), \tag{3.222}$$

在此变换下，Lagrangian

$$\mathcal{L}(x) =: i\overline{\psi}(x)\gamma^\mu \partial_\mu \psi(x) - m\overline{\psi}(x)\psi(x):$$

$$\downarrow \mathcal{P}$$

$$=: i\psi^+(\tilde{x})\gamma^\mu \partial^\mu \gamma^0 \psi(\tilde{x}) - m\psi^+(\tilde{x})\gamma^0\psi(\tilde{x}):$$

$$=: i\overline{\psi}(\tilde{x})\gamma^0\gamma^\mu \partial^\mu \gamma^0 \psi(\tilde{x}) - m\overline{\psi}(\tilde{x})\psi(\tilde{x}):$$

$$=: i\overline{\psi}(\tilde{x})\gamma_\mu \partial^\mu \psi(\tilde{x}) - m\overline{\psi}(\tilde{x})\psi(\tilde{x}):$$

$$= \mathcal{L}(\tilde{x}),$$

因而，作用量

$$I=\int d^4x\mathcal{L}(x)\xrightarrow{\mathcal{P}}\int d^4x\mathcal{L}(\tilde{x})=\int d^4x\mathcal{L}(x)=I,$$

等时对易子在$\mathcal{P}$变换下不变。

寻求Hilbert空间中的幺正算符$\mathcal{P}$，使得

$$\mathcal{P}\psi(x)\mathcal{P}^{-1}=\gamma^0\psi(\tilde{x}). \tag{3.223}$$

利用$\psi(x)$的展开式

$$\psi(x)=\int\tilde{d}k\sum_{\alpha=1,2}[b_\alpha(k)u^{(\alpha)}(k)e^{-ikx}+d_\alpha^+(k)v^{(\alpha)}(k)e^{ikx}],$$

旋量$u^{(\alpha)}(k)$和$v^{(\alpha)}(k)$分别为

$$u^{(\alpha)}(k)=\frac{\not{k}+m}{\sqrt{2m(k^0+m)}}\begin{pmatrix}\varphi^{(\alpha)}\\0\end{pmatrix},$$

$$v^{(\alpha)}(k)=\frac{-\not{k}+m}{\sqrt{2m(k^0+m)}}\begin{pmatrix}0\\ \chi^{(\alpha)}\end{pmatrix},$$

其中$\varphi^{(\alpha)}$和$\chi^{(\alpha)}$取为静止系中σ^3的本征态。

(3.223)的左边和右边分别为

$$左边=\int\tilde{d}k[\mathcal{P}b_\alpha(k)\mathcal{P}^{-1}u^{(\alpha)}(k)e^{-ikx}+\mathcal{P}d_\alpha^+(k)\mathcal{P}^{-1}v^{(\alpha)}(k)e^{ikx}],$$

$$右边=\gamma^0\int\tilde{d}k[b_\alpha(k)u^{(\alpha)}(k)e^{-ik\tilde{x}}+d_\alpha^+(k)v^{(\alpha)}(k)e^{ik\tilde{x}}]$$

$$=\int\tilde{d}k[b_\alpha(k)\gamma^0u^{(\alpha)}(k)e^{-i\tilde{k}x}+d_\alpha^+(k)\gamma^0v^{(\alpha)}(k)e^{i\tilde{k}x}]$$

$$=\int\tilde{d}k[b_\alpha(\tilde{k})\gamma^0u^{(\alpha)}(\tilde{k})e^{-ikx}+d_\alpha^+(\tilde{k})\gamma^0v^{(\alpha)}(\tilde{k})e^{ikx}],$$

由于

$$\gamma^0 u^{(\alpha)}(\tilde{k}) = \gamma^0 \frac{\tilde{\not{k}} + m}{\sqrt{\ }} \begin{pmatrix} \varphi^{(\alpha)} \\ 0 \end{pmatrix} = \frac{\not{k} + m}{\sqrt{\ }} \gamma^0 \begin{pmatrix} \varphi^{(\alpha)} \\ 0 \end{pmatrix}$$

$$= \frac{\not{k} + m}{\sqrt{\ }} \begin{pmatrix} \varphi^{(\alpha)} \\ 0 \end{pmatrix} = u^{(\alpha)}(k),$$

