Introduction to Reinforcement Learning
Week 4

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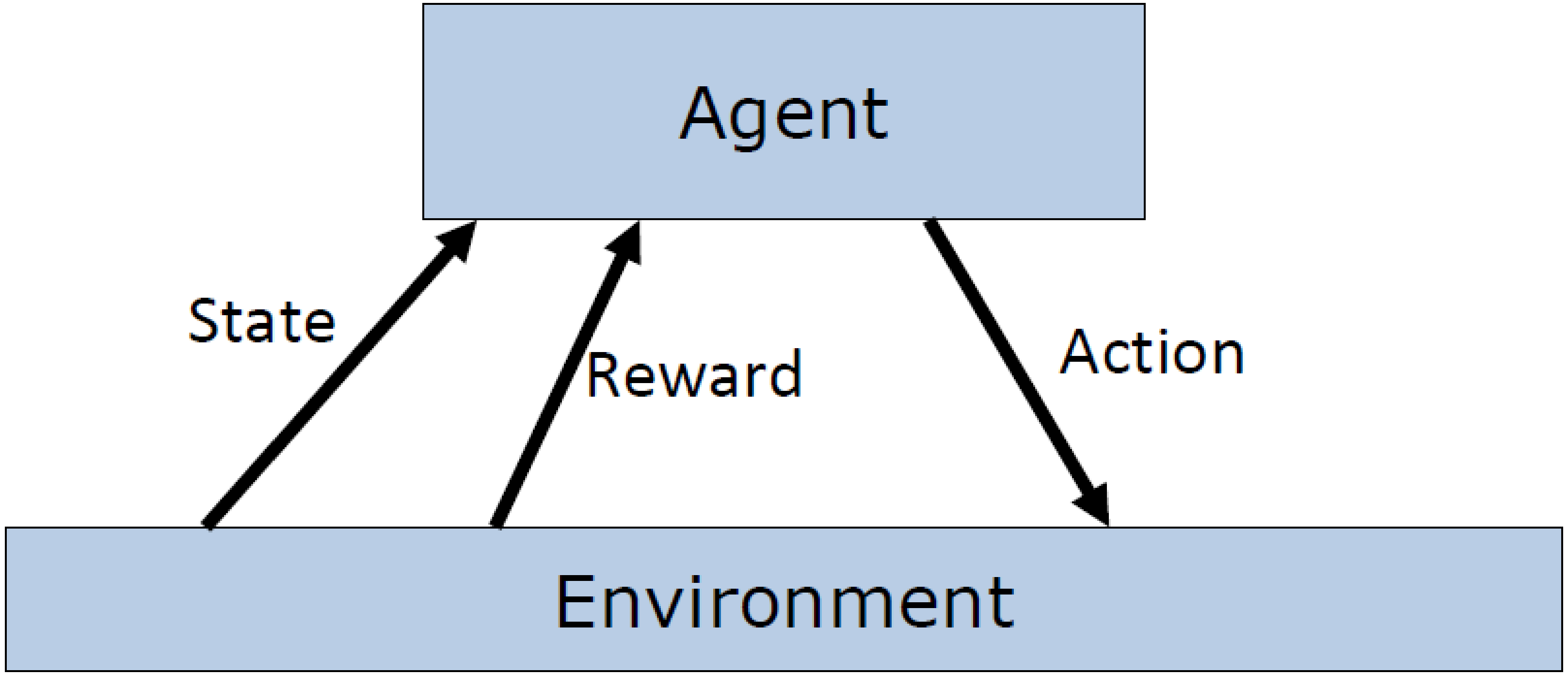
Markov Decision Process

- Definition
  - States: $s \in S$
  - Actions: $a \in A$
  - Rewards: $r \in \mathbb{R}$
  - Transition model: $\Pr(s_t|s_{t-1}, a_{t-1})$
  - Reward model: $\Pr(r_t|s_t, a_t)$
  - Discount factor: $0 \le \gamma \le 1$
    - discounted: $\gamma < 1$ undiscounted: $\gamma = 1$
  - Horizon (i.e., # of time steps): h
    - Finite horizon: $h \in \mathbb{N}$ infinite horizon: $h = \infty$

