알고리즘

04. Dynamic Programming

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미디어소프트웨어학과
민경하

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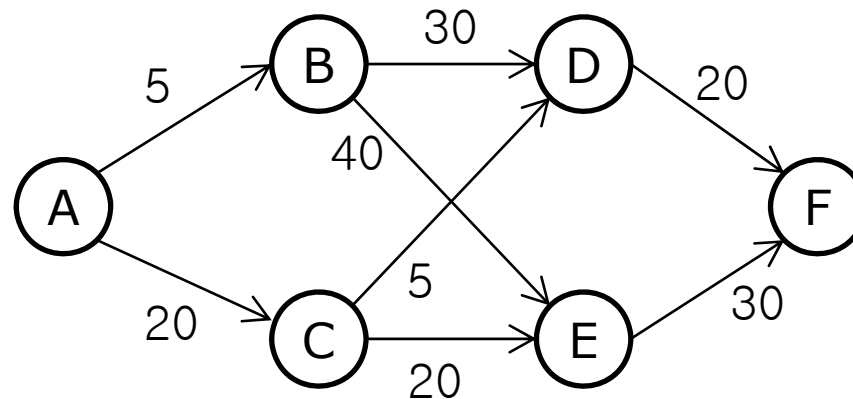
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4.0 Introduction

- Key idea of dynamic programming (1)
  - Comes from greedy algorithm
    - A sequence of selections
    - The counterexample of greedy algorithm
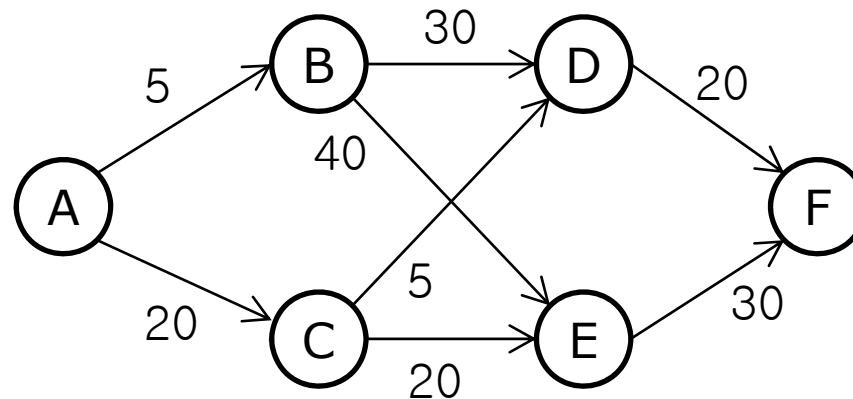   - Find a shortest path from A to F: Cost(A → F)?

Can a greedy algorithm solve this problem?
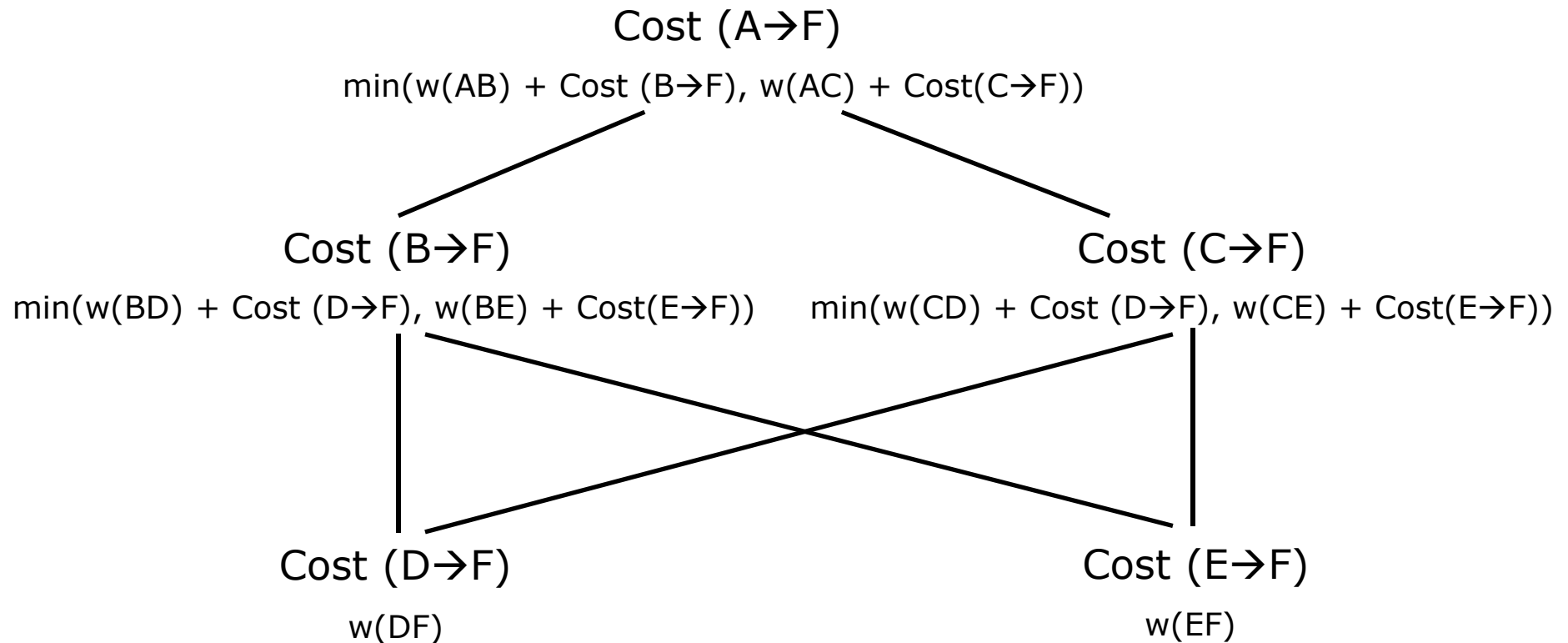
4.0 Introduction

- Key idea of dynamic programming (2)
  - Comes from divide & conquer
    - Decomposing a problem into a set of sub-problems
    - Recursive structure of solution
      - Cost (A → F)?

Cost (A→F) = min(w(AB) + Cost (B→F), w(AC) + Cost(C→F))

- Key idea of dynamic programming (3)
  - Recursive decomposition of a problem

Cost (A→F)

min(w(AB) + Cost (B→F), w(AC) + Cost(C→F))

Cost (B→F)   Cost (C→F)

min(w(BD) + Cost (D→F), w(BE) + Cost(E→F))   min(w(CD) + Cost (D→F), w(CE) + Cost(E→F))

Cost (D→F)   Cost (E→F)

w(DF)   w(EF)

4.0 Introduction

- Principle of optimality

> Whatever the initial state and decision are,
> the remaining decisions must constitute
> an optimal decision sequence with regard to
> the state resulting from the first decision.

- Leads to an recurrence relation
- Two approaches
  - There are two ways to formulate the recurrence relations
  - Forward approach
    - The formulation for decision x_i is made in terms of optimal decision sequences $x_{i+1}, …, x_n$.
  - Backward approach
    - The formulation for decision x_i is made in terms of optimal decision sequences $x_1, …, x_{i-1}$.

4.1 0/1 Knapsack

- Problem:
  - We are given n objects and a knapsack.

  - Object **i** has a weight w_i and a profit p_i, and the knapsack has a capacity **M**.

  - If a fraction x_i, $0 \leq x_i \leq 1$, of object i is placed into the knapsack, then the profit of $p_i x_i$ is earned.

