Journal of Statistical Physics (2021) 185:9
https://doi.org/10.1007/s10955-021-02822-1

Local Pressure of Subsets and Measures

Weisheng Wu[1]

Received: 3 February 2021 / Accepted: 3 September 2021

Abstract
In this paper, we study various notions of local pressure of subsets and measures with respect to a fixed open cover, which are defined by Carathéordory–Pesin construction. We show that all local pressures of an ergodic measure equal the local metric entropy defined by Romagnoli (Ergod Theory Dyn Syst, 23(05):1601–1610, 2003) plus the integral of potential up to a variation of potential with respect to the open cover. In particular, this answers positively a question by Downarowicz (Entropy in dynamical systems, New Mathematical Monographs, vol 18, Cambridge University Press, xii+391, 2011) on variant entropy. As analogs of local variational principles, we also establish variational inequalities for local pressure of subsets.

Keywords Entropy · Pressure · Local variational principle · Carathéordory–Pesin construction

Mathematics Subject Classification Primary: 37A35 · 37B40 · 37A05

1 Introduction

Let (X, T) be a topological dynamical system (TDS for short), that is, X is a compact metrizable space and $T : X \to X$ is a homeomorphism. It is well known that entropy and pressure are the most important invariants of a TDS which measure the complexity of the orbits of the system. The metric entropy of an invariant measure was first introduced by Kolmogorov and Sinai using measurable partitions [18,27], while the first definition of topological entropy for compact invariant sets was given by Adler, Konheim and McAndrew [1] using open covers. In [8], Bowen first noticed the "dimensional" nature of topological entropy and introduced topological entropy for arbitrary (not necessarily compact nor invariant) subsets in a way resembling Hausdorff dimension. Pesin and Pitskel′ [22] further developed Bowen's approach and extended topological pressure to arbitrary subsets. In Pesin's book [21], a general Caratheódory construction is presented, which is called Caratheódory–Pesin

Communicated by Eric A. Carlen.

✉ Weisheng Wu
wuweisheng@xmu.edu.cn

1 School of Mathematical Sciences, Xiamen University, Xiamen 361005, People's Republic of China

Published online: 16 October 2021 Springer construction now. Among the most important Carathеódory–Pesin dimensions are Hausdorff dimension and topological pressure including topological entropy.

In statistical mechanics, it is a physical fact that an equilibrium state μ of a physical system minimizes the free energy, or equivalently, maximizes the metric pressure $h_\mu(T) + \int \varphi d\mu$, where φ is the given potential on the phase space and $h_\mu(T)$ is the metric entropy. Inspired by this, Ruelle [25] and Walters [28] introduced the notion of topological pressure to dynamical systems and established a variational principle for it. Topological and metric pressures, variational principle and equilibrium states constitute the main components of the thermodynamic formalism and play important roles in dimension theory of dynamical systems.

On the other hand, there is demand for the study of entropy and pressure with respect to a fixed open cover $\mathcal{U}$ of X, which has further applications to entropy pairs or tuples, entropy sets and entropy points, and entropy structure, etc. (see [4–7,10,11,13–17,24] and the references therein). Romagnoli [24] proposed two notions of metric entropies relative to $\mathcal{U}$:

$$h_\mu^+(T,\mathcal{U}) := \inf_{\alpha \succeq \mathcal{U}} h_\mu(T,\alpha),$$

$$h_\mu^-(T,\mathcal{U}) := \lim_{n\to\infty} \frac{1}{n} \inf_{\alpha \succeq \mathcal{U}_0^{n-1}} H_\mu(\alpha),$$

where α is a finite Borel partition of X. He showed the following local variational principle:

$$h_{\mathrm{top}}(T,\mathcal{U}) = \max_{\mu\in\mathcal{M}_T(X)} h_\mu^-(T,\mathcal{U}) = \max_{\mu\in\mathcal{E}_T(X)} h_\mu^-(T,\mathcal{U})$$

where $\mathcal{M}(X)$ (resp. $\mathcal{M}_T(X)$, $\mathcal{E}_T(X)$) denotes the set of all (resp. T-invariant, T-ergodic) probability Borel measures on X. Later, Glasner and Weiss [13] proved that if (X, T) is invertible then the local variational principle also holds for $h_\mu^+(T,\mathcal{U})$, i.e.,

$$h_{\mathrm{top}}(T,\mathcal{U}) = \max_{\mu\in\mathcal{M}_T(X)} h_\mu^+(T,\mathcal{U}) = \max_{\mu\in\mathcal{E}_T(X)} h_\mu^+(T,\mathcal{U}).$$

Finally, combining previous results it is observed in [16] that these two local metric entropies coincide:

$$h_\mu^+(T,\mathcal{U}) = h_\mu^-(T,\mathcal{U}),$$

and we denote the common value by $h_\mu(T,\mathcal{U})$. It is easy to see that the classical variational principle follows from the local ones. Another important result along this line is the relative local variational principle proved by Huang, Ye and Zhang in [16]. More precisely, for a factor map π between two TDSs (X, T) and (Y, S), they introduced two kinds of local metric conditional entropies, namely $h_\mu^+(T,\mathcal{U}|Y)$ and $h_\mu^-(T,\mathcal{U}|Y)$, and then showed that

$$h_{\mathrm{top}}(T,\mathcal{U}|Y) = \max_{\mu\in\mathcal{M}_T(X)} h_\mu^+(T,\mathcal{U}|Y) = \max_{\mu\in\mathcal{E}_T(X)} h_\mu^+(T,\mathcal{U}|Y)$$

and $h_\mu^+(T,\mathcal{U}|Y) = h_\mu^-(T,\mathcal{U}|Y) =: h_\mu(T,\mathcal{U}|Y)$. This relative local variational principle generalizes the relative variational principle obtained in [19] and [12].

In this paper, we study the *local metric pressure* and *local metric lower and upper capacities*

$$P_\mu(\varphi,\mathcal{U}), \quad \underline{CP}_\mu(\varphi,\mathcal{U}), \quad \overline{CP}_\mu(\varphi,\mathcal{U})$$

defined by Carathеódory–Pesin construction, with respect to a measure $\mu \in \mathcal{M}(X)$, an open cover $\mathcal{U}$ and potential $\varphi \in C(X,\mathbb{R})$ (see Definition 2.2). Using a method of Glasner and

Weiss [13] and Shapira [26], we show in Theorem 3.1 that $\underline{CP}_\mu(\varphi, \mathcal{U})$ and $\overline{CP}_\mu(\varphi, \mathcal{U})$ equal $h_\mu(T, \mathcal{U}) + \int \varphi d\mu$ up to a constant

$$\gamma(\varphi, \mathcal{U}) := \sup\{|\varphi(x) - \varphi(y)| : x, y \in U \text{ for some } U \in \mathcal{U}\},$$

i.e., the variation of φ with respect to $\mathcal{U}$. In particular, when $\varphi \equiv 0$, we have that $\underline{CP}_\mu(0, \mathcal{U}) = \overline{CP}_\mu(0, \mathcal{U}) = h_\mu(T, \mathcal{U})$. This links the Caratheódory–Pesin construction with the above local entropy theory. Moreover in [11], Downarowicz defined the *variant entropy* of μ with respect to $\mathcal{U}$ and asked whether it is the same as $h_\mu(T, \mathcal{U})$. We answer this question positively in Proposition 3.12, which is one motivation of this paper.

In the above local variational principle, $h_{\mathrm{top}}(T, \mathcal{U})$ equals $\overline{CP}_X(0, \mathcal{U})$, the local upper capacity entropy of X by Caratheódory–Pesin construction. However, the local entropy theory is rarely considered for the local topological pressure $P_Z(\varphi, \mathcal{U})$ of an arbitrary subset $Z \subset X$. Our second result establishes variational inequalities for local topological pressure of noncompact subsets, with an upper bound involving $h_\mu(T, \mathcal{U})$ and a lower bound involving a Brin-Katok local entropy with respect to $\mathcal{U}$ (See Propositions 4.1 and 4.2). It can be viewed as analogs of the above local variational principles. However, it is still an open question if we can improve these variational inequalities to variational principles for local topological pressure of subsets.

The paper is organized as follows. In Sect. 2, we prepare necessary terminology and give precise definitions for local pressures by Caratheódory–Pesin construction. In Sect. 3, we prove the (essential) coincidence of several local metric pressures, and in particular answer the question in [11]. In Sect. 4, we establish the variational inequalities for local topological pressure of subsets. We remark that most results are formulated and proved for the more general relative case, i.e., for a factor map between two TDSs.

2 Definitions of Local Pressure

2.1 Local Topological Pressure of Subsets

We recall the general Carathéodory–Pesin construction which is presented elegantly in [21, Chapter 1].

Let (X, d) be a compact metric space and $\mathcal{B}(X)$ the collection of all Borel subsets of X. A *cover* of X is a finite family of Borel subsets of X, whose union is X. A *partition* of X is a cover of X whose elements are pairwise disjoint. Let $\mathcal{C}_X$ (resp. $\mathcal{P}_X$, $\mathcal{C}_X^o$) denote the set of covers (resp. partitions, open covers) of X. Given two covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, $\mathcal{U}$ is said to be *finer* than $\mathcal{V}$, denoted by $\mathcal{U} \succeq \mathcal{V}$ or $\mathcal{V} \preceq \mathcal{U}$, if each element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. $\mathcal{U}(x)$ will denote the element of $\mathcal{U}$ containing x. Let $\mathcal{U} \vee \mathcal{V} := \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. For $m, n \in \mathbb{Z} \cup \{\pm\infty\}$, denote $\mathcal{U}_m^n := \bigvee_{i=m}^n T^{-i}\mathcal{U}$. For $\mathcal{U} \in \mathcal{C}_X$, diam$\mathcal{U}$ will denote the maximal diameter of elements in $\mathcal{U}$. Let $C(X, \mathbb{R})$ denote the set of all continuous functions $\varphi : X \to \mathbb{R}$ with C^0 norm $\|\cdot\|$.

