

알고리즘

01. Divide & Conquer

2014/09/12

미디어소프트웨어학과
민경하

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1. Divide & Conquer

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1.0 Introduction

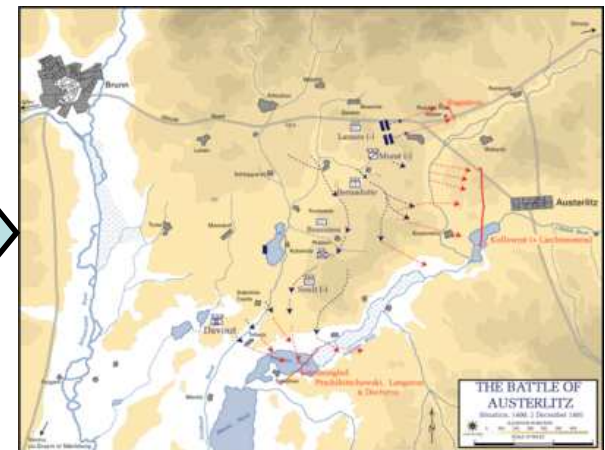
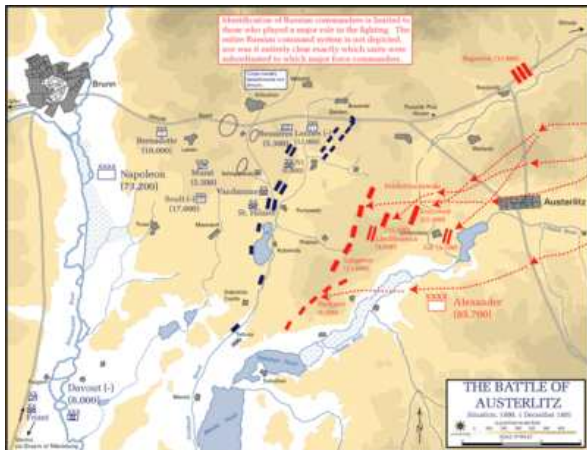
(1) Battle of Austerlitz



1.0 Introduction

(1) Battle of Austerlitz (1805)

- Battle of three emperors
- France (65,000) VS Austria + Russia (80,000)
- The end of the 3rd anti-France aliance



1.0 Introduction

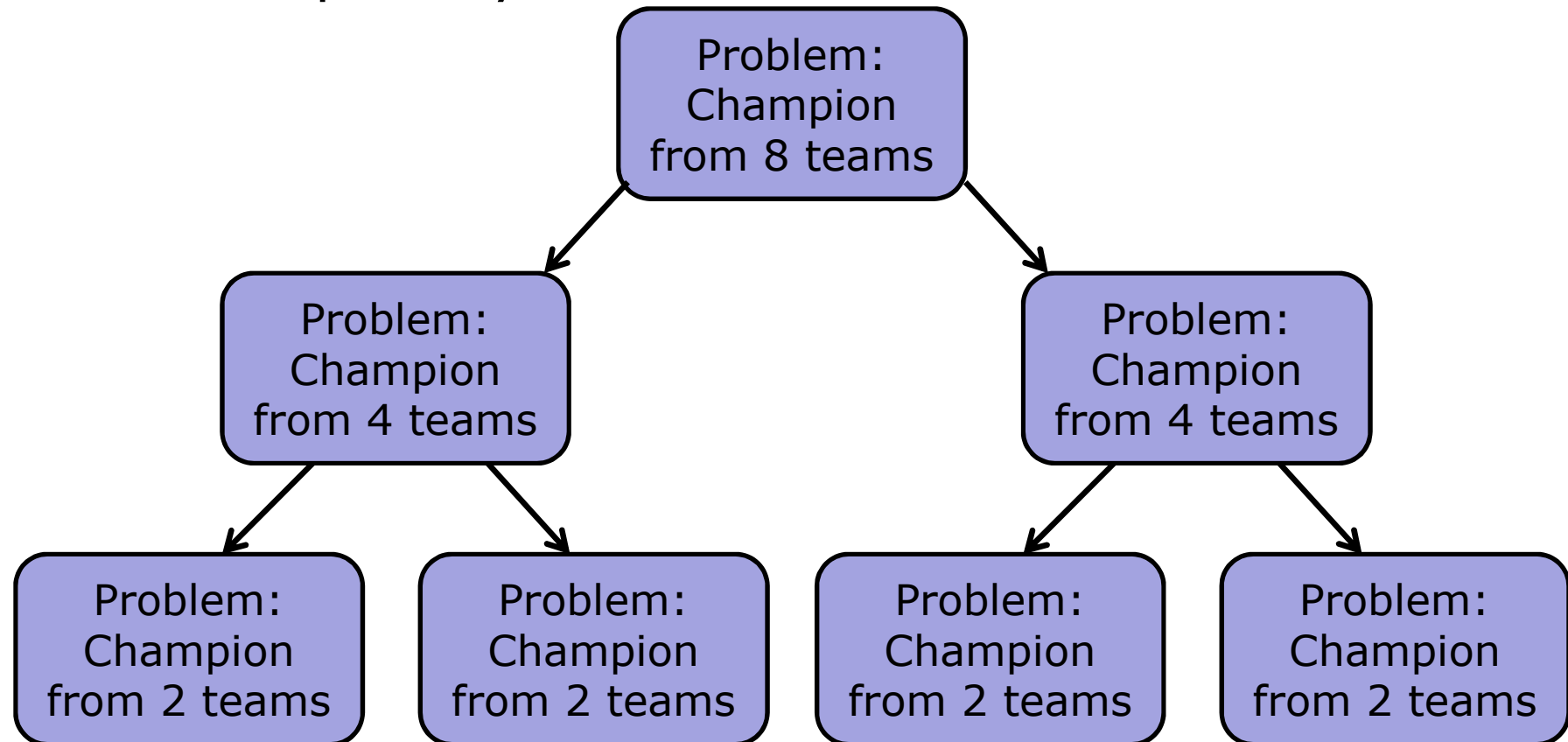
(2) League VS Tournament

- Ex) Elite-8 of Worldcup 2014
 - BRA, COL, FRA, GER, NED, CRC, ARG, BEL
- Champion by league
 - How many games do they play?
- Champion by tournament
 - How many games do they play?

1.0 Introduction

(2) League VS Tournament

– Champion by tournament



1.0 Introduction

(2) League VS Tournament

– Champion by tournament

```
int Champion8 ( {BRA, COL, FRA, GER, NED, CRC, ARG, BEL} )
{
    Lwinner = Champion4 ( {BRA, COL, FRA, GER} );
    Rwinner = Champion4 ( {NED, CRC, ARG, BEL} );

    return Winner ( Lwinner, Rwinner );
}
```


1.0 Introduction

(2) League VS Tournament

– Champion by tournament

```
int Champion4 ( {BRA, COL, FRA, GER} )
{
    Lwinner = Champion2 ( {BRA, COL} );
    Rwinner = Champion2 ( {FRA, GER} );

    return Winner ( Lwinner, Rwinner );
}
```

```
int Champion4 ( {NED, CRC, ARG, BEL} )
{
    Lwinner = Champion2 ( {NED, CRC} );
    Rwinner = Champion2 ( {ARG, BEL} );

    return Winner ( Lwinner, Rwinner );
}
```