  - The objective is to obtain a filling of the knapsack that maximizes the total profit earned.

$$\text{Maximize} \sum_{1 \leq i \leq n} p_i x_i \text{ subject to} \sum_{1 \leq i \leq n} w_i x_i \leq M$$

  - The solution is a set $(x_1, x_2, ..., x_n)$

4.1 0/1 Knapsack

- Strategy:
  - At each step, we include that object which has the maximum profit per unit of capacity.

  - In the order of the ratio p_i / w_i.

- Example:
  - n = 3
  - M = 20
  - $(p_1, p_2, p_3) = (25, 24, 15)$
  - $(w_1, w_2, w_3) = (18, 15, 10)$.

  - What is the solution (x_1, x_2, x_3) ?

4.1 0/1 Knapsack

Review: Knapsack (Greedy algorithm)

- Counter example:
  - Can a greedy algorithm solve the following knapsack problem when x_i is only 0 or 1?

- Example:
  - n = 6,
  - M = 100,
  - $(p_1, p_2, p_3, p_4, p_5, p_6)$ = (40, 35, 18, 4, 10, 2),
  - $(w_1, w_2, w_3, w_4, w_5, w_6)$ = (100, 50, 45, 20, 10, 5).

  - What is the solution ?

4.1 0/1 Knapsack

- Principle of optimality
  - The key background of dynamic programming

  - The property of an optimal sequence of decisions is

  > Whatever the initial state and decision are, the remaining decisions must constitute an optimal decision sequence with regard to the state resulting from the first decision.

- Problem
  - Similar to the knapsack problem except that x_i can be either 0 or 1

  - KNAP (1, n, M)

$$\text{maximize} \sum_{i=1}^{n} p_i x_i$$

$$\text{subject to} \sum_{i=1}^{n} w_i x_i \leq M$$

$$x_i = 0 \text{ or } 1$$

4.1 0/1 Knapsack

- Strategy
  - Build a recurrence relation using KNAP (1, n, M)

  - KNAP (1, n, M)
    - If $x_1 = 0$, then KNAP(1, n, M) = KNAP (2, n, M)

    - If $x_1 = 1$, then
      KNAP(1, n, M) = KNAP (2, n, M − w_1)

- Strategy
  - Let $g_j(y)$ be the value of an optimal solution to KNAP $(j+1, n, y)$.

  - $g_0(M)$ is the value of an optimal solution of KNAP $(1, n, M)$.

$$g_0(M) = \max\{g_1(M),\ g_1(M - w_1)\ +\ p_1\}$$

$$g_i(y) = \max\{g_{i+1}(y),\ g_{i+1}(y - w_{i+1})\ +\ p_{i+1}\}$$

- Strategy
  - Degenerate case

  - $$g_{n-1}(M') = \begin{cases} p_n, & if\ M' \geq w_n \\ 0, & otherwise \end{cases}$$

  - $g_i(M') = 0,\ if\ M' < \min(w_{i+1}, \ldots, w_n)$

  - $g_i(M') = \sum_{k=i+1}^{n} p_k,\ \ if\ \sum_{k=i+1}^{n} w_k \leq M'$
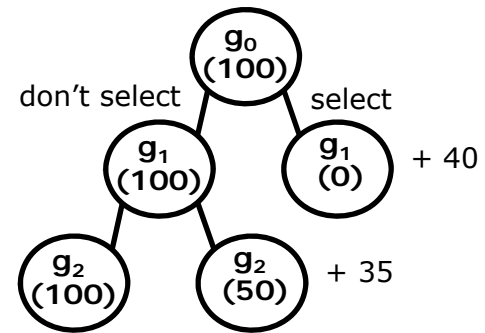
4.1 0/1 Knapsack

- Example
  - $n = 6$,
  - $M = 100$,
  - $(p_1, p_2, p_3, p_4, p_5, p_6) = (40, 35, 18, 4, 10, 2)$,
  - $(w_1, w_2, w_3, w_4, w_5, w_6) = (100, 50, 45, 20, 10, 5)$.

  - What is the solution ?

  - Smart degenerate case?

  $$g_i(M) = \sum_{k=i+1}^{n} p_k, if \sum_{k=i+1}^{n} w_k < M$$

4.1 0/1 Knapsack

4.1 0/1 Knapsack

```
int KNAP( int i, int n, int capacity )
{
    if ( capacity < min_w ( i, n ) )
        return 0;

    if ( capacity > sum_w ( i, n ) )
        return sum_p ( i, n );

    return max ( KNAP (i-1, n, capacity-w[i]) + p[i]),
                 KNAP (i-1, n, capacity );
}
```
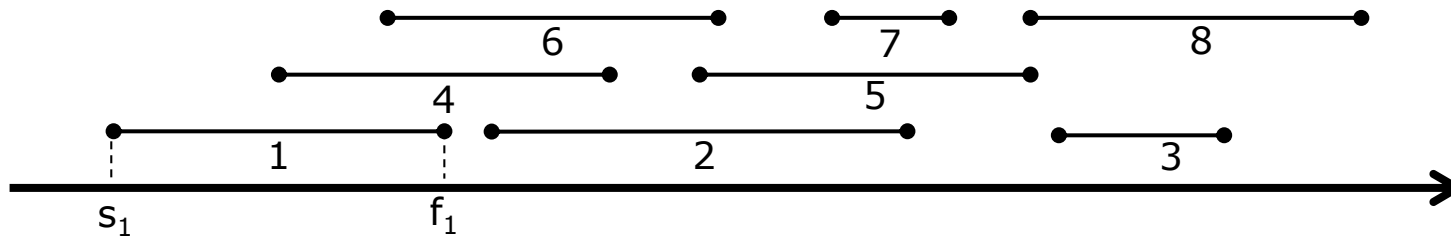
4.2 (Weighted) interval scheduling

- Interval scheduling
  - A set of n request: {1, ..., n}
  - i^{th} request
    - {start time (s_i), finish time (f_i)}
  - Compatible
    - A subset of requests is **compatible**, if no two of them overlap in time.
 - Goal
    - Find a set of maximum compatible subsets
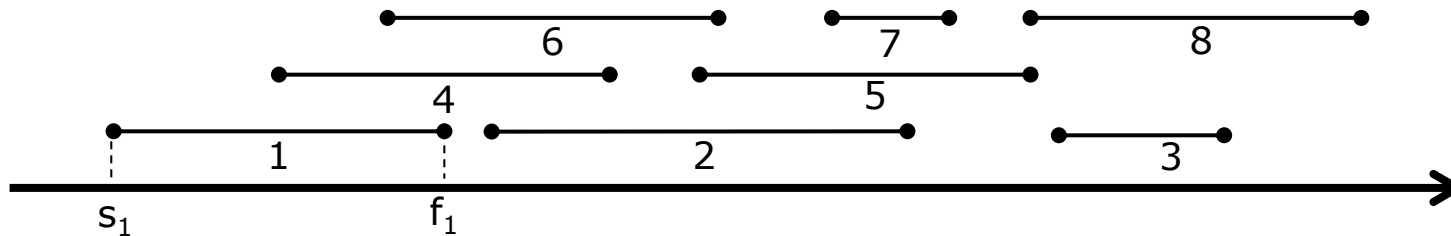
- Designing a greedy algorithm
  - Basic idea
    - Select a first request i_1.
    - Reject all the requests that are not compatible with i_1.
    - Select the next request i_2.
    - Reject all the requests that are not compatible with i_2.
    - Repeat this process until we run out of requests.