Let $T : X \to X$ be a homeomorphism, and $\varphi \in C(X, \mathbb{R})$. Denote the Birkhoff sum by

$$S_n\varphi(x) := \sum_{k=0}^{n-1} \varphi(T^k(x)).$$

Suppose that $\mathcal{U} \in \mathcal{C}_X^o$. For $n \geq 1$, we denote by $\mathcal{W}_n(\mathcal{U})$ the collections of strings $\mathbf{U} = U_{i_0} \cdots U_{i_{n-1}}$ with $U_{i_j} \in \mathcal{U}$, $j = 0, 1, \cdots, n-1$. Given $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$, denote the length of $\mathbf{U}$

by $m(\mathbf{U})=n$, and define

$$X(\mathbf{U}) := \{x \in X : T^j(x) \in U_{i_j},\ j=0,1,\cdots,m(\mathbf{U})-1\}.$$

Let $Z \subset M$ be a nonempty set. For $s \in \mathbb{R}$, define

$$\mathcal{M}_N^s(\mathcal{U},\varphi,Z) := \inf_{\mathcal{G}} \left\{ \sum_{\mathbf{U}\in\mathcal{G}} \exp\left(-sm(\mathbf{U}) + \sup_{x\in X(\mathbf{U})} S_{m(\mathbf{U})}\varphi(x) \right) \right\},$$

where the infimum is taken over all $\mathcal{G} \subset \bigcup_{j\geq N} \mathcal{W}_j(\mathcal{U})$ that covers Z, i.e., $\bigcup_{\mathbf{U}\in\mathcal{G}} X(\mathbf{U}) \supset Z$. Clearly $\mathcal{M}_N^s(\mathcal{U},\varphi,\cdot)$ is a finite outer measure on X. Since $\mathcal{M}_N^s(\mathcal{U},\varphi,Z)$ increases as N increases, we can define $\mathcal{M}^s(\mathcal{U},\varphi,Z)=\lim_{N\to\infty}\mathcal{M}_N^s(\mathcal{U},\varphi,Z)$.

Similarly, for $s \in \mathbb{R}$, define

$$R_N^s(\mathcal{U},\varphi,Z) := \inf_{\mathcal{G}} \left\{ \sum_{\mathbf{U}\in\mathcal{G}} \exp\left(-sN + \sup_{x\in X(\mathbf{U})} S_N\varphi(x) \right) \right\},$$

where the infimum is taken over all $\mathcal{G} \subset \mathcal{W}_N(\mathcal{U})$ that covers Z. Define

$$\underline{R}^s(\mathcal{U},\varphi,Z) = \liminf_{N\to\infty} R_N^s(\mathcal{U},\varphi,Z), \quad \overline{R}^s(\mathcal{U},\varphi,Z) = \limsup_{N\to\infty} R_N^s(\mathcal{U},\varphi,Z).$$

Definition 2.1 The *topological pressure* and *upper capacity topological pressure* of φ on the set Z with respect to $\mathcal{U} \in \mathcal{C}_X^o$ is defined as

$$\begin{aligned}
P_Z(\varphi,\mathcal{U}) &:=n \inf\{s : \mathcal{M}^s(\mathcal{U},\varphi,Z)=0\} = \sup\{s : \mathcal{M}^s(\mathcal{U},\varphi,Z)=+\infty\},\\
\underline{CP}_Z(\varphi,\mathcal{U}) &: = \inf\{s : \underline{R}^s(\mathcal{U},\varphi,Z)=0\} = \sup\{s : \underline{R}^s(\mathcal{U},\varphi,Z)=+\infty\},\\
\overline{CP}_Z(\varphi,\mathcal{U}) &: = \inf\{s : \overline{R}^s(\mathcal{U},\varphi,Z)=0\} = \sup\{s : \overline{R}^s(\mathcal{U},\varphi,Z)=+\infty\}.
\end{aligned}$$

The *topological pressure* and *upper capacity topological pressure* of φ on the set Z is defined as

$$\begin{aligned}
P_Z(\varphi) &:= \sup_{\mathcal{U}\in\mathcal{C}_X^o} P_Z(\varphi,\mathcal{U}),\\
\underline{CP}_Z(\varphi) &: = \sup_{\mathcal{U}\in\mathcal{C}_X^o} \underline{CP}_Z(\varphi,\mathcal{U})\\
\overline{CP}_Z(\varphi) &: = \sup_{\mathcal{U}\in\mathcal{C}_X^o} \overline{CP}_Z(\varphi,\mathcal{U})
\end{aligned}$$

In fact, by [21, Theorem 11.1], all $\sup_{\mathcal{U}\in\mathcal{C}_X^o}$ above can be replaced by $\lim_{\operatorname{diam}\mathcal{U}\to 0}$.

Clearly, $P_Z(\varphi,\mathcal{U}) \leq \underline{CP}_Z(\varphi,\mathcal{U}) \leq \overline{CP}_Z(\varphi,\mathcal{U})$. Moreover,

$$\begin{aligned}
\underline{CP}_Z(\varphi,\mathcal{U}) &= \liminf_{N\to\infty} \frac{1}{N} \log \Lambda(Z,\varphi,\mathcal{U},N),\\
\overline{CP}_Z(\varphi,\mathcal{U}) &= \limsup_{N\to\infty} \frac{1}{N} \log \Lambda(Z,\varphi,\mathcal{U},N),
\end{aligned}$$

where

$$\Lambda(Z,\varphi,\mathcal{U},N) := \inf_{\mathcal{G}} \left\{ \sum_{\mathbf{U}\in\mathcal{G}} \exp\left(\sup_{x\in X(\mathbf{U})} S_N\varphi(x) \right) \right\}$$

and the infimum is taken over all $\mathcal{G} \subset \mathcal{W}_N(\mathcal{U})$ that covers Z.

2.2 Local Metric Pressure

Definition 2.2 Let $\mu \in \mathcal{M}(X)$, $\varphi \in C(X, \mathbb{R})$ and $\mathcal{U} \in \mathcal{C}_X^o$. We define the *local metric pressure* and *local lower and upper capacities* of μ with respect to φ and $\mathcal{U}$ respectively as

$$
\begin{aligned}
&P_\mu(\varphi, \mathcal{U}) := \inf\{P_Z(\varphi, \mathcal{U}) : \mu(Z) = 1\},\\
&\underline{CP}_\mu(\varphi, \mathcal{U}) := \lim_{\delta \to 0} \underline{CP}_\mu(\varphi, \mathcal{U}, \delta) := \lim_{\delta \to 0} \inf\{\underline{CP}_Z(\varphi, \mathcal{U}) : \mu(Z) \geq 1 - \delta\},\\
&\overline{CP}_\mu(\varphi, \mathcal{U}) := \lim_{\delta \to 0} \overline{CP}_\mu(\varphi, \mathcal{U}, \delta) := \lim_{\delta \to 0} \inf\{\overline{CP}_Z(\varphi, \mathcal{U}) : \mu(Z) \geq 1 - \delta\}.
\end{aligned}
$$

The following limit exists and is called *metric pressure* and *lower and upper capacities* of μ with respect to φ respectively:

$$
\begin{aligned}
P_\mu(\varphi) &= \lim_{\operatorname{diam}\mathcal{U} \to 0} P_\mu(\varphi, \mathcal{U}),\\
\underline{CP}_\mu(\varphi) &= \lim_{\operatorname{diam}\mathcal{U} \to 0} \underline{CP}_\mu(\varphi, \mathcal{U}),\\
\overline{CP}_\mu(\varphi) &= \lim_{\operatorname{diam}\mathcal{U} \to 0} \overline{CP}_\mu(\varphi, \mathcal{U}).
\end{aligned}
$$

Definition 2.3 Let $\mu \in \mathcal{M}(X)$, $\varphi \in C(X, \mathbb{R})$ and $\mathcal{U} \in \mathcal{C}_X^o$. We define

$$
\begin{aligned}
\underline{cp}_\mu(\varphi, \mathcal{U}) &:= \lim_{\delta \to 0} \underline{cp}_\mu(\varphi, \mathcal{U}, \delta) := \lim_{\delta \to 0} \liminf_{N \to \infty} \frac{1}{N} \log \Lambda_\mu(\varphi, \mathcal{U}, N, \delta),\\
\overline{cp}_\mu(\varphi, \mathcal{U}) &:= \lim_{\delta \to 0} \overline{cp}_\mu(\varphi, \mathcal{U}, \delta) := \lim_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \Lambda_\mu(\varphi, \mathcal{U}, N, \delta)
\end{aligned}
$$

where

$$
\Lambda_\mu(\varphi, \mathcal{U}, N, \delta) := \inf_{\mathcal{G}} \left\{ \sum_{\mathbf{U} \in \mathcal{G}} \exp\left(\sup_{x \in X(\mathbf{U})} S_N\varphi(x) \right) \right\}
$$

and the infimum is taken over all $\mathcal{G} \subset \mathcal{W}_N(\mathcal{U})$ that covers a subset of X whose μ-measure is at least $1 - \delta$.

Lemma 2.4 *Let $\mu \in \mathcal{M}(X)$, $\varphi \in C(X, \mathbb{R})$ and $\mathcal{U} \in \mathcal{C}_X^o$. We have*

$$
\begin{aligned}
\underline{cp}_\mu(\varphi, \mathcal{U}) &\leq \underline{CP}_\mu(\varphi, \mathcal{U}),\\
\overline{cp}_\mu(\varphi, \mathcal{U}) &\leq \overline{CP}_\mu(\varphi, \mathcal{U}).
\end{aligned}
$$

Proof Given $0 < \delta < 1$, for any $Z \subset X$ with $\mu(Z) \geq 1 - \delta$, we have

$$
\Lambda_\mu(\varphi, \mathcal{U}, N, \delta) \leq \Lambda(Z, \varphi, \mathcal{U}, N).
$$

Then the lemma follows immediately. □

By Proposition 3.9 below, we see that the inequalities in Lemma 2.4 are in fact equalities when $\varphi = 0$ and $\mu \in \mathcal{E}_T(X)$.