1.0 Introduction

(2) League VS Tournament

- Champion by tournament

```
int Champion2 ( {BRA, COL} )
{
    Lwinner = Champion1 ( {BRA} );           //      Unnecessary
    Rwinner = Champion1 ( {COL} );           //      Unnecessary

    return Winner ( BRA, COL);
}
```

```
int Champion2 ( {BRA, COL} )
{
    Lwinner = Champion1 ( {BRA} );           //      Unnecessary
    Rwinner = Champion1 ( {COL} );           //      Unnecessary

    return Winner ( BRA, COL);
}
```

1.0 Introduction

(2) League VS Tournament

– Champion by tournament

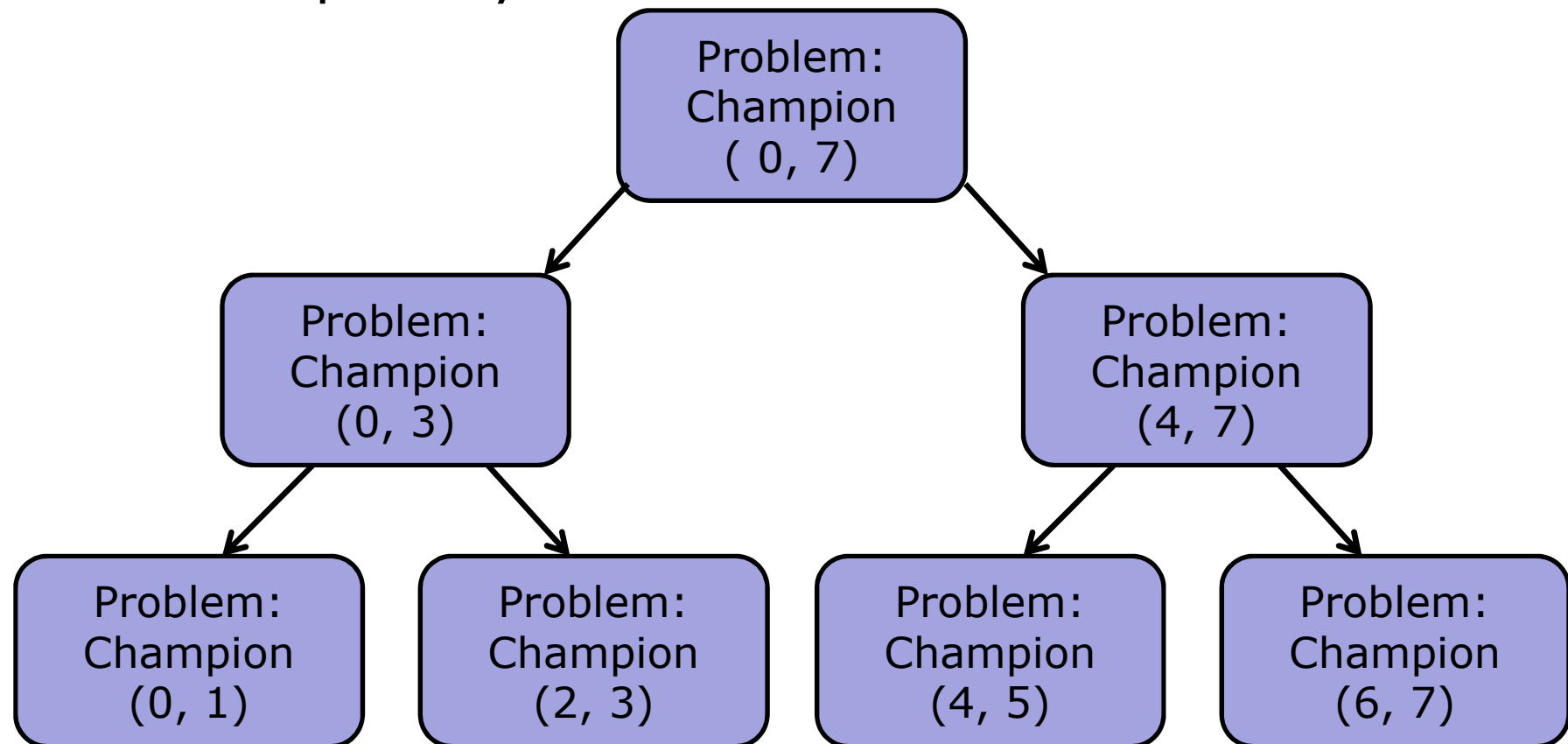
```
char sweet8[8] = {BRA, COL, FRA, GER, NED, CRC, ARG, BEL};
int Champion ( int s, int e )      //initially, s = 0, e = 7
{
    int m = (s + e)/2;
    Lwinner = Champion ( s, m ); //      s = 0, m = 3
    Rwinner = Champion ( m+1, e );//      m+1 = 4, e = 7

    return Winner ( Lwinner, Rwinner );
}
```

1.0 Introduction

(2) League VS Tournament

– Champion by tournament



1.0 Introduction

(2) League VS Tournament

- Champion by tournament

```
char sweet8[8] = {BRA, COL, FRA, GER, NED, CRC, ARG, BEL};  
int Champion ( int s, int e )      //initially, s = 0, e = 7  
{  
    int m = (s + e)/2;  
    Lwinner = Champion ( s, m ); //      s = 0, m = 3  
    Rwinner = Champion ( m+1, e );//      m+1 = 4, e = 7  
  
    return Winner ( Lwinner, Rwinner );  
}
```

- Do we miss something?

1.0 Introduction

(2) League VS Tournament

- Champion by tournament

```
char sweet8[8] = {BRA, COL, FRA, GER, NED, CRC, ARG, BEL};  
int Champion ( int s, int e )      //initially, s = 0, e = 7  
{  
    if ( s == e )  
        return s;  
  
    int m = (s + e)/2;  
    Lwinner = Champion ( s, m ); //      s = 0, m = 3  
    Rwinner = Champion ( m+1, e );//      m+1 = 4, e = 7  
  
    return Winner ( Lwinner, Rwinner );  
}
```

- Do we miss something? → **degenerate case**

1.0 Introduction

(2) League VS Tournament

– Champion by tournament

```
char sweet8[8] = {BRA, COL, FRA, GER, NED, CRC, ARG, BEL};
int Champion ( int s, int e )      //initially, s = 0, e = 7
{
    if ( s == e )                  → Degenerate case
        return s;

    int m = (s + e)/2;             → Divide

    Lwinner = Champion ( s, m );
    Rwinner = Champion ( m+1, e ); → Conquer

    return Winner ( Lwinner, Rwinner ); → Combine
}
```

1.0 Introduction

(3) Key idea of divide & conquer

- Solve a problem of n inputs by splitting the input into k subsets
- Three steps of divide & conquer
 - Divide
 - Breaking a problem into subproblems
 - Conquer
 - Recursively solving these subproblems
 - Combine (optional)
 - Appropriately combining their answers

1.0 Introduction

(4) Abstract algorithm for DnC (recursive)

```
global n, A(1:n);  
DnC ( int p, int q )  
{  
    int m;  
    if ( SMALL (p, q) )  
        return G (p, q);  
    else  
        m ← DIVIDE (p, q);  
        return COMBINE ( DnC (p, m), DnC (m+1, q) );  
}
```

- Most divide & conquer algorithms are implemented using recursive call

1.0 Introduction

(5) Three check points

- Same format
- Reduced problem size
- Degenerate case

1.0 Introduction

- Same format

```
global n, A(1:n);  
DnC ( int p, int q )  
{  
    int m;  
    if ( SMALL (p, q) )  
        return G (p, q);  
    else  
        m ← DIVIDE (p, q);  
        return COMBINE ( DnC (p, m), DnC (m+1, q) );  
}
```

