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曲阜師範大學

硕 士 学 位 论 文

Gronwall-Bellman 型积分不等式的推广及应用

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摘　要

微分方程是数学学科中与应用密切相关的分支, 利用微分方程理论可以描述和解释自然科学和社会科学中的许多现象. 自 1943 年 Gronwall-Bellman 积分不等式被证明以来, 关于这一方面的研究就层出不穷. 近年来, 众多学者建立了许多形式的 Gronwall-Bellman 型积分不等式. 这类不等式为研究微分方程解的存在性, 唯一性, 有界性等定性性质提供了有利的工具.

关于 Gronwall-Bellman 型积分不等式, 开始人们关注的更多的是有关连续函数的, 而有关不连续函数的情形最近才开始被重视. 本文是在已有研究成果的基础上, 对连续函数和不连续函数的积分不等式都进行了推广, 得出了一些新的结果.

根据内容本文分为以下四章:

第一章 建立了一类新型的非连续函数的时滞积分不等式,

$$
\begin{aligned}
u(x) \leq \varphi(x) + q(x)\int_{x_0}^{x} f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)d\tau + q(x)\int_{x_0}^{x} g(\tau,x)u\left(\sigma(\tau)\right)d\tau \\
+ \sum_{x_0<x_j<x}\beta_j u(x_j-0),
\end{aligned}
$$

并对其进行了推广.

第二章 在第一章的基础上, 受文献 [17, 19] 的启发, 对不等式改变了积分限. 如下,

$$
\begin{aligned}
u(x) \leq \varphi(x) + q(x)\int_{x}^{\infty} f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)d\tau + q(x)\int_{x}^{\infty} g(\tau,x)u\left(\sigma(\tau)\right)d\tau \\
+ \sum_{x<x_j<\infty}\beta_j u(x_j-0).
\end{aligned}
$$

第三章 在本章中, 主要研究了以下两个二元积分不等式解的估计,

$$
\begin{aligned}
u^p(x,y) \leq a(x)+b(y)+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\left[f(\tau,s)u^q(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)u^q(\tau,s)\right]dsd\tau \\
+\int_{x}^{\infty}\int_{y}^{\infty}\left[f(\tau,s)u^q(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)u^q(\tau,s)\right]dsd\tau, \\
\varphi\left(u(x,y)\right) \leq a(x)+b(y)+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\left[f(\tau,s)u(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)u(\tau,s)\right]dsd\tau \\
+\int_{x}^{\infty}\int_{y}^{\infty}\left[f(\tau,s)u(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)u(\tau,s)\right]dsd\tau.
\end{aligned}
$$

第四章 在本章中, 主要介绍了一类新的非线性 Volterra-Fredholm 型时滞积分不等式,

$$
\begin{aligned}
& u(x,y) \\
\le\ & k + q(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(u(s,t)\right)\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t} g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds \\
& + q(x,y)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} f(s,t)\varphi\left(u(s,t)\right)\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t} g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds.
\end{aligned}
$$

关键词: 微分方程; 积分方程; 时滞; 非线性积分不等式; 不连续函数.

Abstract

Differential equation is a branch of mathematics and it has closely connection with application. Using the theory of differential equation, we can describe and explain the various problems of many fields of natural and social science. In 1943, the famous Gronwall-Bellman integral inequality was proven. Then the research on this field appears one after another. In recent years, a number of scholars has established many forms of Gronwall-Bellman type integral inequalities, which provides a favorable tool to study the existence, uniqueness, boundedness and other qualitative properties of the solution of the differential and integral equations.

At the beginning, scholars concerned more about the continuous functions rather than the discontinuous functions with regard to Gronwall-Bellman type integral inequalities. Not too long ago, we began to pay attention to the case of discontinuous functions. Based on the existing study, this thesis generalizes and improves some new integral inequalities of both continuous and discontinuous functions and it also obtains some new results.

The thesis is divided into four chapters according to the contents.

Chapter 1 We establish a new type of retarted integral inequality for discontinuous functions,

$$u(x) \leq \varphi(x) + q(x)\int_{x_0}^{x} f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)d\tau + q(x)\int_{x_0}^{x} g(\tau,x)u\left(\sigma(\tau)\right)d\tau + \sum_{x_0<x_j<x}\beta_j u(x_j-0),$$

and extend the above inequality.

Chapter 2 On the basis of Chapter 1 and motivited by [17, 19], we change the limits of integration of the inequalities as follows,

$$u(x) \leq \varphi(x) + q(x)\int_{x}^{\infty} f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)d\tau + q(x)\int_{x}^{\infty} g(\tau,x)u\left(\sigma(\tau)\right)d\tau + \sum_{x<x_j<\infty}\beta_j u(x_j-0).$$

Chapter 3 In this chapter, we mainly study the estimates on the solution of the

following two integral inequalities,

$$\begin{aligned}
u^p(x,y) \leq\ & a(x)+b(y) \\
&+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\left[f(\tau,s)u^q(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)u^q(\tau,s)\right]dsd\tau \\
&+\int_{x}^{\infty}\int_{y}^{\infty}\left[f(\tau,s)u^q(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)u^q(\tau,s)\right]dsd\tau, \\
\varphi\left(u(x,y)\right) \leq\ & a(x)+b(y) \\
&+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\left[f(\tau,s)u(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)u(\tau,s)\right]dsd\tau \\
&+\int_{x}^{\infty}\int_{y}^{\infty}\left[f(\tau,s)u(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)u(\tau,s)\right]dsd\tau.
\end{aligned}$$

Chapter 4 In this chapter, we focus on a class of new nonlinear retarted Volterra-Fredholm type integral inequality,

$$\begin{aligned}
&u(x,y) \\
&\leq k+q(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(x)}f(s,t)\varphi\left(u(s,t)\right)\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds \\
&\quad+q(x,y)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)\varphi\left(u(s,t)\right)\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds.
\end{aligned}$$

Key words: Differential equation; Integral equation; Delay; Nonlinear integral inequalities; Discontinuous function.

目　　录

第一章　一类新型非连续时滞积分不等式

1.1　引言及预备知识

积分不等式在微分方程定性理论的研究中起着极其重要的作用. 自 1943 年 Gronwall-Bellman 不等式被证明以来, 由于它具有广泛的应用, 大量学者对其做出了一系列的推广和改进 $[1-15]$.

本章建立了一类新型有关不连续函数的时滞积分不等式, 对已有研究结果进行了改进和推广. 下面, 我们给出一些引理.

引理 1.1.1[2]　$u \in L^{\infty}[0,r],\ f \in L^{1}[0,r]$ 都是非负函数, 且满足

$$u^2(t) \leq M^2u^2(0) + 2\int_0^t \left[Nf(s)u(s) + Ku^2(s)\right] ds,\ \forall t \in [0,r], \tag{1.1.1}$$

其中 $M, N, K > 0$ 为常数, 则

$$u(t) \leq \left[Mu(0) + N\int_0^r f(s)ds\right] e^{Kr}. \tag{1.1.2}$$

引理 1.1.2[6]　$a(t,s) \in C(R_+^2, R_+), \frac{\partial a(t,s)}{\partial t} \in C(R_+^2, R_+)$. 若 $k, \alpha, w \in C(R_+, R_+)$ 都是非减函数且 $k(0) > 0$, 当 $t > 0$ 时, $w(t) > 0$. 若 $u \in C(R_+, R_+)$ 满足

$$u(t) \leq k(t) + \int_0^{\alpha(t)} a(t,s)w\left(u(s)\right) ds, t \geq 0, \tag{1.1.3}$$

则

$$u(t) \leq \Omega^{-1}\left[\Omega\left(k(t)\right) + \int_0^{\alpha(t)} a(t,s)ds\right], t \geq 0, \tag{1.1.4}$$

其中 $\Omega(t) = \int_1^t \frac{ds}{w(s)}, t > 0$.

1.2　主要结果及证明

定理 1.2.1　$u(x)$ 为非负函数, 定义在区间 $R_+^{x_0} = [x_0, \infty) = \bigcup\limits_{i\geq 1}\left\{x | x \in [x_{i-1}, x_i]\right\}$, $i = 1, 2, 3\ldots$ 上, $u(x)$ 在 $R_+^{x_0}$ 上除了 $x_i\ (i \geq 1)$ 点外连续, $u(x_i - 0) \neq u(x_i + 0)$,

$\lim_{i\to\infty} x_i = \infty, x_i \le x_{i+1}$, 若

$$u(x) \le \varphi(x) + q(x)\int_{x_0}^{x} f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)d\tau + q(x)\int_{x_0}^{x} g(\tau,x)u\left(\sigma(\tau)\right)d\tau$$
$$+\sum_{x_0<x_j<x}\beta_j u(x_j-0),\ x\in[x_{i-1},x_i], \tag{1.2.1}$$

且满足

(1) $x_0 \ge 0,\ \beta_j > 0$ 为常数;

(2) $\forall x \in R_+^{x_0},\ q(x)\ge 1,\ \varphi(x) > 0$ 都是非减函数;

(3) $f(\tau,x),\ g(\tau,x)\in C(R_+^2,R_+),\ \frac{\partial f(\tau,x)}{\partial x},\ \frac{\partial g(\tau,x)}{\partial x}\in C(R_+^2,R)$, 且对于每一个固定的 $\tau\in R_+^{x_0}$, f, g 关于 x 是不减的;

(4) $\sigma(x)\le x$ 是非负连续函数, 且当 $x\in[x_{i-1},x_i]$ 时, $\sigma(x)\le x_i,\ \sigma(x)\ge x_{i-1}$;

(5) $w(u)$ 满足以下条件:

(a) $w(\alpha\beta)\le w(\alpha)w(\beta)$,

(b) $w\in C(R_+,R_+)$, 且当 $x\in(0,\infty)$ 时, $w(x)>0$,

(c) w 为非减函数;

则

$$u(x)\le q(x)\varphi(x)\exp\left(G_{i-1}(x)\right)$$
$$\Phi_{i-1}^{-1}\left[\int_{x_{i-1}}^{x} F_{i-1}(\tau)\exp\left(-G_{i-1}(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_{i-1}(\tau)\right)\right)d\tau\right],\ \forall x\in[x_{i-1},x_i], \tag{1.2.2}$$

其中

$F_{i-1}(x)=\frac{d}{dx}\int_{x_{i-1}}^{x}\frac{f(\tau,x)}{\varphi(\tau)}d\tau,\ G_{i-1}(x)=\int_{x_{i-1}}^{x} g(\tau,x)q(\tau)d\tau,$

$\Phi_{i-1}(r)=\int_{l_{i-1}}^{r}\frac{ds}{w(s)},\ r>0,\ \Phi_{i-1}^{-1}$ 为其反函数, $i=1,2\ldots,$

$l_0=1,$

$l_{i-1}=(1+\beta_{i-1}q(x_{i-1}-0))\exp\left(G_{i-2}(x_{i-1})\right)$
$\Phi_{i-2}^{-1}\left[\int_{x_{i-2}}^{x_{i-1}} F_{i-2}(\tau)\exp\left(-G_{i-2}(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_{i-2}(\tau)\right)\right)d\tau\right],\ i=2,3\ldots.$

证明: 因为 $q(x)\ge 1,\ \varphi(x)>0$ 为非减函数, 由不等式 (1.2.1) 知

$$\frac{u(x)}{\varphi(x)}\le q(x)\left[1+\int_{x_0}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_0}^{x}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau+\sum_{x_0<x_j<x}\beta_j\frac{u(x_j-0)}{\varphi(x_j-0)}\right],$$

令

$$v(x)=1+\int_{x_0}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_0}^{x}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau+\sum_{x_0<x_j<x}\beta_j\frac{u(x_j-0)}{\varphi(x_j-0)},\tag{1.2.3}$$

则 $v(x)$ 非负不减,

$$u(x)\leq q(x)\varphi(x)v(x),\tag{1.2.4}$$

且由 $\sigma(x)\leq x$ 及 (1.2.4) 式得

$$u\left(\sigma(x)\right)\leq q\left(\sigma(x)\right)\varphi\left(\sigma(x)\right)v\left(\sigma(x)\right)\leq q(x)\varphi(x)v(x),\tag{1.2.5}$$

当 $x\in[x_{i-1},x_i]$ 时, 令

$$\widetilde{v}_{i-1}(x)=l_{i-1}+\int_{x_{i-1}}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_{i-1}}^{x}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau,$$

则 $\widetilde{v}_{i-1}(x)$ 在 $[x_{i-1},x_i]$ 上非负不减, 且 $\widetilde{v}_{i-1}(x_{i-1})=l_{i-1}$,
当 $x\in[x_0,x_1]$ 时, 此时

$$\widetilde{v}_0(x)=v(x),\tag{1.2.6}$$

$$\widetilde{v}_0(x)=l_0+\int_{x_0}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_0}^{x}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau,\tag{1.2.7}$$

对 (1.2.7) 式关于 x 求导, 由 (1.2.5) (1.2.6) 式得

$$\begin{aligned}
\widetilde{v}_0'(x)&=\frac{f(x,x)w\left(u\left(\sigma(x)\right)\right)}{\varphi(x)}+\int_{x_0}^{x}\frac{\partial f(\tau,x)}{\partial x}\frac{w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau\\
&\quad+\frac{g(x,x)u\left(\sigma(x)\right)}{\varphi(x)}+\int_{x_0}^{x}\frac{\partial g(\tau,x)}{\partial x}\frac{u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau\\
&\leq\frac{f(x,x)w\left(q(x)\varphi(x)\widetilde{v}_0(x)\right)}{\varphi(x)}+\int_{x_0}^{x}\frac{\partial f(\tau,x)}{\partial x}\frac{w\left(q(\tau)\varphi(\tau)\widetilde{v}_0(\tau)\right)}{\varphi(\tau)}d\tau\\
&\quad+g(x,x)q(x)\widetilde{v}_0(x)+\int_{x_0}^{x}\frac{\partial g(\tau,x)}{\partial x}q(\tau)\widetilde{v}_0(\tau)d\tau\\
&\leq\left[\frac{f(x,x)}{\varphi(x)}+\int_{x_0}^{x}\frac{\partial f(\tau,x)}{\partial x}\frac{1}{\varphi(\tau)}d\tau\right]w\left(q(x)\varphi(x)\widetilde{v}_0(x)\right)\\
&\quad+\left[g(x,x)q(x)+\int_{x_0}^{x}\frac{\partial g(\tau,x)}{\partial x}q(\tau)d\tau\right]\widetilde{v}_0(x)\\
&=\left(\frac{d}{dx}\int_{x_0}^{x}\frac{f(\tau,x)}{\varphi(\tau)}d\tau\right)w\left(q(x)\varphi(x)\widetilde{v}_0(x)\right)+\left(\frac{d}{dx}\int_{x_0}^{x}g(\tau,x)q(\tau)d\tau\right)\widetilde{v}_0(x)\\
&=F_0(x)w\left(q(x)\varphi(x)\widetilde{v}_0(x)\right)+\left(\frac{d}{dx}G_0(x)\right)\widetilde{v}_0(x),
\end{aligned}$$

从而

$$\widetilde{v}_0'(x)-\left(\frac{d}{dx}G_0(x)\right)\widetilde{v}_0(x)\le F_0(x)w\left(q(x)\varphi(x)\widetilde{v}_0(x)\right), \tag{1.2.8}$$

在 (1.2.8) 式两边同乘 $\exp\left(-G_0(x)\right)$ 得

$$\left[\widetilde{v}_0(x)\exp\left(-G_0(x)\right)\right]'\le F_0(x)\exp\left(-G_0(x)\right)w\left(q(x)\varphi(x)\widetilde{v}_0(x)\right), \tag{1.2.9}$$

在 (1.2.9) 式两边从 x_0 到 x 积分, 由 $\widetilde{v}_0(x_0)=0,\ G_0(x_0)=0$ 得

$$\widetilde{v}_0(x)\exp\left(-G_0(x)\right)-\widetilde{v}_0(x_0)\exp\left(-G_0(x_0)\right)\le\int_{x_0}^{x}F_0(\tau)\exp\left(-G_0(\tau)\right)w\left(q(\tau)\varphi(\tau)\widetilde{v}_0(\tau)\right)d\tau,$$

从而

$$\widetilde{v}_0(x)\le\exp\left(G_0(x)\right)\left[l_0+\int_{x_0}^{x}F_0(\tau)\exp\left(-G_0(\tau)\right)w\left(q(\tau)\varphi(\tau)\widetilde{v}_0(\tau)\right)d\tau\right],$$

令

$$p(x)=l_0+\int_{x_0}^{x}F_0(\tau)\exp\left(-G_0(\tau)\right)w\left(q(\tau)\varphi(\tau)\widetilde{v}_0(\tau)\right)d\tau, \tag{1.2.10}$$

则 $p(x_0)=l_0,\ p(x)$ 非负不减, 且

$$\widetilde{v}_0(x)\le\exp\left(G_0(x)\right)p(x), \tag{1.2.11}$$

对 (1.2.10) 式关于 x 求导, 由条件 (5) 得

$$\begin{aligned}p'(x)&=F_0(x)\exp\left(-G_0(x)\right)w\left(q(x)\varphi(x)\widetilde{v}_0(x)\right)\\&\le F_0(x)\exp\left(-G_0(x)\right)w\left(q(x)\varphi(x)\exp\left(G_0(x)\right)\right)w\left(p(x)\right),\end{aligned}$$

从而

$$\frac{p'(x)}{w\left(p(x)\right)}\le F_0(x)\exp\left(-G_0(x)\right)w\left(q(x)\varphi(x)\exp\left(G_0(x)\right)\right), \tag{1.2.12}$$

在 (1.2.12) 式两边从 x_0 到 x 积分得

$$\Phi_0\left(p(x)\right)-\Phi_0\left(p(x_0)\right)\le\int_{x_o}^{x}F_0(\tau)\exp\left(-G_0(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_0(\tau)\right)\right)d\tau,$$

从而由 Φ_0 的定义知

$$p(x)\le\Phi_0^{-1}\left[\int_{x_o}^{x}F_0(\tau)\exp\left(-G_0(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_0(\tau)\right)\right)d\tau\right], \tag{1.2.13}$$

由 (1.2.4) (1.2.6) (1.2.11) (1.2.13) 式可得

$$
\begin{aligned}
&v(x)=\widetilde{v}_0(x)\le\exp\left(G_0(x)\right)p(x)\\
&\le\exp\left(G_0(x)\right)\Phi_0^{-1}\left[\int_{x_0}^{x}F_0(\tau)\exp\left(-G_0(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_0(\tau)\right)\right)d\tau\right],
\end{aligned}
\tag{1.2.14}
$$

$$
\begin{aligned}
&u(x)\le q(x)\varphi(x)v(x)\\
&\le q(x)\varphi(x)\exp\left(G_0(x)\right)\Phi_0^{-1}\left[\int_{x_o}^{x}F_0(\tau)\exp\left(-G_0(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_0(\tau)\right)\right)d\tau\right],
\end{aligned}
\tag{1.2.15}
$$

当 $x\in[x_1,x_2]$ 时, 由 $v(x)$ 的定义及 (1.2.14) 式知

$$
\begin{aligned}
v(x)&=1+\int_{x_0}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_0}^{x}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau+\beta_1\frac{u(x_1-0)}{\varphi(x_1-0)}\\
&=1+\int_{x_0}^{x_1}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_0}^{x_1}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau+\beta_1\frac{u(x_1-0)}{\varphi(x_1-0)}\\
&\quad+\int_{x_1}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_1}^{x}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau\\
&\le\left(1+\beta_1q(x_1-0)\right)\exp\left(G_0(x_1)\right)\\
&\quad\Phi_0^{-1}\left[\int_{x_0}^{x_1}F_0(\tau)\exp\left(-G_0(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_0(\tau)\right)\right)d\tau\right]\\
&\quad+\int_{x_1}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_1}^{x}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau\\
&=l_1+\int_{x_1}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_1}^{x}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau\\
&=\widetilde{v}_1(x),
\end{aligned}
\tag{1.2.16}
$$

与 (1.2.7) 式类似, 可得

$$
v(x)\le\exp\left(G_1(x)\right)\Phi_1^{-1}\left[\int_{x_1}^{x}F_1(\tau)\exp\left(-G_1(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_1(\tau)\right)\right)d\tau\right],
\tag{1.2.17}
$$

假设当 $x\in[x_{i-1},x_i]$ 时,

$$
v(x)\le\exp\left(G_{i-1}(x)\right)\Phi_{i-1}^{-1}\left[\int_{x_{i-1}}^{x}F_{i-1}(\tau)\exp\left(-G_{i-1}(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_{i-1}(\tau)\right)\right)d\tau\right],
\tag{1.2.18}
$$

则当 $x\in[x_i,x_{i+1}]$ 时,

$$\begin{aligned}
v(x)&=1+\int_{x_0}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_0}^{x}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau+\sum_{x_0<x_j<x}\beta_j\frac{u(x_j-0)}{\varphi(x_j-0)}\\
&=1+\int_{x_0}^{x_i}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_0}^{x_i}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau+\sum_{x_0<x_j<x_i}\beta_j\frac{u(x_j-0)}{\varphi(x_j-0)}\\
&\quad+\beta_i\frac{u(x_i-0)}{\varphi(x_i-0)}+\int_{x_i}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_i}^{x}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau\\
&\le(1+\beta_iq(x_i-0))\exp\left(G_{i-1}(x_i)\right)\\
&\quad\Phi_{i-1}^{-1}\left[\int_{x_{i-1}}^{x_i}F_{i-1}(\tau)\exp\left(-G_{i-1}(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_{i-1}(\tau)\right)\right)d\tau\right]\\
&\quad+\int_{x_i}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_i}^{x}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau\\
&=l_i+\int_{x_i}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_i}^{x}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau\\
&=\widetilde{v}_i(x),
\end{aligned}\tag{1.2.19}$$

与 (1.2.7) 式类似, 此时可得

$$v(x)\le\exp\left(G_i(x)\right)\Phi_i^{-1}\left[\int_{x_i}^{x}F_i(\tau)\exp\left(-G_i(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_i(\tau)\right)\right)d\tau\right],\tag{1.2.20}$$

从而

$$u(x)\le q(x)\varphi(x)\exp\left(G_i(x)\right)\Phi_i^{-1}\left[\int_{x_i}^{x}F_i(\tau)\exp\left(-G_i(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_i(\tau)\right)\right)d\tau\right],\tag{1.2.21}$$

从而可证 $u(x)$ 满足 (1.2.2) 式.

