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The Generlization of Bihari's Inequality

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Abstract: Bihari's inequality is one of the most important tools in Differential Equation. In this paper, The generalization of Bihari's inequality which has n nonlinear terms by using inductive method is studied. The obtained results include those of M. Pinto and Sung Kyu Choi etc.. Finally, we consider some more general conditions.

Keywords: Gronwall—Bellman inequality; Bihari's inequality

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1 Introduction

Integral inequalities play a vital role in the study of existence, uniqueness, boundedness, stability and invariant manifolds. The Gronwall—Bellman inequality is well known as following

$$u(t)\leqslant c+\int_{t_0}^{t} h(s)u(s)\mathrm{d}s,\ t\geqslant t_0\geqslant 0 \tag{1}$$

where $u(t)$, $h(t)$ are nonnegative and continuous functions for $t\geqslant t_0$, and c is a positive constant. A lot of results about generalization are obtained, such as [1],[2], [3],[4]. Especially, Bihari's inequality

$$u(t)\leqslant k+\int_{0}^{t} f(s)w(u(s))\mathrm{d}s,\ t\in R^{+} \tag{2}$$

is the most important nonlinear generalization of the Gronwall—Bellman inequality, where $u(t)$, $f(t)$ are nonnegative and continuous functions on R^{+}, $w(u)$ is a continuous and nondecreasing function and $w(u)>0$, for $u>0$, and k is a nonnegative constant. In 1990, Pinto[5] considered the inequality

$$u(t)\leqslant c+\sum_{i=1}^{n}\int_{t_0}^{t}\lambda_i(s)w_i(u(s))\mathrm{d}s$$
$$t\in[a,b] \tag{3}$$

where u, $\{\lambda_i\}$, $i=1,2,\cdots\cdots,n$ are continuous and nonnegative on $[a,b]$, c is a positive constant, w_i, $i=1,\cdots,n$ are continuous, nonnegative and nondecreasing on $[0,\infty)$ and positive on $(0,\infty)$ such that $w_1\propto w_2\propto\cdots\propto w_n$ (See Define 1). In 1997 Sung Kyu Choi [6] studied the inequality

$$u(t)\leqslant c+\int_{t_0}^{t}\lambda_1(s)w(u(s))\mathrm{d}s+\int_{t_0}^{t}\lambda_2(s)\int_{t_0}^{s}\lambda_3(T)w(u(T))\mathrm{d}T\mathrm{d}s$$
$$t\geqslant t_0\geqslant 0 \tag{4}$$

where $u,\lambda_1,\lambda_2,\lambda_3\in C(R^{+})$, $w\in C((0,\infty))$ and $w(u)$ be nondecreasing in u and c is a positive constant, The aim of this paper is to consider the following inequality

$$u(t)\leqslant c+\sum_{i=1}^{n}\int_{t_0}^{t}a_i(s)w_i(u(s))\mathrm{d}s+\sum_{i=1}^{n}\int_{t_0}^{t}b_i(s)\int_{t_0}^{s}d_i(T)w_i(u(T))\mathrm{d}T\mathrm{d}s$$
$$t\geqslant t_0\geqslant 0 \tag{5}$$

The results obtained in this paper include those of [5] and [6].

2 Nonlinear Integral Inequities

In this section, nonlinear integral inequalities are studied. Firstly, we give a definition and a lemma.

Define 1 let $A\subset R$ be a set. For $\omega_1,\omega_2: A\to R-\{0\}$ two functions, we will denote $\omega_1\propto\omega_2$ *if* ω_2/ω_1 is nondecreasing on A.