1.0 Introduction

- Reduced problem size

```
global n, A(1:n);
DnC ( int p, int q )
{
    int m;
    if ( SMALL (p, q) )
        return G (p, q);
    else
        m ← DIVIDE (p, q);
        return COMBINE ( DnC (p, m), DnC (m+1, q) );
}
```

1.0 Introduction

- Degenerate case

```
global n, A(1:n);
DnC ( int p, int q )
{
    int m;
    if ( SMALL (p, q) )
        return G (p, q);
    else
        m ← DIVIDE (p, q);
        return COMBINE ( DnC (p, m), DnC (m+1, q) );
}
```

1.0 Introduction

(6) Performance analysis for DnC

- $T(n)$: time complexity of DnC () for n inputs
- $g(n)$: for small input
- $f_1(n)$: for DIVIDE ()
- $f_2(n)$: for COMBINE ()

$$T(n) = \begin{cases} g(n), & \text{for small } n \\ 2T(n/2) + f_1(n) + f_2(n), & \text{otherwise} \end{cases}$$

$$T(n) = \begin{cases} g(n), & \text{for small } n \\ aT(n/b) + O(n^d), & \text{otherwise} \end{cases}$$

1.0 Introduction

(7) The simplest divide & conquer algorithm

- Binary search:
 - Let $A = \{a_1, \dots, a_n\}$ be a list of elements which are sorted in nondecreasing order.
 - Determine whether x is in A or not. If x is in A , then find i such that $a_i = x$.

1.0 Introduction

- Straightforward search

```
int bsearch_straightforward ( int s, int e, int A[], int x )
{
    for ( i = s; i <= e; i++ ) {
        if ( A[i] == x )
            return i;
    }

    return NONE;
}
```

– Performance analysis?

1.0 Introduction

- Binary search (Divide & Conquer)

```
int bsearch ( int s, int e, int A[], int x )
{
    if ( s == e )
        if ( A[s] == x )
            return s;
        else
            return NONE;
    m ← (s+e)/2;
    if ( A[m] == x )
        return m;
    else if ( A[m] > x )
        return bsearch ( s, m, A, x );
    else
        return bsearch ( m + 1, e, A, x );
}
```

1.0 Introduction

- Binary search → check three key points

```
int bsearch ( int s, int e, int A[], int x )
{
    if ( s == e )
        if ( A[s] == x )
            return s;
        else
            return NONE;
    m ← (s+e)/2;
    if ( A[m] == x )
        return m;
    else if ( A[m] > x )
        return bsearch ( s, m, A, x );
    else
        return bsearch ( m + 1, e, A, x );
}
```

1.0 Introduction

- Binary search (Divide & Conquer)
 - Performance analysis

$$T(n) = \begin{cases} 1, & \text{for } n = 1 \\ T(n/2) + 1, & \text{otherwise} \end{cases}$$

$$T(n) = \log n$$

1.1 Recurrence relation

- Recurrence relation
 - An equation in which each term of the sequence is defined as a function of the preceding terms
 - Examples
 - $a_n = a_{n-1} + 2$
 - $f(n) = n f(n-1)$
 - $f(n) = f(n-1) + f(n-2)$
 - $f(n) = f(n/2) + n$
 - Solutions
 - Repeated substitution or telescoping
 - Guess & verification
 - Master theorem

1.1 Recurrence relation

Solution 1: Repeated substitution

- Continually substitute the recurrence relation on the right hand side
- Substitute a value into the original equation and then derive a previous version of the equation

– Examples

$$(1) \ T(n) = T(n-1) + n$$

$$T(1) = 1$$

$$(2) \ T(n) = 2T(n/2) + n$$

$$T(1) = 1$$

1.1 Recurrence relation

Solution 1: Repeated substitution

$$(1) \quad T(n) = T(n-1) + n$$

$$T(1) = 1$$

$$T(n) = T(n-1) + n$$

$$= (T(n-2) + (n-1)) + n$$

$$= (T(n-3) + (n-2)) + (n-1) + n$$

...

$$= T(1) + 2 + 3 + \dots + n$$

$$= 1 + 2 + \dots + n$$

$$= n(n+1)/2$$

$$= O(n^2)$$

1.1 Recurrence relation

Solution 1: telescoping

$$(2) \quad T(n) = 2T(n/2) + n$$

$$T(1) = 1$$

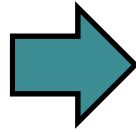
$$T(n) = 2T(n/2) + n$$

$$T(n/2) = 2T(n/4) + n/2$$

$$T(n/4) = 2T(n/8) + n/4$$

...

$$T(2) = 2T(1) + 2$$



$$T(n) = 2T(n/2) + n$$

$$2T(n/2) = 4T(n/4) + n$$

$$4T(n/4) = 8T(n/8) + n$$

...

$$2^{k-1} T(2) = 2^k T(1) + n$$

$(n = 2^k)$

$$\begin{aligned} T(n) &= n + n + \dots + n \\ &= (k+1) n = n \log n + n \\ &= O(n \log n) \end{aligned}$$

1.1 Recurrence relation

Solution 2: Guess & verification

- Guess the solution of the recurrence relation
- Verify that the solution is correct
- Examples

$$T(n) = 2T(n/2) + n$$

1.1 Recurrence relation

Solution 2: Guess & verification

– Examples

$$T(n) = 2T(n/2) + n$$

<Guess>

$$T(n) = O(n \log n), \text{ i.e. } T(n) \leq c n \log n$$

<Verification>

$$\begin{aligned} T(n) &= 2T(n/2) + n \\ &\leq 2c(n/2)\log(n/2) + n \\ &= cn\log n - cn\log 2 + n \\ &= cn\log n + (-c\log 2 + 1)n \\ &\leq cn\log n \end{aligned}$$

1.1 Recurrence relation

Solution 3: Master theorem

– if $T(n)$ is as follows,

$$T(n) = aT(n/b) + O(n^d)$$

then $T(n)$ is

$$T(n) = \begin{cases} O(n^d), & \text{if } d > \log_b a \\ O(n^d \log n), & \text{if } d = \log_b a \\ O(n^{\log_b a}), & \text{if } d < \log_b a. \end{cases}$$

1.1 Recurrence relation

- Performance comparison

Function	n					
	10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

1.1 Recurrence relation

- Exercise 2.5 (P72)

(j) $T(n) = 2T(n-1) + 1$

(k) $T(n) = T(\sqrt{n}) + 1$

- Exercise 2.13 (P73)

2.13. A binary tree is *full* if all of its vertices have either zero or two children. Let B_n denote the number of full binary trees with n vertices.

- (a) By drawing out all full binary trees with 3, 5, or 7 vertices, determine the exact values of B_3 , B_5 , and B_7 . Why have we left out even numbers of vertices, like B_4 ?
- (b) For general n , derive a recurrence relation for B_n .
- (c) Show by induction that B_n is $2^{\Omega(n)}$.

1.2 Multiplication

- Multiplying two integers of n-digit.
 - u & v: n-digit integers
 - Time for adding u & v: $O(n)$

Carry: 1

	1	1	0	1	0	1	(53)
	1	0	0	0	1	1	(35)
1	0	1	1	0	0	0	(88)

1.2 Multiplication

- Multiplying two integers of n-digit.
 - Time for multiplying u & v: $O(n^2)$

				1	1	0	1	
				1	0	1	1	
			×	1	0	1	1	
					1	1	0	1
								(1101 times 1)
				1	1	0	1	
								(1101 times 1, shifted once)
		0	0	0	0			
								(1101 times 0, shifted twice)
+	1	1	0	1				
								(1101 times 1, shifted thrice)
	1	0	0	0	1	1	1	1
								(binary 143)

- Can we improve it by divide & conquer?