定理 1.2.2　$u(x)$ 为非负函数, 定义在区间 $R_+^{x_0}=[x_0,\infty)=\bigcup\limits_{i\ge1}\{x|x\in[x_{i-1},x_i]\}$, $i=1,2,3\ldots$ 上, $u(x)$ 在 $R_+^{x_0}$ 上除了 x_i $(i\ge1)$ 点外连续, $u(x_i-0)\ne u(x_i+0)$, $\lim\limits_{i\to\infty}x_i=\infty$, $x_i\le x_{i+1}$,若

$$\begin{aligned}
u(x)\le{}&\varphi(x)+q(x)\int_{x_0}^{x}f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)d\tau\\
&+q(x)\int_{x_0}^{x}g(\tau,x)\left(u\left(\sigma(\tau)\right)+\int_{x_0}^{\tau}h(\xi,\tau)u\left(\sigma(\xi)\right)d\xi\right)d\tau\\
&+\sum_{x_0<x_j<x}\beta_ju(x_j-0),\ x\in[x_{i-1},x_i],
\end{aligned}\tag{1.2.22}$$

其中 $x_0,\ \beta_j,\ q(x),\ \varphi(x),\ \sigma(x),\ f(\tau,x),\ g(\tau,x),\ w(u)$ 满足定理 1.2.1 的条件, $h(\xi,\tau)$ 为非负连续函数, 则

$$
\begin{aligned}
u(x) \le\ & q(x)\varphi(x)\exp\left(G_{i-1}(x)\right) \\
& \Phi_{i-1}^{-1}\left[\int_{x_{i-1}}^{x} F_{i-1}(\tau)\exp\left(-G_{i-1}(\tau)\right) w\left(q(\tau)\varphi(\tau)\exp\left(G_{i-1}(\tau)\right)\right)d\tau\right], \\
& \forall x\in[x_{i-1},x_i],
\end{aligned}
\tag{1.2.23}
$$

其中

$F_{i-1}(x)=\frac{d}{dx}\int_{x_{i-1}}^{x}\frac{f(\tau,x)}{\varphi(\tau)}d\tau,\ G_{i-1}(x)=\int_{x_{i-1}}^{x} g(\tau,x)\left(q(\tau)+\int_{x_0}^{\tau}h(\xi,\tau)q(\xi)d\xi\right)d\tau,$

$\Phi_{i-1}(r)=\int_{l_{i-1}}^{r}\frac{ds}{w(s)},\ r>0,\ \Phi_{i-1}^{-1}$ 为其反函数,　i=1,2...,

$l_0=1,$

$$
\begin{aligned}
l_{i-1}=\ & (1+\beta_{i-1}q(x_{i-1}-0))\exp\left(G_{i-2}(x_{i-1})\right) \\
& \Phi_{i-2}^{-1}\left[\int_{x_{i-2}}^{x_{i-1}}F_{i-2}(\tau)\exp\left(-G_{i-2}(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_{i-2}(\tau)\right)\right)d\tau\right],\ i=2,3\ldots.
\end{aligned}
$$

证明: 因为 $q(x)\ge 1,\ \varphi(x)>0$ 为非减函数, 由不等式 (1.2.22) 知

$$
\begin{aligned}
\frac{u(x)}{\varphi(x)} \le q(x)\Bigg[& 1+\int_{x_0}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau \\
& +\int_{x_0}^{x}\frac{g(\tau,x)\left(u\left(\sigma(\tau)\right)+\int_{x_0}^{\tau}h(\xi,\tau)u\left(\sigma(\xi)\right)d\xi\right)}{\varphi(\tau)}d\tau+\sum_{x_0<x_j<x}\beta_j\frac{u(x_j-0)}{\varphi(x_j-0)}\Bigg],
\end{aligned}
$$

令

$$
\begin{aligned}
v(x)=1&+\int_{x_0}^{x}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_{x_0}^{x}\frac{g(\tau,x)\left(u\left(\sigma(\tau)\right)+\int_{x_0}^{\tau}h(\xi,\tau)u\left(\sigma(\xi)\right)d\xi\right)}{\varphi(\tau)}d\tau \\
&+\sum_{x_0<x_j<x}\beta_j\frac{u(x_j-0)}{\varphi(x_j-0)},
\end{aligned}
\tag{1.2.24}
$$

则 $v(x)$ 非负不减, 且

$$
u(x)\le q(x)\varphi(x)v(x), \tag{1.2.25}
$$

由 $\sigma(x)\le x$ 及 (1.2.25) 式得

$$
u\left(\sigma(x)\right)\le q\left(\sigma(x)\right)\varphi\left(\sigma(x)\right)v\left(\sigma(x)\right)\le q(x)\varphi(x)v(x), \tag{1.2.26}
$$

当 $x \in [x_0, x_1]$ 时,

$$v(x) = 1 + \int_{x_0}^{x} \frac{f(\tau, x) w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)} d\tau + \int_{x_0}^{x} \frac{g(\tau, x)\left(u\left(\sigma(\tau)\right) + \int_{x_0}^{\tau} h(\xi, \tau) u\left(\sigma(\xi)\right) d\xi\right)}{\varphi(\tau)} d\tau, \tag{1.2.27}$$

对 (1.2.27) 式关于 x 求导

$$\begin{aligned}
v'(x) &= \frac{f(x,x) w\left(u\left(\sigma(x)\right)\right)}{\varphi(x)} + \int_{x_0}^{x} \frac{\partial f(\tau, x)}{\partial x} \frac{w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)} d\tau \\
&\quad + \frac{g(x,x)\left(u\left(\sigma(x)\right) + \int_{x_0}^{x} h(\tau, x) u\left(\sigma(\tau)\right) d\tau\right)}{\varphi(x)} \\
&\quad + \int_{x_0}^{x} \frac{\partial g(\tau, x)}{\partial x} \frac{\left(u\left(\sigma(\tau)\right) + \int_{x_0}^{\tau} h(\xi, \tau) u\left(\sigma(\xi)\right) d\xi\right)}{\varphi(\tau)} d\tau \\
&\le \left[\frac{f(x,x)}{\varphi(x)} + \int_{x_0}^{x} \frac{\partial f(\tau, x)}{\partial x} \frac{1}{\varphi(\tau)} d\tau\right] w\left(q(x)\varphi(x)v(x)\right) \\
&\quad + \left[g(x,x)\left(q(x) + \int_{x_0}^{x} h(\tau, x) q(\tau) d\tau\right)\right. \\
&\quad \left. + \int_{x_0}^{x} \frac{\partial g(\tau, x)}{\partial x}\left(q(\tau) + \int_{x_0}^{\tau} h(\xi, \tau) q(\xi) d\xi\right) d\tau\right] v(x) \\
&= \left(\frac{d}{dx}\int_{x_0}^{x} \frac{f(\tau, x)}{\varphi(\tau)} d\tau\right) w\left(q(x)\varphi(x)v(x)\right) \\
&\quad + \left[\frac{d}{dx}\int_{x_0}^{x} g(\tau, x)\left(q(\tau) + \int_{x_0}^{\tau} h(\xi, \tau) q(\xi) d\xi\right) d\tau\right] v(x) \\
&= F_0(x) w\left(q(x)\varphi(x)v(x)\right) + \left(\frac{d}{dx} G_0(x)\right) v(x),
\end{aligned}$$

从而

$$v'(x) - \left(\frac{d}{dx} G_0(x)\right) v(x) \le F_0(x) w\left(q(x)\varphi(x)v(x)\right), \tag{1.2.28}$$

与定理 1.2.1 的证明类似,可证 $u(x)$ 满足 (1.2.23) 式.

定理 1.2.3　$u(x,y)$ 为非负函数, 定义在区域 $\Omega = \bigcup\limits_{i,j \ge 1} \Omega_{ij}$,

$\Omega_{ij} = \left\{(x,y) | (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]\right\}, i, j = 1, 2, 3 \ldots,$

$u(x,y)$ 在 Ω 上除了 (x_i, y_i) $(i \ge 1)$ 点外连续, $u(x_i - 0, y_i - 0) \ne u(x_i + 0, y_i + 0)$,

$\lim\limits_{i\to\infty} x_i = \infty,\ \lim\limits_{i\to\infty} x_i = \infty,\ x_i \le x_{i+1},\ y_i \le y_{i+1}$, 若

$$\begin{aligned}u(x,y) \le \varphi(x,y) + q(x,y)\int_{x_0}^{x}\int_{y_0}^{y} f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)dsd\tau \\ + q(x,y)\int_{x_0}^{x}\int_{y_0}^{y} g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)dsd\tau \\ + \sum_{(x_0,y_0)<(x_j,y_j)<(x,y)} \beta_j u(x_j-0,y_j-0),\ (x,y)\in\Omega_{ii},\end{aligned} \tag{1.2.29}$$

且满足

(1) $x_0 \ge 0,\ y_0 \ge 0,\ \beta_j > 0$ 为常数;

(2) $\forall (x,y)\in\Omega,\ q(x,y)\ge 1,\ \varphi(x,y)>0$ 都是连续函数, 且关于 $x,\ y$ 都不增;

(3) $f(\tau,s,x,y),\ g(\tau,s,x,y)\in C(R_+^4,R_+),\ \frac{\partial f(\tau,s,x,y)}{\partial x},\ \frac{\partial g(\tau,s,x,y)}{\partial x}\in C(R_+^4,R)$, 对于固定的 $\tau\in R_+,\ f,\ g$ 关于 x 是不减的, 对于固定的 $s\in R_+,\ f,\ g$ 关于 y 是不减的, 且当 $(\tau,s)\in\Omega_{ij},\ i\ne j$ 时, $f(\tau,s,x,y)=g(\tau,s,x,y)=0$;

(4) $\sigma(x)\le x$ 是非负连续函数, 且当 $(x,y)\in\Omega_{ij}$ 时, $(\sigma(x),\sigma(y))\le(x_i,y_j)$, $(\sigma(x),\sigma(y))\ge(x_{i-1},y_{j-1})$;

(5) $w(u)$ 满足以下条件:

(a) $w(\alpha\beta)\le w(\alpha)w(\beta)$,

(b) $w\in C(R_+,R_+)$, 且当 $x\in(0,\infty)$ 时, $w(x)>0$,

(c) w 为非减函数;

则

$$\begin{aligned}u(x,y) \le\ & q(x,y)\varphi(x,y)\exp\left(G_{i-1}(x,y)\right) \\ & \Phi_{i-1}^{-1}\left[\int_{x_{i-1}}^{x} F_{i-1}(\tau,y)\exp\left(-G_{i-1}(\tau,y)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_{i-1}(\tau,y)\right)\right)\right)d\tau\right], \\ & \forall (x,y)\in\Omega_{ii},\end{aligned} \tag{1.2.30}$$

其中

$F_{i-1}(x,y)=\frac{\partial}{\partial x}\int_{x_{i-1}}^{x}\int_{y_{i-1}}^{y}\frac{f(\tau,s,x,y)}{\varphi(\tau,s)}dsd\tau,\ \ G_{i-1}(x,y)=\int_{x_{i-1}}^{x}\int_{y_{i-1}}^{y}g(\tau,s,x,y)q(\tau,s)dsd\tau$,

$\Phi_{i-1}(r)=\int_{l_{i-1}}^{r}\frac{ds}{w(s)},\ r>0,\ \Phi_{i-1}^{-1}$ 为其反函数, $\ i=1,2\ldots,$

$l_0=1,$

$$\begin{aligned}l_{i-1} = & (1+\beta_{i-1}q(x_{i-1}-0,y_{i-1}-0))\exp\left(G_{i-2}(x_{i-1},y_{i-1})\right) \\ & \Phi_{i-2}^{-1}\left[\int_{x_{i-2}}^{x_{i-1}} F_{i-2}(\tau,y_{i-1})\exp\left(-G_{i-2}(\tau,y_{i-1})\right)w\left(q(\tau,y_{i-1})\varphi(\tau,y_{i-1})\exp\left(G_{i-2}(\tau,y_{i-1})\right)\right)d\tau\right],\end{aligned}$$

$i = 2, 3 \ldots.$

证明: 因为 $q(x,y) \geq 1$, $\varphi(x,y) > 0$ 关于 x, y 都是非减函数, 由不等式 (1.2.29) 知

$$\frac{u(x,y)}{\varphi(x,y)} \leq q(x,y)\left[1 + \int_{x_0}^{x}\int_{y_0}^{y}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau \right.$$
$$\left. + \int_{x_0}^{x}\int_{y_0}^{y}\frac{g(\tau,,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau + \sum_{(x_0,y_0)<(x_j,y_j)\leq(x,y)}\beta_j\frac{u(x_j-0,y_j-0)}{\varphi(x_j-0,y_j-0)}\right],$$

令

$$v(x,y) = 1 + \int_{x_0}^{x}\int_{y_0}^{y}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau$$
$$+ \int_{x_0}^{x}\int_{y_0}^{y}\frac{g(\tau,,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau + \sum_{(x_0,y_0)<(x_j,y_j)\leq(x,y)}\beta_j\frac{u(x_j-0,y_j-0)}{\varphi(x_j-0,y_j-0)}, \tag{1.2.31}$$

则 $v(x,y)$ 非负关于 x, y 都不减, 且

$$u(x,y) \leq q(x,y)\varphi(x,y)v(x,y), \tag{1.2.32}$$

由 $\sigma(x) \leq x$ 及 (1.2.32) 式得

$$u\left(\sigma(x),\sigma(y)\right) \leq q\left(\sigma(x),\sigma(y)\right)\varphi\left(\sigma(x),\sigma(y)\right)v\left(\sigma(x),\sigma(y)\right) \leq q(x,y)\varphi(x,y)v(x,y), \tag{1.2.33}$$

当 $(x,y) \in \Omega_{ii}$ 时, 令

$$\widetilde{v}_{i-1}(x,y) = l_{i-1} + \int_{x_{i-1}}^{x}\int_{y_{i-1}}^{y}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}d\tau ds$$
$$+ \int_{x_{i-1}}^{x}\int_{y_{i-1}}^{y}\frac{g(\tau,,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}d\tau ds,$$

则 $\widetilde{v}_{i-1}(x,y)$ 在 Ω_{ii} 上非负关于 x, y 都不减, 且 $\widetilde{v}_{i-1}(x_{i-1},y) = l_{i-1}$, $\widetilde{v}_{i-1}(x,y_{i-1}) = l_{i-1}$, 当 $(x,y) \in \Omega_{11}$ 时, 此时

$$\widetilde{v}_0(x,y) = v(x,y), \tag{1.2.34}$$

$$\widetilde{v}_0(x,y) = l_0 + \int_{x_0}^{x}\int_{y_0}^{y}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}d\tau ds$$
$$+ \int_{x_0}^{x}\int_{y_0}^{y}\frac{g(\tau,,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}d\tau ds, \tag{1.2.35}$$

对 (1.2.35) 式关于 x 求偏导, 由 (1.2.33) (1.2.34) 式得

$$
\begin{aligned}
&\frac{\partial \widetilde{v}_0(x,y)}{\partial x} \\
&= \int_{y_0}^{y} \frac{f(x,s,x,y) w\left(u\left(\sigma(x),\sigma(s)\right)\right)}{\varphi(x,s)} ds + \int_{x_0}^{x}\int_{y_0}^{y} \frac{\partial f(\tau,s,x,y)}{\partial x} \frac{w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)} ds d\tau \\
&\quad + \int_{y_0}^{y} \frac{g(x,s,x,y) u\left(\sigma(x),\sigma(s)\right)}{\varphi(x,s)} + \int_{x_0}^{x}\int_{y_0}^{y} \frac{\partial g(\tau,s,x,y)}{\partial x} \frac{u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)} ds d\tau \\
&\le \int_{y_0}^{y} \frac{f(x,s,x,y) w\left(q(x,s)\varphi(x,s)\widetilde{v}_0(x,s)\right)}{\varphi(x,s)} ds \\
&\quad + \int_{x_0}^{x}\int_{y_0}^{y} \frac{\partial f(\tau,s,x,y)}{\partial x} \frac{w\left(q(\tau,s)\varphi(\tau,s)\widetilde{v}_0(\tau,s)\right)}{\varphi(\tau,s)} ds d\tau \\
&\quad + \int_{y_0}^{y} g(x,s,x,y) q(x,s)\widetilde{v}_0(x,s) ds + \int_{x_0}^{x}\int_{y_0}^{y} \frac{\partial g(\tau,s,x,y)}{\partial x} q(\tau,s)\widetilde{v}_0(\tau,s) ds d\tau \\
&\le \int_{y_0}^{y} \left[\frac{f(x,s,x,y)}{\varphi(x,s)} + \int_{x_0}^{x} \frac{\partial f(\tau,s,x,y)}{\partial x\ \varphi(\tau,s)} d\tau \right] w\left(q(x,s)\varphi(x,s)\widetilde{v}_0(x,s)\right) ds \\
&\quad + \int_{y_0}^{y} \left[g(x,s,x,y) q(x,s) + \int_{x_0}^{x} \frac{\partial g(\tau,s,x,y)}{\partial x} q(\tau,s) d\tau \right] \widetilde{v}_0(x,s) ds \\
&\le \int_{y_0}^{y} \left(\frac{\partial}{\partial x} \int_{x_0}^{x} \frac{f(\tau,s,x,y)}{\varphi(\tau,s)} d\tau \right) ds\ w\left(q(x,y)\varphi(x,y)\widetilde{v}_0(x,y)\right) \\
&\quad + \int_{y_0}^{y} \left(\frac{\partial}{\partial x} \int_{x_0}^{x} g(\tau,s,x,y) q(\tau,s) d\tau \right) ds\ \widetilde{v}_0(x,y) \\
&= \left(\frac{\partial}{\partial x} \int_{x_0}^{x}\int_{y_0}^{y} \frac{f(\tau,s,x,y)}{\varphi(\tau,s)} ds d\tau \right) w\left(q(x,y)\varphi(x,y)\widetilde{v}_0(x,y)\right) \\
&\quad + \left(\frac{\partial}{\partial x} \int_{x_0}^{x}\int_{y_0}^{y} g(\tau,s,x,y) q(\tau,s) d\tau ds \right) \widetilde{v}_0(x,y) \\
&= F_0(x,y) w\left(q(x,y)\varphi(x,y)\widetilde{v}_0(x,y)\right) + \frac{\partial G_0(x,y)}{\partial x} \widetilde{v}_0(x,y),
\end{aligned}
$$

从而

$$
\frac{\partial \widetilde{v}_0(x,y)}{\partial x} - \frac{\partial G_0(x,y)}{\partial x} \widetilde{v}_0(x,y) \le F_0(x,y) w\left(q(x,y)\varphi(x,y)\widetilde{v}_0(x,y)\right), \tag{1.2.36}
$$