Let $T : X \to X$ and $S : Y \to Y$ be two TDSs and $\pi : X \to Y$ a factor map, i.e., a continuous, surjective map such that $\pi \circ T = S \circ \pi$. We proceed by defining the conditional metric entropy of $\mu \in \mathcal{M}_T(X)$ relative to (Y, S).

Let $\mu \in \mathcal{M}(X)$, and ϵ_Y denote the partition of Y into points and $\mathcal{B}(Y)$ the Borel σ-algebra of Y. Then in the Lebesgue space (X, μ), the measurable partition $\pi^{-1}\epsilon_Y$ generates the σ-algebra $\pi^{-1}\mathcal{B}(Y)$ and it is T-invariant. We simply denote by $\mu_x := \mu_x^{\pi^{-1}\epsilon_Y}$ the conditional measures uniquely determined by the measurable partition $\pi^{-1}\epsilon_Y$ for μ-a.e. $x \in X$ (see [23]). We also write $\mu_y := \mu_x$ if $\pi x = y$. Given $\alpha \in \mathcal{P}_X$, the *conditional entropy of* α *relative to* Y is defined as

$$H_\mu(\alpha|Y) := \int_X -\log \mu_x(\alpha(x))d\mu(x).$$

If $\mu \in \mathcal{M}_T(X)$, *the conditional entropy of* α *and* (X,T) *relative to* (Y,S) as

$$h_\mu(T,\alpha|Y) := \lim_{n\to\infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|Y).$$

The above limit exists since the sequence $b_n = H_\mu(\alpha_0^{n-1}|Y)$ is subadditive. Moreover, the *conditional metric entropy of* (X,T,μ) *relative to* (Y,S) *and* $\mathcal{U} \in \mathcal{C}_X^o$ is defined by

$$h_\mu^+(T,\mathcal{U}|Y) : = \inf_{\alpha\succeq\mathcal{U}} h_\mu(T,\alpha|Y),$$

$$h_\mu^-(T,\mathcal{U}|Y) : = \lim_{n\to\infty} \frac{1}{n} \inf_{\alpha\succeq\mathcal{U}_0^{n-1}} H_\mu(\alpha|Y).$$

By [16], we have $h_\mu^+(T,\mathcal{U}|Y) = h_\mu^-(T,\mathcal{U}|Y) =: h_\mu(T,\mathcal{U}|Y)$.

Definition 2.5 Let $\pi : (X,T) \to (Y,S)$ be a factor map between two TDSs, $\mu \in \mathcal{M}(X)$, $Z \subset X$, $\varphi \in C(X,\mathbb{R})$ and $\mathcal{U} \in \mathcal{C}_X^o$. We define the *relative local pressure* of Z, and the *relative local upper capacities* of Z and μ respectively, with respect to Y, φ and $\mathcal{U}$ as

$$\begin{aligned}
P_Z(\varphi,\mathcal{U}|Y) &:= \sup_{y\in Y} P_{Z\cap\pi^{-1}y}(\varphi,\mathcal{U}),\\
\overline{CP}_Z(\varphi,\mathcal{U}|Y) &:= \limsup_{N\to\infty} \frac{1}{N} \log \sup_{y\in Y} \Lambda(\pi^{-1}y\cap Z,\varphi,\mathcal{U},N),\\
\overline{CP}_\mu(\varphi,\mathcal{U}|Y) &:= \lim_{\delta\to 0} \overline{CP}_\mu(\varphi,\mathcal{U},\delta|Y)\\
&:= \lim_{\delta\to 0}\limsup_{N\to\infty} \frac{1}{N} \int \log\inf\{\Lambda(Z,\varphi,\mathcal{U},N) : \mu_x(Z)\geq 1-\delta\}d\mu(x),\\
\overline{cp}_\mu(\varphi,\mathcal{U}|Y) &:= \lim_{\delta\to 0} \overline{cp}_\mu(\varphi,\mathcal{U},\delta|Y)\\
&:= \lim_{\delta\to 0}\limsup_{N\to\infty} \frac{1}{N} \int \log \Lambda_{\mu_x}(\varphi,\mathcal{U},N,\delta)d\mu(x).
\end{aligned}$$

The *relative local lower capacities*

$$\underline{CP}_Z(\varphi,\mathcal{U}|Y),\quad \underline{CP}_\mu(\varphi,\mathcal{U}|Y),\quad \underline{cp}_\mu(\varphi,\mathcal{U}|Y)$$

are defined analogously, replacing the above $\limsup_{N\to\infty}$ by $\liminf_{N\to\infty}$.

Remark 2.6 The relative local pressure $P_Z(\varphi,\mathcal{U}|Y)$ is used in Proposition 4.1 below. However, we are not able to obtain a relative version of Proposition 4.2. The other main results below are formulated and proved for relative case.

The following is a conditional version of Shannon-McMillan-Breiman theorem.

Theorem 2.7 *[11, Theorem B.0.1] Let* $\pi : (X,T) \to (Y,S)$ *be a factor map between two TDSs,* $\mu \in \mathcal{M}_T(X)$ *and* $\alpha \in \mathcal{P}_X$. *Then the following limit exists*

$$h_\mu(T,\alpha|Y,x) := \lim_{n\to\infty} -\frac{1}{n}\log \mu_x(\alpha_0^{n-1}(x)) \quad \mu\text{-a.e. and in } L^1(\mu),$$

and $\int_X h_\mu(T,\alpha|Y,x)d\mu(x) = h_\mu(T,\alpha|Y)$*. Moreover, if* μ *is ergodic then* $h_\mu(T,\alpha|Y,x) = h_\mu(T,\alpha|Y)$ *for* μ*-a.e.* $x \in X$*.*

3 Coincidence of Relative Local Metric Pressure

For $\varphi \in C(X,\mathbb{R})$ and $\mathcal{U} \in \mathcal{C}_X^o$, recall that

$$\gamma(\varphi,\mathcal{U}) = \sup\{|\varphi(x)-\varphi(y)| : x, y \in U \text{ for some } U \in \mathcal{U}\}.$$

Our first main result is:

Theorem 3.1 *Let* $\pi : (X,T) \to (Y,S)$ *be a factor map between two TDSs,* $\mu \in \mathcal{E}_T(X)$*,* $\varphi \in C(X,\mathbb{R})$ *and* $\mathcal{U} \in \mathcal{C}_X^o$*. Then*

$$|\overline{CP}_\mu(\varphi,\mathcal{U}|Y) - (h_\mu(T,\mathcal{U}|Y) + \int \varphi d\mu)| \le \gamma(\varphi,\mathcal{U}),$$

and the above is true if $\overline{CP}_\mu(\varphi,\mathcal{U}|Y)$ *is replaced respectively by*

$$\underline{CP}_\mu(\varphi,\mathcal{U}|Y),\quad \overline{cp}_\mu(\varphi,\mathcal{U}|Y),\quad \underline{cp}_\mu(\varphi,\mathcal{U}|Y).$$

Theorem 3.1 follows immediately from Lemma 2.4, Propositions 3.2 and 3.3 below.

Proposition 3.2 *Let* $\pi : (X,T) \to (Y,S)$ *be a factor map between two TDSs,* $\mu \in \mathcal{E}_T(X)$*,* $\varphi \in C(X,\mathbb{R})$ *and* $\mathcal{U} \in \mathcal{C}_X^o$*. Then for any* $0 < \delta < 1$*,*

$$\overline{CP}_\mu(\varphi,\mathcal{U},\delta|Y) \le h_\mu(T,\mathcal{U}|Y) + \int \varphi d\mu + \gamma(\varphi,\mathcal{U}).$$

Proof Take any finite Borel partition $\alpha \succeq \mathcal{U}$. Let $\mu = \int \mu_y d\nu(y)$ be the disintegration of μ over $\nu = \pi\mu$. According to Theorem 2.7 and the Birkhoff ergodic theorem, as μ is ergodic, there exists $Y_1 \in \mathcal{B}(Y)$ with $\nu(Y_1) = 1$ such that for each $y \in Y_1$ and μ_y-a.e. x,

$$\lim_{n\to\infty} -\frac{1}{n}\log \mu_y(\alpha_0^{n-1}(x)) = h_\mu(T,\alpha|Y),$$

and $\lim_{n\to\infty}\frac{1}{n}S_n\varphi(x) = \int\varphi d\mu$.

Fix $y \in Y_1$. For $N \in \mathbb{N}$ and $\rho > 0$, set $I_N(y)$ to be the set of $x \in \pi^{-1}y$ such that

$$\mu_y(\alpha_0^{n-1}(x)) > \exp(-(h_\mu(T,\alpha|Y)+\rho)n) \tag{1}$$

and

$$|\frac{1}{n}S_n\varphi(x) - \int\varphi d\mu)| \le \rho \tag{2}$$

for any $n \ge N$. Then for any $\rho > 0$, $\lim_{N\to\infty}\mu_y(I_N(y)) = 1$. Thus, for sufficiently large $N \in \mathbb{N}$, we have $\mu_y(I_N(y)) > 1-\delta$.