$$\gamma^0 v^{(\alpha)}(\tilde{k}) = \gamma^0 \frac{-\tilde{\not{k}} + m}{\sqrt{\ }} \begin{pmatrix} 0 \\ \chi^{(\alpha)} \end{pmatrix} = \frac{-\not{k} + m}{\sqrt{\ }} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} 0 \\ \chi^{(\alpha)} \end{pmatrix}$$

$$= \frac{-\not{k} + m}{\sqrt{\ }} \begin{pmatrix} 0 \\ -\chi^{(\alpha)} \end{pmatrix} = -v^{(\alpha)}(k),$$

$$\therefore \ \text{右边} = \int \tilde{d}k [b_\alpha(\tilde{k}) u^{(\alpha)}(k) e^{-ikx} + d_\alpha^+(\tilde{k}) v^{(\alpha)}(k) e^{ikx}],$$

比较(3.223)的左边和右边，得

$$\mathcal{P}\, b_\alpha(k)\mathcal{P}^{-1} = b_\alpha(\tilde{k}) \quad \text{或} \quad \mathcal{P}\, b_\alpha^+(k)\mathcal{P}^{-1} = b_\alpha^+(\tilde{k}), \tag{3.227a}$$

$$\mathcal{P}\, d_\alpha(k)\mathcal{P}^{-1} = -d_\alpha(\tilde{k}) \quad \text{或} \quad \mathcal{P}\, d_\alpha^+(k)\mathcal{P}^{-1} = -d_\alpha^+(\tilde{k}), \tag{3.227b}$$

可见，$(f-\bar{f})$系统的相对内禀宇称是 1。

设$(f-\bar{f})$系统在质心系处于S态，则

$$\mathcal{P}\int \tilde{d}kf(|\vec{k}|)b_\alpha^+(k)d_\beta^+(\tilde{k})|0\rangle$$

$$=\int \tilde{d}kf(|\vec{k}|)\mathcal{P}b_\alpha^+(k)\mathcal{P}^{-1}\mathcal{P}d_\beta^+(\tilde{k})\mathcal{P}^{-1}|0\rangle$$

$$=-\int \tilde{d}kf(|\vec{k}|)b_\alpha^+(\tilde{k})d_\beta^+(k)|0\rangle$$

$$=-\int \tilde{d}kf(|\vec{k}|)b_\alpha^+(k)d_\beta^+(\tilde{k})|0\rangle.$$

又

$$\mathcal{P}^2\psi(x)\mathcal{P}^{-2}=\mathcal{P}\gamma^0\psi(\tilde{x})\mathcal{P}^{-1}=\gamma^0\gamma^0\psi(x)=\psi(x),$$

$$\Rightarrow\qquad \mathcal{P}^2=1.$$

满足(3.227)的$\mathcal{P}$为

$$\mathcal{P}=\exp\{-\frac{i\pi}{2}\int\tilde{d}k\sum_{\alpha=1,2}[b_\alpha^+(k)b_\alpha(k)-b_\alpha^+(k)b_\alpha(\tilde{k})$$

$$+d_\alpha^+(k)d_\alpha(k)+d_\alpha^+(k)d_\alpha(\tilde{k})]\},\quad (3.230)$$

易证实,

$$\mathcal{P}P^\mu\mathcal{P}^{-1}=P_\mu,\qquad \mathcal{P}\left|0\right\rangle=\left|0\right\rangle.$$

Maxwell 场的宇称变换

规定

$$\mathcal{P}A_\mu(x)\mathcal{P}^{-1} = A^\mu(\tilde{x}), \tag{3.233}$$

保证电磁作用的$\mathcal{P}$不变性。

3.6.2 电荷共轭变换（Charge Conjugation）

- K-G 场

$$粒子 \xleftrightarrow{C} 反粒子$$

即

$$Ca(k)C^{-1} = b(k), \qquad Cb(k)C^{-1} = a(k), \tag{3.234}$$

$$\varphi(x) = \int \tilde{d}k[a(k)e^{-ikx} + b^{+}(k)e^{ikx}],$$

$$\varphi^{+}(x) = \int \tilde{d}k[a^{+}(k)e^{ikx} + b(k)e^{-ikx}],$$

$$\therefore \qquad C\varphi(x)C^{-1} = \varphi^{+}(x). \tag{3.235}$$

在此变换下，Lagrangian

$$\mathcal{L}(x) =: \partial_{\mu}\varphi^{+}(x)\partial^{\mu}\varphi(x) - m^{2}\varphi^{+}(x)\varphi(x) \xrightarrow{C} \mathcal{L}(x),$$

