二、最佳平方逼近

插值法：已知 $[a,\ b],\ \{\, x_i,\ f(x_i)\}_{i=0}^{n}$

求：$\varphi(x) \sim f(x)$ 满足：$\varphi(x_i) = f(x_i)\ \ i = 1, 2, 3, \cdots$

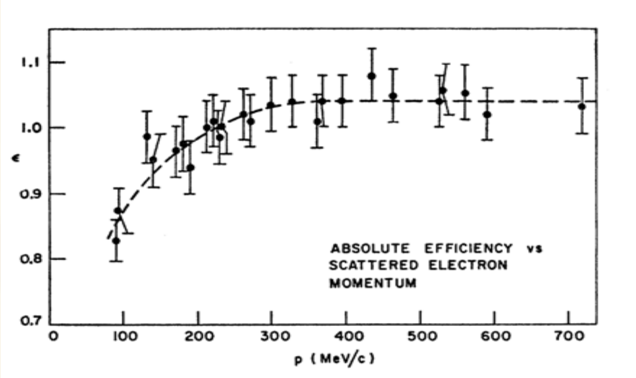
但许多实际问题，并不必要 $\varphi(x_i) = f(x_i)\ \ i = 1, 2, 3, \cdots$

而是在整体上数学意义的逼近 $\varphi(x) \sim f(x)$

如：实验数据点，有误差：$\{\, x_i,\ f(x_i) = y_i,\ \Delta y_i \}_{i=0}^{n}$

最佳平方逼近

$$\sum_{i=0}^{n} \frac{\left|\varphi(x_i) - f(x_i)\right|^2}{(\Delta y_i)^2}$$ 最小



1、数据拟合的最佳平方逼近（最小二乘法）

（1）多项式拟合

设已知数据点 $\{x_i, f(x_i)=y_i\}_{i=0}^{n}$ 求m次多项式

$P_m(x)=a_0+a_1x+\cdots a_mx^m$ 来拟合函数 $y=f(x)$

需要求出多项式的m+1个待定系数，使得：

$$F(a_0,a_1,\cdots a_m)=\sum_{i=0}^{n}\left[y_i-P_m(x_i)\right]^2$$ 最小

要函数值达到最小，求极值有：

$$\frac{\partial F}{\partial a_j} = 2\sum_{i=1}^{n}\left[y_i - P_m(x_i)\right]x_i^{\,j} = 2\sum_{i=0}^{n}\left[y_i - \sum_{k=0}^{m} a_k x_i^k\right]x_i^j = 0$$

即 $\sum_{k=o}^{m} a_k\left(\sum_{i=1}^{n} x_i^{k+j}\right) = \sum_{i=1}^{n} y_i x_i^j$ j=0，1，2，…，m

得方程组

$$\begin{cases} na_0 + a_1\sum_{i=1}^{n} x_i + \cdots + a_m\sum_{i=1}^{n} x_i^m = \sum_{i=1}^{n} y_i \\ a_0\sum_{i=1}^{n} x_i + a_1\sum_{i=1}^{n} x_i^2 + \cdots + a_m\sum x_i^{m+1} = \sum_{i=1}^{n} y_i x_i \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_0\sum_{i=1}^{n} x_i^m + a_1\sum_{i=1}^{n} x_i^{m+1} + \cdots + a_m\sum_{i=1}^{n} x_i^{2m} = \sum_{i=1}^{n} y_i x_i^m \end{cases}$$

可解得： $(a_0, a_1, \cdots a_m)$

（2）指数拟合

若数据点近似一条指数曲线，则考虑用指数函数

$y(x)=ae^{bx}$ 来拟合

由 $\ln y=\ln a+bx$ 可以先做 $\varphi(x)=a^{*}+bx$

即一次的多项式拟合

（3）数据拟合最小二乘法的一般形式

$\Phi = span\{\varphi_0, \varphi_1, \cdots \varphi_m\}$ $\varphi_0, \varphi_1, \cdots \varphi_m$ 为线性无关的基函数

取 $\varphi(x) = a_0\varphi_0(x) + \cdots + a_m\varphi_m(x)$

使 $F(a_0, a_1, \cdots a_m) = \sum_{i=0}^{n} \omega_i [y_i - \varphi(x_i)]^2 = \sum_{i=0}^{n} \omega_i \left[y_i - \sum_{k=1}^{m} a_k \varphi_k(x_i) \right]^2$ 最小

$$\frac{\partial F}{\partial a_j} = 2\sum_{i=1}^{n} \omega_i \left[y_i - \sum_{k=1}^{m} a_k \varphi_k(x_i) \right] \varphi_j = 0 \quad j = 0, 1, \cdots, m$$

即 $\sum_{k=1}^{m} a_k \left[\sum_{i=1}^{n} \omega_i \varphi_k(x_i) \varphi_j(x_i) \right] = \sum_{i=1}^{n} \omega_i y_i \varphi_j(x_i) \quad j = 0, 1, \cdots, m$

记 $(\varphi_k, \varphi_j) = \sum_{i=1}^{n} \omega_i \varphi_k(x_i) \varphi_j(x_i) \quad k, j = 0, 1, \cdots, m$

$$(y, \varphi_j) = \sum_{i=1}^{n} \omega_i y_i \varphi_j(x_i) \quad j = 0, 1, \cdots, m$$

则方程组可写成以下形式

$$\begin{pmatrix} (\varphi_0,\varphi_0) & (\varphi_0,\varphi_1) & \cdots & (\varphi_0,\varphi_m) \\ (\varphi_1,\varphi_0) & (\varphi_1,\varphi_1) & \cdots & (\varphi_1,\varphi_m) \\ \cdots & \cdots & \cdots & \cdots \\ (\varphi_m,\varphi_0) & (\varphi_m,\varphi_1) & \cdots & (\varphi_m,\varphi_m) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} (y,\varphi_0) \\ (y,\varphi_1) \\ \vdots \\ (y,\varphi_m) \end{pmatrix}$$

可解得： $(a_0, a_1, \cdots a_m)$

若选取基函数有（正交）

$$(\varphi_k,\varphi_j) = \sum_{i=1}^{n} \omega_i \varphi_k(x_i) \varphi_j(x_i) = \begin{cases} \alpha_k & j = k \\ 0 & j \neq k \end{cases} \quad k, j = 0,1,\cdots,m$$

则方程易解