Let $n \ge N$. By (1) and the choice of n, the number of elements of the partition α_0^{n-1} intersecting $I_N(y)$ nontrivially does not exceed $\exp((h_\mu(T,\alpha|Y)+\rho)n)$. Let $x \in I_N(y)$. Then there exists a string $\mathbf{U}$ of length n for which $x \in X(\mathbf{U})$. It follows from (2) that

$$|\frac{1}{n}\sup_{z\in X(\mathbf{U})} S_n\varphi(z) - \int\varphi d\mu| \le \rho + \gamma(\varphi,\mathcal{U}).$$

Each element of α_0^{n-1} is contained in $X(\mathbf{U})$ for some string $\mathbf{U}$ of length n. The collection of such strings consists of at most $\exp((h_\mu(T,\alpha|Y)+\rho)n)$ elements which comprise a cover $\mathcal{G}$ of I_N. Then

$$\begin{aligned}\Lambda(I_N(y),\varphi,\mathcal{U},n) &\le \sum_{\mathbf{U}\in\mathcal{G}} \exp\left(\sup_{z\in X(\mathbf{U})} S_n\varphi(z)\right)\\ &\le \exp\left(n(h_\mu(T,\alpha|Y)+\int \varphi d\mu + 2\rho + \gamma(\varphi,\mathcal{U}))\right).\end{aligned}$$

Since $\mu_y(I_N(y)) \ge 1-\delta$, by taking integral over y and then letting $n\to\infty$, we get

$$\overline{CP}_\mu(\varphi,\mathcal{U},\delta|Y) \le h_\mu(T,\alpha|Y) + \int \varphi d\mu + 2\rho + \gamma(\varphi,\mathcal{U}).$$

Letting $\rho\to 0$, and taking infimum over $\alpha \succeq \mathcal{U}$, we have

$$\overline{CP}_\mu(\varphi,\mathcal{U},\delta|Y) \le h_\mu(T,\mathcal{U}|Y) + \int \varphi d\mu + \gamma(\varphi,\mathcal{U}).$$

□

Proposition 3.3 *Let $\pi:(X,T)\to(Y,S)$ be a factor map between two TDSs, $\mu\in\mathcal{E}_T(X)$, $\varphi\in C(X,\mathbb{R})$ and $\mathcal{U}\in\mathcal{C}_X^o$. Then*

$$\underline{cp}_\mu(\varphi,\mathcal{U}|Y) \ge h_\mu(T,\mathcal{U}|Y) + \int \varphi d\mu - \gamma(\varphi,\mathcal{U}).$$

For $\nu\in\mathcal{M}(X)$, $\mathcal{U}\in\mathcal{C}_X$ and $0<\delta<1$, let $N_\nu(\mathcal{U},\delta)$ denote the minimal number of elements of $\mathcal{U}$, needed to cover a subset of X whose ν-measure is at least $1-\delta$. Before proving Proportion 3.3, let us prepare two useful lemmas.

Lemma 3.4 *Let $\pi:(X,T)\to(Y,S)$ be a factor map, $\mu\in\mathcal{M}_T(X)$ and $\mu=\int\mu_y d\nu(y)$ the disintegration of μ over $\nu=\pi\mu$. Then for any $\mathcal{V}\in\mathcal{C}_X$ and $0<\delta<1$, there exists $\beta\in\mathcal{P}_X$ such that $\beta\succeq\mathcal{V}$ and $N_{\mu_y}(\beta,\delta)\le N_{\mu_y}(\mathcal{V},\delta)$ for ν-a.e. $y\in Y$.*

Proof Let $\mathcal{V}=\{V_1,\cdots,V_m\}$. For ν-a.e. $y\in Y$, there exists $I_y\subset\{1,\cdots,m\}$ with cardinality $N_{\mu_y}(\mathcal{V},\delta)$ such that $\bigcup_{i\in I_y} V_i$ covers a subset of $\pi^{-1}y$ up to a set of μ_y-measure less than δ. Hence we can find $y_1,\cdots,y_s\in Y$ such that for ν-a.e. $y\in Y$, $I_y=I_{y_i}$ for some $i\in\{1,\cdots,s\}$. For $i=1,\cdots s$, define

$$D_i=\{y\in Y:\mu_y(\bigcup_{j\in I_{y_i}} V_j)>1-\delta\}.$$

Let $C_1=D_1$, $C_i=D_i\setminus\cup_{j=1}^{i-1}D_j$, $i=2,\cdots,s$.

Fix $i\in\{1,\cdots,s\}$. Assume $I_{y_i}=\{k_1<\cdots<k_{t_i}\}$ where $t_i=N_{\mu_{y_i}}(\mathcal{V},\delta)$. Take $\{W_1(y_i),\cdots,W_{t_i}(y_i)\}$ where

$$W_1(y_i)=V_{k_1},\ W_2(y_i)=V_{k_2}\setminus V_{k_1},\cdots,W_{t_i}(y_i)=V_{k_{t_i}}\setminus\cup_{j=1}^{t_i-1}V_{k_j}.$$

Define $A:=X\setminus\left(\cup_{i=1}^s(\pi^{-1}C_i\cap\cup_{j=1}^{t_i}W_j(y_j)\right)$ and $A_1=A\cap V_1$, $A_l:=A\cap(V_l\setminus\cup_{j=1}^{l-1}V_j)$, $l=2,\cdots,m$. Finally, define

$$\begin{aligned}\beta=\{&\pi^{-1}C_1\cap W_1(y_1),\cdots,\pi^{-1}C_1\cap W_{t_1}(y_1),\cdots,\\ &\pi^{-1}C_s\cap W_1(y_s),\cdots,\pi^{-1}C_s\cap W_{t_s}(y_s),A_1,\cdots,A_m\}.\end{aligned}$$

Then $\beta \succeq \mathcal{V}$ and $N_{\mu_y}(\beta, \delta) \leq N_{\mu_y}(\mathcal{V}, \delta)$ for ν-a.e. $y \in Y$. □

The following strong Rohlin lemma can be found in [26, Lemma 2.5].

Lemma 3.5 *(Strong Rohlin lemma) Let $(X, \mathcal{B}, \mu, T)$ be an ergodic, aperiodic invertible system and let $\alpha \in \mathcal{P}_X$. Then for any $\delta > 0$ and $n \in \mathbb{N}$, one can find a set $B \in \mathcal{B}$ such that $B, TB, \cdots, T^{n-1}B$ are mutually disjoint, $\mu(\cup_{i=0}^{n-1} T^i B) > 1 - \delta$ and the distribution of α is the same as the distribution of the partition $\alpha|_B$ that α induces on B.*

Proof of Proposition 3.3 Let $\mu \in \mathcal{E}_T(X)$. If the system (X, T, μ) is not aperiodic, then μ is supported on a periodic orbit of X. Then $h_\mu(T, \mathcal{U}|Y) = 0$, and $\underline{cp}_\mu(\varphi, \mathcal{U}|Y) \geq \int \varphi d\mu - \gamma(\varphi, \mathcal{U})$ follows from the definition. So let us assume (X, T) is aperiodic.

Fix $n \in \mathbb{N}$. Let β be constructed as in the proof of Lemma 3.4 for $\mathcal{V} \succeq \mathcal{U}_0^{n-1}$. We also use the notation from that proof, for example, A is the subset of X such that $\mu(A) < \delta$ and for any $x \notin A$, $N_{\mu_x}(\beta, \delta) \leq N_{\mu_x}(\mathcal{U}_0^{n-1}, \delta)$. Choose $\rho > 0$ such that $0 < \delta + \rho < 1/4$. By Lemma 3.5, we can construct a strong Rohlin tower with respect to β, with height n and error $< \rho$. Let $\tilde{B}$ denote the base of the tower and $B = \tilde{B} \setminus A$. Clearly, $\mu(B) > (1-\delta)\mu(\tilde{B})$ and $\mu(E) \geq 1 - (\delta + \rho)$ where $E = \cup_{i=0}^{n-1} T^i B$. Consider $\beta|_{\tilde{B}}$ and index its elements by sequences $i_0, \cdots, i_{n-1}$ such that if $B_{i_0,\cdots,i_{n-1}} \in \beta|_{\tilde{B}}$, then $T^j B_{i_0,\cdots,i_{n-1}} \subset U_{i_j}$ for every $0 \leq j \leq n-1$. Let $\hat{\alpha} := \{\hat{A}_1, \cdots, \hat{A}_M\}$ be a partition of E defined by

$$\hat{A}_m := \cup\{T^j B_{i_0,\cdots,i_{n-1}} : 0 \leq j \leq n-1, i_j = m\}.$$

Note that $\hat{A}_m \subset U_m$ for every $1 \leq m \leq M$ where $M = \#\mathcal{U}$. Extend $\hat{\alpha}$ to a partition α of X in some way such that $\alpha \succeq \mathcal{U}$ and $\#\alpha = 2M$.

Set $\eta^4 = \delta + \rho$ and define for every $k > n$ large enough, $f_k(x) = \frac{1}{k}\sum_{i=0}^{k-1} \chi_E(T^i x)$ and $L_k := \{x \in X : f_k(x) > 1 - \eta^2\}$. Then by the Birkhoff ergodic theorem $\int f_k > 1 - \eta^4$, and

$$\eta^2 \cdot \mu(L_k^c) \leq \int_{L_k^c} 1 - f_k \leq \int_X 1 - f_k \leq \eta^4$$

which gives $\mu(L_k) \geq 1 - \eta^2$. Put J_k to be the set of $x \in X$ such that for any $j \geq k$,

$$\mu_x(\alpha_0^{j-1}(x)) < \exp(-(h_\mu(T, \alpha|Y) - \eta)j), \tag{3}$$

$$|\frac{1}{j} S_j\varphi(x) - \int \varphi d\mu| \leq \eta, \tag{4}$$

and

$$|\frac{1}{j}\sum_{i=0}^{j-1} \log N_{\mu_{T^i x}}(\mathcal{U}_0^{n-1}, \delta)\chi_B(T^i x) - \int_B \log N_{\mu_z}(\mathcal{U}_0^{n-1}, \delta) d\mu(z)| \leq \eta. \tag{5}$$

By Theorem 2.7 and the Birkhoff ergodic theorem, $\mu(J_k) > 1 - \eta^2$ for k large enough. Set $G_k = L_k \cap J_k$ and then $\mu(G_k) > 1 - 2\eta^2$. Define $\tilde{G}_k = \{x \in G_k : \mu_x(G_k) \geq 1 - 4\eta\}$, then

$$\tilde{G}_k^c = \{x \in G_k : \mu_x(G_k) < 1 - 4\eta\} \cup G_k^c = \{x \in G_k : \mu_x(G_k^c) > 4\eta\} \cup G_k^c.$$

Therefore,

$$\mu(\tilde{G}_k^c) \cdot 4\eta \leq \int \mu_x(G_k^c) d\mu(x) + \mu(G_k^c) = 2\mu(G_k^c) \leq 4\eta^2,$$

i.e., $\mu(\tilde{G}_k^c) \leq \eta$.