等时对易关系在C变换下不变。C的显示形式为

$$C = \exp\{\frac{-i\pi}{2}\int \tilde{d}k[a^+(k) - b^+(k)][a(k) - b(k)]\}, \qquad (3.236)$$

C幺正且满足

$$C|0\rangle = |0\rangle. \qquad (3.237)$$

对于实标量场

$$C\varphi(x)C^{-1} = \varphi(x). \qquad (3.238)$$

▪ Maxwell 场

规定

$$\mathcal{C} A_\mu(x) \mathcal{C}^{-1} = -A_\mu(x). \tag{3.239}$$

在此变换下，Lagrangian

$$\mathcal{L}(x) =: -\frac{1}{4} F^2 - \frac{\lambda}{2} (\partial \cdot A)^2 : \xrightarrow{\mathcal{C}} \mathcal{L}(x),$$

等时对易关系在$\mathcal{C}$变换下也不变。

▪ Dirac 场

$$\psi(x) \to \psi^C(x) = C\bar{\psi}^T(x),$$
$$\bar{\psi}(x) \to \bar{\psi}^C(x) = \psi^T(x)C, \tag{3.240}$$

T — 旋量转置，

$$C = i\gamma^2\gamma^0,$$

C矩阵的性质：

(1) $C^+ = C^T = C^{-1} = -C, \quad C^2 = -1;$

(2) $C\gamma_\mu^T C^{-1} = -\gamma_\mu, \quad C\gamma_5^T C^{-1} = \gamma_5.$

在C变换下，

$$\mathcal{L} =: i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi :$$

$$\downarrow$$

$$\mathcal{L}^C =: i\overline{\psi}^C \gamma^\mu \partial_\mu \psi^C - m\overline{\psi}^C \psi^C :$$

$$=: i\psi^T C\gamma^\mu \partial_\mu C\overline{\psi}^T - m\psi^T CC\overline{\psi}^T :$$

$$=: i\psi^T \gamma^{\mu T} \partial_\mu \overline{\psi}^T + m\psi^T \overline{\psi}^T :$$

$$=: -i(\partial_\mu \overline{\psi}\gamma^\mu \psi)^T - m(\overline{\psi}\psi)^T :$$

$$=: -i(\partial_\mu \overline{\psi})\gamma^\mu \psi - m\overline{\psi}\psi :$$

$$=: -i\partial_\mu(\overline{\psi}\gamma^\mu \psi) + i\overline{\psi}\gamma^\mu \partial_\mu \psi - m\overline{\psi}\psi :$$

$= \mathcal{L} + 4$散度项,

等时反对易子也不变。

选择Hillbert空间中的幺正算符C，使

$$C\psi(x)C^{-1}=C\overline{\psi}^{T}(x). \tag{3.243}$$

选择与螺旋态相应的湮灭和产生算符

$$b(k,\pm),\ d(k,\pm)\quad 和\quad b^{+}(k,\pm),\ d^{+}(k,\pm),$$

正频解取为：

$$u(k,\pm)=\frac{\not{k}+m}{\sqrt{2m(k^{0}+m)}}\begin{pmatrix}\varphi_{\pm}(\hat{k})\\ 0\end{pmatrix}, \tag{3.244}$$

其中$\varphi_{\pm}(\hat{k})\equiv\varphi^{(\alpha)}(\hat{k})$，满足

$$\vec{\sigma}\cdot\hat{k}\varphi_{\pm}(\hat{k})=\pm\varphi_{\pm}(\hat{k}), \tag{3.245}$$

$$\varphi_{\varepsilon}^{+}(\hat{k})\varphi_{\varepsilon'}(\hat{k})=\delta_{\varepsilon\varepsilon'},\quad \varepsilon,\varepsilon'=\pm. \tag{3.246}$$

负频解取为：

$$v(k,\pm)=C\overline{u}^{T}(k,\pm)=\frac{-\not{k}+m}{\sqrt{2m(k^{0}+m)}}\begin{pmatrix}0\\ \chi_{\pm}(\hat{k})\end{pmatrix}, \qquad (3.247)$$

$$\overline{u}^{T}(k,\pm)=\{u^{+}\gamma^{0}\}^{T}=\left\{\left[\frac{-\not{k}+m}{\sqrt{}}\begin{pmatrix}\varphi_{\pm}\\ 0\end{pmatrix}\right]^{+}\gamma^{0}\right\}^{T},$$