在 (1.2.36) 式两边同乘 $\exp\left(-G_0(x,y)\right)$ 得

$$
\frac{\partial \left[\widetilde{v}_0(x,y) \exp\left(-G_0(x,y)\right)\right]}{\partial x} \le F_0(x,y) \exp\left(-G_0(x,y)\right) w\left(q(x,y)\varphi(x,y)\widetilde{v}_0(x,y)\right), \tag{1.2.37}
$$

在 (1.2.37) 式两边从 x_0 到 x 积分得

$$\begin{aligned}&\widetilde{v}_0(x,y)\exp\left(-G_0(x,y)\right)-\widetilde{v}_0(x_0,y)\exp\left(-G_0(x_0,y)\right)\\ \leq&\int_{x_0}^{x}F_0(\tau,y)\exp\left(-G_0(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\widetilde{v}_0(\tau,y)\right)d\tau,\end{aligned}$$

从而

$$\widetilde{v}_0(x,y)\leq\exp\left(G_0(x,y)\right)\left[l_0+\int_{x_0}^{x}F_0(\tau,y)\exp\left(-G_0(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\widetilde{v}_0(\tau,y)\right)d\tau\right],$$

令

$$p(x,y)=l_0+\int_{x_0}^{x}F_0(\tau,y)\exp\left(-G_0(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\widetilde{v}_0(\tau,y)\right)d\tau,\tag{1.2.38}$$

则 $p(x_0,y)=l_0$, $p(x,y)$ 关于 x 非负不减, 且

$$\widetilde{v}_0(x,y)\leq\exp\left(G_0(x,y)\right)p(x,y),\tag{1.2.39}$$

对 (1.2.38) 式关于 x 求偏导

$$\begin{aligned}\frac{\partial p(x,y)}{\partial x}&=F_0(x,y)\exp\left(-G_0(x,y)\right)w\left(q(x,y)\varphi(x,y)\widetilde{v}_0(x,y)\right)\\&\leq F_0(x,y)\exp\left(-G_0(x,y)\right)w\left(q(x,y)\varphi(x,y)\exp\left(G_0(x,y)\right)\right)w\left(p(x,y)\right),\end{aligned}$$

从而

$$\frac{\frac{\partial p(x,y)}{\partial x}}{w\left(p(x,y)\right)}\leq F_0(x,y)\exp\left(-G_0(x,y)\right)w\left(q(x,y)\varphi(x,y)\exp\left(G_0(x,y)\right)\right),\tag{1.2.40}$$

在 (1.2.40) 式两边从 x_0 到 x 积分得

$$\Phi_0\left(p(x,y)\right)-\Phi_0\left(p(x_0,y)\right)\leq\int_{x_0}^{x}F_0(\tau,y)\exp\left(-G_0(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_0(\tau,y)\right)\right)d\tau,$$

从而

$$p(x,y)\leq\Phi_0^{-1}\left[\int_{x_0}^{x}F_0(\tau,y)\exp\left(-G_0(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_0(\tau,y)\right)\right)d\tau\right],\tag{1.2.41}$$

由 (1.2.32) (1.2.34) (1.2.39) (1.2.41) 式可得

$$\begin{aligned}&v(x,y)\leq\widetilde{v}_0(x,y)\leq\exp\left(G_0(x,y)\right)p(x,y)\\ \leq&\exp\left(G_0(x,y)\right)\Phi_0^{-1}\left[\int_{x_0}^{x}F_0(\tau,y)\exp\left(-G_0(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_0(\tau,y)\right)\right)d\tau\right],\end{aligned}\tag{1.2.42}$$

$$
\begin{aligned}
&u(x,y) \leq q(x,y)\varphi(x,y)v(x,y) \\
&\leq q(x,y)\varphi(x,y)\exp\left(G_0(x,y)\right) \\
&\quad \Phi_0^{-1}\left[\int_{x_0}^{x} F_0(\tau,y)\exp\left(-G_0(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_0(\tau,y)\right)\right)d\tau\right],
\end{aligned} \tag{1.2.43}
$$

当 $(x,y)\in\Omega_{22}$ 时, 由 $v(x)$ 的定义及 (1.2.42) 式知

$$
\begin{aligned}
&v(x,y) \\
&= 1+\int_{x_0}^{x}\int_{y_0}^{y}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau \\
&\quad +\int_{x_0}^{x}\int_{y_0}^{y}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau+\beta_1\frac{u(x_1-0,y_1-0)}{\varphi(x_1-0,y_1-0)} \\
&= 1+\int_{x_0}^{x_1}\int_{y_0}^{y_1}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau \\
&\quad +\int_{x_0}^{x_1}\int_{y_0}^{y_1}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau+\beta_1\frac{u(x_1-0,y_1-0)}{\varphi(x_1-0,y_1-0)} \\
&\quad +\int_{x_1}^{x}\int_{y_1}^{y}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau+\int_{x_1}^{x}\int_{y_1}^{y}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau \\
&\leq \left(1+\beta_1 q(x_1-0,y_1-0)\right)\exp\left(G_0(x_1,y_1)\right) \\
&\quad \Phi_0^{-1}\left[\int_{x_0}^{x_1}F_0(\tau,y_1)\exp\left(-G_0(\tau,y_1)\right)w\left(q(\tau,y_1)\varphi(\tau,y_1)\exp\left(G_0(\tau,y_1)\right)\right)d\tau\right] \\
&\quad +\int_{x_1}^{x}\int_{y_1}^{y}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau+\int_{x_1}^{x}\int_{y_1}^{y}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau \\
&= l_1+\int_{x_1}^{x}\int_{y_1}^{y}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau \\
&\quad +\int_{x_1}^{x}\int_{y_1}^{y}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau=\widetilde{v}_1(x,y),
\end{aligned} \tag{1.2.44}
$$

与 (1.2.35) 式类似,可得

$$
v(x,y)\leq\exp\left(G_1(x,y)\right)\Phi_1^{-1}\left[\int_{x_1}^{x}F_1(\tau,y)exp\left(-G_1(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_1(\tau,y)\right)\right)d\tau\right], \tag{1.2.45}
$$

假设当 $(x,y)\in\Omega_{ii}$ 时,

$$
\begin{aligned}
&v(x,y)\leq\exp\left(G_{i-1}(x,y)\right) \\
&\qquad \Phi_{i-1}^{-1}\left[\int_{x_{i-1}}^{x}F_{i-1}(\tau,y)\exp\left(-G_{i-1}(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_{i-1}(\tau,y)\right)\right)d\tau\right],
\end{aligned} \tag{1.2.46}
$$

则当 $(x,y)\in\Omega_{i+1\ i+1}$ 时,

$$
\begin{aligned}
&v(x,y)\\
&=1+\int_{x_0}^{x}\int_{y_0}^{y}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau+\int_{x_0}^{x}\int_{y_0}^{y}\frac{g(\tau,s,x,y)u\left(\sigma(\tau)\right)}{\varphi(\tau,s)}dsd\tau\\
&\quad+\sum_{(x_0,y_0)<(x_j,y_j)<(x,y)}\beta_j\frac{u(x_j-0,y_j-0)}{\varphi(x_j-0,y_j-0)}\\
&=l_0+\int_{x_0}^{x_i}\int_{y_0}^{y_i}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau\\
&\quad+\int_{x_0}^{x_i}\int_{y_0}^{y_i}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau\\
&\quad+\sum_{(x_0,y_0)<(x_j,y_j)<(x_i,y_i)}\beta_j\frac{u(x_j-0,y_j-0)}{\varphi(x_j-0,y_j-0)}+\beta_i\frac{u(x_i-0,y_i-0)}{\varphi(x_i-0,y_i-0)}\\
&\quad+\int_{x_i}^{x}\int_{y_i}^{y}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau+\int_{x_i}^{x}\int_{y_i}^{y}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau\\
&\le(1+\beta_iq(x_i-0,y_i-0))\exp\left(G_{i-1}(x_i,y_i)\right)\\
&\quad\Phi_{i-1}^{-1}\left[\int_{x_{i-1}}^{x_i}F_{i-1}(\tau,y_i)exp\left(-G_{i-1}(\tau,y_i)\right)w\left(q(\tau,y_i)\varphi(\tau,y_i)\exp\left(G_{i-1}(\tau,y_i)\right)\right)d\tau\right]\\
&\quad+\int_{x_i}^{x}\int_{y_i}^{y}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau+\int_{x_i}^{x}\int_{y_i}^{y}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau\\
&=l_i+\int_{x_i}^{x}\int_{y_i}^{y}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau+\int_{x_i}^{x}\int_{y_i}^{y}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau\\
&=\widetilde{v}_i(x,y),
\end{aligned}
\tag{1.2.47}
$$

与 (1.2.35) 式类似, 此时可得

$$
v(x,y)\le\exp\left(G_i(x,y)\right)\Phi_i^{-1}\left[\int_{x_i}^{x}F_i(\tau,y)\exp\left(-G_i(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_i(\tau,y)\right)\right)d\tau\right],
\tag{1.2.48}
$$

从而

$$
\begin{aligned}
u(x,y)\le\ &q(x,y)\varphi(x,y)\exp\left(G_i(x,y)\right)\\
&\Phi_i^{-1}\left[\int_{x_i}^{x}F_i(\tau,y)\exp\left(-G_i(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,\exp\left(G_i(\tau,y)\right)\right)d\tau\right],
\end{aligned}
\tag{1.2.49}
$$

从而可证 (1.2.30) 式.

1.3 应用

这一部分我们举例来说明本章结果, 考虑积分方程

$$u(x)=\varphi(x)+q(x)\int_{x_0}^{x}H(\tau,x,u)d\tau+\sum_{x_0<x_j<x}\beta_j u(x_j-0), \tag{1.3.1}$$

其中 $u(x)$ 在 $R_+^{x_0}$ 上除了 x_i $(i\geq 1)$ 点外连续, $H\in C(R_+^2\times R,R)$, $x_0\geq 0$, $\beta_j>0$ 为常数, $\sigma(x)\leq x$ 是非负连续函数, 且当 $x\in[x_{i-1},x_i]$ 时, $\sigma(x)\leq x_i,\sigma(x)\geq x_{i-1}$, $\forall x\in R_+^{x_0}$, $q(x)\geq 1$, $\varphi(x)>0$ 都是非减函数.

定理 1.3.1 设 $u(x)$ 是方程 (1.3.1) 的一个解, 且

$$\begin{cases}|H(\tau,x,u)|\leq f(\tau,x)|u|^m+g(\tau,x)|u|,\\ w(\alpha)=\alpha^m,\ \alpha\in R_+,\end{cases} \tag{1.3.2}$$

其中 $0<m<1$, $f(\tau,x)$, $g(\tau,x)\in C(R_+^2,R_+)$, $\frac{\partial f(\tau,x)}{\partial x}$, $\frac{\partial g(\tau,x)}{\partial x}\in C(R_+^2,R)$, 且对于每一个固定的 $\tau\in R_+^{x_0}$, f, g 关 x 是非负不减的, 则

$$\begin{aligned}|u(x)|\leq{}& q(x)\varphi(x)\exp\left(G_{i-1}(x)\right)\\ &\Phi_{i-1}^{-1}\left[\int_{x_{i-1}}^{x}F_{i-1}(\tau)\exp\left(-G_{i-1}(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_{i-1}(\tau)\right)\right)d\tau\right],\\ &\forall x\in[x_{i-1},x_i],\end{aligned} \tag{1.3.3}$$

其中 $\Phi_{i-1},F_{i-1},G_{i-1}$ 的定义与定理 1.2.1 相同.

证明: 由 (1.3.1) (1.3.2) 式知

$$\begin{aligned}|u(x)|\leq{}&\varphi(x)+q(x)\int_{x_0}^{x}f(\tau,x)|u\left(\sigma(\tau)\right)|^m d\tau+q(x)\int_{x_0}^{x}g(\tau,x)|u\left(\sigma(\tau)\right)|d\tau\\ &+\sum_{x_0<x_j<x}\beta_j|u(x_j-0)|\\ ={}&\varphi(x)+q(x)\int_{x_0}^{x}f(\tau,x)w\left(|u\left(\sigma(\tau)\right)|\right)d\tau+q(x)\int_{x_0}^{x}g(\tau,x)|u\left(\sigma(\tau)\right)|d\tau\\ &+\sum_{x_0<x_j<x}\beta_j|u(x_j-0)|,\end{aligned} \tag{1.3.4}$$

这与 (1.2.1) 式形式相同, 同理可证 $|u(x)|$ 满足 (1.3.3) 式.

定理 1.3.2 在定理 1.3.1 的条件下, 还可得

$$
\begin{aligned}
|u(x)| \leq\ & q(x)\varphi(x)\exp\left(G_{i-1}(x)\right) \\
& \left[(1-m)\int_{x_{i-1}}^{x} F_{i-1}(\tau)\exp\left(-G_{i-1}(\tau)q^m(\tau)\varphi^m(\tau)\exp\left(mG_{i-1}(\tau)\right)\right)d\tau + l_{i-1}^{1-m}\right]^{\frac{1}{1-m}}, \\
& \forall x \in [x_{i-1}, x_i]. \qquad (1.3.5)
\end{aligned}
$$

证明: 由于 $w(\alpha)=\alpha^m, 0<m<1$ 及

$$
\Phi_{i-1}(r)=\int_{l_{i-1}}^{r}\frac{ds}{w(s)}=\int_{l_{i-1}}^{r}\frac{ds}{s^m}=\frac{1}{1-m}(r^{1-m}-l_{i-1}^{1-m}), \qquad (1.3.6)
$$

则

$$
\Phi_{i-1}^{-1}(r)=\left[(1-m)r+l_{i-1}^{1-m}\right]^{\frac{1}{1-m}}, \qquad (1.3.7)
$$

结合 (1.3.3) (1.3.7) 式,可得 $|u(x)|$ 满足 (1.3.5) 式.

第二章　一类积分上限为无穷的非连续积分不等式

2.1　引言及预备知识

Gronwall-Bellman 积分不等式已有许多形式的推广, 但目前关于积分上限为无穷的不连续函数的情形的研究还相对较少. 在 [16] 中, Mate和Nevai给出了积分不等式来研究一类微分方程解的渐近性. 之后Gao和Meng [17], Zheng [19] 等又对其做出了一些研究.

本文是在文献 [19] 的基础上, 将其中的不等式进行了推广, 参考 [17, 19] 中的方法, 得出了更为广泛的结果.

引理 2.1.1[17] 设 $x(t) \in C(R_+, R_+)$, $f(t,s)$, $g(t,s) \in C(R_+^2, R_+)$ 且 $f(t,s)$, $g(t,s)$ 对于固定的 s 关于 t 是单调非增的, $\Omega \in C(R_+, R_+)$ 为递增的且为次可乘的, 若对常数 $c \geq 0$ 及 $t \in R_+$, 有

$$x(t) \leq c + \int_t^\infty f(t,s)x(s)ds + \int_t^\infty g(t,s)\Omega\left(x(s)\right)ds, \tag{2.1.1}$$

则对 $0 \leq T \leq t < \infty$ 有

$$x(t) \leq \exp\left(\int_t^\infty f(t,s)ds\right)G^{-1}\left(G(c) + \int_t^\infty g(t,s)\Omega\left(\exp\int_s^\infty f(t,\xi)d\xi\right)ds\right), \tag{2.1.2}$$

其中 $G(z) = \int_{z_0}^z \frac{ds}{\Omega(s)}$, $z \geq z_0 > 0$, G^{-1} 是 G 的反函数, $t \in R_+$ 满足

$$G(c) + \int_t^\infty g(t,s)\Omega\left(\exp\int_s^\infty f(t,\xi)d\xi\right)ds \in Dom(G^{-1}), T \leq t < \infty.$$

2.2　主要结果及证明

定理 2.2.1　$u(x)$ 为非负函数, 定义在区间 $R_+^{x_0} = [x_0, \infty)$ 上, $u(x)$ 在 $R_+^{x_0}$ 上除了 x_i $(i = 1, 2, \ldots, n)$ 点外连续, $u(x_i - 0) \neq u(x_i + 0)$, $x_i \leq x_{i+1}$,
则 $R_+^{x_0} = \bigcup\limits_{i=1}^{n+1}\{x|x \in [x_{i-1}, x_i)\}$, 其中 $x_{n+1} = \infty$, 若

$$\begin{aligned} u(x) \leq \varphi(x) &+ q(x)\int_x^\infty f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)d\tau + q(x)\int_x^\infty g(\tau,x)u\left(\sigma(\tau)\right)d\tau \\ &+ \sum_{x<x_j<\infty}\beta_j u(x_j - 0),\ x \in [x_{i-1}, x_i), \end{aligned} \tag{2.2.1}$$

且满足

(1) $x_0 \geq 0,\ \beta_j > 0$ 为常数;

(2) $\forall x \in R_+^{x_0},\ q(x) \geq 1,\ \varphi(x) > 0$ 都是非增函数;

(3) $f(\tau, x),\ g(\tau, x) \in C(R_+^2, R_+),\ \frac{\partial f(\tau,x)}{\partial x},\ \frac{\partial g(\tau,x)}{\partial x} \in C(R_+^2, R)$, 且对于每一个固定的 $\tau \in R_+^{x_0},\ f,\ g$ 关于 x 是不增的;

(4) $\sigma(x) \geq x$ 是非负连续函数, 且当 $x \in [x_{i-1}, x_i)$ 时, $\sigma(x) \leq x_i,\ \sigma(x) \geq x_{i-1}$;

(5) $w(u)$ 满足以下条件:

(a) $w(\alpha\beta) \leq w(\alpha)w(\beta)$,

(b) $w \in C(R_+, R_+)$, 且当 $x \in (0, \infty)$ 时, $w(x) > 0$,

(c) w 为非减函数;

则

$$\begin{aligned} u(x) \leq\ & q(x)\varphi(x)\exp\left(G_i(x)\right) \\ & \Phi_i^{-1}\left[-\int_x^{x_i} F_i(\tau)\exp\left(-G_i(\tau)\right) w\left(q(\tau)\varphi(\tau)\exp\left(G_i(\tau)\right)\right) d\tau\right], \\ & \forall x \in [x_{i-1}, x_i), \end{aligned} \tag{2.2.2}$$

其中

$F_i(x) = \frac{d}{dx}\int_x^{x_i} \frac{f(\tau,x)}{\varphi(\tau)} d\tau,\ G_i(x) = \int_x^{x_i} g(\tau, x) q(\tau) d\tau$,

$\Phi_i(r) = \int_{l_i}^r \frac{ds}{w(s)},\ r > 0,\ \Phi_i^{-1}$ 为其反函数, $i = 1, 2, \ldots, n+1$,

$l_{n+1} = 1$,

$l_i = (1 + \beta_i q(x_i - 0))\exp\left(G_{i+1}(x_i)\right)$

$\Phi_{i+1}^{-1}\left[-\int_{x_i}^{x_{i+1}} F_{i+1}(\tau)\exp\left(-G_{i+1}(\tau)\right) w\left(q(\tau)\varphi(\tau)\exp\left(G_{i+1}(\tau)\right)\right) d\tau\right],\ i = 1, 2, \ldots, n.$

证明: 因为 $q(x) \geq 1,\ \varphi(x) > 0$ 为非增函数, 由不等式 (2.2.1) 知

$$\frac{u(x)}{\varphi(x)} \leq q(x)\left[1 + \int_x^\infty \frac{f(\tau,x) w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)} d\tau + \int_x^\infty \frac{g(\tau,x) u\left(\sigma(\tau)\right)}{\varphi(\tau)} d\tau + \sum_{x < x_j < \infty} \beta_j \frac{u(x_j - 0)}{\varphi(x_j - 0)}\right],$$

令

$$v(x) = 1 + \int_x^\infty \frac{f(\tau,x) w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)} d\tau + \int_x^\infty \frac{g(\tau,x) u\left(\sigma(\tau)\right)}{\varphi(\tau)} d\tau + \sum_{x < x_j < \infty} \beta_j \frac{u(x_j - 0)}{\varphi(x_j - 0)}, \tag{2.2.3}$$

则 $v(x)$ 非负不增, 且

$$u(x) \leq q(x)\varphi(x)v(x), \tag{2.2.4}$$

由 $\sigma(x)\geq x$ 及 (2.2.4) 式得

$$u\left(\sigma(x)\right)\leq q\left(\sigma(x)\right)\varphi\left(\sigma(x)\right)v\left(\sigma(x)\right)\leq q(x)\varphi(x)v(x), \tag{2.2.5}$$