1.2 Multiplication

- Gauss's original suggestion
 - $(a + b i) (c + d i) = ac - bd + (ad + bc) i$
 - How many multiplications?
 - Actually, 3 instead of 4
 - $ad + bc = (a + b)(c + d) - ac - bd.$

1.2 Multiplication

- Two binary numbers x & y with length n

$$x = [x_L][x_R] = 2^s x_L + x_R$$

$$y = [y_L][y_R] = 2^s y_L + y_R, \text{ where } s = \left\lceil \frac{n}{2} \right\rceil$$

$$xy = (2^s x_L + x_R)(2^s y_L + y_R)$$

$$= 2^n x_L y_L + 2^s (x_L y_R + x_R y_L) + x_R y_R$$

$$- T(n) \rightarrow 4 T(n/2) + O(n)$$

$$x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R$$

$$- T(n) \rightarrow 3 T(n/2) + O(n)$$

1.2 Multiplication

- Divide u & v such that

$$u = w \times 10^s + x$$

$$v = y \times 10^s + z, \text{ where } s = \left\lceil \frac{n}{2} \right\rceil$$

$$uv = (w \times 10^s + x)(y \times 10^s + z)$$

$$= wy \times 10^{2s} + (wz + xy) \times 10^s + xz$$

$$- T(n) \rightarrow 4 T(n/2) + n$$

$$(wz + xy) = (w + x)(y + z) - wy - xz$$

$$- T(n) \rightarrow 3 T(n/2) + n$$

1.2 Multiplication

- Algorithm

```
void multiply ( int u, int v )  
{  
    Degenerate case  
    Divide  
    Conquer  
    Combine  
}
```

1.2 Multiplication

- Algorithm

```
void multiply ( int u, int v )
{
    n ← min ( digit of u, digit of v );
    if ( n is small enough )
        return u * v;
    s ← n div 2;
    w ← u div 10s;
    x ← u mod 10s;
    y ← v div 10s;
    z ← v mod 10s;

    r ← multiply ( w + x, y + z );
    p ← multiply ( w, y );
    q ← multiply ( x, z );

    return p * 102s + (r - p - q) * 10s + q;
}
```

1.2 Multiplication

- Algorithm → Check three points

```
void multiply ( int u, int v )
{
    n ← min ( digit of u, digit of v );
    if ( n is small enough )
        return u * v;
    s ← n div 2;
    w ← u div 10s;
    x ← u mod 10s;
    y ← v div 10s;
    z ← v mod 10s;

    r ← multiply ( w + x, y + z );
    p ← multiply ( w, y );
    q ← multiply ( x, z );

    return p * 102s + (r - p - q) * 10s + q;
}
```

1.2 Multiplication

- Performance analysis
 - Recurrence relation

$$T(n) = 3T(n/2) + O(n)$$

- $a = 3, b = 2, k = 1.$
- $a = 3 > b^k = 2^1,$

$$T(n) = O(n^{\log_2 3})$$

1.2 Multiplication

- Comparison

	degenerate case	divide	conquer	combine	performance
tournament	$n = 1$ ($s = e$)	$m = (s+e)/2$	champ (s, m); champ ($m+1, e$);	win (LW, RW);	$2T(n/2) + O(1)$ $= O(n)$
binary search	$n = 1$ ($s = e$)	$m = (s+e)/2$	bs ($s, m-1$); or bs ($m+1, e$);	-	$T(n/2) + O(1)$ $= O(\log n)$
integer multiplication	$n = 1$	$s = n/2$; $w = u \text{ div } 2^s$;	mult ($w+x, y+z$); mult (w, y); mult (x, z);	$P 10^n +$ $(r - p - q) 10^s +$ q ;	$3T(n/2) + O(n)$ $= O(n^{\log_2 3})$
Merge sort					
quick sort					
median					
matrix multiplication					

1.3 Sorting

- Problem:
 - Arrange the elements of a set in an ascending (or descending) order
 - $[2, 1, 4, 3, 7, 5, 6] \rightarrow [1, 2, 3, 4, 5, 6, 7]$
- Sorting algorithms
 - $O(n^2)$ algorithms for pairwise comparison
 - Bubble sort
 - Insertion sort
 - Selection sort
 -
 - Can it be faster? \rightarrow Use divide & conquer
 - Merge sort
 - Quick sort

1.3 Sorting

- Merge sort
 - Given a sequent of n elements $\{ a_1, a_2, \dots, a_n \}$, split them into two set.
 - Sort each set individually and merge them to produce a single sorted sequence of n elements.
 - Three steps
 - Divide
 - Split the list into two halves
 - Conquer
 - Recursively sort each half
 - Combine
 - Merge the two sorted sublists

1.3 Sorting

- Merge sort (Divide & Conquer)

```
void msort( int s, int e, int A[] )
{
    if ( s == e )
        return;
    m ← (s+e)/2;
    msort ( s, m, A );
    msort ( m+1, e, A );
    merge ( s, m, e, A );
}
```

1.3 Sorting

- Merge sort (Divide & Conquer)

```
void merge( int s, int m, int e, int A[] )
{
    int lpitr = s, rpitr = m+1, bptr = 0;
    int *B = (int *) calloc ( e - s + 1, sizeof(int) );

    while ( lpitr <= m && rpitr <= e ) {
        if ( A[lpitr] < A[rpitr] )
            B[bptr++] = A[lpitr++];
        else
            B[bptr++] = A[rpitr++];
    }
    if ( lpitr > m )
        for ( int i = rpitr; i <= e; i++ )
            B[bptr++] = A[i];
    if ( rpitr > e )
        for ( int i = lpitr; i <= m; i++ )
            B[bptr++] = A[i];
    A ← B;
}
```

1.3 Sorting

- Merge sort (Check three points)

```
void msort( int s, int e, int A[] )
{
    if ( s == e )
        return;
    m ← (s+e)/2;
    msort ( s, m, A );
    msort ( m+1, e, A );
    merge ( s, m, e, A );
}
```

1.3 Sorting

- Merge sort (Example)

16	12	5	38	19	4	20	27
----	----	---	----	----	---	----	----

1.3 Sorting

- Merge sort (Performance analysis)
 - Recurrence relation

$$T(n) = \begin{cases} 2T(n/2) + cn, & n > 1 \\ a, & n = 1 \end{cases}$$

$$T(n) = an + cn \log n$$

1.3 Sorting

- Iterative merge sort
 - A merge sort without recursive call

```
function iterative-mergesort( $a[1 \dots n]$ )  
Input: elements  $a_1, a_2, \dots, a_n$  to be sorted  
  
 $Q = []$  (empty queue)  
for  $i = 1$  to  $n$ :  
    inject( $Q, [a_i]$ )  
while  $|Q| > 1$ :  
    inject( $Q, \text{merge}(\text{eject}(Q), \text{eject}(Q))$ )  
return eject( $Q$ )
```