Given any $y \in \tilde{G}_k$, we shall cover $G_k \cap \pi^{-1}\pi y$ by elements of α_0^{k-1}. Partition $G_k \cap \pi^{-1}\pi y$ into subsets as follows: For every point x in each subset, there are common values of $0 \leq i \leq k-n$ such that $T^i x \in B$. Note that if $x \in G_k \cap \pi^{-1}\pi y$ and $0 \leq i_1 < \cdots < i_m \leq k-n$ are the times at which x visits B, then the collection $\{[i_j, i_j+n-1]\}_{j=1}^m$ covers all but at most $\eta^2 k + 2n$ elements of $[0, k-1]$. So the number of elements in the partition of $G_k \cap \pi^{-1}\pi y$ is at most

$$\sum_{j<\eta^2 k+2n} \binom{k}{j} \leq e^{kH(\eta^2+2n/k)}$$

where $H(t) := -t\log t - (1-t)\log(1-t)$.

We fix an element C_y of this partition of $G_k \cap \pi^{-1}\pi y$ and want to estimate the number of elements of α_0^{n-1} needed to cover it. If $0 \leq i_1 < \cdots < i_m \leq k-n$ are the times elements of C_y visit B, then we need at most $N_{\mu_{T^{i_j}y}}(\mathcal{U}_0^{n-1}, \delta)$ $\alpha_{i_j}^{i_j+n-1}$-elements for every $1 \leq j \leq m$ to cover C_y. Because the size of $[0, k-1] \setminus \cup_j [i_j, i_j+n-1]$ is at most $\eta^2 k + 2n$, we need at most $\prod_{j=1}^m N_{\mu_{T^{i_j}y}}(\mathcal{U}_0^{n-1}, \delta) \cdot (2M)^{\eta^2 k+2n}$ α_0^{k-1}-elements to cover C_y. Finally, in view of (5), we know that $G_k \cap \pi^{-1}\pi y$ can be covered by no more than

$$e^{kH(\eta^2+2n/k)} \cdot (2M)^{\eta^2 k+2n} \cdot e^{k(\int_B \log N_{\mu_z}(\mathcal{U}_0^{n-1},\delta)d\mu(z)+\eta)}$$

α_0^{k-1}-elements. Since $y \in G_k \subset J_k$, any $V \in \alpha_0^{k-1}$ intersecting nontrivially with $G_k \cap \pi^{-1}\pi y$ has μ_y-measure less than $\exp(-(h_\mu(T, \alpha|Y) - \eta)k)$ by (3). Thus we have

$$\begin{aligned} 1 - 4\eta &\leq \mu_y(G_k \cap \pi^{-1}\pi y) \\ &\leq e^{-(h_\mu(T,\alpha|Y)-\eta)k} e^{kH(\eta^2+2n/k)} \cdot (2M)^{\eta^2 k+2n} \cdot e^{k(\int_B \log N_{\mu_z}(\mathcal{U}_0^{n-1},\delta)d\mu(z)+\eta)}. \end{aligned} \tag{6}$$

Recall that for any $z \in X$, by (4)

$$\Lambda_{\mu_z}(\varphi, \mathcal{U}, n, \delta) \geq N_{\mu_z}(\mathcal{U}_0^{n-1}, \delta) \cdot \exp\left(n\left(\int \varphi d\mu - \eta - \gamma(\varphi, \mathcal{U})\right)\right), \tag{7}$$

the distribution of β is the same as the distribution of the partition $\beta|_{\tilde{B}}$ and $z \mapsto N_{\mu_z}(\mathcal{U}_0^{n-1}, \delta)$ is constant on each atom of $\beta|_{X\setminus A}$ by Lemma 3.4. Then combining (6), (7), and setting $k \to \infty$, we get

$$\begin{aligned} &h_\mu(T, \alpha|Y) \\ &\quad\leq \eta + H(\eta^2) + \eta^2\log(2M) + \int_B \log N_{\mu_z}(\mathcal{U}_0^{n-1}, \delta)d\mu(z) + \eta \\ &\quad\leq 2\eta + H(\eta^2) + \eta^2\log(2M) + \frac{1}{n}\int \log N_{\mu_z}(\mathcal{U}_0^{n-1}, \delta)d\mu(z) \\ &\quad\leq 3\eta + H(\eta^2) + \eta^2\log(2M) + \frac{1}{n}\int \log \Lambda_{\mu_z}(\varphi, \mathcal{U}, n, \delta)d\mu(z) - \int \varphi d\mu + \gamma(\varphi, \mathcal{U}). \end{aligned}$$

By letting $\rho \to 0$, we obtain

$$\begin{aligned} &h_\mu(T, \alpha|Y) + \int \varphi d\mu - \gamma(\varphi, \mathcal{U}) \\ &\quad\leq 3\delta^{\frac{1}{4}} + H(\delta^{\frac{1}{2}}) + \delta^{\frac{1}{2}}\log(2M) + \frac{1}{n}\int \log \Lambda_{\mu_z}(\varphi, \mathcal{U}, n, \delta)d\mu(z). \end{aligned}$$

Taking $\liminf_{n\to\infty}$ and then $\lim_{\delta\to 0}$, we have

$$\begin{aligned} &h_\mu(T,\mathcal{U}|Y)+\int\varphi d\mu-\gamma(\varphi,\mathcal{U})\\ &\le h_\mu(T,\alpha|Y)+\int\varphi d\mu-\gamma(\varphi,\mathcal{U})\\ &\le \underline{cp}_\mu(\varphi,\mathcal{U}|Y). \end{aligned}$$

□

Combining Theorem 3.1 and the relative local variational principle for pressure [20, Theorem 1.3], we have:

Corollary 3.6 *Under the setting of Theorem 3.1, one has*

$$|\overline{CP}_X(\varphi,\mathcal{U}|Y)-\sup_{\mu\in\mathcal{E}_T(X)}\overline{CP}_\mu(\varphi,\mathcal{U}|Y)|\le\gamma(\varphi,\mathcal{U}),$$

and the above is true if $\overline{CP}_\mu(\varphi,\mathcal{U}|Y)$ *is replaced respectively by*

$$\underline{CP}_\mu(\varphi,\mathcal{U}|Y),\quad \overline{cp}_\mu(\varphi,\mathcal{U}|Y),\quad \underline{cp}_\mu(\varphi,\mathcal{U}|Y).$$

3.1 Variant Entropy

In this subsection, we consider various notions of local metric entropy. Let us introduce the *variant entropy* of μ with respect to $\mathcal{U}\in\mathcal{C}_X^o$ defined in Downarowicz [11].

Definition 3.7 [11, Definition 8.3.12] For any $\mu\in\mathcal{E}_T(X)$, the *variant entropy* of μ with respect to $\mathcal{U}\in\mathcal{C}_X^o$ is defined by

$$\mathbf{h}_\mu(T,\mathcal{U})=\inf\{\overline{CP}_Z(0,\mathcal{U}):\mu(Z)>0\}.$$

It is proved that $h_\mu(T)=\sup_{\mathcal{U}\in\mathcal{C}_X^o}\mathbf{h}_\mu(T,\mathcal{U})$ in [11, Lemma 8.4.13]. Then it is natural to ask:

Question 3.8 *[11, Question 8.3.13] For any $\mu\in\mathcal{E}_T(X)$ and $\mathcal{U}\in\mathcal{C}_X^o$, do we have*

$$\boldsymbol{h}_\mu(T,\mathcal{U})=h_\mu(T,\mathcal{U})?$$

To answer this question, let us summarize a corollary of Theorem 3.1.

Proposition 3.9 *We have for any $\mu\in\mathcal{E}_T(X)$ and $\mathcal{U}\in\mathcal{C}_X^o$*

$$h_\mu(T,\mathcal{U}|Y)=\underline{cp}_\mu(0,\mathcal{U}|Y)=\overline{cp}_\mu(0,\mathcal{U}|Y)=\underline{CP}_\mu(0,\mathcal{U}|Y)=\overline{CP}_\mu(0,\mathcal{U}|Y).$$

In the classical case, Shapira [26] has shown a stronger result:

Proposition 3.10 *[26, Theorems 4.1 and 4.4] For any $\mu\in\mathcal{E}_T(X)$, $\mathcal{U}\in\mathcal{C}_X^o$ and every $0<\delta<1$,*

$$h_\mu(T,\mathcal{U})=\underline{cp}_\mu(0,\mathcal{U},\delta)=\overline{cp}_\mu(0,\mathcal{U},\delta).$$

Thus in fact we have

Corollary 3.11 *For any $\mu\in\mathcal{E}_T(X)$, $\mathcal{U}\in\mathcal{C}_X^o$ and every $0<\delta<1$,*

$$h_\mu(T,\mathcal{U})=\underline{cp}_\mu(0,\mathcal{U},\delta)=\overline{cp}_\mu(0,\mathcal{U},\delta)=\underline{CP}_\mu(0,\mathcal{U},\delta)=\overline{CP}_\mu(0,\mathcal{U},\delta).$$

Proof By Propositions 3.10 and 3.9, we have

$$h_\mu(T,\mathcal{U}) = \underline{cp}_\mu(0,\mathcal{U},\delta) \le \underline{CP}_\mu(0,\mathcal{U},\delta) \\ \le \overline{CP}_\mu(0,\mathcal{U},\delta) \le \overline{CP}_\mu(0,\mathcal{U}) = h_\mu(T,\mathcal{U}).$$

So all inequalities above become equalities, and the corollary follows. □

Proposition 3.12 *For any $\mu \in \mathcal{E}_T(X)$ and $\mathcal{U} \in \mathcal{C}_X^o$, we have*

$$h_\mu(T,\mathcal{U}) = \boldsymbol{h}_\mu(T,\mathcal{U}).$$

Proof By definition $\mathbf{h}_\mu(T,\mathcal{U}) \le \overline{CP}_\mu(0,\mathcal{U},\delta)$ for any $\delta > 0$. Hence $\mathbf{h}_\mu(T,\mathcal{U}) \le h_\mu(T,\mathcal{U})$ by Corollary 3.11.