当 $x\in[x_{i-1},x_i)$ 时, 令

$$\widetilde{v}_i(x)=l_i+\int_x^{x_i}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_x^{x_i}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau,$$

则 $\widetilde{v}_i(x)$ 在 $[x_{i-1},x_i)$ 上非负不增, 且 $\widetilde{v}_i(x)=l_i$,

当 $x\in[x_n,\infty)$ $(x_{n+1}=\infty)$ 时,

$$\widetilde{v}_{n+1}(x)=v(x), \tag{2.2.6}$$

$$\widetilde{v}_{n+1}(x)=l_{n+1}+\int_x^{\infty}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_x^{\infty}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau, \tag{2.2.7}$$

对 (2.2.7) 式关于 x 求导, 由 (2.2.5) (2.2.6) 式得

$$\begin{aligned}
\widetilde{v}'_{n+1}(x)=&-\frac{f(x,x)w\left(u\left(\sigma(x)\right)\right)}{\varphi(x)}+\int_x^{\infty}\frac{\partial f(\tau,x)}{\partial x}\frac{w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau\\
&-\frac{g(x,x)u\left(\sigma(x)\right)}{\varphi(x)}+\int_x^{\infty}\frac{\partial g(\tau,x)}{\partial x}\frac{u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau\\
\geq&-\frac{f(x,x)w\left(q(x)\varphi(x)\widetilde{v}_{n+1}(x)\right)}{\varphi(x)}+\int_x^{\infty}\frac{\partial f(\tau,x)}{\partial x}\frac{w\left(q(\tau)\varphi(\tau)\widetilde{v}_{n+1}(\tau)\right)}{\varphi(\tau)}d\tau\\
&-g(x,x)q(x)\widetilde{v}_{n+1}(x)+\int_x^{\infty}\frac{\partial g(\tau,x)}{\partial x}q(\tau)\widetilde{v}_{n+1}(\tau)d\tau\\
\geq&\left[-\frac{f(x,x)}{\varphi(x)}+\int_x^{\infty}\frac{\partial f(\tau,x)}{\partial x}\frac{1}{\varphi(\tau)}d\tau\right]w\left(q(x)\varphi(x)\widetilde{v}_{n+1}(x)\right)\\
&+\left[-g(x,x)q(x)+\int_x^{\infty}\frac{\partial g(\tau,x)}{\partial x}q(\tau)d\tau\right]\widetilde{v}_{n+1}(x)\\
=&\left(\frac{d}{dx}\int_x^{\infty}\frac{f(\tau,x)}{\varphi(\tau)}d\tau\right)w\left(q(x)\varphi(x)\widetilde{v}_{n+1}(x)\right)+\left(\frac{d}{dx}\int_x^{\infty}g(\tau,x)q(\tau)d\tau\right)\widetilde{v}_{n+1}(x)\\
=&F_{n+1}(x)w\left(q(x)\varphi(x)\widetilde{v}_{n+1}(x)\right)+\left(\frac{d}{dx}G_{n+1}(x)\right)\widetilde{v}_{n+1}(x),
\end{aligned}$$

从而

$$\widetilde{v}'_{n+1}(x)-\left(\frac{d}{dx}G_{n+1}(x)\right)\widetilde{v}_{n+1}(x)\geq F_{n+1}(x)w\left(q(x)\varphi(x)\widetilde{v}_{n+1}(x)\right), \tag{2.2.8}$$

在 (2.2.8) 式两边同乘 $\exp\left(-G_{n+1}(x)\right)$ 得

$$\left[\widetilde{v}_{n+1}(x)\exp\left(-G_{n+1}(x)\right)\right]'\geq F_{n+1}(x)\exp\left(-G_{n+1}(x)\right)w\left(q(x)\varphi(x)\widetilde{v}_{n+1}(x)\right), \tag{2.2.9}$$

在 (2.2.9) 式两边从 x 到 ∞ 积分得

$$\begin{aligned}&\widetilde{v}_{n+1}(\infty)\exp\left(-G_{n+1}(\infty)\right)-\widetilde{v}_{n+1}(x)\exp\left(-G_{n+1}(x)\right)\\ \geq&\int_x^\infty F_{n+1}(\tau)\exp\left(-G_{n+1}(\tau)\right)w\left(q(\tau)\varphi(\tau)\widetilde{v}_{n+1}(\tau)\right)d\tau,\end{aligned}$$

从而

$$\widetilde{v}_{n+1}(x)\leq\exp\left(G_{n+1}(x)\right)\left[l_{n+1}-\int_x^\infty F_{n+1}(\tau)\exp\left(-G_{n+1}(\tau)\right)w\left(q(\tau)\varphi(\tau)\widetilde{v}_{n+1}(\tau)\right)d\tau\right],$$

令

$$p(x)=l_{n+1}-\int_x^\infty F_{n+1}(\tau)\exp\left(-G_{n+1}(\tau)\right)w\left(q(\tau)\varphi(\tau)\widetilde{v}_{n+1}(\tau)\right)d\tau,\tag{2.2.10}$$

则 $p(\infty)=l_{n+1}$, $p(x)$ 非负不增, 且

$$\widetilde{v}_{n+1}(x)\leq\exp\left(G_{n+1}(x)\right)p(x),\tag{2.2.11}$$

对 (2.2.10) 式关于 x 求导,

$$\begin{aligned}p'(x)&=F_{n+1}(x)\exp\left(-G_{n+1}(x)\right)w\left(q(x)\varphi(x)\widetilde{v}_{n+1}(x)\right)\\&\geq F_{n+1}(x)\exp\left(-G_{n+1}(x)\right)w\left(q(x)\varphi(x)\exp\left(G_{n+1}(x)\right)\right)w\left(p(x)\right),\end{aligned}$$

从而

$$\frac{p'(x)}{w\left(p(x)\right)}\geq F_{n+1}(x)\exp\left(-G_{n+1}(x)\right)w\left(q(x)\varphi(x)\exp\left(G_{n+1}(x)\right)\right),\tag{2.2.12}$$

在 (2.2.12) 式两边从 x 到 ∞ 积分得

$$\Phi_{n+1}\left(p(\infty)\right)-\Phi_{n+1}\left(p(x)\right)\geq\int_x^\infty F_{n+1}(\tau)\exp\left(-G_{n+1}(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_{n+1}(\tau)\right)\right)d\tau,$$

从而

$$p(x)\leq\Phi_{n+1}^{-1}\left[-\int_x^\infty F_{n+1}(\tau)\exp\left(-G_{n+1}(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_{n+1}(\tau)\right)\right)d\tau\right],\tag{2.2.13}$$

由 (2.2.4) (2.2.6) (2.2.11) (2.2.13) 式可得

$$\begin{aligned}v(x)&=\widetilde{v}_{n+1}(x)\leq\exp\left(G_{n+1}(x)\right)p(x)\\&\leq\exp\left(G_{n+1}(x)\right)\Phi_{n+1}^{-1}\left[-\int_x^\infty F_{n+1}(\tau)\exp\left(-G_{n+1}(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_{n+1}(\tau)\right)\right)d\tau\right],\end{aligned}\tag{2.2.14}$$

$$
\begin{aligned}
&u(x) \leq q(x)\varphi(x)v(x) \\
\leq\ & q(x)\varphi(x)\exp\left(G_{n+1}(x)\right) \\
&\Phi_{n+1}^{-1}\left[-\int_x^{\infty} F_{n+1}(\tau)\exp\left(-G_{n+1}(\tau)\right) w\left(q(\tau)\varphi(\tau)\exp\left(G_{n+1}(\tau)\right)\right) d\tau\right],
\end{aligned}
\tag{2.2.15}
$$

当 $x \in [x_{n-1}, x_n)$ 时, 由 $v(x)$ 的定义及 (2.2.14) 式知

$$
\begin{aligned}
v(x) &= 1 + \int_x^{\infty} \frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)} d\tau + \int_x^{\infty} \frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)} d\tau + \beta_n \frac{u(x_n - 0)}{\varphi(x_n - 0)} \\
&= 1 + \int_{x_n}^{\infty} \frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)} d\tau + \int_{x_n}^{\infty} \frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)} d\tau + \beta_n \frac{u(x_n - 0)}{\varphi(x_n - 0)} \\
&\quad + \int_x^{x_n} \frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)} d\tau + \int_x^{x_n} \frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)} d\tau \\
&\leq \left(1 + \beta_n q(x_n - 0)\right)\exp\left(G_{n+1}(x_n)\right) \\
&\quad \Phi_{n+1}^{-1}\left[-\int_{x_n}^{\infty} F_{n+1}(\tau)\exp\left(-G_{n+1}(\tau)\right) w\left(q(\tau)\varphi(\tau)\exp\left(G_{n+1}(\tau)\right)\right) d\tau\right] \\
&\quad + \int_x^{x_n} \frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)} d\tau + \int_x^{x_n} \frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)} d\tau \\
&= l_n + \int_x^{x_n} \frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)} d\tau + \int_x^{x_n} \frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)} d\tau = \widetilde{v}_n(x),
\end{aligned}
\tag{2.2.16}
$$

与 (2.2.7) 式的证明过程类似, 可得

$$
v(x) \leq \exp\left(G_n(x)\right)\Phi_n^{-1}\left[-\int_x^{x_n} F_n(\tau)\exp\left(-G_n(\tau)\right) w\left(q(\tau)\varphi(\tau)\exp\left(G_n(\tau)\right)\right) d\tau\right],
\tag{2.2.17}
$$

假设当 $x \in [x_{k-1}, x_k), k = i+1, \ldots, n-1$ 时,

$$
v(x) \leq \exp\left(G_k(x)\right)\Phi_k^{-1}\left[-\int_x^{x_k} F_k(\tau)\exp\left(-G_k(\tau)\right) w\left(q(\tau)\varphi(\tau)\exp\left(G_k(\tau)\right)\right) d\tau\right],
\tag{2.2.18}
$$

则当 $x \in [x_{i-1}, x_i)$ 时,

$$
\begin{aligned}
v(x) &= 1 + \int_x^{\infty} \frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)} d\tau + \int_x^{\infty} \frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)} d\tau + \sum_{x<x_j<\infty} \beta_j \frac{u(x_j - 0)}{\varphi(x_j - 0)} \\
&= 1 + \int_{x_i}^{\infty} \frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)} d\tau + \int_{x_i}^{\infty} \frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)} d\tau + \sum_{x_i<x_j<\infty} \beta_j \frac{u(x_j - 0)}{\varphi(x_j - 0)} \\
&\quad + \beta_i \frac{u(x_i - 0)}{\varphi(x_i - 0)} + \int_x^{x_i} \frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)} d\tau + \int_x^{x_i} \frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)} d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1+\beta_i q(x_i-0))\exp\left(G_{i+1}(x_i)\right)\\
&\quad \Phi_{i+1}^{-1}\left[-\int_{x_i}^{x_{i+1}} F_{i+1}(\tau)\exp\left(-G_{i+1}(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_{i+1}(\tau)\right)\right)d\tau\right]\\
&\quad +\int_x^{x_i}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_x^{x_i}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau\\
&= l_{i+1}+\int_x^{x_i}\frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau+\int_x^{x_i}\frac{g(\tau,x)u\left(\sigma(\tau)\right)}{\varphi(\tau)}d\tau\\
&= \widetilde{v}_i(x),
\end{aligned}
\tag{2.2.19}
$$

与 (2.2.7) 式的证明类似, 可得

$$
v(x)\leq \exp\left(G_i(x)\right)\Phi_i^{-1}\left[-\int_x^{x_i}F_i(\tau)\exp\left(-G_i(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_i(\tau)\right)\right)d\tau\right],
\tag{2.2.20}
$$

从而

$$
u(x)\leq q(x)\varphi(x)\exp\left(G_i(x)\right)\Phi_i^{-1}\left[-\int_x^{x_i}F_i(\tau)\exp\left(-G_i(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_i(\tau)\right)\right)d\tau\right],
\tag{2.2.21}
$$

从而可证 $u(x)$ 满足 (2.2.2) 式.

定理 2.2.2　$u(x)$ 为非负函数, 定义在区间 $R_+^{x_0}=[x_0,\infty)$ 上, $u(x)$ 在 $R_+^{x_0}$ 上除了 $x_i\ (i=1,2,\ldots,n)$ 点外连续, $u(x_i-0)\neq u(x_i+0), x_i\leq x_{i+1}$,
则 $R_+^{x_0}=\bigcup\limits_{i=1}^{n+1}\{x|x\in[x_{i-1},x_i)\}$, 其中 $x_{n+1}=\infty$, 若

$$
\begin{aligned}
u(x)\leq{}&\varphi(x)+q(x)\int_x^{\infty}f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)d\tau\\
&+q(x)\int_x^{\infty}g(\tau,x)\left(u\left(\sigma(\tau)\right)+\int_\tau^{\infty}h(\xi,\tau)u\left(\sigma(\xi)\right)d\xi\right)d\tau\\
&+\sum_{x<x_j<\infty}\beta_j u(x_j-0),\ x\in[x_{i-1},x_i),
\end{aligned}
\tag{2.2.22}
$$

其中 x_0, β_j, $q(x)$, $\varphi(x)$, $\sigma(x)$, $f(\tau,x)$, $g(\tau,x)$, $w(u)$ 满足定理 2.2.1 的条件, $h(\xi,\tau)$ 为非负连续函数, 则

$$
\begin{aligned}
u(x)\leq{}&q(x)\varphi(x)\exp\left(G_i(x)\right)\\
&\Phi_i^{-1}\left[-\int_x^{x_i}F_i(\tau)\exp\left(-G_i(\tau)\right)w\left(q(\tau)\varphi(\tau)\exp\left(G_i(\tau)\right)\right)d\tau\right],\ \forall x\in[x_{i-1},x_i),
\end{aligned}
\tag{2.2.23}
$$

其中

$F_i(x) = \frac{d}{dx}\int_x^{x_i} \frac{f(\tau,x)}{\varphi(\tau)} d\tau$, $G_i(x) = \int_x^{x_i} g(\tau,x)\left(q(\tau) + \int_\tau^\infty h(\xi,\tau)q(\xi)d\xi\right)d\tau$,

$\Phi_i(r) = \int_{l_i}^r \frac{ds}{w(s)}$, $r > 0$, Φ_i^{-1} 为其反函数, $i = 1, 2, \ldots, n+1$,

$l_{n+1} = 1$,

$l_i = (1 + \beta_i q(x_i - 0)) \exp\left(G_{i+1}(x_i)\right)$

$\Phi_{i+1}^{-1}\left[-\int_{x_i}^{x_{i+1}} F_{i+1}(\tau)\exp\left(-G_{i+1}(\tau)\right) w\left(q(\tau)\varphi(\tau)\exp\left(G_{i+1}(\tau)\right)\right)d\tau\right]$, $i = 1, 2, \ldots, n$.

证明: 因为 $q(x) \geq 1, \varphi(x) > 0$ 为非增函数, 由不等式 (2.2.22) 知

$$\frac{u(x)}{\varphi(x)} \leq q(x)\left[1 + \int_x^\infty \frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau + \int_x^\infty \frac{g(\tau,x)\left(u\left(\sigma(\tau)\right) + \int_\tau^\infty h(\xi,\tau)u\left(\sigma(\xi)\right)d\xi\right)}{\varphi(\tau)}d\tau + \sum_{x<x_j<\infty}\beta_j\frac{u(x_j-0)}{\varphi(x_j-0)}\right],$$

令

$$v(x) = 1 + \int_x^\infty \frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau + \int_x^\infty \frac{g(\tau,x)\left(u\left(\sigma(\tau)\right) + \int_\tau^\infty h(\xi,\tau)u\left(\sigma(\xi)\right)d\xi\right)}{\varphi(\tau)}d\tau + \sum_{x<x_j<\infty}\beta_j\frac{u(x_j-0)}{\varphi(x_j-0)}, \tag{2.2.24}$$

则 $v(x)$ 非负不增, 且

$$u(x) \leq q(x)\varphi(x)v(x), \tag{2.2.25}$$

由 $\sigma(x) \geq x$ 及 (2.2.25) 式得

$$u\left(\sigma(x)\right) \leq q\left(\sigma(x)\right)\varphi\left(\sigma(x)\right)v\left(\sigma(x)\right) \leq q(x)\varphi(x)v(x), \tag{2.2.26}$$

当 $x \in [x_n, \infty)$ 时,

$$v(x) = 1 + \int_x^\infty \frac{f(\tau,x)w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau + \int_x^\infty \frac{g(\tau,x)\left(u\left(\sigma(\tau)\right) + \int_\tau^\infty h(\xi,\tau)u\left(\sigma(\xi)\right)d\xi\right)}{\varphi(\tau)}d\tau, \tag{2.2.27}$$

对 (2.2.27) 式关于 x 求导, 由 (2.2.26) 式得

$$\begin{aligned} v'(x) = &-\frac{f(x,x)w\left(u\left(\sigma(x)\right)\right)}{\varphi(x)} + \int_x^\infty \frac{\partial f(\tau,x)}{\partial x}\frac{w\left(u\left(\sigma(\tau)\right)\right)}{\varphi(\tau)}d\tau \\ &- \frac{g(x,x)\left(u\left(\sigma(x)\right) + \int_x^\infty h(\tau,x)u\left(\sigma(\tau)\right)d\tau\right)}{\varphi(x)} \\ &+ \int_x^\infty \frac{\partial g(\tau,x)}{\partial x}\frac{u\left(\sigma(\tau)\right) + \int_\tau^\infty h(\xi,\tau)u\left(\sigma(\xi)\right)d\xi}{\varphi(\tau)}d\tau \end{aligned}$$

$$
\begin{aligned}
&\geq \left[-\frac{f(x,x)}{\varphi(x)}+\int_x^\infty \frac{\partial f(\tau,x)}{\partial x\ \varphi(\tau)}d\tau\right] w\left(q(x)\varphi(x)v(x)\right) \\
&\quad + \left[- g(x,x)\left(q(x)+\int_x^\infty h(\tau,x)q(\tau)d\tau\right)\right. \\
&\quad \left. + \int_x^\infty \frac{\partial g(\tau,x)}{\partial x}\left(q(\tau)+\int_\tau^\infty h(\xi,\tau)q(\xi)d\xi\right)d\tau\right]v(x) \\
&= \left(\frac{d}{dx}\int_x^\infty \frac{f(\tau,x)}{\varphi(\tau)}d\tau\right) w\left(q(x)\varphi(x)v(x)\right) \\
&\quad + \left[\frac{d}{dx}\int_x^\infty g(\tau,x)\left(q(\tau)+\int_\tau^\infty h(\xi,\tau)q(\xi)d\xi\right)d\tau\right]v(x) \\
&= F_{n+1}(x)w\left(q(x)\varphi(x)\widetilde{v}_{n+1}(x)\right)+\left(\frac{d}{dx}G_{n+1}(x)\right)v(x),
\end{aligned}
$$

从而

$$
v'(x)-\left(\frac{d}{dx}G_{n+1}(x)\right)v(x)\geq F_{n+1}(x)w\left(q(x)\varphi(x)v(x)\right), \tag{2.2.28}
$$

与定理 2.2.1 的证明类似, 可证 $u(x)$ 满足 (2.2.23) 式.