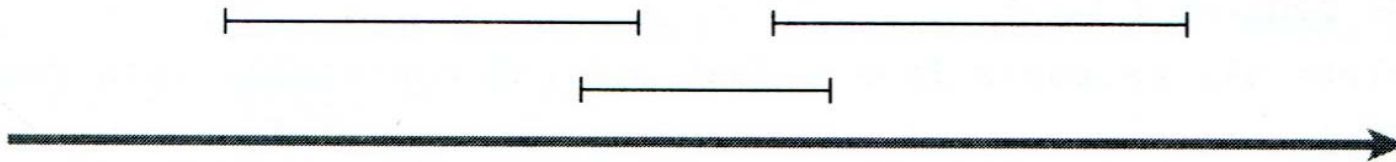 - Greedy algorithm
    - Which rule to select the requests.

- Designing a greedy algorithm
  - Rule 1.
    - Choose the earliest starting interval.
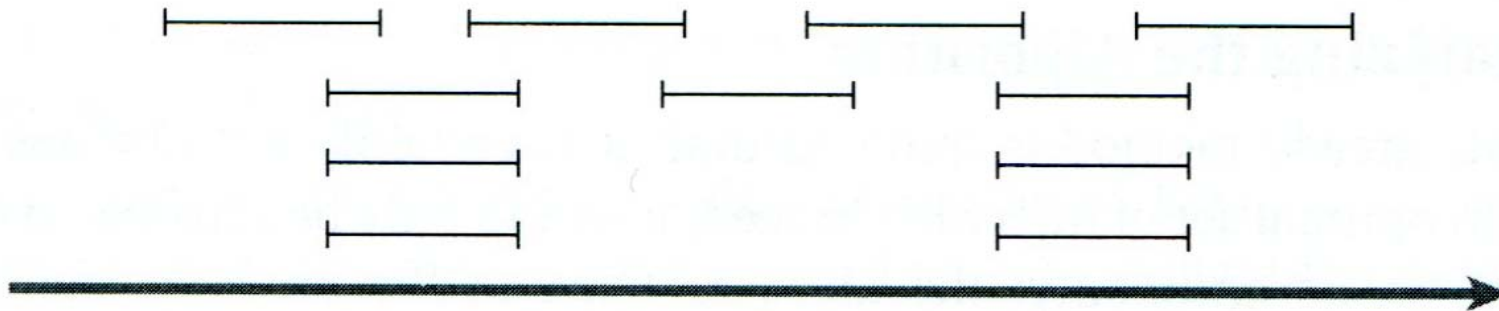   - Choose an interval i such that s_i is minimum.

    - Counterexample
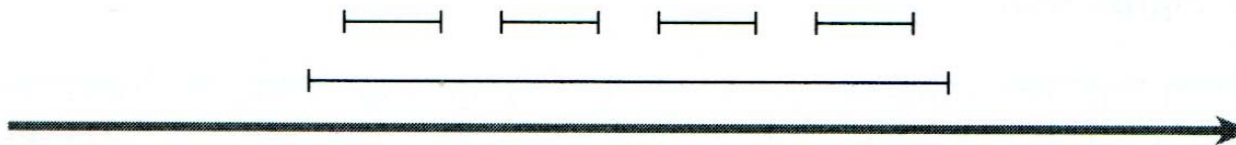
4.2 (Weighted) interval scheduling

- Designing a greedy algorithm
  - Rule 2.
    - Choose the shortest interval.
    - Choose an interval i such that $f_i - s_i$ is minimum.

    - Counterexample

- Designing a greedy algorithm
  - Rule 3.
    - Choose the interval with least conflicts.
    - Choose an interval i that overlaps least intervals.

    - Counterexample

4.2 (Weighted) interval scheduling

- Designing a greedy algorithm
  - Rule 4.
    - Choose the earliest finishing interval.
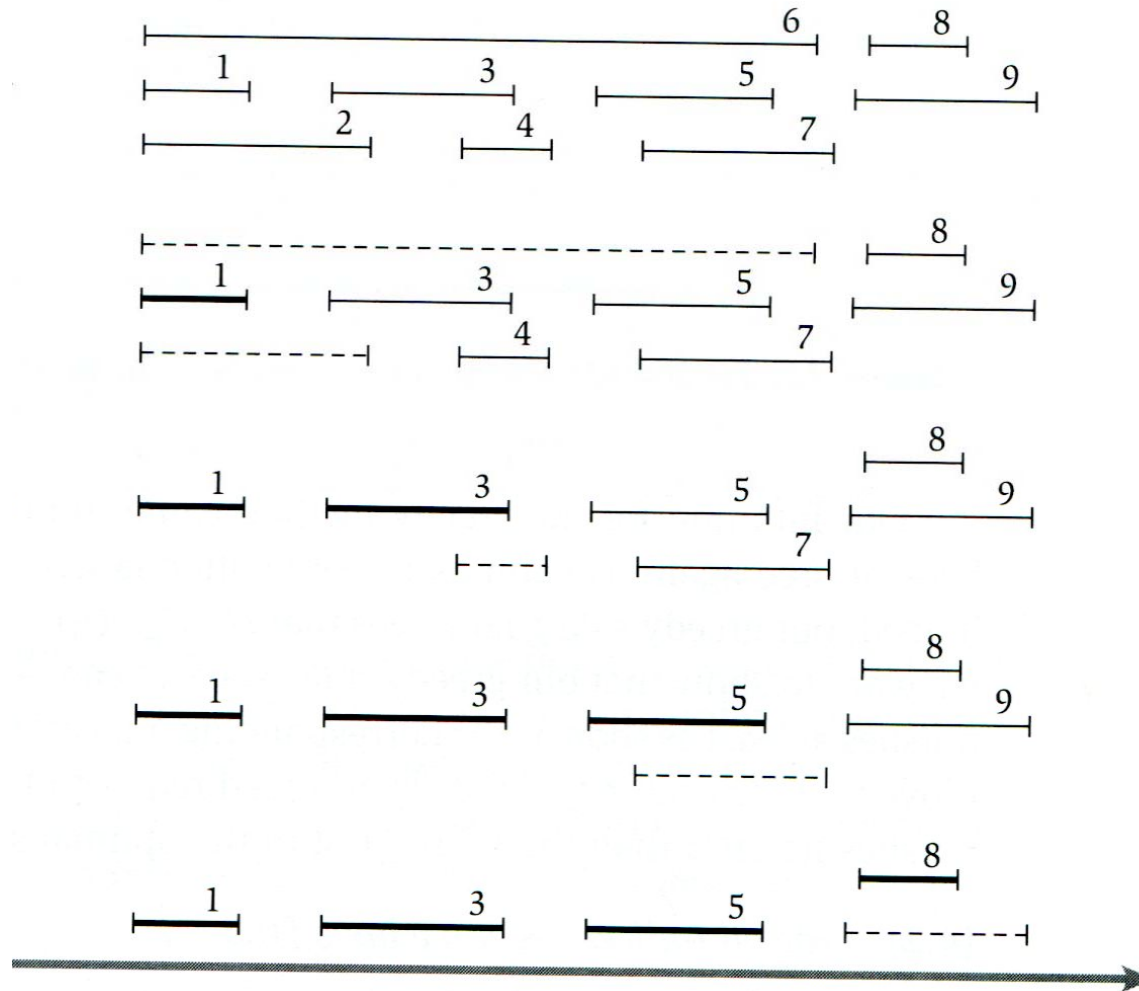   - Choose an interval i such that f_i is minimum.

```
Interval Scheduling( int n, request R )
{
    request A; //       solution
    A ← Φ;
    while ( R is not Φ )
        choose a request i from R whose finish time is minimum;
        add i to A;
        delete all the requests from R that are not compatible with i;

    return A;
}
```
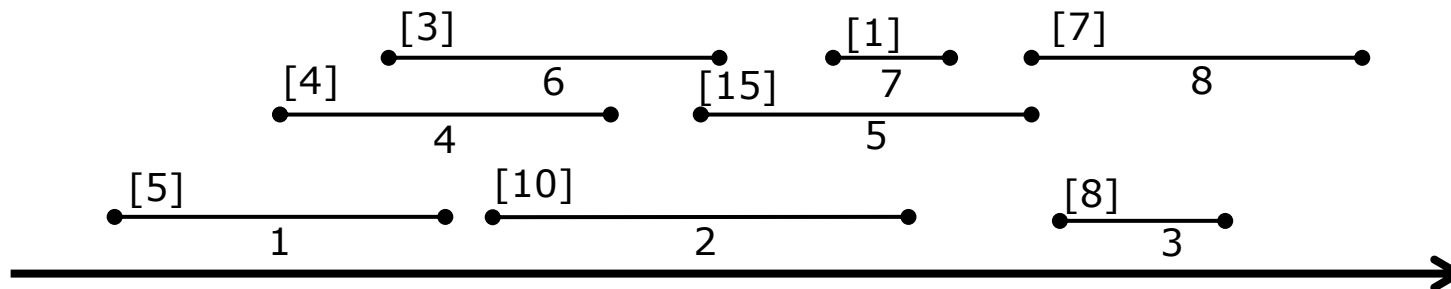
- Example