- Goal: find optimal policy π^* such that

$$\pi^* = argmax_\pi \sum_{t=0}^{h} \gamma^t E_\pi[r_t]$$

Reinforcement Learning Problem



Goal: Learn to choose actions that maximize rewards

Reinforcement Learning

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  - Actions: $a \in A$
  - Rewards: $r \in \mathbb{R}$
  - Transition model: $\Pr(s_t|s_{t-1}, a_{t-1})$
  - Reward model: $\Pr(r_t|s_t, a_t)$

  unknown model

  - Discount factor: $0 \leq \gamma \leq 1$
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- Goal: find optimal policy π^* such that

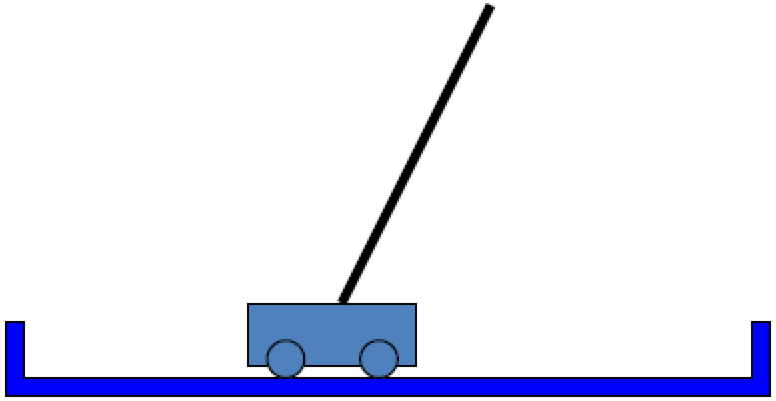
$$\pi^* = argmax_\pi \sum_{t=0}^{h} \gamma^t E_\pi[r_t]$$

Policy optimization

- Markov Decision Process:
  - Find optimal policy given transition and reward model
  - Execute policy found
- Reinforcement learning:
  - Learn an optimal policy while interacting with the environment

Example: Inverted Pendulum

- State:
  $x(t), x'(t), \theta(t), \theta'(t)$
- Action: Force F
- Reward: 1 for any step where pole balanced



Problem: Find $\pi: S \to A$ that maximizes rewards

Important Components in RL

RL agents may or may not include the following components:

- Model: $\Pr(s'|s,a)\,, \Pr(r|s,a)$
  - Environment dynamics and rewards
- Policy: $\pi(s)$
  - Agent action choices
- Value function: $V(s)$
  - Expected total rewards of the agent policy

Categorizing RL agents

Value based

- No policy (implicit)
- Value function

Policy based

- Policy
- No value function

Actor critic

- Policy
- Value function

Model based

- Transition and reward model

Model free

- No transition and no reward model (implicit)

Toy Maze Example

3	r	r	r	+1
2	u		u	-1
1	u	l	l	l
	1	2	3	4

Start state: (1,1)
Terminal states: (4,2), (4,3)
No discount: $\gamma = 1$

Reward is -0.04 for non-terminal states

Four actions: up (u), left (l), right (r), down (d)
Do not know the transition probabilities

What is the value $V(s)$ of being in state s?

Monte-Carlo Reinforcement Learning

- MC methods learn directly from episodes of experience
- MC is model-free: no knowledge of MDP transitions / rewards
- MC learns from complete episodes: no bootstrapping
- MC uses the simplest possible idea: value = mean return
- Caveat: can only apply MC to episodic MDPs
    - All episodes must terminate

Monte-Carlo policy evaluation uses empirical mean return instead of expected return

Model free evaluation

- Given a policy π, estimate $V^{\pi}(s)$ without any transition or reward model

- **Monte Carlo** evaluation

$$V^{\pi}(s) = E_{\pi}[\sum_t \gamma^t r_t]$$

$$\approx \frac{1}{n(s)} \sum_{k=1}^{n(s)} \left[\sum_t \gamma^t r_t^{(k)}\right] \quad \text{(sample approximation)}$$

- **Temporal difference (TD)** evaluation

$$V^{\pi}(s) = E[r|s, \pi(s)] + \gamma \sum_{s'} \Pr(s'|s, \pi(s))\, V^{\pi}(s')$$

$$\approx r + \gamma V^{\pi}(s') \quad \text{(one sample approximation)}$$

Monte Carlo Evaluation

- Let G_k be a one-trajectory Monte Carlo target

$$G_k = \sum_t \gamma^t r_t^{(k)}$$

- Approximate value function

$$\begin{aligned} V_n^\pi(s) &\approx \frac{1}{n(s)} \sum_{k=1}^{n(s)} G_k \\ &= \frac{1}{n(s)} \left(G_{n(s)} + \sum_{k=1}^{n(s)-1} G_k \right) \\ &= \frac{1}{n(s)} \left(G_{n(s)} + (n(s)-1) V_{n-1}^\pi(s) \right) \\ &= V_{n-1}^\pi(s) + \frac{1}{n(s)} \left(G_{n(s)} - V_{n-1}^\pi(s) \right) \end{aligned}$$