定理 2.2.3　$u(x,y)$ 为非负函数, 定义在区域 $\Omega=R_+^{x_0}\times R_+^{y_0}, u(x,y)$ 在 Ω 上除了 (x_i,y_i) $(i=1,2,\ldots,n)$ 点外连续, $u(x_i-0,y_i-0)\neq u(x_i+0,y_i+0)$, $x_i\leq x_{i+1}$, $y_i\leq y_{i+1}$, 则 $\Omega=\bigcup\limits_{i,j=1}^{n+1}\Omega_{ij}=\bigcup\limits_{i,j=1}^{n+1}\{(x,y)|(x,y)\in[x_{i-1},x_i)\times[y_{j-1},y_j)\}$, 其中 $x_{n+1}=\infty$, $y_{n+1}=\infty$,若

$$
\begin{aligned}
u(x,y)\leq{}& \varphi(x,y)+q(x,y)\int_x^\infty\int_y^\infty f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)dsd\tau \\
&+q(x,y)\int_x^\infty\int_y^\infty g(\tau,,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)dsd\tau \\
&+\sum_{x<x_j<\infty,y<y_j<\infty}\beta_j u(x_j-0,y_j-0),\ (x,y)\in\Omega_{ii},
\end{aligned}
\tag{2.2.29}
$$

且满足

(1)　$x_0\geq 0$, $y_0\geq 0$, $\beta_j>0$ 为常数;

(2)　$\forall(x,y)\in\Omega$, $q(x,y)\geq 1$, $\varphi(x,y)>0$ 都是连续函数, 且关于 x, y 都不增;

(3)　$f(\tau,s,x,y)$, $g(\tau,s,x,y)\in C(R_+^4,R_+)$, $\frac{\partial f(\tau,s,x,y)}{\partial x}$, $\frac{\partial g(\tau,s,x,y)}{\partial x}\in C(R_+^4,R)$, 对于固定的 $\tau\in R_+$, f, g 关于 x 是不增的, 对于固定的 $s\in R_+$, f, g 关于 y 是不增的, 且当 $(\tau,s)\in\Omega_{ij},i\neq j$ 时, $f(\tau,s,x,y)=g(\tau,s,x,y)=0$;

(4)　$\sigma(x)\leq x$ 是非负连续函数, 且当 $(x,y)\in\Omega_{ij}$ 时, $(\sigma(x),\sigma(y))\leq(x_i,y_j)$,

$(\sigma(x),\sigma(y))\geq(x_{i-1},y_{j-1})$;

(5) $w(u)$ 满足以下条件:

(a) $w(\alpha\beta)\leq w(\alpha)w(\beta)$,

(b) $w\in C(R_+,R_+)$, 且当 $x\in(0,\infty)$ 时, $w(x)>0$,

(c) w 为非减函数;

则

$$\begin{aligned}u(x,y)\leq{}& q(x,y)\varphi(x,y)\exp\left(G_i(x,y)\right)\\&\Phi_i^{-1}\left[-\int_x^{x_i}F_i(\tau,y)\exp\left(-G_i(\tau,y)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_i(\tau,y)\right)\right)\right)d\tau\right],\\&\forall(x,y)\in\Omega_{ii},\end{aligned}\tag{2.2.30}$$

其中

$F_i(x,y)=\frac{\partial}{\partial x}\int_x^{x_i}\int_y^{y_i}\frac{f(\tau,s,x,y)}{\varphi(\tau,s)}dsd\tau,\ \ G_i(x,y)=\int_x^{x_i}\int_y^{y_i}g(\tau,s,x,y)q(\tau,s)dsd\tau,$

$\Phi_i(r)=\int_{l_i}^r\frac{ds}{w(s)},\ r>0,\ \Phi_i^{-1}$ 为其反函数, $\ i=1,2\ldots,n+1,$

$l_{n+1}=1,$

$l_i=(1+\beta_iq(x_i-0,y_i-0))\exp\left(G_{i+1}(x_i,y_i)\right)$

$\quad\Phi_{i+1}^{-1}\left[-\int_{x_i}^{x_{i+1}}F_{i+1}(\tau,y_i)\exp\left(-G_{i+1}(\tau,y_i)\right)w\left(q(\tau,y_i)\varphi(\tau,y_i)\exp\left(G_{i+1}(\tau,y_i)\right)\right)d\tau\right],$

$\quad i=1,2,\ldots,n.$

证明: 因为 $q(x,y)\geq1$, $\varphi(x,y)>0$ 关于 x, y 都是非增函数, 由不等式 (2.2.29) 知

$$\begin{aligned}\frac{u(x,y)}{\varphi(x,y)}\leq q(x,y)\Bigg[&1+\int_x^\infty\int_y^\infty\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau\\&+\int_x^\infty\int_y^\infty\frac{g(\tau,,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau+\sum_{x<x_j<\infty,y<y_j<\infty}\beta_j\frac{u(x_j-0,y_j-0)}{\varphi(x_j-0,y_j-0)}\Bigg],\end{aligned}$$

令

$$\begin{aligned}v(x,y)={}&1+\int_x^\infty\int_y^\infty\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau\\&+\int_x^\infty\int_y^\infty\frac{g(\tau,,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau+\sum_{x<x_j<\infty,y<y_j<\infty}\beta_j\frac{u(x_j-0,y_j-0)}{\varphi(x_j-0,y_j-0)},\end{aligned}\tag{2.2.31}$$

则 $v(x,y)$ 非负关于 x, y 都不增, 且

$$u(x,y)\leq q(x,y)\varphi(x,y)v(x,y),\tag{2.2.32}$$

由 $\sigma(x)\geq x$ 及 (2.2.32) 式得

$$u\left(\sigma(x),\sigma(y)\right)\leq q\left(\sigma(x),\sigma(y)\right)\varphi\left(\sigma(x),\sigma(y)\right)v\left(\sigma(x),\sigma(y)\right)\leq q(x,y)\varphi(x,y)v(x,y), \tag{2.2.33}$$

当 $(x,y)\in\Omega_{ii}$ 时, 令

$$\begin{aligned}\widetilde{v}_i(x,y)=l_i&+\int_x^{x_i}\int_y^{y_i}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau\\&+\int_x^{x_i}\int_y^{y_i}\frac{g(\tau,,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau,\end{aligned}$$

则 $\widetilde{v}_i(x,y)$ 在 Ω_{ii} 上非负关于 x, y 都不增, 且 $\widetilde{v}_i(x_i,y)=l_i$, $\widetilde{v}_i(x,y_i)=l_i$,
当 $(x,y)\in\Omega_{n+1\ n+1}=[x_n,\infty)\times[y_n,\infty),(x_{n+1}=\infty,y_{n+1}=\infty)$ 时,

$$\widetilde{v}_{n+1}(x,y)=v(x,y), \tag{2.2.34}$$

$$\begin{aligned}\widetilde{v}_{n+1}(x,y)=l_{n+1}&+\int_x^{\infty}\int_y^{\infty}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau\\&+\int_x^{\infty}\int_y^{\infty}\frac{g(\tau,,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau,\end{aligned} \tag{2.2.35}$$

在 (2.2.35) 式关于 x 求偏导, 由 (2.2.33) (2.2.34) 式得

$$\begin{aligned}&\frac{\partial\widetilde{v}_{n+1}(x,y)}{\partial x}\\&=-\int_y^{\infty}\frac{f(x,s,x,y)w\left(u\left(\sigma(x),\sigma(s)\right)\right)}{\varphi(x,s)}ds\\&\quad+\int_x^{\infty}\int_y^{\infty}\frac{\partial f(\tau,s,x,y)}{\partial x}\frac{w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau\\&\quad-\int_y^{\infty}\frac{g(x,s,x,y)u\left(\sigma(x),\sigma(s)\right)}{\varphi(x,s)}+\int_x^{\infty}\int_y^{\infty}\frac{\partial g(\tau,s,x,y)}{\partial x}\frac{u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau\\&\geq-\int_y^{\infty}\frac{f(x,s,x,y)w\left(q(x,s)\varphi(x,s)\widetilde{v}_{n+1}(x,s)\right)}{\varphi(x,s)}ds\\&\quad+\int_x^{\infty}\int_y^{\infty}\frac{\partial f(\tau,s,x,y)}{\partial x}\frac{w\left(q(\tau,s)\varphi(\tau,s)\widetilde{v}_{n+1}(\tau,s)\right)}{\varphi(\tau,s)}dsd\tau\\&\quad-\int_y^{\infty}g(x,s,x,y)q(x,s)\widetilde{v}_{n+1}(x,s)ds+\int_x^{\infty}\int_y^{\infty}\frac{\partial g(\tau,s,x,y)}{\partial x}q(\tau,s)\widetilde{v}_{n+1}(\tau,s)dsd\tau\end{aligned}$$

$$
\begin{aligned}
&\geq \int_y^\infty \left[-\frac{f(x,s,x,y)}{\varphi(x,s)} + \int_x^\infty \frac{\partial f(\tau,s,x,y)}{\partial x\ \varphi(\tau,s)} d\tau\right] w\left(q(x,s)\varphi(x,s)\widetilde{v}_{n+1}(x,s)\right) ds \\
&\quad + \int_y^\infty \left[-g(x,s,x,y)q(x,s) + \int_x^\infty \frac{\partial g(\tau,s,x,y)}{\partial x} q(\tau,s) d\tau\right] \widetilde{v}_{n+1}(x,s) ds \\
&\geq \int_y^\infty \left(\frac{\partial}{\partial x}\int_x^\infty \frac{f(\tau,s,x,y)}{\varphi(\tau,s)} d\tau\right) ds\ w\left(q(x,y)\varphi(x,y)\widetilde{v}_{n+1}(x,y)\right) \\
&\quad + \int_y^\infty \left(\frac{\partial}{\partial x}\int_x^\infty g(\tau,s,x,y)q(\tau,s) d\tau\right) ds\ \widetilde{v}_{n+1}(x,y) \\
&= \left(\frac{\partial}{\partial x}\int_x^\infty \int_y^\infty \frac{f(\tau,s,x,y)}{\varphi(\tau,s)} ds d\tau\right) w\left(q(x,y)\varphi(x,y)\widetilde{v}_{n+1}(x,y)\right) \\
&\quad + \left(\frac{\partial}{\partial x}\int_x^\infty \int_y^\infty g(\tau,s,x,y)q(\tau,s) ds d\tau\right) \widetilde{v}_{n+1}(x,y) \\
&= F_{n+1}(x,y) w\left(q(x,y)\varphi(x,y)\widetilde{v}_{n+1}(x,y)\right) + \frac{\partial G_{n+1}(x,y)}{\partial x}\widetilde{v}_{n+1}(x,y),
\end{aligned}
$$

从而

$$
\frac{\partial \widetilde{v}_{n+1}(x,y)}{\partial x} - \frac{\partial G_{n+1}(x,y)}{\partial x}\widetilde{v}_{n+1}(x,y) \geq F_{n+1}(x,y) w\left(q(x,y)\varphi(x,y)\widetilde{v}_{n+1}(x,y)\right), \tag{2.2.36}
$$

在 (2.2.36) 式两边同乘 $\exp\left(-G_{n+1}(x,y)\right)$ 得

$$
\frac{\partial\left[\widetilde{v}_{n+1}(x,y)\exp\left(-G_{n+1}(x,y)\right)\right]}{\partial x} \geq F_{n+1}(x,y)\exp\left(-G_{n+1}(x,y)\right) w\left(q(x,y)\varphi(x,y)\widetilde{v}_{n+1}(x,y)\right), \tag{2.2.37}
$$

在 (2.2.37) 式两边从 x 到 ∞ 积分得

$$
\begin{aligned}
&\widetilde{v}_{n+1}(\infty,y)\exp\left(-G_{n+1}(\infty,y)\right) - \widetilde{v}_{n+1}(x,y)\exp\left(-G_{n+1}(x,y)\right) \\
&\geq \int_x^\infty F_{n+1}(\tau,y)\exp\left(-G_{n+1}(\tau,y)\right) w\left(q(\tau,y)\varphi(\tau,y)\widetilde{v}_{n+1}(\tau,y)\right) d\tau,
\end{aligned}
$$

从而

$$
\begin{aligned}
\widetilde{v}_{n+1}(x,y) \leq{} & \exp\left(G_{n+1}(x,y)\right) \\
& \left[l_{n+1} - \int_x^\infty F_{n+1}(\tau,y)\exp\left(-G_{n+1}(\tau,y)\right) w\left(q(\tau,y)\varphi(\tau,y)\widetilde{v}_{n+1}(\tau,y)\right) d\tau\right],
\end{aligned}
$$

令

$$
p(x,y) = l_{n+1} - \int_x^\infty F_{n+1}(\tau,y)\exp\left(-G_{n+1}(\tau,y)\right) w\left(q(\tau,y)\varphi(\tau,y)\widetilde{v}_{n+1}(\tau,y)\right) d\tau, \tag{2.2.38}
$$

则 $p(\infty,y)=l_{n+1}$，$p(x,y)$ 关于 x 非负不增, 且

$$\widetilde{v}_{n+1}(x,y)\le \exp\left(G_{n+1}(x,y)\right)p(x,y), \tag{2.2.39}$$

对 (2.2.38) 式关于 x 求偏导

$$\begin{aligned}\frac{\partial p(x,y)}{\partial x}&=F_{n+1}(x,y)\exp\left(-G_{n+1}(x,y)\right)w\left(q(x,y)\varphi(x,y)\widetilde{v}_{n+1}(x,y)\right)\\&\ge F_{n+1}(x,y)\exp\left(-G_{n+1}(x,y)\right)w\left(q(x,y)\varphi(x,y)\exp\left(G_{n+1}(x,y)\right)\right)w\left(p(x,y)\right),\end{aligned}$$

从而

$$\frac{\frac{\partial p(x,y)}{\partial x}}{w\left(p(x,y)\right)}\ge F_{n+1}(x,y)\exp\left(-G_{n+1}(x,y)\right)w\left(q(x,y)\varphi(x,y)\exp\left(G_{n+1}(x,y)\right)\right), \tag{2.2.40}$$

在 (1.2.40) 式两边从 x 到 ∞ 积分得

$$\begin{aligned}&\Phi_{n+1}\left(p(\infty,y)\right)-\Phi_{n+1}\left(p(x,y)\right)\\&\ge\int_x^\infty F_{n+1}(\tau,y)\exp\left(-G_{n+1}(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_{n+1}(\tau,y)\right)\right)d\tau,\end{aligned}$$

从而

$$p(x,y)\le\Phi_{n+1}^{-1}\left[-\int_x^\infty F_{n+1}(\tau,y)\exp\left(-G_{n+1}(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_{n+1}(\tau,y)\right)\right)d\tau\right], \tag{2.2.41}$$

由 (2.2.32) (2.2.34) (2.2.39) (2.2.41) 式可得

$$\begin{aligned}&v(x,y)\le\widetilde{v}_{n+1}(x,y)\le\exp\left(G_{n+1}(x,y)\right)p(x,y)\\&\le\exp\left(G_{n+1}(x,y)\right)\\&\quad\Phi_{n+1}^{-1}\left[-\int_x^\infty F_{n+1}(\tau,y)\exp\left(-G_{n+1}(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_{n+1}(\tau,y)\right)\right)d\tau\right],\end{aligned} \tag{2.2.42}$$

$$\begin{aligned}&u(x,y)\le q(x,y)\varphi(x,y)v(x,y)\\&\le q(x,y)\varphi(x,y)\exp\left(G_{n+1}(x,y)\right)\\&\quad\Phi_{n+1}^{-1}\left[-\int_x^\infty F_{n+1}(\tau,y)\exp\left(-G_{n+1}(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_{n+1}(\tau,y)\right)\right)d\tau\right],\end{aligned} \tag{2.2.43}$$

当 $(x,y)\in\Omega_{nn}$ 时, 由 $v(x,y)$ 的定义及 (2.2.42) 式知

$$\begin{aligned}
&v(x,y)\\
&=1+\int_x^\infty\int_y^\infty\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau\\
&\quad+\int_x^\infty\int_y^\infty\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau+\beta_n\frac{u(x_n-0,y_n-0)}{\varphi(x_n-0,y_n-0)}\\
&=1+\int_{x_n}^\infty\int_{y_n}^\infty\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau\\
&\quad+\int_{x_n}^\infty\int_{y_n}^\infty\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau+\beta_n\frac{u(x_n-0,y_n-0)}{\varphi(x_n-0,y_n-0)}\\
&\quad+\int_x^{x_n}\int_y^{y_n}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau\\
&\quad+\int_x^{x_n}\int_y^{y_n}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau\\
&\le\left(1+\beta_nq(x_n-0,y_n-0)\right)\exp\left(G_{n+1}(x_n,y_n)\right)\\
&\quad\Phi_{n+1}^{-1}\left[-\int_{x_n}^\infty F_{n+1}(\tau,y_n)\exp\left(-G_{n+1}(\tau,y_n)\right)w\left(q(\tau,y_n)\varphi(\tau,y_n)\exp\left(G_{n+1}(\tau,y_n)\right)\right)d\tau\right]\\
&\quad+\int_x^{x_n}\int_y^{y_n}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau+\int_x^{x_n}\int_y^{y_n}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau\\
&=l_n+\int_x^{x_n}\int_y^{y_n}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau\\
&\quad+\int_x^{x_n}\int_y^{y_n}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau=\widetilde{v}_n(x,y),
\end{aligned}\tag{2.2.44}$$

与 (2.2.35) 式类似, 可得

$$\begin{aligned}
&v(x,y)\le\exp\left(G_n(x,y)\right)\\
&\qquad\Phi_n^{-1}\left[-\int_x^{x_n}F_n(\tau,y)\exp\left(-G_n(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_n(\tau,y)\right)\right)d\tau\right],
\end{aligned}\tag{2.2.45}$$

假设当 $(x,y)\in\Omega_{kk},k=i+1,\dots,n-1$ 时,

$$\begin{aligned}
&v(x,y)\le\exp\left(G_k(x,y)\right)\\
&\qquad\Phi_k^{-1}\left[-\int_x^{x_k}F_k(\tau,y)\exp\left(-G_k(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_k(\tau,y)\right)\right)d\tau\right],
\end{aligned}\tag{2.2.46}$$

则当 $(x,y)\in\Omega_{ii}$ 时,

$$
\begin{aligned}
&v(x,y)\\
&=1+\int_x^\infty\int_y^\infty\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau+\int_x^\infty\int_y^\infty\frac{g(\tau,s,x,y)u\left(\sigma(\tau)\right)}{\varphi(\tau,s)}dsd\tau\\
&\quad+\sum_{x<x_j<\infty,y<y_j<\infty}\beta_j\frac{u(x_j-0,y_j-0)}{\varphi(x_j-0,y_j-0)}\\
&=1+\int_{x_i}^\infty\int_{y_i}^\infty\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau\\
&\quad+\int_{x_i}^\infty\int_{y_i}^\infty\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau\\
&\quad+\sum_{x_i<x_j<\infty,y_i<y_j<\infty}\beta_j\frac{u(x_j-0,y_j-0)}{\varphi(x_j-0,y_j-0)}+\beta_i\frac{u(x_i-0,y_i-0)}{\varphi(x_i-0,y_i-0)}\\
&\quad+\int_x^{x_i}\int_y^{y_i}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau+\int_x^{x_i}\int_y^{y_i}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau\\
&\le(1+\beta_iq(x_i-0,y_i-0))\exp\left(G_{i+1}(x_i,y_i)\right)\\
&\quad\Phi_{i+1}^{-1}\left[-\int_{x_i}^{x_{i+1}}F_{i+1}\exp\left(-G_{i+1}(\tau,y_i)\right)w\left(q(\tau,y_i)\varphi(\tau,y_i)\exp\left(G_{i+1}(\tau,y_i)\right)\right)d\tau\right]\\
&\quad+\int_x^{x_i}\int_y^{y_i}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau+\int_x^{x_i}\int_y^{y_i}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau\\
&=l_i+\int_x^{x_i}\int_y^{y_i}\frac{f(\tau,s,x,y)w\left(u\left(\sigma(\tau),\sigma(s)\right)\right)}{\varphi(\tau,s)}dsd\tau\\
&\quad+\int_x^{x_i}\int_y^{y_i}\frac{g(\tau,s,x,y)u\left(\sigma(\tau),\sigma(s)\right)}{\varphi(\tau,s)}dsd\tau=\widetilde{v}_i(x,y),
\end{aligned}\tag{2.2.47}
$$

与 (1.2.35) 式类似, 此时可得

$$
\begin{aligned}
v(x,y)\le{}&\exp\left(G_i(x,y)\right)\\
&\Phi_i^{-1}\left[-\int_x^{x_i}F_i(\tau,y)\exp\left(-G_i(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,y)\exp\left(G_i(\tau,y)\right)\right)d\tau\right],
\end{aligned}\tag{2.2.48}
$$

从而

$$
\begin{aligned}
u(x,y)\le{}&q(x,y)\varphi(x,y)\exp\left(G_i(x,y)\right)\\
&\Phi_i^{-1}\left[-\int_x^{x_i}F_i(\tau,y)\exp\left(-G_i(\tau,y)\right)w\left(q(\tau,y)\varphi(\tau,\exp\left(G_i(\tau,y)\right)\right)d\tau\right],
\end{aligned}\tag{2.2.49}
$$

从而可证 (1.2.30) 式.