1.3 Sorting

- Comparison

	degenerate case	divide	conquer	combine	performance
tournament	$n = 1$ ($s = e$)	$m = (s+e)/2$	champ (s, m); champ ($m+1, e$);	win (LW, RW);	$2T(n/2) + O(1)$ $= O(n)$
binary search	$n = 1$ ($s = e$)	$m = (s+e)/2$	bs ($s, m-1$); or bs ($m+1, e$);	-	$T(n/2) + O(1)$ $= O(\log n)$
integer multiplication	$n = 1$	$s = n/2$ $w = u \text{ div } 2^s$;	mult ($w+x, y+z$); mult (w, y); mult (x, z);	$P 10^n +$ $(r - p - q) 10^s +$ q ;	$3T(n/2) + O(n)$ $= O(n^{\log_2 3})$
Merge sort	$n = 1$ ($s = e$)	$m = (s+e)/2$	ms (s, m); ms ($m+1, e$);	merge (s, m, e);	$2T(n/2) + O(n)$ $= O(n \log n)$
quick sort					
median					
matrix multiplication					

속제 1. String generation

```
char *alpha;
void kstring(int n, int k)
{
    if (n <= 0) {
        cnt++;
        if (strcmp(str, A) == 0) {
            printf("%d matched\n", cnt);
            return;
        }
    }
    for (int i = 0; i < k; i++) {
        A[N - n] = alpha[i];
        kstring(n - 1, k);
    }
}
```

1.3 Sorting

- Quick sort
 - Given a sequence of n elements $\{ a_1, a_2, \dots, a_n \}$, split them into two set.
 - Rearrange the elements so that merging later is not necessary.
 - Rearrange such that $a_i \leq a_j$ for all i between 1 and m and j between $m+1$ and $n \rightarrow$ partitioning
 - Two steps
 - Divide
 - Split the list into two halves (with partitioning)
 - Conquer
 - Recursively sort each half

1.3 Sorting

- partition

```
int partition ( int s, int e, int A[] )
{
    int pivot, left, right;
    left = s+1;                right = e;                pivot = A[s];
    while (left < right) {
        while ((A[right] >= pivot) && (left < right))
            right--;

        while ((A[left] <= pivot) && (left < right))
            left++;

        if ( left < right )
            swap ( A[left], A[right] );
    }
    A[left] = pivot;
    return left;
}
```

1.3 Sorting

- Quick sort (Divide & Conquer)

```
void quick_sort( int s, int e, int A[] )
{
    if ( s >= e )
        return;
    int m = partition ( s, e, A );
    quick_sort ( s, m-1, A );
    quick_sort ( m+1, e, A );
}
```

1.3 Sorting

- Quick sort (Example)

16	12	5	38	19	4	20	27
----	----	---	----	----	---	----	----

1.3 Sorting

- Quick sort (Performance analysis)
 - Recurrence relation

$$T(n) = (n + 1) + \frac{1}{n} \sum_{1 \leq k \leq n} (T(k - 1) + T(n - k))$$

1.3 Sorting

- Quick sort (Performance analysis)
 - Comparison to merge sort

	Example	Quick sort	Merge sort
Best case	5 1 6 3 4 8 7 2	$O(n \log n)$	$O(n \log n)$
Worst case	1 2 3 4 5 6 7 8 8 7 6 5 4 3 2 1	$O(n^2)$	$O(n \log n)$
Average case	4 1 3 6 5 2 9 8	$O(n \log n)$	$O(n \log n)$

1.3 Sorting

- Comparison

	degenerate case	divide	conquer	combine	performance
tournament	$n = 1$ ($s = e$)	$m = (s+e)/2$	champ (s, m); champ ($m+1, e$);	win (LW, RW);	$2T(n/2) + O(1)$ $= O(n)$
binary search	$n = 1$ ($s = e$)	$m = (s+e)/2$	bs ($s, m-1$); or bs ($m+1, e$);	-	$T(n/2) + O(1)$ $= O(\log n)$
integer multiplication	$n = 1$	$s = n/2$	mult ($w+x, y+z$); mult (w, y); mult (x, z);	$P 10^n +$ $(r - p - q) 10^s +$ q ;	$3T(n/2) + O(n)$ $= O(n^{\log_2 3})$
Merge sort	$n = 1$ ($s = e$)	$m = (s+e)/2$	ms (s, m); ms ($m+1, e$);	merge (s, m, e);	$2T(n/2) + O(n)$ $= O(n \log n)$
quick sort	$n = 1$ ($s = e$)	$m =$ partition ();	qs ($s, m-1$); qs ($m+1, e$);	-	$O(n \log n)$
median					
matrix multiplication					

1.4 Median (Select k-th)

- Median
 - The 50th percentile element of a list
 - Ex: Median of [45, 1, 10, 30, 25] → 25
- Generalized problem:
 - Select the k-th smallest element among a set of n unsorted elements $\{a_1, a_2, \dots, a_n\}$.
- Key idea:
 - Use **partition ()** in quick sort algorithm
 - The partition () returns the position of the pivot → m
 - If $k < m$, then select k-th in the left subset
 - Else, find select (k-m)-th in the right subset

1.4 Median (Select k-th)

- Select (Divide & Conquer)

```
void select_kth ( int k, int s, int e )
{
    if ( s == e )
        return A[s];

    int m = partition ( s, e );

    if ( k < m )
        select_kth ( k, s, m - 1 );
    else if ( k > m )
        select_kth ( k-m, m + 1, e );
    else
        return A[k];
}
```

1.4 Median (Select k-th)

- Selection (Performance analysis)
 - Recurrence relation

$$T(n) = n + T(n / 2)$$

1.4 Median (Select k-th)

- Comparison

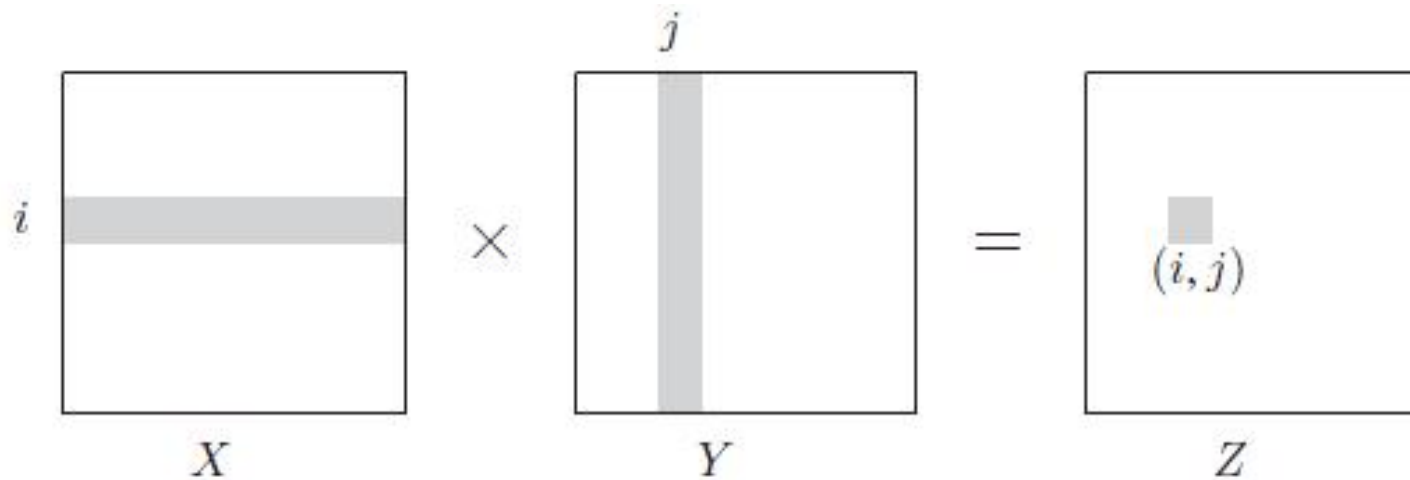
	degenerate case	divide	conquer	combine	performance
tournament	$n = 1$ ($s = e$)	$m = (s+e)/2$	champ (s, m); champ ($m+1, e$);	win (LW, RW);	$2T(n/2) + O(1)$ $= O(n)$
binary search	$n = 1$ ($s = e$)	$m = (s+e)/2$	bs ($s, m-1$); or bs ($m+1, e$);	-	$T(n/2) + O(1)$ $= O(\log n)$
integer multiplication	$n = 1$	$s = n/2$	mult ($w+x, y+z$); mult (w, y); mult (x, z);	$P \cdot 10^n +$ $(r - p - q) \cdot 10^s$ $+ q$;	$3T(n/2) + O(n)$ $= O(n^{\log_2 3})$
merge sort	$n = 1$ ($s = e$)	$m = (s+e)/2$	ms (s, m); ms ($m+1, e$);	merge (s, m, e);	$2T(n/2) + O(n)$ $= O(n \log n)$
quick sort	$n = 1$ ($s = e$)	$m =$ partition ();	qs ($s, m-1$); qs ($m+1, e$);	-	$2T(n/2) + O(n)$ $= O(n \log n)$
median (find k-th)	$n = 1$ ($s = e$)	$m =$ partition ();	select ($k, s, m-1$); or select($k-m,$ $m+1, e$);	-	$T(n/2) + O(n)$ $= O(n)$
matrix multiplication					