利用$\gamma^{0}\not{k}^{+}\gamma^{0}=\not{k}$，可得

$$\overline{u}^{T}(k,\pm)=\frac{\not{k}^{T}+m}{\sqrt{}}\gamma^{0T}\begin{pmatrix}\varphi_{\pm}^{*}\\ 0\end{pmatrix},$$

$$\therefore\quad C\overline{u}^{T}(k,\pm)=C\frac{\gamma^{\mu T}k_{\mu}+m}{\sqrt{}}\gamma^{0T}\begin{pmatrix}\varphi_{\pm}^{*}\\ 0\end{pmatrix}$$

$$C\bar{u}^T(k,\pm) = \frac{1}{\sqrt{\ }}[C\gamma^{\mu T}C^{-1}k_\mu + m]C\gamma^{0T}\begin{pmatrix}\varphi_\pm^*\\ 0\end{pmatrix}$$

$$= \frac{1}{\sqrt{\ }}[-\not{k} + m]C\gamma^{0T}\begin{pmatrix}\varphi_\pm^*\\ 0\end{pmatrix}$$

而Dirac表象中，

$$C\gamma^{0T} = \begin{pmatrix}0 & i\sigma^2\\ -i\sigma^2 & 0\end{pmatrix},$$

$$\therefore\ \ C\bar{u}^T(k,\pm) = \frac{-\not{k} + m}{\sqrt{\ }}\begin{pmatrix}0\\ -i\sigma^2\varphi_\pm^*\end{pmatrix},$$

上式与(3.247)比较，得到

$$\chi_\pm(\hat{k}) = -i\sigma^2\varphi_\pm^{\ *}(\hat{k}), \tag{3.248}$$

利用(3.248)和$\sigma^2\vec{\sigma}^*\sigma^2=-\vec{\sigma}$，由(3.245)可导出

$$-\vec{\sigma}\cdot\hat{k}\chi_{\pm}(\hat{k})=\pm\chi_{\pm}(\hat{k}). \tag{3.249}$$

比较：

$$\vec{\sigma}\cdot\hat{k}\varphi_{\pm}(\hat{k})=\pm\varphi_{\pm}(\hat{k}). \tag{3.245}$$

电荷共轭时，

$$v(k,\pm)=C\overline{u}^T(k,\pm),\qquad u(k,\pm)=C\overline{v}^T(k,\pm), \tag{3.250}$$

态的螺旋性不变。

采用上述选择，$\psi(x),\overline{\psi}(x)$的展开式成为

$$\psi(x)=\int\widetilde{d}k[b(k,\pm)u(k,\pm)e^{-ikx}+d^+(k,\pm)v(k,\pm)e^{ikx}],$$

$$\overline{\psi}(x)=\int\widetilde{d}k[b^+(k,\pm)\overline{u}(k,\pm)e^{ikx}+d(k,\pm)\overline{v}(k,\pm)e^{-ikx}],$$

代入(3.243)，其左边和右边分别为

$$\text{左边} = \int \tilde{d}k[\mathcal{C}b(k,\pm)\mathcal{C}^{-1}u(k,\pm)e^{-ikx} + \mathcal{C}d^{+}(k,\pm)\mathcal{C}^{-1}v(k,\pm)e^{ikx}]$$

$$\text{右边} = \int \tilde{d}k[b^{+}(k,\pm)C\overline{u}^{T}(k,\pm)e^{ikx} + d(k,\pm)C\overline{v}^{T}(k,\pm)e^{-ikx}]$$

$$= \int \tilde{d}k[b^{+}(k,\pm)v(k,\pm)e^{ikx} + d(k,\pm)u(k,\pm)e^{-ikx}],$$

$$\Rightarrow \quad \mathcal{C}b(k,\pm)\mathcal{C}^{-1} = d(k,\pm), \quad \mathcal{C}d^{+}(k,\pm)\mathcal{C}^{-1} = b^{+}(k,\pm). \qquad (3.252)$$

$\mathcal{C}$的显示式为

$$\mathcal{C} = \exp\{\frac{-i\pi}{2}\int \tilde{d}k\sum_{\varepsilon=\pm}[b^{+}(k,\varepsilon) - d^{+}(k,\varepsilon)][b(k,\varepsilon) - d(k,\varepsilon)]\},$$