2.3　应用

考虑下面的脉冲扰动微分方程:

$$\begin{cases} \frac{\partial^2 u(x,y)}{\partial x \partial y} = M(x,y,u), \quad (x,y) \in \Omega, (x,y) \neq (x_i, y_i), \\ u(x,\infty) = a_1(x),\ u(\infty,y) = a_2(y), \quad (x,\infty), (\infty,y) \in \Omega, \\ \Delta u|_{(x,y)=(x_i,y_i)} = \beta_i u(x_i - 0, y_i - 0), \quad i = 1,2,\ldots,n, \end{cases} \tag{2.3.1}$$

其中 $u(x,y)$ 在 Ω 上除了 (x_i,y_i) $i=1,2,\ldots,n$ 点外连续, $\Omega = \bigcup\limits_{i,j=1}^{n+1} \Omega_{ij}$,
$\Omega_{ij} = \{(x_i,y_i)|(x_i,y_i) \in [x_{i-1},x_i) \times [y_{i-1},y_i)\}$, $M \in C(R_+^2 \times R, R)$, $\sigma(x) \geq x$,
且当 $x \in [x_{i-1}, x_i)$ 时, $\sigma(x) \leq x_i$, $\sigma(x) \geq x_{i-1}$, $a_1(x)$, $a_2(y) \in C(R_+, R_+)$ 为非增函数, 且 $a_1(\infty) = a_2(\infty) = 0$, $\beta_i \geq 0$ 为常数.

定理 2.3.1　设 $u(x,y)$ 是方程 (2.3.1) 的一个解, 且

$$\begin{cases} |M(x,y,u)| \leq f(x,y)|u|^m + g(x,y)|u|, \\ w(\alpha) = \alpha^m,\ \alpha \in R_+, \end{cases} \tag{2.3.2}$$

其中 $0 < m < 1$, $f(x,y)$, $g(x,y)$ 为非负连续函数, 则

$$\begin{aligned} u(x,y) \leq{}& \varphi(x,y) \exp\left(\widetilde{g}_i(x,y)\right) \\ & \Phi_i^{-1}\left[-\int_x^{x_i} \widetilde{f}_i(\tau,y) \exp\left(-\widetilde{g}_i(\tau,y)\right) w\left(\varphi(\tau,y)\exp\left(\widetilde{g}_i(\tau,y)\right)\right) d\tau\right], \end{aligned} \tag{2.3.3}$$

其中 $\varphi(x,y) = a_1(x) + a_2(y)$, $\widetilde{f}_i(x,y) = -\int_y^{y_i} f(x,s)ds$, $\widetilde{g}_i(x,y) = \int_x^{x_i}\int_y^{y_i} g(\tau,s)dsd\tau$,
$\Phi_i(r) = \int_{l_i}^r \frac{ds}{w(s)}$, $r > 0$, Φ_i^{-1} 为其反函数, $i = 1,2,\ldots,n+1$,
$l_{n+1} = 1$,

$$\begin{aligned} l_i ={}& (1 + \beta_i q(x_i - 0, y_i - 0)) \exp\left(\widetilde{g}_{i+1}(x_i,y_i)\right) \\ & \Phi_{i+1}^{-1}\left[-\int_{x_i}^{x_{i+1}} \widetilde{f}_{i+1}(\tau,y_i) \exp\left(-\widetilde{g}_{i+1}(\tau,y_i)\right) w\left(q(\tau,y_i)\varphi(\tau,y_i)\exp\left(\widetilde{g}_{i+1}(\tau,y_i)\right)\right) d\tau\right], \\ & i = 1,2,\ldots,n. \end{aligned}$$

证明: 微分方程 (2.3.1) 的积分方程是

$$\begin{aligned} u(x,y) = a_1(x) + b_2(y) + \int_x^\infty \int_y^\infty M\left(\tau, s, u\left(\sigma(\tau), \sigma(s)\right)\right) dsd\tau \\ + \sum_{x < x_j < \infty, y < y_j < \infty} \beta_j u(x_j - 0, y_j - 0), \end{aligned}$$

由 (2.3.2) 式知

$$
\begin{aligned}
|u(x,y)| &\le a_1(x)+b_2(y)+\int_x^\infty\int_y^\infty f(\tau,s)|u(\sigma(\tau),\sigma(s))|^n dsd\tau \\
&\quad+\int_x^\infty\int_y^\infty g(\tau,s)|u(\sigma(\tau),\sigma(s))|dsd\tau+\sum_{x<x_j<\infty,y<y_j<\infty}\beta_j|u(x_j-0,y_j-0)| \\
&=\varphi(x)+\int_x^\infty\int_y^\infty f(\tau,s)w(|u(\sigma(\tau),\sigma(s))|)dsd\tau \\
&\quad+\int_x^\infty\int_y^\infty g(\tau,s)|u(\sigma(\tau),\sigma(s))|dsd\tau+\sum_{x<x_j<\infty,y<y_j<\infty}\beta_j|u(x_j-0,y_j-0)|,
\end{aligned}
\tag{2.3.4}
$$

令 $b(x,y)=|u(x,y)|$, 则 (2.3.4) 式是 (2.2.29) 式的特殊情形, 类似可证得 (2.3.3) 式.

定理 2.3.2　在定理 2.3.1 的条件下, 还可得

$$
\begin{aligned}
u(x,y) &\le \varphi(x,y)\exp(\widetilde{g}_i(x,y)) \\
&\quad\left\{(1-m)\left[-\int_x^{x_i}\widetilde{f}_i(\tau,y)\exp(-\widetilde{g}_i(\tau,y))\,w(\varphi(\tau,y)\exp(\widetilde{g}_i(\tau,y)))\,d\tau\right]+l_i^{1-n}\right\}^{\frac{1}{1-m}}, \\
&\quad\forall x\in[x_{i-1},x_i),\ i=1,2,\ldots,n+1.
\end{aligned}
\tag{2.3.5}
$$

证明: 由于 $w(\alpha)=\alpha^m$ 及 $\Phi_i(r)=\int_{l_i}^r\frac{ds}{w(s)}$, 可得

$$
\Phi_i^{-1}(r)=\left[(1-m)r+l_i^{1-m}\right]^{\frac{1}{1-m}}, \tag{2.3.6}
$$

由 (2.3.3) (2.3.6) 式知, $|u(x,y)|$ 满足 (2.3.5) 式.

第三章　几类两个变量的积分不等式及应用

3.1　引言及预备知识

积分不等式在研究微分方程的定性性质理论的过程中起着十分重要的作用. 近年来, 从实际应用出发, 许多学者已经建立了大量相关的积分不等式.本章是在文献 [17] 和 [21] 的基础上, 建立了新的含有两个变量的广义积分不等式, 给出了主要结果的证明, 并将所得结论运用到偏微分方程解的研究中去.

下面, 我们将给出引理.

引理 3.1.1[20]　设 x, $f \in C(R_+, R_+)$, w 是 R_+ 上单调递增的连续函数, 且当 $u > 0$ 时, $w(u) > 0$, $c \geq 0$ 为常数, 若

$$x(t) \leq c + \int_t^{\infty} f(s) w\left(x(s)\right) ds,\ t \in R_+, \tag{3.1.1}$$

则对 $0 \leq T \leq t < \infty$, 有

$$x(t) \leq G^{-1}\left(G(c) + \int_t^{\infty} f(s) ds\right), \tag{3.1.2}$$

其中 $G(z) = \int_{z_0}^{z} \frac{ds}{w(s)}$, $z \geq z_0 > 0$, G^{-1} 是 G 的反函数, $T \in R_+$ 满足
$G(c) + \int_t^{\infty} f(s) ds \in Dom(G^{-1})$, $T \leq t < \infty$.

3.2　主要结果及证明

定理 3.2.1　$u(x, y)$, $f(x, y)$, $g(x, y) \in C(R_+^{x_0} \times R_+^{y_0}, R_+)$, 若

$$\begin{aligned} u^p(x,y) \leq\ & a(x) + b(y) + \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} \left[f(\tau,s) u^q(\tau,s) w\left(u(\tau,s)\right) + g(\tau,s) u^q(\tau,s)\right] ds d\tau \\ & + \int_x^{\infty} \int_y^{\infty} \left[f(\tau,s) u^q(\tau,s) w\left(u(\tau,s)\right) + g(\tau,s) u^q(\tau,s)\right] ds d\tau, \\ & \forall (x,y) \in R_+^{x_0} \times R_+^{y_0}, \end{aligned} \tag{3.2.1}$$

且满足

(1)　$x_0 \geq 0$, $y_0 \geq 0$, $p > q > 0$ 为常数;

(2)　$a(x) > 0$, $b(y) > 0$, $a'(x) \leq 0$, $b'(y) \leq 0$, 且 $\lim\limits_{x\to\infty} a(x)$, $\lim\limits_{y\to\infty} b(y)$ 存在;

(3)　$\alpha(x)$, $\beta(y) \in C^1(R_+, R_+)$是不减的, 且在 R_+ 上, $\alpha(x) \geq x$, $\beta(y) \geq y$;

(4) $w \in C(R_+, R_+)$ 是不减的, 且当 $x \in (0, \infty)$ 时, $w(x) > 0$;
则

$$u(x,y) \le \left\{ \Phi^{-1}\left[\Phi\left(G(x,y)\right) + \frac{p-q}{p} F(x,y) \right] \right\}^{\frac{1}{p-q}}, \ \forall (x,y) \in R_+^{x_0} \times R_+^{y_0}, \qquad (3.2.2)$$

其中
$\Phi(r) = \int_{r_0}^{r} \frac{ds}{w(s^{\frac{1}{p-q}})}$, $r > r_0 > 0$, Φ^{-1} 为其反函数,
$G(x,y) = (a(\infty) + b(y))^{\frac{p-q}{p}} + (a(x) + b(\infty))^{\frac{p-q}{p}} + (a(\infty) + b(\infty))^{\frac{p-q}{p}}$
$\quad + \frac{p-q}{p}\left[\int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} g(\tau,s) ds d\tau + \int_x^{\infty} \int_y^{\infty} g(\tau,s) ds d\tau \right]$,
$F(x,y) = \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} f(\tau,s) ds d\tau + \int_x^{\infty} \int_y^{\infty} f(\tau,s) ds d\tau.$

证明：令

$$\begin{aligned} z(x,y) = a(x) + b(y) &+ \int_{\alpha(x)}^{\infty} \int_{\beta(y)}^{\infty} \left[f(\tau,s) u^q(\tau,s) w\left(u(\tau,s)\right) + g(\tau,s) u^q(\tau,s) \right] ds d\tau \\ &+ \int_x^{\infty} \int_y^{\infty} \left[f(\tau,s) u^q(\tau,s) w\left(u(\tau,s)\right) + g(\tau,s) u^q(\tau,s) \right] ds d\tau, \end{aligned} \qquad (3.2.3)$$

则 $z(x,y)$ 非负关于 x, y 都不增, $z(x,\infty) = a(x) + b(\infty)$, $z(\infty, y) = a(\infty) + b(y)$, $\frac{\partial z(x,\infty)}{\partial x} = a'(x)$, $\frac{\partial z(\infty,y)}{\partial y} = b'(y)$, $z(x,y) \ge a(x) + b(y)$, 且

$$u(x,y) \le z^{\frac{1}{p}}(x,y), \qquad (3.2.4)$$

从而

$$u\left(\alpha(x), \beta(y)\right) \le z^{\frac{1}{p}}\left(\alpha(x), \beta(y)\right) \le z^{\frac{1}{p}}(x,y), \qquad (3.2.5)$$

对 (3.2.3) 式求二阶偏导, 由 (3.2.4) (3.2.5) 式得

$$\begin{aligned} \frac{\partial^2 z(x,y)}{\partial x \partial y} &= \alpha'(x)\beta'(y)\left[f\left(\alpha(x), \beta(y)\right) w\left(u\left(\alpha(x), \beta(y)\right)\right) + g\left(\alpha(x), \beta(y)\right) \right] u^q\left(\alpha(x), \beta(y)\right) \\ &\quad + \left[f(x,y) w\left(u(x,y)\right) + g(x,y) \right] u^q(x,y) \\ &\le \alpha'(x)\beta'(y)\left[f\left(\alpha(x), \beta(y)\right) w\left(u\left(\alpha(x), \beta(y)\right)\right) + g\left(\alpha(x), \beta(y)\right) \right] z^{\frac{q}{p}}(x,y) \\ &\quad + \left[f(x,y) w\left(u(x,y)\right) + g(x,y) \right] z^{\frac{q}{p}}(x,y), \end{aligned}$$

从而

$$\begin{aligned} \frac{\frac{\partial^2 z(x,y)}{\partial x \partial y}}{z^{\frac{q}{p}}(x,y)} &\le \alpha'(x)\beta'(y)\left[f\left(\alpha(x), \beta(y)\right) w\left(u\left(\alpha(x), \beta(y)\right)\right) + g\left(\alpha(x), \beta(y)\right) \right] \\ &\quad + \left[f(x,y) w\left(u(x,y)\right) + g(x,y) \right], \end{aligned} \qquad (3.2.6)$$

又

$$\frac{\partial}{\partial y}\left(\frac{\frac{\partial z(x,y)}{\partial x}}{z^{\frac{q}{p}}(x,y)}\right)\leq\frac{\frac{\partial^2 z(x,y)}{\partial x\partial y}}{z^{\frac{q}{p}}(x,y)},\tag{3.2.7}$$

由 (3.2.6) (3.2.7) 式

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\frac{\partial z(x,y)}{\partial x}}{z^{\frac{q}{p}}(x,y)}\right)\leq{}&\alpha'(x)\beta'(y)\left[f\left(\alpha(x),\beta(y)\right)w\left(u\left(\alpha(x),\beta(y)\right)\right)+g\left(\alpha(x),\beta(y)\right)\right]\\&+\left[f(x,y)w\left(u(x,y)\right)+g(x,y)\right],\end{aligned}\tag{3.2.8}$$

在 (3.2.8) 式两边从 y 到 ∞ 积分

$$\begin{aligned}\frac{\frac{\partial z(x,\infty)}{\partial x}}{z^{\frac{q}{p}}(x,\infty)}-\frac{\frac{\partial z(x,y)}{\partial x}}{z^{\frac{q}{p}}(x,y)}\leq{}&\alpha'(x)\int_{\beta(y)}^{\infty}\left[f\left(\alpha(x),s\right)w\left(u\left(\alpha(x),s\right)\right)+g\left(\alpha(x),s\right)\right]ds\\&+\int_y^{\infty}\left[f(x,s)w\left(u(x,s)\right)+g(x,s)\right]ds,\end{aligned}$$

从而

$$\begin{aligned}\frac{\frac{\partial z(x,y)}{\partial x}}{z^{\frac{q}{p}}(x,y)}\geq{}&\frac{a'(x)}{\left(a(x)+b(\infty)\right)^{\frac{q}{p}}}-\alpha'(x)\int_{\beta(y)}^{\infty}\left[f\left(\alpha(x),s\right)w\left(u\left(\alpha(x),s\right)\right)+g\left(\alpha(x),s\right)\right]ds\\&-\int_y^{\infty}\left[f(x,s)w\left(u(x,s)\right)+g(x,s)\right]ds,\end{aligned}\tag{3.2.9}$$

在 (3.2.9) 式两边从 x 到 ∞ 积分得

$$\begin{aligned}&\frac{p}{p-q}z^{\frac{p-q}{p}}(\infty,y)-\frac{p}{p-q}z^{\frac{p-q}{p}}(x,y)\\\geq{}&\int_x^{\infty}\frac{a'(\tau)}{\left(a(\tau)+b(\infty)\right)^{\frac{q}{p}}}d\tau-\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\left[f(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)\right]dsd\tau\\&-\int_x^{\infty}\int_y^{\infty}\left[f(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)\right]dsd\tau\\={}&\frac{p}{p-q}\left(a(\infty)+b(\infty)\right)^{\frac{p}{p-q}}-\frac{p}{p-q}\left(a(x)+b(\infty)\right)^{\frac{p}{p-q}}\\&-\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\left[f(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)\right]dsd\tau\\&-\int_x^{\infty}\int_y^{\infty}\left[f(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)\right]dsd\tau,\end{aligned}$$

从而

$$
\begin{aligned}
& z^{\frac{p-q}{p}}(x,y) \\
\le & (a(\infty)+b(y))^{\frac{p}{p-q}}+(a(x)+b(\infty))^{\frac{p}{p-q}}-(a(\infty)+b(\infty))^{\frac{p}{p-q}} \\
& +\frac{p-q}{p}\left[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} g(\tau,s)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty} g(\tau,s)dsd\tau\right] \\
& +\frac{p-q}{p}\left[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(\tau,s)w\left(u(\tau,s)\right)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty} f(\tau,s)w\left(u(\tau,s)\right)dsd\tau\right] \\
= & G(x,y)+\frac{p-q}{p}\left[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(\tau,s)w\left(u(\tau,s)\right)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty} f(\tau,s)w\left(u(\tau,s)\right)dsd\tau\right],
\end{aligned}
$$

由于 $G(x,y)$ 非负关于 x, y 都不增, 从而 $\forall(x,y)\in R_{+}^{x_0}\times R_{+}^{y_0}$, $\exists(X,Y)\in R_{+}^{x_0}\times R_{+}^{y_0}$ $(X\le x, Y\le y)$, 使得 $G(x,y)\le G(X,Y)$, 从而当 $(x,y)\in[X,\infty)\times[Y,\infty)$ 时

$$
\begin{aligned}
& z^{\frac{p-q}{p}}(x,y) \\
\le & G(X,Y)+\frac{p-q}{p}\left[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(\tau,s)w\left(u(\tau,s)\right)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty} f(\tau,s)w\left(u(\tau,s)\right)dsd\tau\right],
\end{aligned}
$$

令

$$
\begin{aligned}
& v(x,y) \\
= & G(X,Y)+\frac{p-q}{p}\left[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} f(\tau,s)w\left(u(\tau,s)\right)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty} f(\tau,s)w\left(u(\tau,s)\right)dsd\tau\right],
\end{aligned}
\tag{3.2.10}
$$

则 $v(x,y)$ 非负关于 x, y 都不增, $v(\infty,y)=G(X,Y)$, 且

$$
z(x,y)\le v^{\frac{p}{p-q}}(x,y),\ u(x,y)\le v^{\frac{1}{p}}(x,y)\le v^{\frac{1}{p-q}}(x,y),
\tag{3.2.11}
$$

对 (3.2.10) 式关于 x 求偏导, 由 (3.2.5) (3.2.11) 式得

$$
\begin{aligned}
\frac{\partial v(x,y)}{\partial x} &= \frac{p-q}{p}\left[-\alpha'(x)\int_{\beta(y)}^{\infty} f\left(\alpha(x),s\right)w\left(u\left(\alpha(x),s\right)\right)ds-\int_{y}^{\infty} f(x,s)w\left(u(x,s)\right)ds\right] \\
&\ge \frac{p-q}{p}\left[-\alpha'(x)\int_{\beta(y)}^{\infty} f\left(\alpha(x),s\right)ds-\int_{y}^{\infty} f(x,s)ds\right]w\left(v^{\frac{1}{p-q}}(x,y)\right),
\end{aligned}
$$

从而

$$
\frac{\frac{\partial v(x,y)}{\partial x}}{w\left(v^{\frac{1}{p-q}}(x,y)\right)}\ge\frac{p-q}{p}\left[-\alpha'(x)\int_{\beta(y)}^{\infty} f\left(\alpha(x),s\right)ds-\int_{y}^{\infty} f(x,s)ds\right],
\tag{3.2.12}
$$

在 (3.2.12) 式两边从 x 到 ∞ 积分得

$$\Phi\left(v(\infty,y)\right)-\Phi\left(v(x,y)\right)\geq\frac{p-q}{p}\left[-\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}f(\tau,s)dsd\tau-\int_{x}^{\infty}\int_{y}^{\infty}f(\tau,s)dsd\tau\right],$$

从而

$$\begin{aligned}v(x,y)&\leq\Phi^{-1}\left\{\Phi\left(G(X,Y)\right)+\frac{p-q}{p}\left[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}f(\tau,s)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty}f(\tau,s)dsd\tau\right]\right\}\\&=\Phi^{-1}\left[\Phi\left(G(X,Y)\right)+\frac{p-q}{p}F(x,y)\right],\end{aligned}$$

令 $x=X,\ y=Y$, 则

$$v(X,Y)\leq\Phi^{-1}\left[\Phi\left(G(X,Y)\right)+\frac{p-q}{p}F(X,Y)\right],$$

由 $X,\ Y$ 的任意性可得

$$v(x,y)\leq\Phi^{-1}\left[\Phi\left(G(x,y)\right)+\frac{p-q}{p}F(x,y)\right],\ \forall(x,y)\in R_+^{x_0}\times R_+^{y_0},\tag{3.2.13}$$

从而可证 $u(x,y)$ 满足 (3.2.2) 式.