On the other hand, pick any Z with $\mu(Z) > 0$. Let $\delta = \mu(Z)$. Then $\overline{CP}_Z(0,\mathcal{U}) \ge \overline{CP}_\mu(0,\mathcal{U},\delta) = h_\mu(T,\mathcal{U})$ by Corollary 3.11. We are done by definition of $\mathbf{h}_\mu(T,\mathcal{U})$. □

To end this section, we define one more version of local metric entropy following Brink-Katok's approach [9].

Definition 3.13 Let $\pi : (X,T) \to (Y,S)$ be a factor map between two TDSs, $\mu \in \mathcal{M}_T(X)$ and $\mathcal{U} \in \mathcal{C}_X^o$. Define

$$\underline{h}_\mu^{BK}(T,\mathcal{U}|Y) := \int \underline{h}_\mu^{BK}(T,\mathcal{U}|Y,x)d\mu(x), \\ \overline{h}_\mu^{BK}(T,\mathcal{U}|Y) := \int \overline{h}_\mu^{BK}(T,\mathcal{U}|Y,x)d\mu(x),$$

where

$$\underline{h}_\mu^{BK}(T,\mathcal{U}|Y,x) := \liminf_{n\to\infty} \inf_{\mathbf{U}} -\frac{1}{n}\log \mu_x(X(\mathbf{U})), \\ \overline{h}_\mu^{BK}(T,\mathcal{U}|Y,x) := \limsup_{n\to\infty} \sup_{\mathbf{U}} -\frac{1}{n}\log \mu_x(X(\mathbf{U})),$$

and the infimum and supremum are taken over all strings $\mathbf{U}$ for which $x \in X(\mathbf{U})$ and $m(\mathbf{U}) = n$.

Let $\mu \in \mathcal{E}_T(X)$. Then $\underline{h}_\mu^{BK}(T,\mathcal{U}|Y,x) \ge \underline{h}_\mu^{BK}(T,\mathcal{U}|Y,Tx)$. By the Birkhoff ergodic theorem and ergodicity of μ,

$$\underline{h}_\mu^{BK}(T,\mathcal{U}|Y,x) = \underline{h}_\mu^{BK}(T,\mathcal{U}|Y) \tag{8}$$

for μ-a.e. x. Similar result holds for $\overline{h}_\mu^{BK}(T,\mathcal{U}|Y,x)$. Combining with Brin-Katok entropy formula [9,29], we have

$$h_\mu(T|Y) = \lim_{\mathrm{diam}\mathcal{U}\to 0} \underline{h}_\mu^{BK}(T,\mathcal{U}|Y,x) = \lim_{\mathrm{diam}\mathcal{U}\to 0} \overline{h}_\mu^{BK}(T,\mathcal{U}|Y,x),$$

see also [21, (11.16)] for the classical case.

4 Variational Inequalities for Local Pressure of Subsets

Given a T-invariant (not necessarily compact) Borel measurable set $Z \subseteq X$, denote also by $\mathcal{M}_T(Z) \subseteq \mathcal{M}_T(X)$ and $\mathcal{E}_T(Z) \subseteq \mathcal{E}_T(X)$ the subsets of measures μ for which $\mu(Z) = 1$ respectively. For $x \in X$ and $n \geq 0$, we define probability measures $\mu_{x,n}$ on X by

$$\mu_{x,n} = \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k x},$$

where δ_y denotes the Dirac measure supported at the point $y \in X$. Denote by $V_T(x)$ the set of limit measures (in weak* topology) of the sequence of measures $\{\mu_{x,n}\}_{n\in\mathbb{N}}$. It is easy to see that $\emptyset \neq V_T(x) \subseteq \mathcal{M}_T(X)$ and $V_T(x)$ is connected for each $x \in X$. Put

$$\mathcal{L}(Z) = \{x \in Z : V_T(x) \cap \mathcal{E}_T(Z) \neq \emptyset\}.$$

The following statements establish variational inequalities for local topological pressure on noncompact sets with respect to a fixed open cover. They generalize Pesin and Pitskel's result in [22] (see also [21, Theorem A.2.1]).

Proposition 4.1 *Let $\pi : (X, T) \to (Y, S)$ be a factor map between two TDSs and $Z \subseteq X$ be a T-invariant Borel measurable set. Then for any $\varphi \in C(X, \mathbb{R})$,*

$$P_{\mathcal{L}(Z)}(\varphi, \mathcal{U}|Y) \geq \sup_{\mu \in \mathcal{E}_T(Z)} \left(\underline{h}_\mu^{BK}(T, \mathcal{U}|Y) + \int_Z \varphi d\mu \right) - \gamma(\varphi, \mathcal{U}).$$

Proposition 4.2 *Let $T : X \to X$ be a TDS and $Z \subseteq X$ be a T-invariant Borel measurable set. Then for any $\varphi \in C(X, \mathbb{R})$,*

$$P_{\mathcal{L}(Z)}(\varphi, \mathcal{U}) \leq \sup_{\mu \in \mathcal{E}_T(Z)} \left(h_\mu(T, \mathcal{U}) + \int_Z \varphi d\mu \right) + \gamma(\varphi, \mathcal{U}).$$

Remark 4.3 If we define

$$\mathcal{L}_1(Z) := \{x \in Z : V_T(x) \cap \mathcal{M}_T(Z) \neq \emptyset\}$$

for a Borel measurable subset $Z \subset X$, then $\mathcal{L}(Z) \subset \mathcal{L}_1(Z)$. The proof of Proposition 4.2 below can be modified to show that

$$\begin{aligned} P_{\mathcal{L}_1(Z)}(\varphi, \mathcal{U}) &\leq \sup_{\mu \in \mathcal{M}_T(Z)} \left(h_\mu(T, \mathcal{U}) + \int_Z \varphi d\mu \right) + \gamma(\varphi, \mathcal{U}) \\ &= \sup_{\mu \in \mathcal{E}_T(Z)} \left(h_\mu(T, \mathcal{U}) + \int_Z \varphi d\mu \right) + \gamma(\varphi, \mathcal{U}) \end{aligned}$$

where the last equality follows from [16, Theorem 5.3]. See [2,3] for variational principles for $P_{\mathcal{L}_1(Z)}(\varphi, \mathcal{U})$.

We state some most interesting corollaries of Propositions 4.1 and 4.2:

(1) *Variational principle for topological pressure of subsets*: Letting diam$\mathcal{U} \to 0$, we get [21, Theorem A.2.1]:

$$P_{\mathcal{L}(Z)}(\varphi) = \sup_{\mu \in \mathcal{E}_T(Z)} \left(h_\mu(T) + \int_Z \varphi d\mu \right).$$

(2) *Inequality for local topological pressure of subsets:* For any T-invariant Borel measurable set $Z \subseteq X$, any $\varphi \in C(X, \mathbb{R})$ and $\mu \in \mathcal{E}_T(Z)$,

$$\underline{h}_\mu^{BK}(T, \mathcal{U}|Y) + \int_Z \varphi d\mu - \gamma(\varphi, \mathcal{U}) \leq P_Z(\varphi, \mathcal{U}|Y);$$

(3) *Local variational principle for topological pressure on compact invariant sets:* for any $\varphi \in C(X, \mathbb{R})$ and a compact invariant set $Z \subset X$,

$$\begin{aligned}&\sup_{\mu\in\mathcal{E}_T(Z)} \Big(\underline{h}_\mu^{BK}(T, \mathcal{U}) + \int_Z \varphi d\mu\Big) - \gamma(\varphi, \mathcal{U}),\\ &\leq P_Z(\varphi, \mathcal{U}) = \underline{CP}_Z(\varphi, \mathcal{U}) = \overline{CP}_Z(\varphi, \mathcal{U})\\ &\leq \sup_{\mu\in\mathcal{E}_T(Z)} \Big(h_\mu(T, \mathcal{U}) + \int_Z \varphi d\mu\Big) + \gamma(\varphi, \mathcal{U});\end{aligned}$$

By [17, Theorem 2], a stronger result than the last inequality holds:

$$\overline{CP}_Z(\varphi, \mathcal{U}) = \sup_{\mu\in\mathcal{E}_T(Z)} \Big(h_\mu(T, \mathcal{U}) + \int_Z \varphi d\mu\Big).$$

(4) *Generalization of Bowen's formula* [8, Theorem 3]: For any T-invariant Borel measurable set $Z \subseteq X$, any $\varphi \in C(X, \mathbb{R})$ and $\mu \in \mathcal{E}_T(Z)$,

$$\begin{aligned}P_{G_\mu}(\varphi, \mathcal{U}|Y) \geq& \underline{h}_\mu^{BK}(T, \mathcal{U}|Y) + \int_Z \varphi d\mu - \gamma(\varphi, \mathcal{U}),\\ P_{G_\mu}(\varphi, \mathcal{U}) \leq& h_\mu(T, \mathcal{U}) + \int_Z \varphi d\mu + \gamma(\varphi, \mathcal{U}),\end{aligned}$$

where G_μ is the set of all forward generic points of the measure μ (i.e., the points for which the Birkhoff ergodic theorem holds for any continuous function on X);