Lemma 1[7] let u, λ be continuous and nonnegative on $[0,\infty)$ and c be a positive constant, and ω be continuous, nonnegative and nondecreasing on $[0,\infty)$ and positive on $(0,\infty)$. Suppose that

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$$u(t)\leqslant c+\int_{t_0}^{t}\lambda(s)\omega(u(s))ds,0\leqslant t_0\leqslant t$$

then

$$u(t)\leqslant W^{-1}[W(c)+\int_{t_0}^{t}\lambda(s)ds],t_0\leqslant t\leqslant b_1$$

where $W(u)=\int_{u_0}^{u}\frac{dz}{\omega(z)}$, $W^{-1}(u)$ is the inverse of $W(u)$ and $b_1=\sup\{t\geqslant t_0: W(c)+\int_{t_0}^{t}(\lambda s)ds\in$ $\mathrm{dom}\,W^{-1}\}$

Theorem 1 Suppose the following two conditions.

(H) The functions ω_i, $i=1,\cdots,n$ are continuous, nonnegative and nondecreasing on $[0,\infty)$ and positive on $[0,\infty)$ such that $\omega_1\propto\omega_2\propto\cdots\propto\omega_n$.

(H_1) The functions u, a_i,b_i,d_i are continuous and nonnegative on $[0,\infty)$ for $i=1,\cdots,n$, and c is a positive constant.

If (5) is valid for $t_0\geqslant 0$, then

$$u(t)\leqslant W_n^{-1}[W_n(c_n-1)+\int_{t_0}^{t}\lambda_n(s)ds]$$

for, $t_0\leqslant t<b_1$ (6)

Where $\lambda_i(t)=a_i(t)+b_i(t)\int_{t_0}^{t}d_i(T)dT$

$$W_k(u)=\int_{u_k}^{u}\frac{dz}{\omega_k(z)},u>u_k>0$$
$$k=1,\cdots,n \tag{7}$$

and W_k^{-1} is the inverse function of W_k, the constant c_k are given by $c_0=c$ and

$$c_k=W_k^{-1}[W_k(c_k-1)+\|\lambda_k\|b_1]$$
$$k=1,\cdots,n-1 \tag{8}$$

the number b_1 is the largest number such that

$$\|\lambda_k\|b_1\overset{\text{def}}{=}\int_{t_0}^{b_1}\lambda_k(s)ds\leqslant\int_{c_{k-1}}^{\infty}\frac{d_z}{\omega_k(z)}$$
$$k=1,\cdots,n \tag{9}$$

Proof

let

$$z(t)=c+\sum_{i=1}^{n}\int_{t_0}^{t}a_i(s)\omega_i(u(s))ds+\sum_{i=1}^{n}\int_{t_0}^{t}b_i(s)\int_{t_0}^{s}d_i(T)\omega_i(u(T))dTds.$$

It is obvious that $z(t_0)=c$, $u(t)\leqslant z(t)$ for $t_0\leqslant t$ and

$$z'(t)=\sum_{i=1}^{n}a_i(t)\omega_i(u(t))+\sum_{i=1}^{n}b_i(t)\int_{t_0}^{t}d_i(T)\omega_i(u(T))dT$$
$$\leqslant\sum_{i=1}^{n}a_i(t)\omega_i(z(t))+\sum_{i=1}^{n}b_i(t)\int_{t_0}^{t}d_i(T)\omega_i(z(T))dT$$
$$\leqslant\sum_{i=1}^{n}(a_i(t)+b_i(t)\int_{t_0}^{t}d_i(T)dT)\omega_i(z(T))$$
$$=\sum_{i=1}^{n}\lambda_i(t)\omega_i(z(t))$$

$$z(t)\leqslant z(t_0)+\sum_{i=1}^{n}\int_{t_0}^{t}\lambda_i(s)\omega_i(z(s))ds=c+\sum_{i=1}^{n}\int_{t_0}^{t}\lambda_i(s)\omega_i(z(s))ds \tag{10}$$

Since $z(t)$ is nondecreasing

If $n=1$, we have

$$z(t)\leqslant c+\int_{t_0}^{t}\lambda_1(s)\omega_1(z(s))ds$$

Using Lemma 1, we get

$$z(t)\leqslant W_1^{-1}[W_1(c_0)+\int_{t_0}^{t}\lambda_1(s)ds$$

Where $c_0=c$.

Now, we use the induction on n for (10).