1.5 Matrix Multiplication

- Multiplying two matrices: $Z = XY$
 - X & Y : $n \times n$ matrices

- Time complexity: $O(n^3)$

$$Z_{i,j} = \sum_{k=1}^n X_{ik} Y_{kj}$$



1.5 Matrix Multiplication

- Multiplying two matrices: $Z = XY$
 - X & Y: $n \times n$ matrices

```
void mat_mult( int **Z, int **X, int **Y, int n )
{
    int i, j, k;

    for ( i = 0; i < n; i++ )
        for ( j = 0; j < n; j++ )
            for ( k = 0, Z[i][j] = 0; k < n; k++ )
                Z[i][j] += X[i][k]*Y[k][j];
}
```

1.5 Matrix Multiplication

- Improve the performance using DnC
 - Divide X & Y into 2 × 2 groups

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

$$Z = XY = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

- 8 (n/2) × (n/2) multiplications
- Matrix addition → O(n²)

1.5 Matrix Multiplication

- Performance analysis
 - 8 $(n/2) \times (n/2)$ multiplications $\rightarrow 8 T(n/2)$
 - Matrix addition $\rightarrow O(n^2)$

$$T(n) = 8T(n/2) + O(n^2)$$

$$= O(n^3)$$

- No improvement !!

1.5 Matrix Multiplication

- Decomposing and assembling multiplications

$$P_1 = A(F - H)$$

$$P_5 = (A + D)(E + H)$$

$$P_2 = (A + B)H$$

$$P_6 = (B - D)(G + H)$$

$$P_3 = (C + D)E$$

$$P_7 = (A - C)(E + F)$$

$$P_4 = D(G - E)$$

$$XY = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

$$= \begin{bmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_1 + P_5 - P_3 - P_7 \end{bmatrix}$$

1.5 Matrix Multiplication

- Improvement of performance
 - 7 $(n/2) \times (n/2)$ multiplications $\rightarrow 7 T(n/2)$
 - Matrix addition $\rightarrow O(n^2)$

$$T(n) = 7T(n/2) + O(n^2)$$

$$= O(n^{\log_2 7})$$

$$\approx O(n^{2.81})$$

1.4 Median (Select k-th)

- Comparison

	degenerate case	divide	conquer	combine	performance
tournament	$n = 1$ ($s = e$)	$m = (s+e)/2$	champ (s, m); champ ($m+1, e$);	win (LW, RW);	$2T(n/2) + O(1)$ $= O(n)$
binary search	$n = 1$ ($s = e$)	$m = (s+e)/2$	bs ($s, m-1$); or bs ($m+1, e$);	-	$T(n/2) + O(1)$ $= O(\log n)$
integer multiplication	$n = 1$	$s = n/2$	mult ($w+x, y+z$); mult (w, y); mult (x, z);	$P 10^n +$ $(r - p - q) 10^s$ $+ q$;	$3T(n/2) + O(n)$ $= O(n^{\log_2 3})$
merge sort	$n = 1$ ($s = e$)	$m = (s+e)/2$	ms (s, m); ms ($m+1, e$);	merge (s, m, e);	$2T(n/2) + O(n)$ $= O(n \log n)$
quick sort	$n = 1$ ($s = e$)	$m =$ partition ();	qs ($s, m-1$); qs ($m+1, e$);	-	$2T(n/2) + O(n)$ $= O(n \log n)$
median (find k-th)	$n = 1$ ($s = e$)	$m =$ partition ();	select ($k, s, m-1$); or select($k-m,$ $m+1, e$);	-	$T(n/2) + O(n)$ $= O(n)$
matrix multiplication	$n = 1$	$X \rightarrow A, B, C, D$ $Y \rightarrow E, F, G, H$	P_1, \dots, P_7	combine $P_1 \sim P_7$	$7T(n/2) +$ $O(n^2)$

1.6 Finding the closest pair of points

- Problem (2.32 at P78)

CLOSEST PAIR

Input: A set of points in the plane, $\{p_1 = (x_1, y_1), \dots, p_n = (x_n, y_n)\}$

Output: The closest pair of points: that is, the pair $p_i \neq p_j$ for which the distance between p_i and p_j , that is,

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2},$$

is minimized.

For simplicity, assume that n is a power of two, and that all the x -coordinates x_i are distinct, as are the y -coordinates.

1.6 Finding the closest pair of points

Here's a high-level overview of the algorithm:

- Find a value x for which exactly half the points have $x_i < x$, and half have $x_i > x$. On this basis, split the points into two groups, L and R .
- Recursively find the closest pair in L and in R . Say these pairs are $p_L, q_L \in L$ and $p_R, q_R \in R$, with distances d_L and d_R respectively. Let d be the smaller of these two distances.
- It remains to be seen whether there is a point in L and a point in R that are less than distance d apart from each other. To this end, discard all points with $x_i < x - d$ or $x_i > x + d$ and sort the remaining points by y -coordinate.
- Now, go through this sorted list, and for each point, compute its distance to the *seven* subsequent points in the list. Let p_M, q_M be the closest pair found in this way.
- The answer is one of the three pairs $\{p_L, q_L\}$, $\{p_R, q_R\}$, $\{p_M, q_M\}$, whichever is closest.