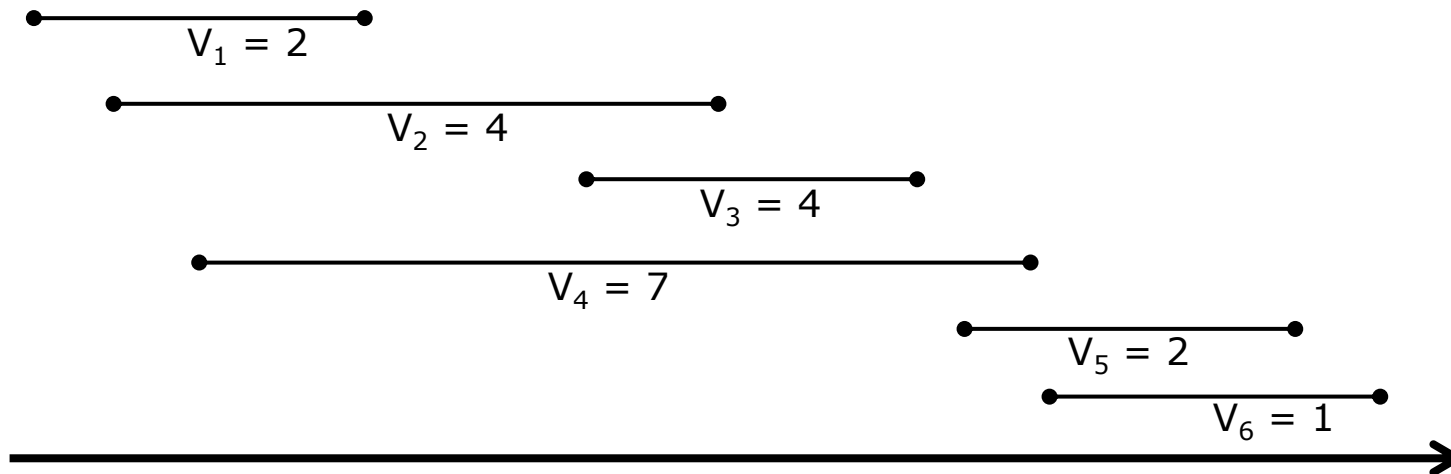# 4.2 Weighted interval scheduling

- Extended problem
  → Weighted interval scheduling
  - Interval with different weights
  - Complete requests with maximum weights

Q1. Is there a greedy algorithm that solves this problem?
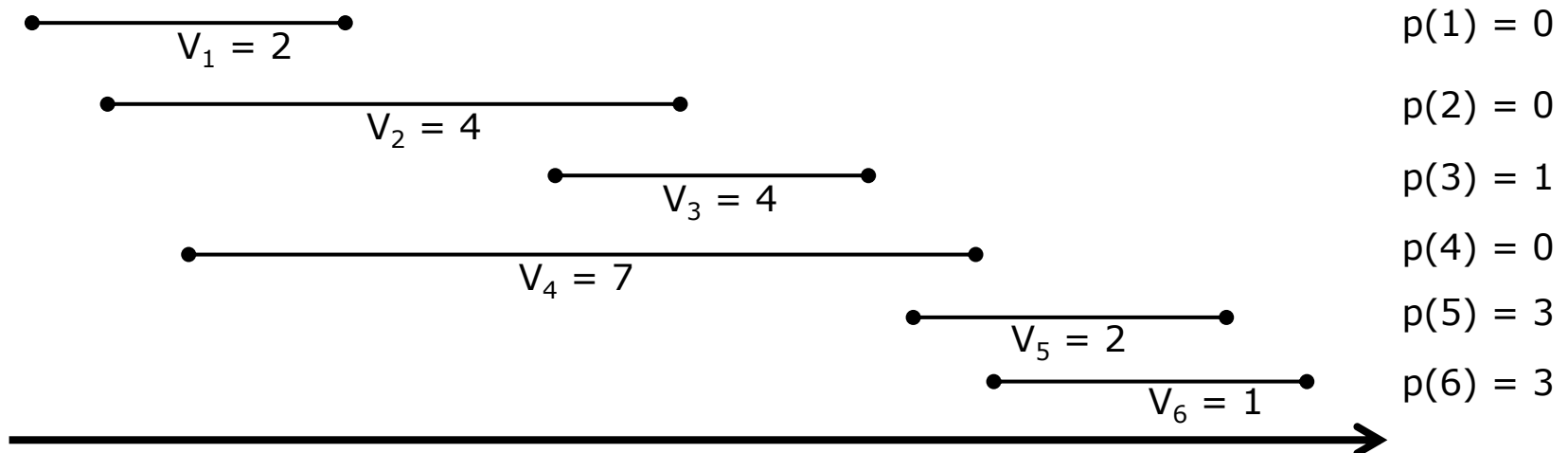
4.2 Weighted interval scheduling

- Problem
  - Each interval is defined by three values:
    {start time (s_i), finish time (f_i), weight (v_i)}
  - Goal
    - Find a set of maximum compatible subsets (Greedy)
      ➔ **Find a set of compatible intervals such that the sum of the weights is maximum**