它是幺正的并保持真空不变性，即

$$\mathcal{C}|0\rangle = |0\rangle.$$

3.6.3 时间反演（Time Reveisal）

$$\begin{array}{c} A \to B \\ \downarrow \mathcal{T} \\ A \leftarrow B \end{array}$$

- K-G 场

$$\varphi(\vec{x},t) \xrightarrow{\mathcal{T}} \pm\varphi(\vec{x},-t), \qquad (3.255)$$

在此变换下, Lagrangian

$$\mathcal{L}(x) = \partial_\mu \varphi^+(x)\partial^\mu \varphi(x) - m^2 \varphi^+(x)\varphi(x) \xrightarrow{\mathcal{T}} \mathcal{L}(\vec{x},-t),$$

因而，作用量

$$I = \int d^4x \mathcal{L}(x) \xrightarrow{\mathcal{T}} \int d^4x \mathcal{L}(\vec{x},-t) = \int d^4x \mathcal{L}(x) = I,$$

等时对易子

$$[\varphi(\vec{x},t),\dot{\varphi}^{+}(\vec{x}',t)]=i\delta^{3}(\vec{x}-\vec{x}'),$$

$$\downarrow \mathcal{T}$$

$$[\pm\varphi(\vec{x},-t),\pm\frac{\partial}{\partial(-t)}\varphi^{+}(\vec{x}',-t)]=-i\delta^{3}(\vec{x}-\vec{x}'),$$

需要寻找Hilbert空间中的时间反演算符，使得

$$\mathcal{T}\varphi(\vec{x},t)\mathcal{T}^{-1}=\pm\varphi(\vec{x},-t),\tag{3.256}$$

$\mathcal{T}$中必须包括一个复共轭运算K，满足

$$K\lambda=\lambda^{*}K,\quad(\lambda\text{是}c\text{-数}),\tag{3.257}$$

则等时对易关系变为：

$$\mathcal{T}[\varphi(\vec{x},t),\dot{\varphi}(\vec{x}',t)]\mathcal{T}^{-1}=\mathcal{T}i\mathcal{T}^{-1}\delta^{3}(\vec{x}-\vec{x}'),$$

导致

$$[\pm\varphi(\vec{x},-t),\pm\frac{\partial}{\partial t}\varphi^{+}(\vec{x}',-t)]=-i\delta^{3}(\vec{x}-\vec{x}'),$$

或 $$[\varphi(\vec{x},-t),\frac{\partial}{\partial(-t)}\varphi^{+}(\vec{x}',-t)]=i\delta^{3}(\vec{x}-\vec{x}'),$$

因而等时对易子在$\mathcal{T}$变换下不变。于是可写

$$\mathcal{T}=\mathcal{U}K, \tag{3.258}$$

其中$\mathcal{U}$是一个待定的幺正算符 则(3.256)的左边和右边分别成为

$$\text{左边}=\mathcal{T}\varphi(\vec{x},t)\mathcal{T}^{-1}$$

$$=\mathcal{U}K\int\tilde{d}k[a(k)e^{-ikx}+b^{+}(k)e^{ikx}]K^{-1}\mathcal{U}^{-1}$$

$$左边=\int \tilde{d}k[\mathcal{U}a(k)\mathcal{U}^{-1}e^{ikx}+\mathcal{U}b^{+}(k)\mathcal{U}^{-1}e^{-ikx}],$$

$$右边=\pm\int \tilde{d}k[a(k)e^{ik\tilde{x}}+b^{+}(k)e^{-ik\tilde{x}}]$$

$$=\pm\int \tilde{d}k[a(k)e^{i\tilde{k}x}+b^{+}(k)e^{-i\tilde{k}x}]$$

$$=\pm\int \tilde{d}k[a(\tilde{k})e^{ikx}+b^{+}(\tilde{k})e^{-ikx}],$$

比较，得到

$$\mathcal{U}a(k)\mathcal{U}^{-1}=\pm a(\tilde{k}),\quad 或\quad \mathcal{U}a^{+}(k)\mathcal{U}^{-1}=\pm a^{+}(\tilde{k}),\tag{3.259a}$$

$$\mathcal{U}b^{+}(k)\mathcal{U}^{-1}=\pm b^{+}(\tilde{k}),\quad 或\quad \mathcal{U}b(k)\mathcal{U}^{-1}=\pm b(\tilde{k}),\tag{3.259b}$$