(5) Given a Borel measurable T-invariant (not necessarily compact) subset $Z \subseteq X$ with the property that $V_T(x) \cap \mathcal{E}_T(Z) \neq \emptyset$ for any $x \in Z$, we have

$$\begin{aligned}P_Z(\varphi, \mathcal{U}|Y) &\geq \sup_{\mu\in\mathcal{E}_T(Z)} \Big(\underline{h}_\mu^{BK}(T, \mathcal{U}|Y) + \int_Z \varphi d\mu\Big) - \gamma(\varphi, \mathcal{U}),\\ P_Z(\varphi, \mathcal{U}) &\leq \sup_{\mu\in\mathcal{E}_T(Z)} \Big(h_\mu(T, \mathcal{U}) + \int_Z \varphi d\mu\Big) + \gamma(\varphi, \mathcal{U}).\end{aligned}$$

Proof of Proposition 4.1 We first show that for any invariant set $Z \subseteq X$, any $\varphi \in C(X, \mathbb{R})$ and any $\mu \in \mathcal{E}_T(Z)$,

$$\underline{h}_\mu^{BK}(T, \mathcal{U}|Y) + \int_Z \varphi d\mu - \gamma(\varphi, \mathcal{U}) \leq P_Z(\varphi, \mathcal{U}|Y).$$

Fix $\rho > 0$. For each $k \geq 1$, define Z_k to be the set of all $x \in Z$ such that for any $n \geq k$,

$$\inf_{\mathbf{U}} \frac{-\log \mu_x(X(\mathbf{U}))}{n} \geq \underline{h}_\mu^{BK}(T, \mathcal{U}|Y) - \rho,$$

and

$$|\frac{1}{n} S_n\varphi(x) - \int_Z \varphi d\mu| < \rho, \tag{9}$$

where the infimum is taken over all strings $\mathbf{U}$ for which $x \in X(\mathbf{U})$ and $m(\mathbf{U}) = n$. Then $Z_k \subset Z_{k+1}$, and $Z = \bigcup_{k=1}^{\infty} Z_k \mod \mu$ by (8). We fix some $k \geq 1$ such that $\mu(Z_k) > \frac{1}{2}\mu(Z) > 0$. One has

$$\mu_x(X(\mathbf{U})) \leq e^{-n(\underline{h}_{\mu}^{BK}(T,\mathcal{U}|Y)-\rho)} \tag{10}$$

for all $x \in Z_k$, $n \geq k$ and any string $\mathbf{U}$ for which $x \in X(\mathbf{U})$ and $m(\mathbf{U}) = n$. Clearly, there exists $x \in X$ such that $\mu_x(Z_k) > 0$. Fix such x. Note that if $y \in \pi^{-1}\pi x$, then $\mu_y = \mu_x$.

Fix a sufficiently large $N > k$ and consider $\mathcal{G} \subset \bigcup_{j \geq N} \mathcal{W}_j(\mathcal{U})$ that covers $Z_k \cap \pi^{-1}\pi x$, i.e., $(Z_k \cap \pi^{-1}\pi x) \subset \bigcup_{\mathbf{U} \in \mathcal{G}} X(\mathbf{U})$. Let $\mathcal{G}_n \subset \mathcal{G}$ be a subcollection of strings for which $m(\mathbf{U}) = n$ and $X(\mathbf{U}) \cap (Z_k \cap \pi^{-1}\pi x) \neq \emptyset$. Denote $s = \underline{h}_{\mu}^{BK}(T,\mathcal{U}|Y) + \int_Z \varphi d\mu - \gamma(\varphi, \mathcal{U}) - 2\rho$. By (9) and (10),

$$\begin{aligned}
&\sum_{\mathbf{U} \in \mathcal{G}} \exp\left(-sm(\mathbf{U}) + \sup_{x \in X(\mathbf{U})} S_{m(\mathbf{U})}\varphi(x)\right) \\
&\geq \sum_{n=N}^{\infty} \sum_{\mathbf{U} \in \mathcal{G}_n} \exp\left((-s + \int_Z \varphi d\mu - \gamma(\varphi, \mathcal{U}) - \rho)n\right) \\
&= \sum_{n=N}^{\infty} \sum_{\mathbf{U} \in \mathcal{G}_n} \exp\left((-\underline{h}_{\mu}^{BK}(T,\mathcal{U}|Y) + \rho)n\right) \\
&\geq \sum_{n=N}^{\infty} \sum_{\mathbf{U} \in \mathcal{G}_n} \mu_x(X(\mathbf{U})) \geq \mu_x(Z_k) > 0.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathcal{M}^s(\mathcal{U}, \varphi, Z \cap \pi^{-1}\pi x) &\geq \mathcal{M}_N^s(\mathcal{U}, \varphi, Z \cap \pi^{-1}\pi x) \\
&\geq \mathcal{M}_N^s(\mathcal{U}, \varphi, Z_k \cap \pi^{-1}\pi x) > 0.
\end{aligned}$$

Thus $P_{Z \cap \pi^{-1}\pi x}(\varphi, \mathcal{U}) \geq s$. Letting $\rho \to 0$, we have

$$\underline{h}_{\mu}^{BK}(T,\mathcal{U}|Y) + \int_Z \varphi d\mu - \gamma(\varphi, \mathcal{U}) \leq P_Z(\varphi, \mathcal{U}|Y).$$

Now consider a T-invariant subset $Z \subset X$. For any $\mu \in \mathcal{E}_T(Z)$, denote $Z_\mu := \{x \in Z : V_T(x) = \{\mu\}\}$. Then $\mu(Z_\mu) = 1$ and $Z_\mu \subset \mathcal{L}(Z)$. Thus

$$\underline{h}_{\mu}^{BK}(T,\mathcal{U}|Y) + \int_Z \varphi d\mu - \gamma(\varphi, \mathcal{U}) \leq P_{Z_\mu}(\varphi, \mathcal{U}|Y) \leq P_{\mathcal{L}(Z)}(\varphi, \mathcal{U}|Y).$$

□

Proof of Proposition 4.2 The proof is a slight modification of the second part of the proof of [21, Theorem A.2.1]. We sketch the proof for completeness.

We only need to show the following: Given a Borel T-invariant (not necessarily compact) subset $Y \subseteq X$ with the property that $V_T(x) \cap \mathcal{E}_T(Y) \neq \emptyset$ for any $x \in Y$, we have

$$P_Y(\varphi, \mathcal{U}) \leq \sup_{\mu \in \mathcal{E}_T(Y)} \left(h_\mu(T, \mathcal{U}) + \int_Y \varphi d\mu\right) + \gamma(\varphi, \mathcal{U}).$$

Let E be a finite set and $\underline{a} = (a_0, \cdots, a_{k-1}) \in E^k$. Define the measure $\mu_{\underline{a}}(e) = \frac{1}{k} \cdot$ (the number of those j for which $a_j = e$). Set

$$H(\underline{a}) = -\sum_{e \in E} \mu_{\underline{a}} \log \mu_{\underline{a}}(e).$$

Consider the set

$$R(k, h, E) = \{\underline{a} \in E^k : H(\underline{a}) \le h\}.$$

The following statement describes the asymptotic growth in k of the number of elements in the set $R(k, h, E)$.

Lemma 4.4 [21, Lemma 3] *We have*

$$\limsup_{k \to \infty} \frac{1}{k} \log |R(k, h, E)| \le h.$$

Let us fix $\epsilon > 0$. Suppose that $\mathcal{U} = \{U_1, \cdots, U_r\}$.

Lemma 4.5 [21, Lemma 4] *Given $x \in Y$ and $\mu \in V_T(x) \cap \mathcal{E}_T(Y)$, there exists a number $m > 0$ such that for any $n > 0$ one can find $N > n$ and a string $\mathbf{U}$ with $m(\mathbf{U}) = N$ satisfying:*

(1) $x \in X(\mathbf{U})$;

(2) $\sup_{y \in X(\mathbf{U})} \sum_{k=0}^{m(\mathbf{U})-1} \varphi(T^k y) \le N\Big(\int_Y \varphi d\mu + \gamma(\varphi, \mathcal{U}) + \epsilon\Big)$;

(3) the string $\mathbf{U}$ contains a substring $\mathbf{U}'$ of length $m(\mathbf{U}') = km \ge N - m$ which, being written as $\underline{a} = (a_0, \cdots, a_{k-1})$, satisfies the inequality

$$\frac{1}{m} H(\underline{a}) \le h_\mu(T, \mathcal{U}) + \epsilon. \tag{11}$$

Proof The lemma is [21, Lemma 4] except that in item (3) we have $h_\mu(T, \mathcal{U})$ instead of $h_\mu(T)$. By the proof of [21, Lemma 4] item (3) holds for $h_\mu(T, \zeta)$, for any $\zeta = \{C_1, \cdots, C_r\} \in \mathcal{P}_X$ with $\overline{C_i} \subset U_i, i = 1, \cdots r$. We claim that $C_i \subset U_i$ is enough for that proof. So item (3) holds for $h_\mu(T, \zeta)$ for any $\zeta = \{C_1, \cdots, C_r\} \in \mathcal{P}_X$ with $C_i \subset U_i, i = 1, \cdots r$. By [17, Lemma 3.1], it holds for $h_\mu(T, \mathcal{U})$ as required.