- **Incremental update**

$$V_n^\pi(s) \leftarrow V_{n-1}^\pi(s) + \alpha_n \left(G_n - V_{n-1}^\pi(s) \right)$$

learning rate $1/n(s)$

Temporal Difference Evaluation

- Approximate value function: $V^{\pi}(s) \approx r + \gamma V^{\pi}(s')$
- **Incremental update**

$$V_n^{\pi}(s) \leftarrow V_{n-1}^{\pi}(s) + \alpha_n\left(r + \gamma V_{n-1}^{\pi}(s') - V_{n-1}^{\pi}(s)\right)$$

- **Theorem:** If α_n is appropriately decreased with number of times a state is visited then $V_n^{\pi}(s)$ converges to correct value

- **Sufficient conditions** for α_n:

  (1) $\sum_n \alpha_n \to \infty$ (2) $\sum_n (\alpha_n)^2 < \infty$

- Often $\alpha_n(s) = 1/n(s)$
  - Where $n(s)$ = # of times s is visited

Temporal Difference (TD) evaluation

TDevaluation(π, V^π)
  Repeat
    Execute $\pi(s)$
    Observe s' and r
    Update counts: $n(s) \leftarrow n(s) + 1$
    Learning rate: $\alpha \leftarrow 1/n(s)$
    Update value: $V^\pi(s) \leftarrow V^\pi(s) + \alpha(r + \gamma V^\pi(s') - V^\pi(s))$
    $s \leftarrow s'$
  Until convergence of V^π
  Return V^π

Comparison

- Monte Carlo evaluation:
  - Unbiased estimate
  - High variance
  - Needs many trajectories

- Temporal difference evaluation:
  - Biased estimate
  - Lower variance
  - Needs less trajectories

Model Free Control

- Instead of evaluating the state value fn, $V^\pi(s)$, evaluate the state-action value fn, $Q^\pi(s,a)$

  $Q^\pi(s,a)$: value of executing a followed by π

  $Q^\pi(s,a) = E[r|s,a] + \gamma \sum_{s'} Pr(s'|s,a) V^\pi(s')$

- Greedy policy π':

  $\pi'(s) = argmax_a Q^\pi(s,a)$

Bellman's Equation

- Optimal state value function $V^*(s)$

$$V^*(s) = \max_a E[r|s,a] + \gamma \sum_{s'} Pr(s'|s,a) V^*(s')$$

- Optimal state-action value function $Q^*(s,a)$

$$Q^*(s,a) = E[r|s,a] + \gamma \sum_{s'} Pr(s'|s,a) \max_{a'} Q^*(s',a')$$

where $V^*(s) = max_a Q^*(s,a)$

$\pi^*(s) = argmax_a Q^*(s,a)$

Monte Carlo Control

- Let G_k^a be a one-trajectory Monte Carlo target

$$G_k^a = \underbrace{r_0^{(k)}}_{\text{a}} + \underbrace{\sum_{t=1} \gamma^t r_t^{(k)}}_{\pi}$$

- Alternate between
  - **Policy evaluation**

$$Q_n^{\pi}(s,a) \leftarrow Q_{n-1}^{\pi}(s,a) + \alpha_n \left(G_n^a - Q_{n-1}^{\pi}(s,a) \right)$$

  - **Policy improvement**

$$\pi'(s) \leftarrow argmax_a Q^{\pi}(s,a)$$

Temporal Difference Control

- Approximate Q-function:

$$Q^*(s,a) = E[r|s,a] + \gamma \sum_{s'} \Pr(s'|s,a) \max_{a'} Q^*(s',a')$$

$$\approx r + \gamma \max_{a'} Q^*(s',a')$$

- **Incremental update**

$$Q_n^*(s,a) \leftarrow Q_{n-1}^*(s,a) + \alpha_n \left(r + \gamma \max_{a'} Q_{n-1}^*(s',a') - Q_{n-1}^*(s,a) \right)$$

Q-Learning

Qlearning(s, Q^*)