$V_1 = 2$

$V_2 = 4$

$V_3 = 4$

$V_4 = 7$

$V_5 = 2$

$V_6 = 1$

- Strategy
  - Sort intervals according to the decreasing order of finish time

    $\{6, 5, 4, 3, 2, 1\}$
  - Define $p(j)$ for an interval j
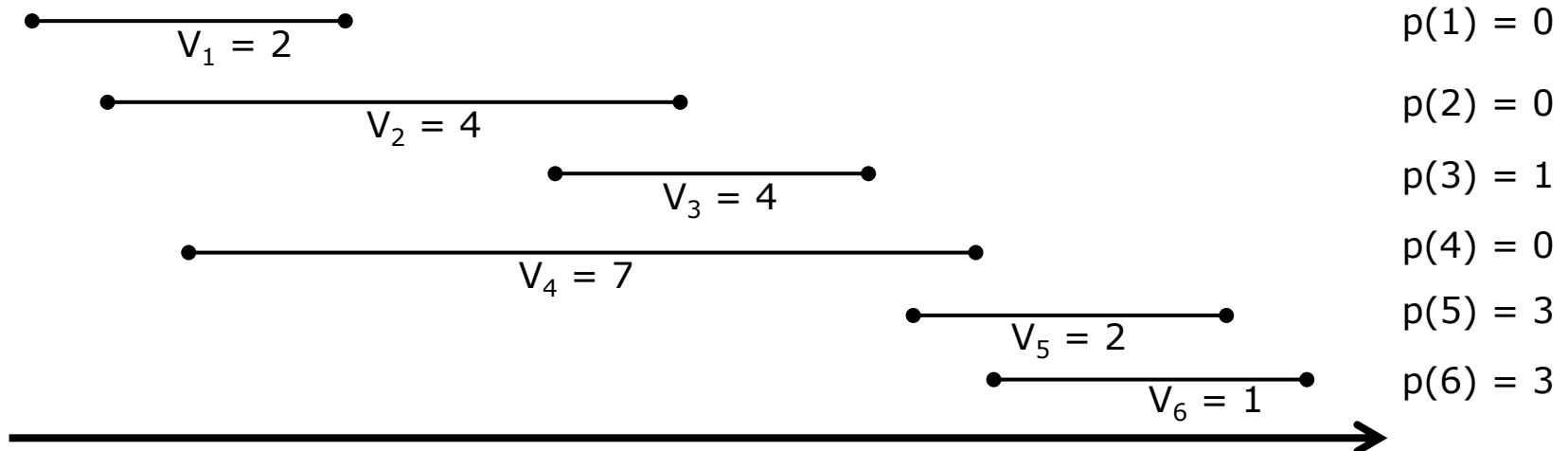   - $p(j)$ = the largest index $i < j$ such that intervals i & j are disjoint

$V_1 = 2$

$p(1) = 0$

$V_2 = 4$

$p(2) = 0$

$V_3 = 4$

$p(3) = 1$

$V_4 = 7$

$p(4) = 0$

$V_5 = 2$

$p(5) = 3$

$V_6 = 1$

$p(6) = 3$

- Solution
  - OPT(j)
    - The optimal solution of j intervals

$$OPT(j) = max\{v_j + OPT(p(j)), OPT(j-1)\}$$

Solution with selecting j-th interval

Solution without selecting j-th interval

$V_1 = 2$

$V_2 = 4$
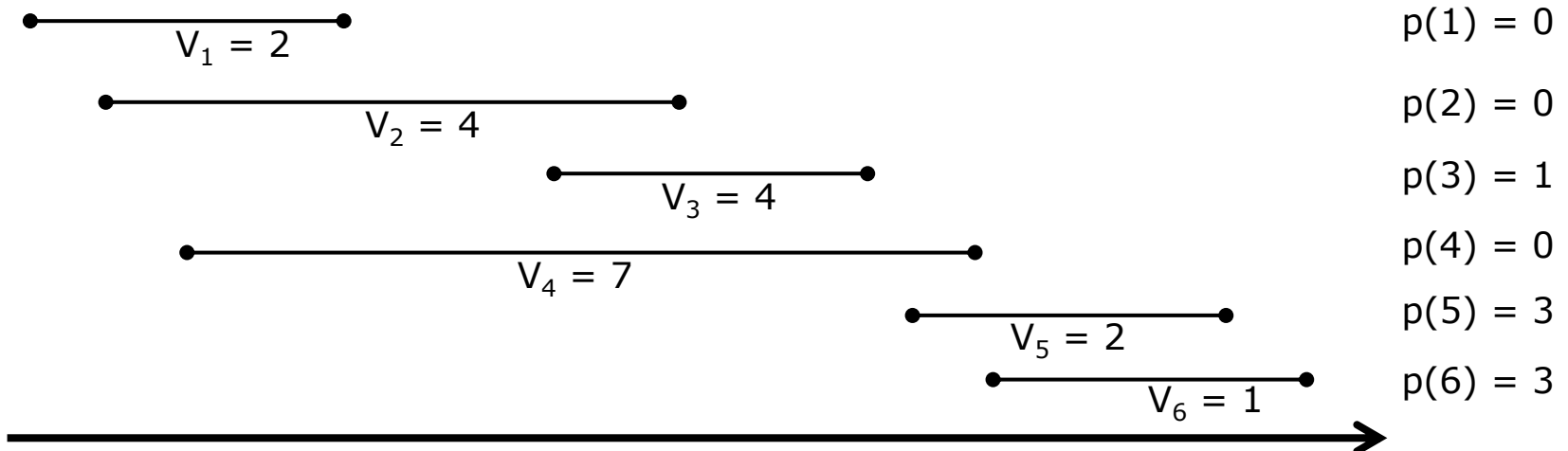
$V_3 = 4$

$V_4 = 7$

$V_5 = 2$

$V_6 = 1$

$p(1) = 0$

$p(2) = 0$

$p(3) = 1$

$p(4) = 0$

$p(5) = 3$

$p(6) = 3$

4.2 Weighted interval scheduling

- Example $OPT(j) = max\{v_j + OPT(p(j)), OPT(j-1)\}$
  - 6 intervals
    - $OPT(6) = max\{v_6 + OPT(p(6)), OPT(5)\}$
    $\qquad\;\; = max\{1 + OPT(3), OPT(5)\}$
    - $OPT(5) = max\{2 + OPT(3), OPT(4)\}$
    - $OPT(4) = max\{7 + OPT(0), OPT(3)\}$
    - $OPT(3) = max\{4 + OPT(1), OPT(2)\}$
    - $OPT(2) = max\{4 + OPT(0), OPT(1)\}$
    - $OPT(1) = max\{2 + OPT(0), OPT(0)\}$

$V_1 = 2$ $p(1) = 0$

$V_2 = 4$ $p(2) = 0$

$V_3 = 4$ $p(3) = 1$

$V_4 = 7$ $p(4) = 0$

$V_5 = 2$ $p(5) = 3$

$V_6 = 1$ $p(6) = 3$

4.2 Weighted interval scheduling

- Algorithm (recursive)

```
int OPT ( int j )
{
    if ( j == 0 )
        return 0;

    return max( v[j] + OPT(P[j]), OPT(j-1) );
}
```

- Problem?
  - Redundantly calling OPT(x)
  - Use an extra memory to void redundant recursive call

4.2 Weighted interval scheduling

- Algorithm (memorized & recursive)

```
int OPT ( int j )
{
    if ( j == 0 )
        return 0;

    if ( M[j] is not empty )
        return M[j];

    M[j] = max( v[j] + OPT(P[j]), OPT(j-1) );
    return M[j];
}
```

- Time complexity?

4.2 Weighted interval scheduling

- Algorithm (memorized & non-recursive)

```
int OPT ( int j )
{
    int M[];

    M[0] = 0;

    for ( i = 1; i <= n; i++ )
        M[i] = max( v[i] + M[P[i]], M[i-1] );
}
```