定理 3.2.2　$u(x,y),\ f(x,y),\ g(x,y)\in C(R_+^{x_0}\times R_+^{y_0},R_+)$, 若

$$\begin{aligned}\varphi\left(u(x,y)\right)\leq a(x)+b(y)&+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\left[f(\tau,s)u(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)u(\tau,s)\right]dsd\tau\\&+\int_{x}^{\infty}\int_{y}^{\infty}\left[f(\tau,s)u(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)u(\tau,s)\right]dsd\tau,\\&\forall(x,y)\in R_+^{x_0}\times R_+^{y_0},\end{aligned}\tag{3.2.14}$$

且满足

(1) $x_0\geq0,\ y_0\geq0$ 为常数, $a(x)>0,\ b(y)>0,\ a'(x)\leq0,\ b'(y)\leq0$, 且 $\lim\limits_{x\to\infty}a(x),\ \lim\limits_{y\to\infty}b(y)$ 存在;

(2) $\alpha(x),\ \beta(y)\in C^1(R_+,R_+)$是不减的, 且在 R_+ 上, $\alpha(x)\geq x,\ \beta(y)\geq y$;

(3) $\varphi\in C(R_+,R_+)$ 是不减的, 且当 $x\in(0,\infty)$ 时, $\varphi(x)>0$;

(4) $w\in C(R_+,R_+)$ 是不减的, 且当 $x\in(0,\infty)$ 时, $w(x)>0$;

则

$$u(x,y)\leq\varphi^{-1}\left\{\Phi^{-1}\left\{\Psi^{-1}\left[\Psi\left(G(x,y)\right)+F(x,y)\right]\right\}\right\},\ \forall(x,y)\in R_+^{x_0}\times R_+^{y_0},\tag{3.2.15}$$

其中

$\Phi(r)=\int_{r_0}^{r}\frac{ds}{\varphi^{-1}(s)},\ r>r_0>0,\ \Psi(z)=\int_{z_0}^{z}\frac{ds}{w\{\varphi^{-1}[\Phi^{-1}(s)]\}},\ z>z_0>0,\ \Phi^{-1},\ \Psi^{-1}$ 分别为其反函数,

$G(x,y)=\Phi\left(a(\infty)+b(y)\right)-\int_{x}^{\infty}\frac{a'(\tau)}{\varphi^{-1}(a(\tau)+b(\infty))}d\tau+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}g(\tau,s)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty}g(\tau,s)dsd\tau,$

$F(x,y)=\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}f(\tau,s)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty}f(\tau,s)dsd\tau.$

证明：令

$$\begin{aligned}z(x,y)=a(x)+b(y)&+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\left[f(\tau,s)u(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)u(\tau,s)\right]dsd\tau\\&+\int_{x}^{\infty}\int_{y}^{\infty}\left[f(\tau,s)u(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)u(\tau,s)\right]dsd\tau,\end{aligned}\tag{3.2.16}$$

则 $z(x,y)$ 非负关于 x, y 都不增, $z(x,\infty)=a(x)+b(\infty)$, $z(\infty,y)=a(\infty)+b(y)$, $\frac{\partial z(x,\infty)}{\partial x}=a'(x)$, $\frac{\partial z(\infty,y)}{\partial y}=b'(y)$, $z(x,y)\geq a(x)+b(y)$, 且

$$u(x,y)\leq\varphi^{-1}\left(z(x,y)\right),\tag{3.2.17}$$

从而

$$u\left(\alpha(x),\beta(y)\right)\leq\varphi^{-1}\left(z\left(\alpha(x),\beta(y)\right)\right)\leq\varphi^{-1}\left(z(x,y)\right),\tag{3.2.18}$$

对 (3.2.16) 式求二阶偏导, 由 (3.2.17) (3.2.18) 式得

$$\begin{aligned}\frac{\partial^2 z(x,y)}{\partial x\partial y}&=\alpha'(x)\beta'(y)\left[f\left(\alpha(x),\beta(y)\right)w\left(u\left(\alpha(x),\beta(y)\right)\right)+g\left(\alpha(x),\beta(y)\right)\right]u\left(\alpha(x),\beta(y)\right)\\&\quad+\left[f(x,y)w\left(u(x,y)\right)+g(x,y)\right]u(x,y)\\&\leq\alpha'(x)\beta'(y)\left[f\left(\alpha(x),\beta(y)\right)w\left(u\left(\alpha(x),\beta(y)\right)\right)+g\left(\alpha(x),\beta(y)\right)\right]\varphi^{-1}\left(z(x,y)\right)\\&\quad+\left[f(x,y)w\left(u(x,y)\right)+g(x,y)\right]\varphi^{-1}\left(z(x,y)\right),\end{aligned}$$

从而

$$\begin{aligned}\frac{\frac{\partial^2 z(x,y)}{\partial x\partial y}}{\varphi^{-1}\left(z(x,y)\right)}\leq\alpha'(x)\beta'(y)&\left[f\left(\alpha(x),\beta(y)\right)w\left(u\left(\alpha(x),\beta(y)\right)\right)+g\left(\alpha(x),\beta(y)\right)\right]\\&+\left[f(x,y)w\left(u(x,y)\right)+g(x,y)\right],\end{aligned}\tag{3.2.19}$$

又

$$\frac{\partial}{\partial y}\left(\frac{\frac{\partial z(x,y)}{\partial x}}{\varphi^{-1}\left(z(x,y)\right)}\right)\leq\frac{\frac{\partial^2 z(x,y)}{\partial x\partial y}}{\varphi^{-1}\left(z(x,y)\right)},\tag{3.2.20}$$

由 (3.2.19) (3.2.20) 式

$$\frac{\partial}{\partial y}\left(\frac{\frac{\partial z(x,y)}{\partial x}}{\varphi^{-1}\left(z(x,y)\right)}\right) \leq \alpha'(x)\beta'(y)\left[f\left(\alpha(x),\beta(y)\right)w\left(u\left(\alpha(x),\beta(y)\right)\right)+g\left(\alpha(x),\beta(y)\right)\right] + \left[f(x,y)w\left(u(x,y)\right)+g(x,y)\right], \tag{3.2.21}$$

在 (3.2.21) 式两边从 y 到 ∞ 积分

$$\frac{\frac{\partial z(x,\infty)}{\partial x}}{\varphi^{-1}\left(z(x,\infty)\right)} - \frac{\frac{\partial z(x,y)}{\partial x}}{\varphi^{-1}\left(z(x,y)\right)} \leq \alpha'(x)\int_{\beta(y)}^{\infty}\left[f\left(\alpha(x),s\right)w\left(u\left(\alpha(x),s\right)\right)+g\left(\alpha(x),s\right)\right]ds + \int_{y}^{\infty}\left[f(x,s)w\left(u(x,s)\right)+g(x,s)\right]ds,$$

从而

$$\frac{\frac{\partial z(x,y)}{\partial x}}{\varphi^{-1}\left(z(x,y)\right)} \geq \frac{a'(x)}{\varphi^{-1}\left(a(x)+b(\infty)\right)} - \alpha'(x)\int_{\beta(y)}^{\infty}\left[f\left(\alpha(x),s\right)w\left(u\left(\alpha(x),s\right)\right)+g\left(\alpha(x),s\right)\right]ds - \int_{y}^{\infty}\left[f(x,s)w\left(u(x,s)\right)+g(x,s)\right]ds, \tag{3.2.22}$$

在 (3.2.22) 式两边从 x 到 ∞ 积分得

$$\begin{aligned}&\Phi\left(z(\infty,y)\right)-\Phi\left(z(x,y)\right)\\ \geq&\int_{x}^{\infty}\frac{a'(\tau)}{\varphi^{-1}\left(a(\tau)+b(\infty)\right)}d\tau-\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}\left[f(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)\right]dsd\tau\\ &-\int_{x}^{\infty}\int_{y}^{\infty}\left[f(\tau,s)w\left(u(\tau,s)\right)+g(\tau,s)\right]dsd\tau,\end{aligned}$$

从而

$$\begin{aligned}z(x,y) \leq{}& \Phi^{-1}\Bigg[\Phi\left(a(\infty)+b(y)\right)-\int_{x}^{\infty}\frac{a'(\tau)}{\varphi^{-1}\left(a(\tau)+b(\infty)\right)}d\tau\\ &+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}g(\tau,s)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty}g(\tau,s)dsd\tau\\ &+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}f(\tau,s)w\Bigg(u(\tau,s)\Bigg)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty}f(\tau,s)w\left(u(\tau,s)\right)dsd\tau\Bigg]\\ ={}&\Phi^{-1}\left[G(x,y)+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}f(\tau,s)w\left(u(\tau,s)\right)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty}f(\tau,s)w\left(u(\tau,s)\right)dsd\tau\right],\end{aligned}$$

由于 $G(x,y)$ 非负关于 x, y 不增, 从而 $\forall(x,y)\in R_+^{x_0}\times R_+^{y_0}$, $\exists(X,Y)\in R_+^{x_0}\times R_+^{y_0}$ $(X\le x, Y\le y)$, 使得 $G(x,y)\le G(X,Y)$, 从而当 $(x,y)\in[X,\infty)\times[Y,\infty)$ 时

$$z(x,y)\le\Phi^{-1}\Bigg[G(X,Y)+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}f(\tau,s)w\left(u(\tau,s)\right)dsd\tau + \int_{x}^{\infty}\int_{y}^{\infty}f(\tau,s)w\left(u(\tau,s)\right)dsd\tau\Bigg],$$

令

$$v(x,y)=G(X,Y)+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}f(\tau,s)w\left(u(\tau,s)\right)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty}f(\tau,s)w\left(u(\tau,s)\right)dsd\tau, \tag{3.2.23}$$

则 $v(x,y)$ 非负关于 x, y 都不增, $v(\infty,y)=G(X,Y)$, 且

$$z(x,y)\le\Phi^{-1}\left(v(x,y)\right),\ u(x,y)\le\varphi^{-1}\left(z(x,y)\right)\le\varphi^{-1}\left[\Phi^{-1}\left(v(x,y)\right)\right], \tag{3.2.24}$$

对 (3.2.23) 式关于 x 求偏导, 由 (3.2.18) (3.2.24) 式得

$$\begin{aligned}\frac{\partial v(x,y)}{\partial x}&=-\alpha'(x)\int_{\beta(y)}^{\infty}f\left(\alpha(x),s\right)w\left(u\left(\alpha(x),s\right)\right)ds-\int_{y}^{\infty}f(x,s)w\left(u(x,s)\right)ds\\&\ge\left[-\alpha'(x)\int_{\beta(y)}^{\infty}f\left(\alpha(x),s\right)ds-\int_{y}^{\infty}f(x,s)ds\right]w\left\{\varphi^{-1}\left[\Phi^{-1}\left(v(x,y)\right)\right]\right\},\end{aligned}$$

从而

$$\frac{\frac{\partial v(x,y)}{\partial x}}{w\left\{\varphi^{-1}\left[\Phi^{-1}\left(v(x,y)\right)\right]\right\}}\ge-\alpha'(x)\int_{\beta(y)}^{\infty}f\left(\alpha(x),s\right)ds-\int_{y}^{\infty}f(x,s)ds, \tag{3.2.25}$$

在 (3.2.25) 式两边从 x 到 ∞ 积分得

$$\Psi\left(v(\infty,y)\right)-\Psi\left(v(x,y)\right)\ge-\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}f(\tau,s)dsd\tau-\int_{x}^{\infty}\int_{y}^{\infty}f(\tau,s)dsd\tau,$$

从而

$$\begin{aligned}v(x,y)&\le\Psi^{-1}\left[\Psi\left(G(X,Y)\right)+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}f(\tau,s)dsd\tau+\int_{x}^{\infty}\int_{y}^{\infty}f(\tau,s)dsd\tau\right]\\&=\Psi^{-1}\left[\Psi\left(G(X,Y)\right)+F(x,y)\right],\end{aligned}$$

令 $x=X$, $y=Y$, 则

$$v(X,Y)\le\Psi^{-1}\left[\Psi\left(G(X,Y)\right)+F(X,Y)\right],$$

由 X, Y 的任意性可得

$$v(x,y)\le\Psi^{-1}\left[\Psi\left(G(x,y)\right)+F(x,y)\right],\ \forall(x,y)\in R_+^{x_0}\times R_+^{y_0},\tag{3.2.26}$$

从而可证 $u(x,y)$ 满足 (3.2.15) 式.

3.3　应用

考虑偏微分方程

$$\begin{cases} \quad pu^{p-1}(x,y)\frac{\partial^2 u(x,y)}{\partial x\partial y}+p(p-1)u^{p-2}(x,y)\frac{\partial u(x,y)}{\partial x}\frac{\partial u(x,y)}{\partial y} \\ =\alpha'(x)\beta'(y)h\left(\alpha(x),\beta(y),u(\alpha,\beta)\right)+h(x,y,u), \\ \quad u(x,\infty)=m(x),\ u(\infty,y)=n(y),\ u(\infty,\infty)=0, \end{cases}\tag{3.3.1}$$

其中 $u(x,y)\in C(R_+^{x_0}\times R_+^{y_0},R)$, $p>1$ 为常数, $\alpha(x)$, $\beta(y)\in C^1(R_+,R_+)$ 是不减的, 且在 R_+ 上, $\alpha(x)\ge x$, $\beta(y)\ge y$, $h(x,y,u)\in C(R_+^2\times R,R)$, $m(x)\in C(R_+^{x_0},R)$, $n(y)\in C(R_+^{y_0},R)$.

定理 3.3.1　$u(x,y)$ 为方程 (3.3.1)　的一个解, 若

$$|h(x,y,u)|\le f(x,y)|u|w(|u|)+g(x,y)|u|,\tag{3.3.2}$$

其中 $f(x,y),g(x,y)\in C(R_+^2,R_+)$, $w\in C(R_+^2,R_+)\in$ 是不减的, 且当 $x\in(0,\infty)$ 时, $w(x)>0$, 则

$$|u(x,y)|\le\left\{\Phi^{-1}\left[\Phi\left(G(x,y)\right)+\frac{p-1}{p}F(x,y)\right]\right\}^{\frac{1}{p-1}},\ \forall(x,y)\in R_+^{x_0}\times R_+^{y_0},\tag{3.3.3}$$

其中

$\Phi(r)=\int_{r_0}^{r}\frac{ds}{w(s^{\frac{1}{p-1}})}$, $r>r_0>0$, Φ^{-1} 为其反函数,

$G(x,y)=a^{\frac{p-1}{p}}(x)+b^{\frac{p-1}{p}}(y)+\frac{p-1}{p}\left[\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}g(\tau,s)dsd\tau+\int_x^{\infty}\int_y^{\infty}g(\tau,s)dsd\tau\right]$,

$a(x)=|m(x)|^p$, $b(y)=|n(y)|^p$,

$F(x,y)=\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}f(\tau,s)dsd\tau+\int_x^{\infty}\int_y^{\infty}f(\tau,s)dsd\tau$.

证明: 偏微分方程 (3.3.1) 的积分方程是

$$u^p(x,y)=m^p(x)+n^p(y)+\int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty}h\left(\tau,s,u(\tau,s)\right)dsd\tau+\int_x^{\infty}\int_y^{\infty}h\left(\tau,s,u(\tau,s)\right)dsd\tau,$$

由 (3.3.2) 式知

$$
\begin{aligned}
|u(x,y)|^p \le |m(x)|^p + |n(y)|^p &+ \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} \left[f(\tau,s)|u(\tau,s)|w\left(|u(\tau,s)|\right) + g(\tau,s)|u(\tau,s)|\right] ds d\tau \\
&+ \int_{x}^{\infty}\int_{y}^{\infty} \left[f(\tau,s)|u(\tau,s)|w\left(|u(\tau,s)|\right) + g(\tau,s)|u(\tau,s)|\right] ds d\tau \\
= a(x) + b(y) &+ \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} \left[f(\tau,s)|u(\tau,s)|w\left(|u(\tau,s)|\right) + g(\tau,s)|u(\tau,s)|\right] ds d\tau \\
&+ \int_{x}^{\infty}\int_{y}^{\infty} \left[f(\tau,s)|u(\tau,s)|w\left(|u(\tau,s)|\right) + g(\tau,s)|u(\tau,s)|\right] ds d\tau,
\end{aligned}
$$

令 $v(x,y) = |u(x,y)|$, 则 $v(x,y) \in C(R_+^2, R_+)$, 从而

$$
\begin{aligned}
v^p(x,y) \le a(x) + b(y) &+ \int_{\alpha(x)}^{\infty}\int_{\beta(y)}^{\infty} \left[f(\tau,s)v(\tau,s)w\left(v(\tau,s)\right) + g(\tau,s)v(\tau,s)\right] ds d\tau \\
&+ \int_{x}^{\infty}\int_{y}^{\infty} \left[f(\tau,s)v(\tau,s)w\left(v(\tau,s)\right) + g(\tau,s)v(\tau,s)\right] ds d\tau, \qquad (3.3.4)
\end{aligned}
$$

这就是定理 3.2.1 $q=1$ 的情形, 从而

$$
v(x,y) \le \left\{\Phi^{-1}\left[\Phi\left(G(x,y)\right) + \frac{p-1}{p}F(x,y)\right]\right\}^{\frac{1}{p-1}}, \qquad (3.3.5)
$$

从而可证 $u(x,y)$ 满足 (3.3.3) 式.

第四章　一类非线性 Volterra-Fredholm 型时滞积分不等式

4.1　引言

Volterra-Fredholm 积分不等式在微分方程的研究中有着重要的作用, 许多学者也已经建立了一些 Volterra-Fredholm 型时滞积分不等式. 本章主要是在文献 [30] 和 [32] 的基础上, 建立了一类新的两个变量的 Volterra-Fredholm 型时滞积分分不等式, 运用微分方程研究中的一些数学方法, 给出了不等式中未知函数的估计. 最后, 还给出了相关应用.

4.2　主要结果及证明

定理 4.2.1　$u(x,y)\in C(R_+^2,R_+)$, 若 $\forall(x,y)\in[x_0,X]\times[y_0,Y]$, 有

$$\begin{aligned}&u(x,y)\\&\le k+q(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)\varphi\left(u(s,t)\right)\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds\\&\quad+q(x,y)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)\varphi\left(u(s,t)\right)\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds,\end{aligned}\tag{4.2.1}$$

且满足

(1)　$x_0\ge0,\ y_0\ge0,\ k>0$ 为常数, $q(x,y)\in C(R_+^2,R_+)$　关于 $x,\ y$ 都是不减的;

(2)　$f(x,y),\ g(x,y)\in C(R_+^2,R_+)$;

(3)　$\alpha(x),\ \beta(y)\in C^1(R_+,R_+)$ 为非减函数, 且 $\alpha(x)\le x,\ \beta(y)\le y$;

(4)　$\varphi(u),\ \psi(u)\in C(R_+,R_+),\ \varphi(u),\ \psi(u)$ 和 $\frac{\varphi(u)}{\psi(u)}$ 都是非减函数, 且当 $u>0$ 时, $\varphi(u)>0,\ \psi(u)>0$;

则

$$\begin{aligned}u(x,y)\le\Psi^{-1}\Bigg\{\Phi^{-1}\Bigg[&\Phi\Bigg(\Psi\Bigg(H^{-1}\Bigg(q(X,Y)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)dtds\Bigg)\Bigg)\\&+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}g(s,t)dtds\Bigg)+q(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)dtds\Bigg]\Bigg\},\\&\forall(x,y)\in[x_0,X]\times[y_0,Y],\end{aligned}\tag{4.2.2}$$

其中

$\Psi(r)=\int_{r_0}^{r}\frac{ds}{\psi(s)},\ r>r_0>0,\ \Phi(m)=\int_{m_0}^{m}\frac{\psi\left(\Psi^{-1}(n)\right)dm}{\varphi(\Psi^{-1}(m))\Psi^{-1}(m)},\ m>m_0>0,$

$\Psi^{-1},\ \Phi^{-1}$ 分别为其反函数,

$H(u)=\Phi\left[\Psi(2u-k)\right]-\Phi\left[\Psi(u)+\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}g(s,t)dtds\right]$, $H(u)$ 在 $[k,\infty)$ 上关于 u 是严格增的, H^{-1} 为其反函数.