1.6 Finding the closest pair of points

- (a) In order to prove the correctness of this algorithm, start by showing the following property: any square of size $d \times d$ in the plane contains at most four points of L .
- (b) Now show that the algorithm is correct. The only case which needs careful consideration is when the closest pair is split between L and R .
- (c) Write down the pseudocode for the algorithm, and show that its running time is given by the recurrence:

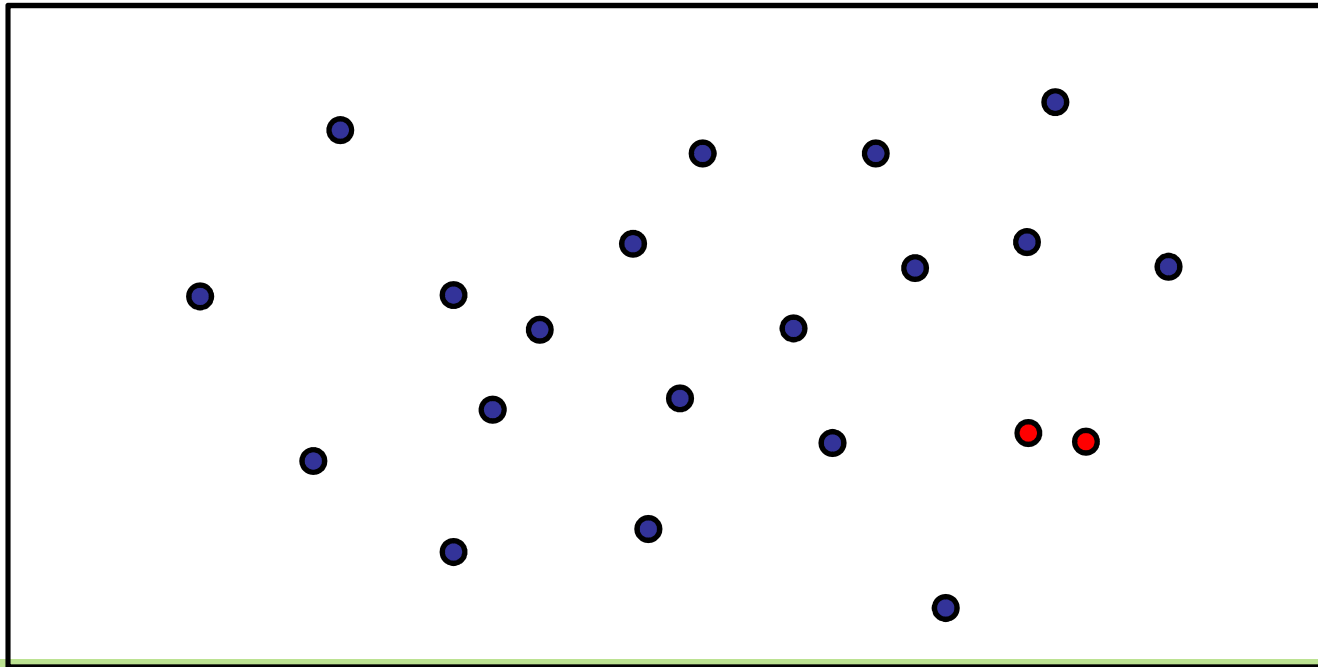
$$T(n) = 2T(n/2) + O(n \log n).$$

Show that the solution to this recurrence is $O(n \log^2 n)$.

- (d) Can you bring the running time down to $O(n \log n)$?

1.6 Finding the closest pair of points

- Problem
 - Closest pair
 - Given n points in 2D: $\{p_1, \dots, p_n\}$, where $p_i = (x_i, y_i)$.
 - Distance between two points: $d(p_i, p_j)$
 - Find i & j such that $d(p_i, p_j)$ is minimum among all i & j



1.6 Finding the closest pair of points

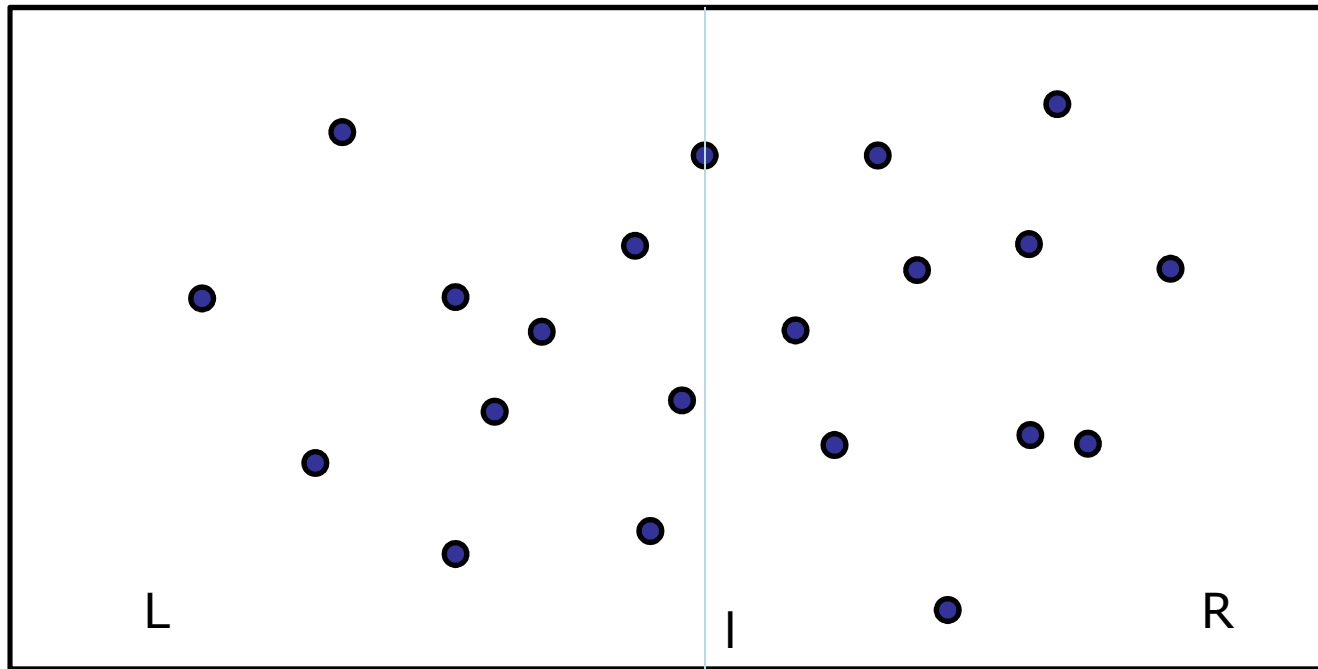
- Solution
 - Bruteforce algorithm
 - Compare pairwise

```
void closest_pair( int n, Point p )
{
    float min_dist = MAX;
    int min_pair[2];
    for ( i = 0; i < n; i++ ) {
        for ( j = i+1; j < n; j++ ) {
            if ( d (p[i], p[j]) < min_dist )
                min_dist = d (p[i], p[j]);
            min_pair = (i, j);
        }
    }
}
```

- Time complexity?