4.3 Multistage graph

- Definition of multistage graph
  - A directed graph
    - The vertices are partitioned into $k \geq 2$ disjoint sets V_i, $1 \leq i \leq k$.
    - If $<u, v>$ is an edge in E, then $u \in V_i$ and $v \in V_{i+1}$, for some i.
    - $|V_1| = |V_k| = 1$.
    - If $s \in V_1$, then s is the source
    - If $t \in V_k$, then t is the sink
  - Let c(i, j) be the cost of edge $<i, j>$.
    - The cost of a path from s to t is the sum of the costs of the edges on the path.
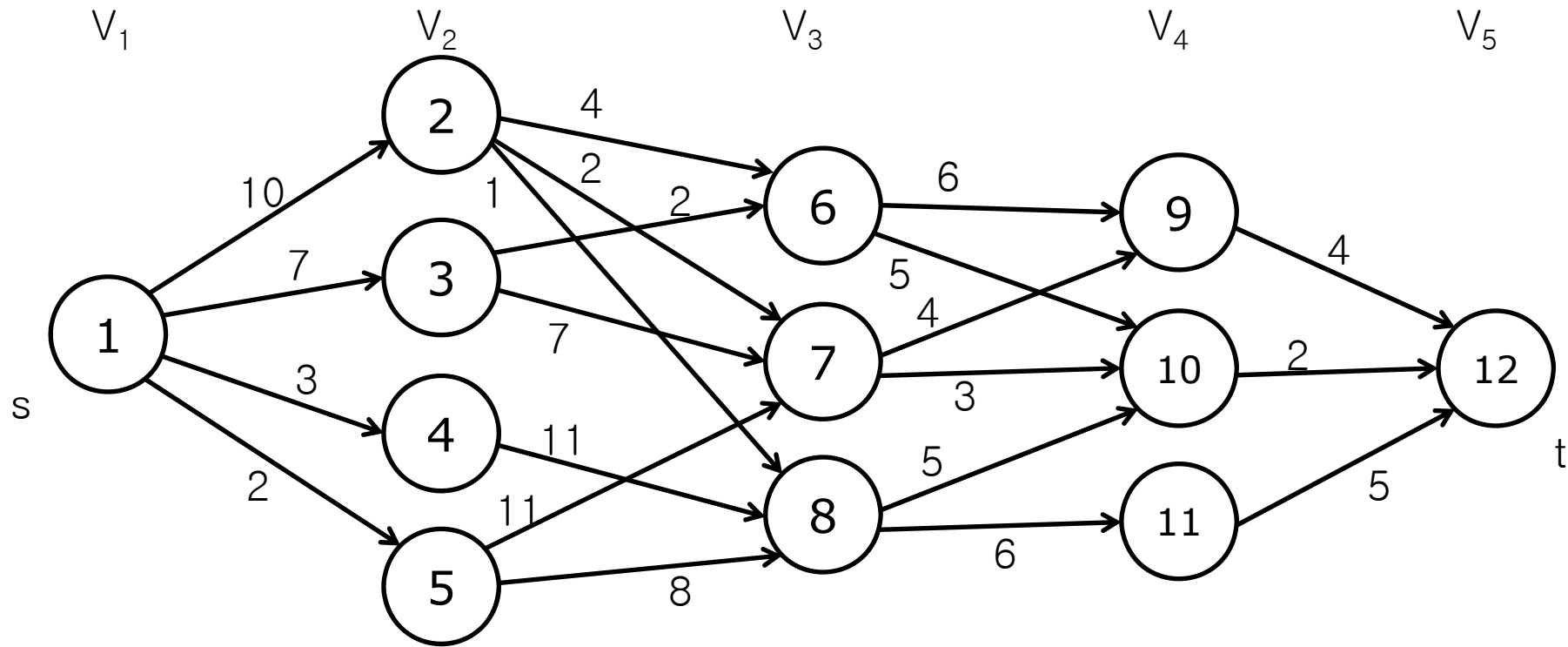   - The multistage graph problem is to find a minimum cost path from s to t.

4.3 Multistage graph

- Definition of multistage graph
  - An example of a multistage graph
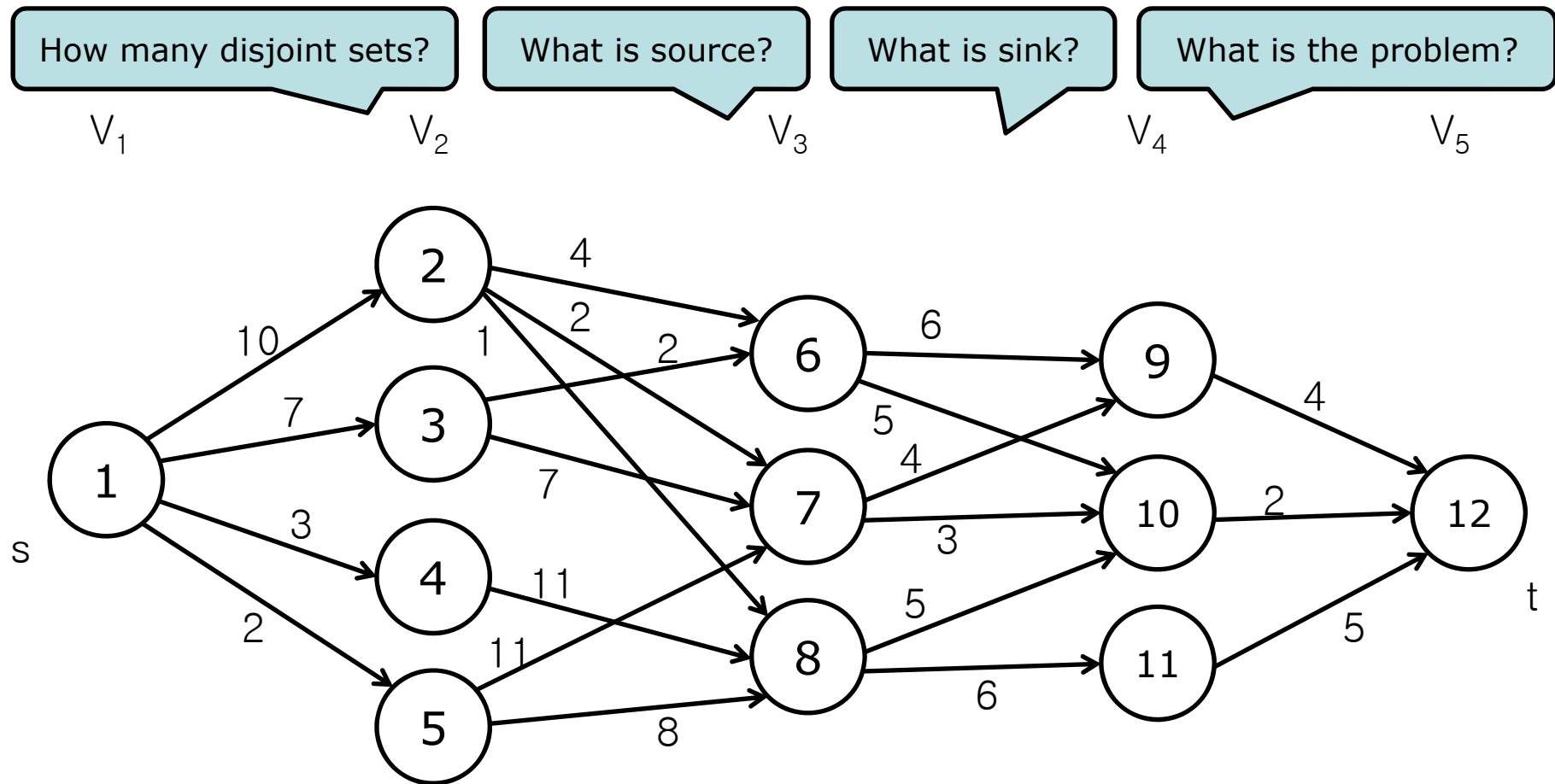
4.3 Multistage graph

- Problem
  - Find an optimal path from source to sink

- Strategy
  - P(i, j)
    - minimum cost path from vertex j in V_i to sink
  - COST(i, j)
    - the cost of P(i, j)

    Algorithm

$$COST(i, j) = \min_{l \in V_{i+1}} \{ c(j,l) + COST(i+1,l) \}$$

  - What is the problem to solve?
    - P(1, 1)
    - COST(1, 1)