Repeat

Select and execute a

Observe s' and r

Update counts: $n(s,a) \leftarrow n(s,a) + 1$

Learning rate: $\alpha \leftarrow 1/n(s,a)$

Update Q-value:

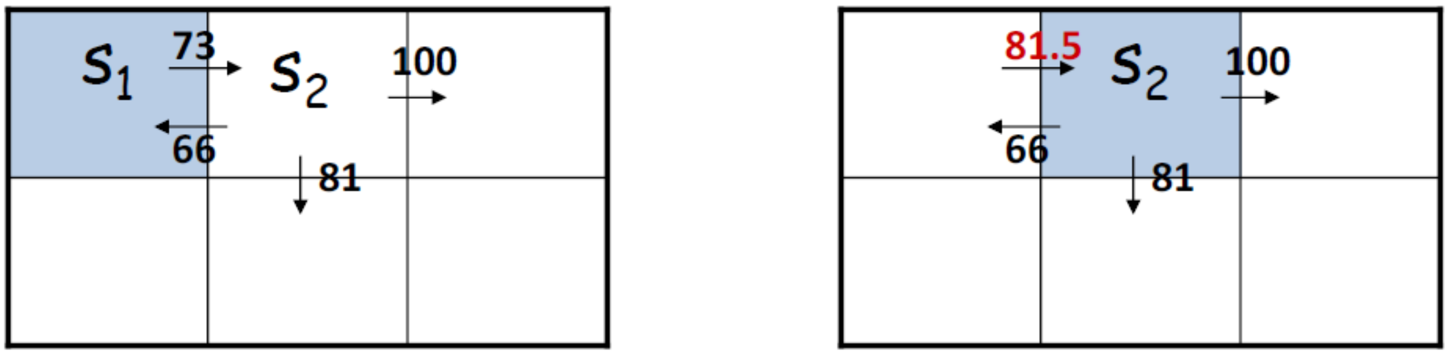
$$Q^*(s,a) \leftarrow Q^*(s,a) + \alpha\left(r + \gamma \max_{a'} Q^*(s',a') - Q^*(s,a)\right)$$

$s \leftarrow s'$

Until convergence of Q^*

Return Q^*

Q-learning example



$\gamma = 0.9, \quad \alpha = 0.5, \quad r = 0$ for non-terminal states

$$
\begin{aligned}
Q(s_1, right) &= Q(s_1, right) + \alpha\left(r + \gamma \max_{a'} Q(s_2, a') - Q(s_1, right)\right) \\
&= 73 + 0.5(0 + 0.9 \max\{66{,}81{,}100\} - 73) \\
&= 73 + 0.5(17) \\
&= 81.5
\end{aligned}
$$

Q-Learning

Qlearning(s, Q^*)

Repeat

Select and execute a

Observe s' and r

Update counts: $n(s,a) \leftarrow n(s,a) + 1$

Learning rate: $\alpha \leftarrow 1/n(s,a)$

Update Q-value:

$$Q^*(s,a) \leftarrow Q^*(s,a) + \alpha\left(r + \gamma \max_{a'} Q^*(s',a') - Q^*(s,a)\right)$$

$s \leftarrow s'$

Until convergence of Q^*

Return Q^*

Exploration vs Exploitation

- If an agent always chooses the action with the highest value then it is exploiting
  - The learned model is not the real model
  - Leads to suboptimal results
- By taking random actions (pure exploration) an agent may learn the model
  - But what is the use of learning a complete model if parts of it are never used?
- Need a balance between exploitation and exploration

Common exploration methods

- ε-greedy:
  - With probability ϵ execute random action
  - Otherwise execute best action a^*

$$a^* = argmax_a\, Q(s,a)$$

- Boltzmann exploration

$$\Pr(a) = \frac{e^{\frac{Q(s,a)}{T}}}{\sum_a e^{\frac{Q(s,a)}{T}}}$$

Exploration and Q-learning

- Q-learning converges to optimal Q-values if
  - Every state is visited infinitely often (due to exploration)
  - The action selection becomes greedy as time approaches infinity
  - The learning rate α is decreased fast enough, but not too fast (sufficient conditions for α):

$$(1)\ \sum_n \alpha_n \to \infty \qquad (2)\ \sum_n (\alpha_n)^2 < \infty$$

Summary

- We can optimize a policy by RL when the transition and reward functions are unknown
- Model free, value based agent:
  - Monte Carlo learning (unbiased, but lots of data)
  - Temporal difference learning (low variance, less data)
- Active learning:
  - Exploration/exploitation dilemma