比较标量场的$\mathcal{P}$变换：

$$\mathcal{P}_s a(k)\mathcal{P}_s^{-1}=a(\tilde{k}),\qquad \mathcal{P}_{ps}a(k)\mathcal{P}_{ps}^{-1}=-a(\tilde{k}),$$

$$\therefore \qquad \mathcal{U}=\begin{cases}\mathcal{P}_s & \text{对}+\text{号}\\ \mathcal{P}_{ps} & \text{对}-\text{号}\end{cases}. \tag{3.260}$$

4动量算符的空间分量$\vec{P}$在$\mathcal{T}$变换下成为：

$$\mathcal{T}\vec{P}\mathcal{T}^{-1}=\mathcal{U}\vec{P}\mathcal{U}^{-1}=-\vec{P}. \tag{3.261}$$

K为反幺正算符，满足

$$\langle K\alpha|K\beta\rangle=\langle\alpha|\beta\rangle^{*}, \tag{3.262}$$

幺正算符$\mathcal{U}$满足：

$$\langle\mathcal{U}\alpha|\mathcal{U}\beta\rangle=\langle\alpha|\beta\rangle, \tag{3.263}$$

$$\because\langle\mathcal{T}\alpha|\mathcal{T}\beta\rangle=\langle\mathcal{U}K\alpha|\mathcal{U}K\beta\rangle=\langle K\alpha|\mathcal{U}^{+}\mathcal{U}|K\beta\rangle=\langle K\alpha|K\beta\rangle=\langle\alpha|\beta\rangle^{*},$$

$\therefore\mathcal{T}=\mathcal{U}K$也是反幺正的。

▪ Dirac 场

$$\mathcal{T} = \mathcal{U}K,$$

$$\psi(x) \xrightarrow{\mathcal{T}} \psi'(x) \equiv \mathcal{T}\psi(x)\mathcal{T}^{-1} = T\psi(-\tilde{x}), \qquad (3.267)$$

其中T是一个非奇异的4×4矩阵。

假定自由Dirac场是$\mathcal{T}$不变的，则有

$$\mathcal{T}\mathcal{L}(x)\mathcal{T}^{-1} = \mathcal{L}(-\tilde{x}), \qquad (3.268)$$

又

$$\mathcal{L} =: i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi : \qquad (3.177)$$

(3.177)代入(3.268)，并利用(3.267)，得到

$$:-i\psi^{+}(-\tilde{x})T^{+}\gamma^{0^*}\gamma^{\mu^*}\partial_{\mu}T\psi(-\tilde{x})-m\psi^{+}(-\tilde{x})T^{+}\gamma^{0^*}T\psi(-\tilde{x}):$$

$$=:i\psi^{+}(-\tilde{x})\gamma^{0}\gamma^{\mu}(-\tilde{\partial}_{\mu})\psi(-\tilde{x})-m\psi^{+}(-\tilde{x})\gamma^{0}\psi(-\tilde{x}):$$

其中$\tilde{\partial}_{\mu}=(\partial_0,-\partial_i)$。上式成立要求：

$$T^{+}\gamma^{0^*}T=\gamma^{0},$$

$$T^{+}\gamma^{0^*}\gamma^{\mu^*}T\partial_{\mu}=\gamma^{0}\gamma^{\mu}\tilde{\partial}_{\mu},$$

它们等价于：

$$T^{-1}\gamma^{\mu^*}T=\tilde{\gamma}^{\mu}=\gamma_{\mu}, \tag{3.271}$$

$$T^{-1}=T^{+}, \tag{3.272}$$

因此，只要T满足条件(3.271)和(3.272)，则(3.268)成立，作用量是T不变的，等时反对易子也是T不变的。

容易证明：

$$T=-i\gamma_5 C=i\gamma^1\gamma^3,\ (\gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3) \tag{3.273}$$

满足条件(3.271)和(3.272)，且有

$$T^T=-T, \quad T^*=-T^+. \tag{3.274}$$

$$\mathcal{T}^2\psi(x)\mathcal{T}^{-2}=\mathcal{T}T\psi(-\tilde{x})\mathcal{T}^{-1}=\mathcal{T}T\mathcal{T}^{-1}\mathcal{T}\psi(-\tilde{x})\mathcal{T}^{-1}$$

$$=T^*T\psi(x)=-T^+T\psi(x)=-\psi(x).$$

- Maxwell 场

规定：

$$\mathcal{T} A_\mu(x) \mathcal{T}^{-1} = A^\mu(-\tilde{x}).$$