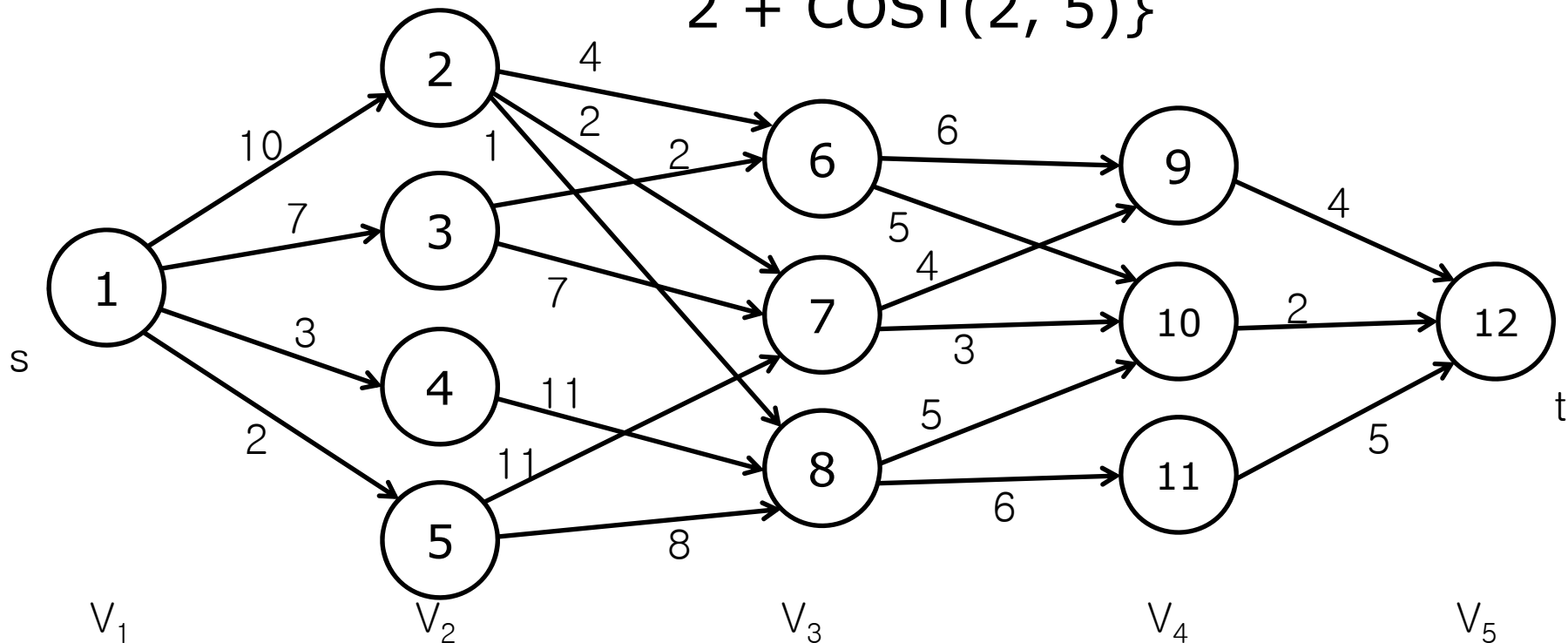# 4.3 Multistage graph

- Algorithm

```
void MULTIGRAPH ( Graph G)
{
    int **COST, **D;

    estimate COST(k - 1);
    for ( i = k – 2; i >= 1; i-- ) {
        for all vertices j at stage i
            COST(i, j) ← min(cost(j, l) + COST(i+1, l));
            D(i, j) ← l;
    }
}
```

4.3 Multistage graph

- Example (At V_1)
  - COST(1, 1) = min{10 + COST(2, 2),
    7 + COST(2, 3),
    3 + COST(2, 4),
    2 + COST(2, 5)}

4.3 Multistage graph

- Example (At V_2)
  - COST(2, 2) = min{4 + COST(3, 6),
    - 2 + COST(3, 7),
    - 1 + COST(3, 8)}

  - COST(2, 3) = min{2 + COST(3, 6),
    - 7 + COST(3, 7)}

  - COST(2, 4) = min{11 + COST(3, 8)}

  - COST(2, 5) = min{11 + COST(3, 7),
    - 8 + COST(3, 8)}

- Example (At V_3)
  - COST(3, 6) = min{6 + COST(4, 9),
    5 + COST(4, 10)}

  - COST(3, 7) = min{4 + COST(4, 9),
    3 + COST(4, 10)}

  - COST(3, 8) = min{5 + COST(4, 10),
    6 + COST(4, 11)}

4.3 Multistage graph

- Example (At V_4)
  - COST(4, 9) = 4

  - COST(4, 10) = 2

  - COST(4, 11) = 5

4.3 Multistage graph

- Example (Again at V_3)
  - COST(3, 6) = min{6 + COST(4, 9),
                        5 + COST(4, 10)}
               = min{6 + 4, 5 + 2} = 7

  - COST(3, 7) = min{4 + COST(4, 9),
                        3 + COST(4, 10)}
               = min{4 + 4, 3 + 2} = 5

  - COST(3, 8) = min{5 + COST(4, 10),
                        6 + COST(4, 11)}
               = min{5 + 2, 6 + 5} = 7

- Example (Again at V_2)
  - COST(2, 2) = min{4 + COST(3, 6),
    2 + COST(3, 7),
    1 + COST(3, 8)}
    = min{4 + 7, 2 + 5, 1 + 7} = 7

  - COST(2, 3) = min{2 + COST(3, 6),
    7 + COST(3, 7)}
    = min{2 + 7, 7 + 5} = 9

  - COST(2, 4) = min{11 + COST(3, 8)} = 11 + 7 = 18

  - COST(2, 5) = min{11 + COST(3, 7),
    8 + COST(3, 8)}
    = min{11 + 5, 8 + 7} = 15

- Example (Again at V_1)
  - COST(1, 1) = min{10 + COST(2, 2),
    7 + COST(2, 3),
    3 + COST(2, 4),
    2 + COST(2, 5)}
    = min{10 + 7, 7 + 9, 3 + 18, 2 + 15}
    = 16

 - P(1, 1) = 1 → 3 → 6 → 10 → 12

4.3 Multistage graph

- Application: Resource allocation problem
  - Allocating n units of resources to r projects
  - $N(i, j)$
    - The net profit of allocating j units of resources to project i.
  - How to maximize total net profit?
  - Formulate this problem using multistage graph

4.3 Multistage graph

- Application: Resource allocation problem
  - Multistage graph
    - (r+1) stage graph problem
    - Stage i represents project i.
    - (n+1) vertices V(i, j) are associated with stage i.
    - Stage 1 and r+1 has one vertex: V(1, 0) = s and V(r+1, n) = t.
    - Vertex V(i, j) represents the stage in which a total of j units of resources have been allocated to projects 1, 2, …, i-1.
    - The edges are of the form <V(i, j), V(i+1, l)>.
      - The cost of the edges is N(i, l-j).