To prove the claim, choose any $\zeta = \{C_1, \cdots, C_r\} \in \mathcal{P}_X$ with $C_i \subset U_i, i = 1, \cdots r$. Following the notations in the proof of [21, Lemma 4], let $\xi := \zeta_0^{m-1} = \{D_1, \cdots, D_t\}$. Fix $\beta > 0$ small enough. For each D_i one can find a compact set $K_i \subset D_i$ such that $\mu(D_i \setminus K_i) \le \beta$. Each D_i is contained in some element B_i of the open cover $\mathcal{V} := \mathcal{U}_0^{m-1}$. Since K_i are disjoint compact sets, one can find disjoint open sets V_i such that $K_i \subset V_i \subset B_i$. Moreover, there exists Borel subsets V_i^* comprising a Borel partition of X such that $V_i \subset V_i^* \subset B_i$. We emphasize that the construction of V_i and V_i^* does not require $\overline{D_i} \subset B_i$. Then for $p_i^{(j)}$ defined on page 92 in [21], we have

$$\begin{aligned}
\liminf_{j \to \infty} p_i^{(j)} &= \liminf_{j \to \infty} \mu_{x,n_j}(V_i^*) \ge \liminf_{j \to \infty} \mu_{x,n_j}(V_i) \\
&\ge \mu(V_i) \ge \mu(K_i) \ge \mu(D_i) - \beta, \\
\limsup_{j \to \infty} p_i^{(j)} &= \limsup_{j \to \infty} \mu_{x,n_j}(V_i^*) \le \limsup_{j \to \infty} \mu_{x,n_j}(X \setminus (\cup_{j \ne i} V_i)) \\
&\le \mu(X \setminus (\cup_{j \ne i} V_i)) \le \mu(X \setminus (\cup_{j \ne i} K_i)) \\
&\le \mu(D_i) + \sum_{j \ne i} \mu(D_j \setminus K_j) \le \mu(D_i) + t\beta.
\end{aligned}$$

So we can proceed exactly as in the proof of [21, Lemma 4] to prove the claim. □

Given a number $m > 0$, denote by Y_m the set of points $y \in Y$ for which Lemma 4.5 holds for this m and for some $\mu \in V_T(x) \cap \mathcal{E}_T(Y)$. Then $Y = \cup_{m>0} Y_m$. Denote also by $Y_{m,u}$ the set of points $y \in Y_m$ for which Lemma 4.5 holds for some measure $\mu \in V_T(x) \cap \mathcal{E}_T(Y)$ satisfying $\int_Y \varphi d\mu \in [u-\epsilon, u+\epsilon]$. Set

$$c = \sup_{\mu \in \mathcal{E}_T(Y)} (h_\mu(T, \mathcal{U}) + \int_Y \varphi d\mu).$$

Note that if $x \in Y_{m,u}$ then the corresponding measure μ satisfies $h_\mu(T, \mathcal{U}) \leq c - u + \epsilon$. Let $\mathcal{G}_{m,u}$ be the collection of all strings $\mathbf{U}$ described in Lemma 4.5 that correspond to all $x \in Y_{m,u}$ and all N exceeding some number N_0. It follows form (11) that for any $x \in Y_{m,u}$ the substring constructed in Lemma 4.5 is contained in $R(k, m(h+\epsilon), \mathcal{U}^m)$, where $h = c - u + \epsilon$. Therefore, the total number of the strings constructed in Lemma 4.5 does exceed $b(N) = |\mathcal{U}|^m |R(k, m(h+\epsilon), \mathcal{U}^m)|$. By Lemma 4.4 we obtain that

$$\limsup_{N\to\infty} \frac{\log b(N)}{N} \leq h + \epsilon. \tag{12}$$

Since the collection of strings $\mathcal{G}_{m,u}$ covers the set $Y_{m,u}$ we conclude using Lemma 4.5 and (12) that

$$\begin{aligned}\mathcal{M}^\lambda_{N_0}(\mathcal{U}, \varphi, Y_{m,u}) &\leq \sum_{N=N_0}^{\infty} b(N) \exp(-\lambda m(\mathbf{U}) + \sup_{y \in X(U)} \sum_{k=0}^{m(U)-1} \varphi(T^k y)) \\ &\leq \sum_{N=N_0}^{\infty} b(N) \exp(-\lambda m(\mathbf{U}) + N(\int_Y \varphi d\mu + \gamma(\varphi, \mathcal{U}) + \epsilon)).\end{aligned}$$

If N_0 is sufficiently large, we have that $b(N) \leq \exp(N(h + 2\epsilon))$. Hence,

$$\mathcal{M}^\lambda_{N_0}(\mathcal{U}, \varphi, Y_{m,u}) \leq \frac{\beta^{N_0}}{1-\beta}, \tag{13}$$

where

$$\beta = \exp(-\lambda + h + \int_Y \varphi d\mu + \gamma(\varphi, \mathcal{U}) + 3\epsilon).$$

It follows from (13) that if $\lambda > c + \gamma(\varphi, \mathcal{U}) + 5\epsilon$ then $\mathcal{M}^\lambda(\mathcal{U}, \varphi, Y_{m,u}) = 0$. Hence $\lambda \geq P_{Y_{m,u}}(\varphi, \mathcal{U})$. Assume that points $u_1, \cdots, u_r$ form an ϵ-net of the interval $[-\|\varphi\|, \|\varphi\|]$. Then

$$Y = \cup_{m=1}^{\infty} \cup_{i=1}^{r} Y_{m,u_i}.$$

We have that $\lambda \geq P_{Y_{m,u_i}}(\varphi, \mathcal{U})$ for any m and i. Therefore,

$$\lambda \geq \sup_{m,i} P_{Y_{m,u_i}}(\varphi, \mathcal{U}) = P_Y(\varphi, \mathcal{U}).$$

This implies that $c + \gamma(\varphi, \mathcal{U}) + 5\epsilon \geq P_Y(\varphi, \mathcal{U})$. Since ϵ can be chosen arbitrarily small it follows that $c + \gamma(\varphi, \mathcal{U}) \geq P_Y(\varphi, \mathcal{U})$ and the desired result follows. □

Acknowledgements This work is supported by NSFC Nos. 12071474 and 11701559.

References

1. Adler, R.L., Konheim, A.G., McAndrew, M.H.: Topological entropy. Trans. Am. Math. Soc. **114**(2), 309–319 (1965)
2. Barreira, L.: A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems. Ergodic Theory Dynam. Systems **16**(5), 871–927 (1996)
3. Barreira, L.: Thermodynamic formalism and applications to dimension theory. Progress in Mathematics, 294. Birkhäuser/Springer Basel AG, Basel, 2011. xii+297 pp
4. Blanchard, F.: Fully positive topological entropy and topological mixing. Symbolic Dynamics and Its Applications (AMS Contemporary Mathematics, 135). Ed. P. Walters, American Mathematical Society, Providence, 1992, 95–105
5. Blanchard, F.: A disjointness theorem involving topological entropy. Bull. Soc. Math. France **121**, 565–578 (1993)
6. Blanchard, F., Glasner, E., Host, B.: A variation on the variational principle and applications to entropy pairs. Ergod. Theory Dyn. Syst. **17**, 29–43 (1997)
7. Blanchard, F., Host, B., Maass, A., Martinez, S., Rudolph, D.: Entropy pairs for a measure. Ergod. Theory Dyn. Syst. **15**, 621–632 (1995)
8. Bowen, R.: Topological entropy for noncompact sets. Trans. Am. Math. Soc. **184**, 125–136 (1973)
9. Brin, M., Katok, A.: On local entropy. Geometric Dynamics, Lecture Notes in Math.,vol.1007, Springer, Berlin, 1983, pp.30–38
10. Dooley, A., Zhang, G.: Local entropy theory of a random dynamical system. Memoirs of the American Mathematical Society, **233** (1099) (2015), vi+106pp
11. Downarowicz, T.: Entropy in dynamical systems. New Mathematical Monographs, 18, Cambridge University Press, 2011, xii+391 pp
12. Downarowicz, T., Serafin, J.: Fiber entropy and conditional variational principles in compact non-metrizable spaces. Fund. Math. **172**, 217–247 (2002)
13. Glasner, E., Weiss, B.: On the interplay between measurable and topological dynamics. Handbook of Dynamical Systems. Vol. 1B. Eds. Hasselblatt and Katok. North-Holland, Amsterdam, 2005, pp. 597–648
14. Glasner, E., Ye, X.: Local entropy theory. Ergod. Theory Dyn. Syst. **29**(2), 321–356 (2009)
15. Huang, W., Ye, X.: A local variational relation and applications. Israel J. Math. **151**(1), 237–279 (2006)
16. Huang, W., Ye, X., Zhang, G.: A local variational principle for conditional entropy. Ergod. Theory Dyn. Syst. **26**(01), 219–245 (2006)
17. Huang, W., Yi, Y.: A local variational principle of pressure and its applications to equilibrium states. Israel J. Math. **161**(1), 29–74 (2007)
18. Kolmogorov, A.N.: A new metric invariant of transient dynamical systems and automorphisms in Lebesgue spaces. Doklady Akad. Nauk SSSR **119**, 861–864 (1958)
19. Ledrappier, F., Walters, P.: A relativised variational principle for continuous transformations. J. Lond. Math. Soc **16**(2), 568–576 (1977)
20. Ma, X., Chen, E.: *Variational principles for relative local pressure with subadditive potentials*, J. Math. Phys. **54**(3), 2701, 25 pp (2013)
21. Pesin, Ya. B.: Dimension theory in dynamical systems: contemporary views and applications. Chicago Lectures in Mathematics, University of Chicago Press (2008)
22. Y. B. Pesin and B. S. Pitskel′, Topological pressure and the variational principle for noncompact sets. Funct. Anal. Appl. **18**(4): 307-318 (1984)
23. Rohlin, V.A.: On the fundamental ideas of measure theory. Am. Math. Soc. Transl. No. 71, 55 (1952)
24. Romagnoli, P.: A local variational principle for the topological entropy. Ergod. Theory Dyn. Syst. **23**(05), 1601–1610 (2003)
25. Ruelle, D.: Statistical mechanics on a compact set with $\mathbb{N}^{\nu}$ action satisfying expansiveness and specification. Trans. Am. Math. Soc. **187**, 237–251 (1973)
26. Shapira, U.: Measure theoretical entropy of covers. Israel J. Math. **158**(1), 225–247 (2007)
27. Sinai, Y.: On the concept of entropy for a dynamic system. Dokl. Akad. Nauk SSSR **124**, 768–771 (1959)
28. Walters, P.: A variational principle for the pressure of continuous transformations. Am. J. Math. **97**(4), 937–971 (1975)
29. Zhou, X.: A formula of conditional entropy and some applications. Discrete Contin. Dyn. Syst. Ser. A **36**(7), 4063–4075 (2016)