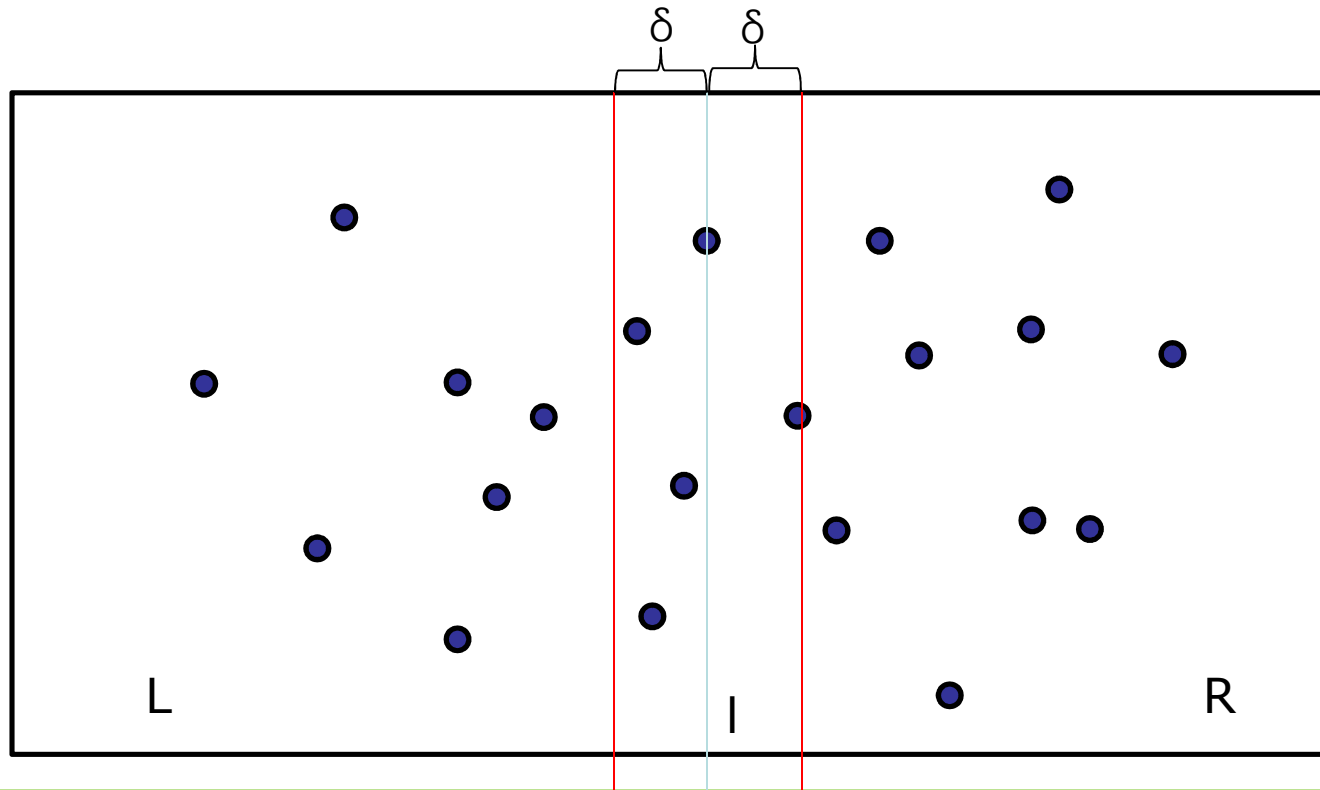
1.6 Finding the closest pair of points

- Solution
 - Divide and conquer
 - Sort the points in x-coordinate
 - Set a line $l = x_{\text{median}}$
 - Divide points as $L = \{p_i \mid x_i < l\}$ and $R = \{p_i \mid x_i > l\}$



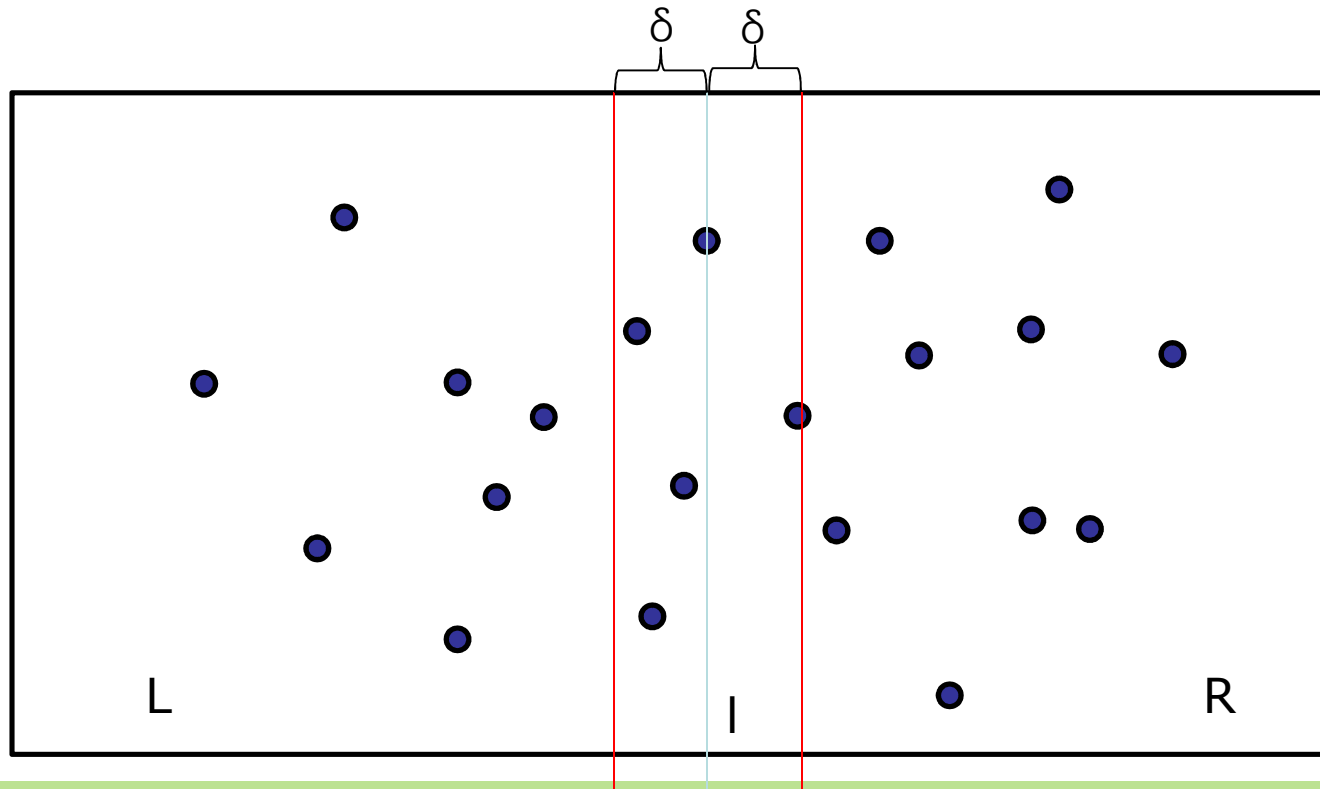
1.6 Finding the closest pair of points

- Solution
 - Divide and conquer
 - $\text{Min_pair}(P) = \text{Min_pair}(L) + \text{Min_Pair}(R) + \text{Merge}$
 - $\delta = \min(\text{Min_pair}(L), \text{Min_pair}(R))$



1.6 Finding the closest pair of points

- Solution
 - Divide and conquer
 - Merge finds closest pair only for points in $(l - \delta, l + \delta)$
 - It takes $O(n)$



1.6 Finding the closest pair of points

- Solution
 - Divide and conquer algorithm

```
float closest_pair( int n, Point p )
{
    if ( n == 2 )
        return d (P[0], P[1]);
    l = median_X ( P );
    float lmin = closest_pair ( L );
    float rmin = closest_pair ( R );
    float delta = min (lmin, rmin);
    float mmin = merge ( delta );
    return min ( mmin, delta );
}
```

1.6 Finding the closest pair of points

- Analysis
 - Time complexity
 - Sorting in x-coordinate: $O(n \log n)$
 - Closest-pair

$$\begin{aligned} T(n) &= 2T(n/2) + O(n) \\ &= O(n \log n) \end{aligned}$$

- In total, $O(n \log n)$

1. Divide & Conquer

1.0 Introduction

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