- Application: Resource allocation problem
  - Multistage graph of 4 units to 3 teams
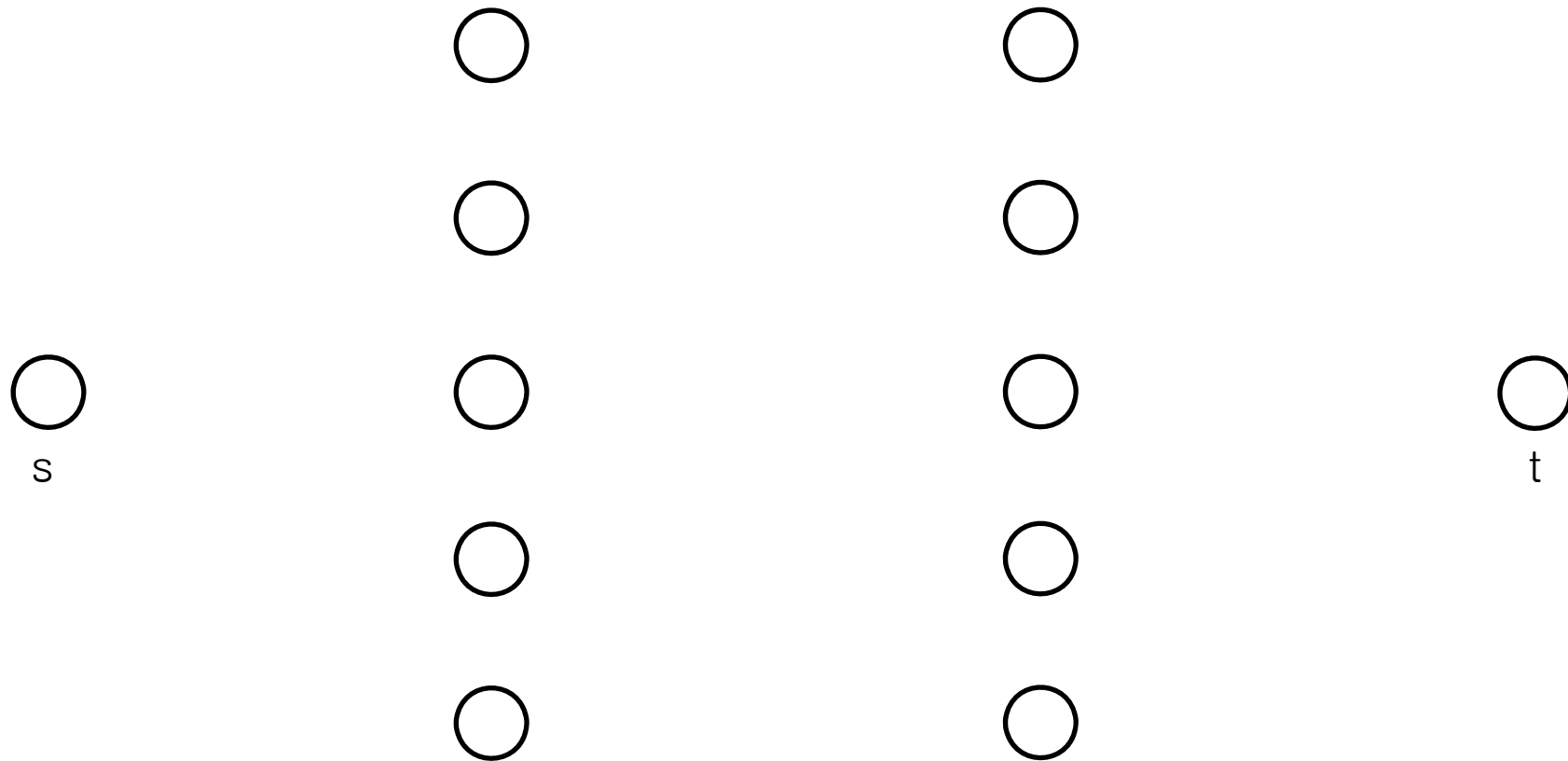
4.3 Multistage graph

- Example
  - 4 units to 3 teams

	1st team	2nd team	3rd team
N=0	0	5	3
N=1	10	15	5
N=2	20	20	10
N=3	25	22	15
N=4	25	25	25

- Problem: All-pairs shortest path
  - The problem of finding shortest paths for every two vertices *v* to *u*.
  - Solving single-source shortest path for all vertices in G

- Example
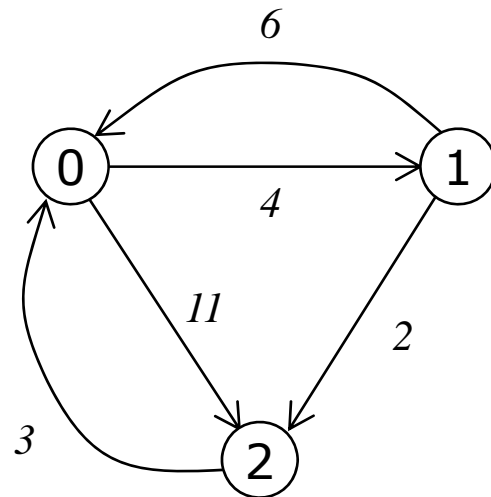
4.4 All Pairs Shortest path

- Strategy
  - Suppose we wish to find a shortest path from vertex i to vertex j.
  - Let A_i be the vertices adjacent to i.
  - Which of the vertices in A_i should be the second vertex?
  - Principle of optimality

  Whatever the initial state and decision are, the remaining decisions must constitute an optimal decision sequence

    - Let $A_1 = \{i_x\}$.
    - The optimal path will be i $\rightarrow$ i_x $\rightarrow$ ... $\rightarrow$ j.
    - Whatever i_x will be, $i_x \rightarrow$ j must be optimal.

$$Cost(i, i_1, i_2, \ldots, j) = \min(Cost(i, i_x) + Cost(i_x, \ldots, j))$$

4.4 All Pairs Shortest path

- Strategy
  - Let k be an intermediate vertex on a shortest path from i to j such that {i $\rightarrow$ i_1 $\rightarrow$ i_2 $\rightarrow$ ... $\rightarrow$ k $\rightarrow$ p_1 $\rightarrow$ p_2 $\rightarrow$ ... $\rightarrow$ j}.
  - The path {i $\rightarrow$ i_1 $\rightarrow$ i_2 $\rightarrow$ ... $\rightarrow$ k} and {k $\rightarrow$ p_1 $\rightarrow$ p_2 $\rightarrow$ ... $\rightarrow$ j} should be shortest from i to k and k to j.

$$Cost(i, i_1, i_2, ..., k, p_1, p_2, ..., j)$$
$$= Cost(i, i_1, i_2, ..., k,) + Cost(k, p_1, p_2, ..., j)$$

4.4 All Pairs Shortest path

- Algorithm: Floyd's algorithm
  - Finding the all-pair's shortest path.
    - Input: adjacency matrix of a graph.
    - Output: minimum cost distance + path.
  - The weight of a path between two vertices is the sum of the weights of the edges along that path.
    - Negative weight is allowed.
    - Negative cycle is not allowed.

4.4 All Pairs Shortest path

- Algorithm: Floyd's algorithm
  - $A^k[i][j]$:
    - The cost of the shortest path from vertex i to j, using only those intermediate vertices with an index $\leq$ k.

  - $A^{-1}[i][j]$: the weight of an edge connecting vertex i and vertex j

Minimum cost with those vertices less than k

Minimum cost from i to k those vertices less than k

Minimum cost from k to j those vertices less than k

$$A^k[i][j] = \min \{ A^{k-1}[i][j], A^{k-1}[i][k] + A^{k-1}[k][j] \}$$
$$A^{-1}[i][j] = w[i][j]$$

All about Dynamic Programming

Type	Stepwise approach	Recursive structure	Approach	Time	Specialty
0/1 KNAP	Choosing objects	KNAP(1, n, M) = max {KNAP(2, n, M), KNAP(2, n, M-w_1)+p_1}	Forward	$O(2^n)$	Smart Degenerate Case
Weighted Interval Scheduling	Selecting intervals	OPT(j) = max {OPT(p(j)) + v_j, OPT(j-1)}	Backward	$O(n)$	Auxiliary memory
Multistage Graph	Determining vertex in the next stage	COST(i, j) = $min_{l \in V_{i+1}}$ {c(j,l) + COST(i+1, l)}	Forward	$O(m)$	Resource allocation problem
All Pairs Shortest Path	Comparing paths with /without v_k	$A^k[i][j]$ = min { $A^{k-1}[i][j]$, $A^{k-1}[i][k]$ + $A^{k-1}[k][j]$ }	Forward	$O(n^3)$	Floyd's algorithm