证明: $\forall(M,N)\in R_+^2$ 满足 $x_0\le M\le X,\ y_0\le N\le Y$, 当 $(x,y)\in[x_0,M]\times[y_0,N]$ 时, 由 (4.2.1) 式知

$$\begin{aligned}u(x,y)\le k&+q(M,N)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)\varphi\left(u(s,t)\right)\\&\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds\\&+q(M,N)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)\varphi\left(u(s,t)\right)\\&\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds,\end{aligned}$$

令

$$\begin{aligned}z(x,y)=k&+q(M,N)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)\varphi\left(u(s,t)\right)\\&\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds\\&+q(M,N)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)\varphi\left(u(s,t)\right)\\&\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds,\end{aligned}\tag{4.2.3}$$

则 $z(x,y)$ 在 $[x_0,M]\times[y_0,N]$ 上为正的关于 $x,\ y$ 都不减,

$$\begin{aligned}&z(x_0,y)=z(x,y_0)=z(x_0,y_0)\\&=k+q(M,N)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)\varphi\left(u(s,t)\right)\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds,\end{aligned}\tag{4.2.4}$$

且

$$u(x,y)\le z(x,y),\tag{4.2.5}$$

对 (4.2.3) 式求二阶偏导, 由 (4.2.5) 式得

$$\begin{aligned}\frac{\partial^2 z(x,y)}{\partial x\partial y} &= \alpha'(x)\beta'(y)q(M,N)f(\alpha(x),\beta(y))\varphi(u(\alpha(x),\beta(y)))\\&\quad\left[u(\alpha(x),\beta(y))+\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}g(s,t)\psi(u(s,t))\,dtds\right]\\&\le \alpha'(x)\beta'(y)q(M,N)f(\alpha(x),\beta(y))\varphi(z(\alpha(x),\beta(y)))\\&\quad\left[z(\alpha(x),\beta(y))+\int_{x_0}^{\alpha(x)}\int_{y_0}^{\beta(y)}g(s,t)\psi(z(s,t))\,dtds\right]\\&\le \alpha'(x)\beta'(y)q(M,N)f(\alpha(x),\beta(y))\varphi(z(\alpha(x),\beta(y)))\\&\quad\left[z(x,y)+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}g(s,t)\psi(z(s,t))\,dtds\right],\end{aligned}\tag{4.2.6}$$

且 $\frac{\partial^2 z(x,y)}{\partial x\partial y}\ge 0$, 令

$$z_1(x,y)=z(x,y)+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}g(s,t)\psi(z(s,t))\,dtds,\tag{4.2.7}$$

则 $z_1(x,y)$ 在 $[x_0,M]\times[y_0,N]$ 上为正的关于 x, y 都不减,

$$\begin{aligned}&z_1(x_0,y)=z_1(x,y_0)=z_1(x_0,y_0)=z(x_0,y)=z(x,y_0)=z(x_0,y_0)\\&=k+q(M,N)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)\varphi(u(s,t))\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}g(\xi,\eta)\psi(u(\xi,\eta))\,d\eta d\xi\right]dtds,\end{aligned}\tag{4.2.8}$$

$$z(x,y)\le z_1(x,y),\tag{4.2.9}$$

且 $\frac{\partial z_1(x,y_0)}{\partial x}=0$, 对 (4.2.7) 式求二阶偏导, 由 (4.2.9) 式得

$$\begin{aligned}\frac{\partial^2 z_1(x,y)}{\partial x\partial y}&=\frac{\partial^2 z(x,y)}{\partial x\partial y}+\alpha'(x)\beta'(y)g(\alpha(x),\beta(y))\psi(z(\alpha(x),\beta(y)))\\&\le \alpha'(x)\beta'(y)q(M,N)f(\alpha(x),\beta(y))\varphi(z_1(\alpha(x),\beta(y)))z_1(x,y)\\&\quad+\alpha'(x)\beta'(y)g(\alpha(x),\beta(y))\psi(z_1(\alpha(x),\beta(y))),\end{aligned}\tag{4.2.10}$$

不等式 (4.2.10) 两边同除以 $\psi\left(z_1\left(\alpha(x), \beta(y)\right)\right)$, 由 ψ 和 z_1 的单调性知

$$\begin{aligned}
&\frac{\frac{\partial^2 z_1(x,y)}{\partial x \partial y}}{\psi\left(z_1(x,y)\right)} \leq \frac{\frac{\partial^2 z_1(x,y)}{\partial x \partial y}}{\psi\left(z_1\left(\alpha(x), \beta(y)\right)\right)} \\
&\leq \alpha'(x)\beta'(y)\left[q(M,N) f\left(\alpha(x), \beta(y)\right) \frac{\varphi\left(z_1\left(\alpha(x), \beta(y)\right)\right)}{\psi\left(z_1\left(\alpha(x), \beta(y)\right)\right)} z_1(x,y) + g\left(\alpha(x), \beta(y)\right)\right],
\end{aligned} \tag{4.2.11}$$

又

$$\frac{\partial}{\partial y}\left(\frac{\frac{\partial z_1(x,y)}{\partial x}}{\psi\left(z_1(x,y)\right)}\right) \leq \frac{\frac{\partial^2 z_1(x,y)}{\partial x \partial y}}{\psi\left(z_1(x,y)\right)}, \tag{4.2.12}$$

由 (4.2.11) (4.2.12) 式知

$$\begin{aligned}
&\frac{\partial}{\partial y}\left(\frac{\frac{\partial z_1(x,y)}{\partial x}}{\psi\left(z_1(x,y)\right)}\right) \\
&\leq \alpha'(x)\beta'(y)\left[q(M,N) f\left(\alpha(x), \beta(y)\right) \frac{\varphi\left(z_1\left(\alpha(x), \beta(y)\right)\right)}{\psi\left(z_1\left(\alpha(x), \beta(y)\right)\right)} z_1(x,y) + g\left(\alpha(x), \beta(y)\right)\right],
\end{aligned} \tag{4.2.13}$$

在 (4.2.13) 式两边从 y_0 到 y 积分得

$$\begin{aligned}
&\frac{\frac{\partial z_1(x,y)}{\partial x}}{\psi\left(z_1(x,y)\right)} - \frac{\frac{\partial z_1(x,y_0)}{\partial x}}{\psi\left(z_1(x,y_0)\right)} \\
&\leq \int_{\beta(y_0)}^{\beta(y)} \alpha'(x)\left[q(M,N) f\left(\alpha(x), t\right) \frac{\varphi\left(z_1\left(\alpha(x), t\right)\right)}{\psi\left(z_1\left(\alpha(x), t\right)\right)} z_1(x, \beta^{-1}(t)) + g\left(\alpha(x), t\right)\right] dt,
\end{aligned}$$

从而

$$\frac{\frac{\partial z_1(x,y)}{\partial x}}{\psi\left(z_1(x,y)\right)} \leq \int_{\beta(y_0)}^{\beta(y)} \alpha'(x)\left[q(M,N) f\left(\alpha(x), t\right) \frac{\varphi\left(z_1\left(\alpha(x), t\right)\right)}{\psi\left(z_1\left(\alpha(x), t\right)\right)} z_1\left(x, \beta^{-1}(t)\right) + g\left(\alpha(x), t\right)\right] dt, \tag{4.2.14}$$

在 (4.2.14) 式两边从 x_0 到 x 积分得

$$\begin{aligned}
&\Psi\left(z_1(x,y)\right) - \Psi\left(z_1(x_0,y)\right) \\
&\leq \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} \left[q(M,N) f(s,t) \frac{\varphi\left(z_1(s,t)\right)}{\psi\left(z_1(s,t)\right)} z_1\left(\alpha^{-1}(s), \beta^{-1}(t)\right) + g(s,t)\right] dt ds,
\end{aligned}$$

从而

$$
\begin{aligned}
&\Psi\left(z_1(x,y)\right)\\
\le&\Psi\left(z_1(x_0,y)\right)+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}\left[q(M,N)f(s,t)\frac{\varphi\left(z_1(s,t)\right)}{\psi\left(z_1(s,t)\right)}z_1\left(\alpha^{-1}(s),\beta^{-1}(t)\right)+g(s,t)\right]dtds\\
\le&\Psi\left(z_1(x_0,y_0)\right)+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)}g(s,t)dtds\\
&+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}q(M,N)f(s,t)\frac{\varphi\left(z_1(s,t)\right)}{\psi\left(z_1(s,t)\right)}z_1\left(\alpha^{-1}(s),\beta^{-1}(t)\right)dtds,
\end{aligned}
$$

令

$$
\begin{aligned}
z_2(x,y)=&\Psi\left(z_1(x_0,y_0)\right)+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)}g(s,t)dtds\\
&+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}q(M,N)f(s,t)\frac{\varphi\left(z_1(s,t)\right)}{\psi\left(z_1(s,t)\right)}z_1\left(\alpha^{-1}(s),\beta^{-1}(t)\right)dtds,
\end{aligned}\tag{4.2.15}
$$

则 $z_2(x,y)$ 在 $[x_0,M]\times[y_0,N]$ 上为正的关于 x, y 都不减,

$$
z_2(x_0,y)=z_2(x,y_0)=z_2(x_0,y_0)=\Psi\left(z_1(x_0,y_0)\right)+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)}g(s,t)dtds,\tag{4.2.16}
$$

$$
z_1(x,y)\le\Psi^{-1}\left(z_2(x,y)\right),\tag{4.2.17}
$$

且 $\frac{\partial z_2(x,y_0)}{\partial x}=0$, 对 (4.2.15) 式求二阶偏导, 由 (4.2.17) 式及 $\frac{\varphi}{\psi}$ 的单调性知

$$
\begin{aligned}
\frac{\partial^2 z_2(x,y)}{\partial x\partial y}=&\alpha'(x)\beta'(y)q(M,N)f\left(\alpha(x),\beta(y)\right)\frac{\varphi\left(z_1\left(\alpha(x),\beta(y)\right)\right)}{\psi\left(z_1\left(\alpha(x),\beta(y)\right)\right)}z_1(x,y)\\
\le&\alpha'(x)\beta'(y)q(M,N)f\left(\alpha(x),\beta(y)\right)\frac{\varphi\left(\Psi^{-1}\left(z_2\left(\alpha(x),\beta(y)\right)\right)\right)}{\psi\left(\Psi^{-1}\left(z_2\left(\alpha(x),\beta(y)\right)\right)\right)}\Psi^{-1}\left(z_2(x,y)\right),
\end{aligned}
$$

从而由 $\psi(\Psi^{-1})$ 与 $\varphi(\Psi^{-1})$ 的单调性可知

$$
\begin{aligned}
\frac{\psi\left(\Psi^{-1}\left(z_2(x,y)\right)\right)\frac{\partial^2 z_2(x,y)}{\partial x\partial y}}{\varphi\left(\Psi^{-1}\left(z_2(x,y)\right)\right)\Psi^{-1}\left(z_2(x,y)\right)}\le&\frac{\psi\left(\Psi^{-1}\left(z_2\left(\alpha(x),\beta(y)\right)\right)\right)\frac{\partial^2 z_2(x,y)}{\partial x\partial y}}{\varphi\left(\Psi^{-1}\left(z_2\left(\alpha(x),\beta(y)\right)\right)\right)\Psi^{-1}\left(z_2(x,y)\right)}\\
\le&\alpha'(x)\beta'(y)q(M,N)f\left(\alpha(x),\beta(y)\right),
\end{aligned}\tag{4.2.18}
$$

又

$$
\frac{\partial}{\partial y}\left(\frac{\psi\left(\Psi^{-1}\left(z_2(x,y)\right)\right)\frac{\partial z_2(x,y)}{\partial x}}{\varphi\left(\Psi^{-1}\left(z_2(x,y)\right)\right)\Psi^{-1}\left(z_2(x,y)\right)}\right)\le\frac{\psi\left(\Psi^{-1}\left(z_2(x,y)\right)\right)\frac{\partial^2 z_2(x,y)}{\partial x\partial y}}{\varphi\left(\Psi^{-1}\left(z_2(x,y)\right)\right)\Psi^{-1}\left(z_2(x,y)\right)},\tag{4.2.19}
$$

由 (4.2.18) (4.2.19) 式得

$$\frac{\partial}{\partial y}\left(\frac{\psi\left(\Psi^{-1}\left(z_2(x,y)\right)\right)\frac{\partial z_2(x,y)}{\partial x}}{\varphi\left(\Psi^{-1}\left(z_2(x,y)\right)\right)\Psi^{-1}\left(z_2(x,y)\right)}\right)\le \alpha'(x)\beta'(y)q(M,N)f\left(\alpha(x),\beta(y)\right), \quad (4.2.20)$$

在 (4.2.20) 式两边从 y_0 到 y 积分得

$$\begin{aligned}&\frac{\psi\left(\Psi^{-1}\left(z_2(x,y)\right)\right)\frac{\partial z_2(x,y)}{\partial x}}{\varphi\left(\Psi^{-1}\left(z_2(x,y)\right)\right)\Psi^{-1}\left(z_2(x,y)\right)}-\frac{\psi\left(\Psi^{-1}\left(z_2(x,y_0)\right)\right)\frac{\partial z_2(x,y_0)}{\partial x}}{\varphi\left(\Psi^{-1}\left(z_2(x,y_0)\right)\right)\Psi^{-1}\left(z_2(x,y_0)\right)}\\&\le\int_{\beta(y_0)}^{\beta(y)}\alpha'(x)q(M,N)f\left(\alpha(x),t\right)dt,\end{aligned}$$

从而

$$\frac{\psi\left(\Psi^{-1}\left(z_2(x,y)\right)\right)\frac{\partial z_2(x,y)}{\partial x}}{\varphi\left(\Psi^{-1}\left(z_2(x,y)\right)\right)\Psi^{-1}\left(z_2(x,y)\right)}\le\int_{\beta(y_0)}^{\beta(y)}\alpha'(x)q(M,N)f\left(\alpha(x),t\right)dt, \quad (4.2.21)$$

在 (4.2.21) 式两边从 x_0 到 x 积分得

$$\Phi\left(z_2(x,y)\right)-\Phi\left(z_2(x_0,y)\right)\le q(M,N)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)dtds,$$

从而

$$z_2(x,y)\le\Phi^{-1}\left[\Phi\left(z_2(x_0,y)\right)+q(M,N)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)dtds\right], \quad (4.2.22)$$

由 (4.2.5) (4.2.9) (4.2.17) (4.2.22) 式可得, 当 $(x,y)\in[x_0,M]\times[y_0,N]$ 时,

$$\begin{aligned}u(x,y)&\le z(x,y)\le z_1(x,y)\le\Psi^{-1}\left(z_2(x,y)\right)\\&\le\Psi^{-1}\Bigg\{\Phi^{-1}\Bigg[\Phi\left(\Psi\left(z(x_0,y_0)\right)+\int_{\alpha(x_0)}^{\alpha(M)}\int_{\beta(y_0)}^{\beta(N)}g(s,t)dtds\right)\\&\qquad+q(M,N)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)dtds\Bigg]\Bigg\}, \quad (4.2.23)\end{aligned}$$

由 (M,N) 的任意性可得, 当 $(x,y)\in[x_0,X]\times[y_0,Y]$ 时,

$$\begin{aligned}u(x,y)\le z(x,y)\le\Psi^{-1}\Bigg\{\Phi^{-1}\Bigg[\Phi\left(\Psi\left(z(x_0,y_0)\right)+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}g(s,t)dtds\right)\\+q(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)dtds\Bigg]\Bigg\}, \quad (4.2.24)\end{aligned}$$

又由 (4.2.4) 式

$$
\begin{aligned}
&2z(x_0,y_0)-k\\
&=k+2q(M,N)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)\varphi\left(u(s,t)\right)\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds\\
&\le k+2q(X,Y)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)\varphi\left(u(s,t)\right)\left[u(s,t)+\int_{x_0}^{s}\int_{y_0}^{t}g(\xi,\eta)\psi\left(u(\xi,\eta)\right)d\eta d\xi\right]dtds\\
&=z(X,Y)\\
&\le\Psi^{-1}\left\{\Phi^{-1}\left[\Phi\left(\Psi\left(z(x_0,y_0)\right)+\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}g(s,t)dtds\right)\right.\right.\\
&\qquad\left.\left.+q(X,Y)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)dtds\right]\right\},
\end{aligned}
$$

即

$$
\begin{aligned}
H\left(z(x_0,y_0)\right)&=\Phi\left[\Psi\left(2z(x_0,y_0)-k\right)\right]-\Phi\left[\Psi\left(z(x_0,y_0)\right)+\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}g(s,t)dtds\right]\\
&\le q(X,Y)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)dtds, \qquad (4.2.25)
\end{aligned}
$$

从而

$$
z(x_0,y_0)\le H^{-1}\left(q(X,Y)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)dtds\right), \qquad (4.2.26)
$$

将 (4.2.26) 式带入 (4.2.24) 式得

$$
\begin{aligned}
u(x,y)\le\Psi^{-1}&\left\{\Phi^{-1}\left[\Phi\left(\Psi\left(H^{-1}\left(q(X,Y)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)}f(s,t)dtds\right)\right)\right.\right.\right.\\
&\left.\left.\left.+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}g(s,t)dtds\right)+q(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}f(s,t)dtds\right]\right\},\\
&\forall(x,y)\in[x_0,X]\times[y_0,Y],
\end{aligned}
$$

即 (4.2.2) 式得证.

4.3　应用

考虑下面的Volterra-Fredholm型积分方程

$$u(x,y) = u_0 + q(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} F\left(s,t,u,\int_{x_0}^{s}\int_{y_0}^{t} G\left(\xi,\eta,u\right)d\eta d\xi\right)dtds$$
$$+ q(x,y)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} F\left(s,t,u,\int_{x_0}^{s}\int_{y_0}^{t} G\left(\xi,\eta,u\right)d\eta d\xi\right)dtds, \qquad (4.3.1)$$

其中 $x_0 \geq 0$, $y_0 \geq 0$, $\alpha(x)$, $\beta(y)$, $q(x,y)$ 满足定理 4.2.1 的条件, $u(x,y) \in C(R_+^2, R)$, $F \in C(R_+^2 \times R \times R, R)$, $G \in C(R_+^2 \times R, R)$.

定理 4.3.1　设 $u(x,y)$ 是方程 (4.3.1) 的一个解, 若 (4.3.1) 式中的 F, G 满足

$$|F(s,t,u,v)| \leq f(s,t)\varphi(|u|)(|u|+|v|), \quad |G(\xi,\eta,u)| \leq g(\xi,\eta)\psi(|u|), \qquad (4.3.2)$$

其中 f, g, φ, ψ 满足定理 4.2.1 的条件. 若 $u(x,y)$ 为方程 (4.3.1) 的一个解, 则

$$|u(x,y)| \leq \Psi^{-1}\Bigg\{\Phi^{-1}\Bigg[\Phi\Bigg(\Psi\Bigg(H^{-1}\Bigg(q(X,Y)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} f(s,t)dtds\Bigg)\Bigg)$$
$$+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} g(s,t)dtds\Bigg) + q(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)dtds\Bigg]\Bigg\},$$
$$\forall (x,y) \in [x_0,X]\times[y_0,Y], \qquad (4.3.3)$$

其中 Ψ, Φ 的定义与定理 4.2.1 相同,
$H(u) = \Phi\left[\Psi(2|u|-|u_0|)\right] - \Phi\left[\Psi(|u|) + \int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} g(s,t)dtds\right]$, $H(u)$ 在 $[|u_0|,\infty)$ 上关于 u 是严格增的, H^{-1} 为其反函数.

证明: 由 (4.3.1) (4.3.2) 式可得, 当 $(x,y) \in [x_0,X]\times[y_0,Y]$ 时

$$|u(x,y)| \leq |u_0| + q(x,y)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} f(s,t)\varphi\left(|u(s,t)|\right)$$
$$\left[|u(s,t)| + \int_{x_0}^{s}\int_{y_0}^{t} g(\xi,\eta)\psi\left(|u(\xi,\eta)|\right)d\eta d\xi\right]dtds$$
$$+ q(x,y)\int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} f(s,t)\varphi\left(|u(s,t)|\right)$$
$$\left[|u(s,t)| + \int_{x_0}^{s}\int_{y_0}^{t} g(\xi,\eta)\psi\left(|u(\xi,\eta)|\right)d\eta d\xi\right]dtds, \qquad (4.3.4)$$

由定理 4.2.1 可得 $|u(x,y)|$ 满足 (4.3.3) 式.

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攻读硕士学位期间完成的主要学术论文

1. Sun Di, Meng Fanwei. A new type of retarted discontinuous integral inequality.
2. Sun Di, Meng Fanwei. A class of discontinuous integral inequality containing integration on infinite intervals.
3. Sun Di, Meng Fanwei. Some generalized integral inequalities with two variables and their applications.
4. Sun Di, Meng Fanwei. A generalized nonlinear Volterra-Fredholm type integral inequality delay.

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