Suppose that the result is valid for $n=m$, namely,

$$z(t)\leqslant W_m^{-1}[W_m(c_m-1)+\int_{t_0}^{t}\lambda_m(s)ds]$$

Where the number c_k are

$$c_k=W_k^{-1}[W_k(c_{k-1})+\|\lambda_k\|b_1],$$
$$k=1,\cdots,m-1,$$

the number b_1 is the largest number such that

$$\|\lambda_k\|b_1\overset{\text{def}}{=}\int_{t_0}^{b_1}\lambda_k(s)ds\leqslant\int_{c_{k-1}}^{\infty}\frac{d_z}{\omega_k(z)},k=1,\cdots,m.$$

We use (10) for $n=m+1$, and obtain

$$W'_1(z(t))=\frac{z'(t)}{\omega_1(z(t))}\leqslant\lambda_1(t)+\sum_{i=2}^{m+1}\lambda_i(t)\omega_{i,1}(z(t)),\ \omega_{i,1}(t)=\omega_i(t)/\omega_1(t)$$

and

$$W_1(z(t))\leqslant W_1(c)+\|\lambda_1\|b_1+\sum_{i=2}^{m+1}\int_{t_0}^{t}\lambda_i(s)\omega_{i,1}(z(s))ds=\tilde{c}+\sum_{i=1}^{m}\int_{t_0}^{t}\lambda_{i+1}(s)\omega_{i+1,1}(z(s))ds$$

where $\tilde{c}=W_1(c)+\|\lambda_1\|b_1$. Let $v_1=W_1(z)$, and we have

$$v_1(t)\leqslant\tilde{c}+\sum_{i=1}^{m}\int_{t_0}^{t}\lambda_{i+1}(s)(\omega_{i+1,1}\circ W_1^{-1}(v_1(s))ds \tag{11}$$

where $w_{i+1,1}(t)=w_{i+1}(t)/w_1(t)$. Obviously, $w_{i+1,1},(t)$ is nondecreasing on $[0,\infty)$ by (H).

Define

$$\widetilde{W}_{k+1}(u)=\int_{\tilde{u}_{k+1}}^{u}\frac{dz}{w_{k+1,1}(W_1^{-1}(z))},\ \tilde{u}_{k+1}=W_1(u_{k+1}),\quad k=1,\cdots,m$$

Using induction, we have

$$v_1(t)\leqslant \widetilde{W}_{m+1}^{-1}[\widetilde{W}_{m+1}(\tilde{c}_{m-1})+\int_{t_0}^{t}\lambda_{m+1}(s)ds]$$

Where $\tilde{c}_0=\tilde{c}=W_1(c)+\|\lambda_1\|_{b_1}$

$$\tilde{c}_k=\widetilde{W}_{k+1}^{-1}[\widetilde{W}_{k+1}(\tilde{c}_{k-1})+\|\lambda_{k+1}\|_{b_0}]$$

$$k=1,\cdots,m-1$$

and b_0 is the largest number $b_0\geqslant t_0$ such that

$$\|\lambda_{k+1}\|_{b_0}\overset{\text{def}}{=}\int_{t_0}^{b_0}\lambda_{k+1}(s)ds$$

$$\leqslant\int_{\tilde{c}_{k-1}}^{\infty}\frac{dz}{w_{k+1,1}(W_1^{-1}(z))}$$

It is clear that $b_1=b_0$ by the transformation $z=W_1(T)$. Because $\widetilde{W}_{k+1}=W_{k+1}\cdot W_1^{-1}$ and $z=W_1^{-1}(v_1)$, we have

$$z(t)=W_1^{-1}(v_1(t))\leqslant W_1^{-1}[W_1\cdot W_{m+1}^{-1}[W_{m+1}(W_1^{-1}\tilde{c}_{m-1})+\int_{t_0}^{t}\lambda_{m+1}(s)ds]]$$

$$=W_{m+1}^{-1}[W_{m+1}\approx{c}_{m-1}))+\int_{t_0}^{t}\lambda_{m+1}(s)ds]$$

Where $\approx{c}_{m-1}=W_1^{-1}(\tilde{c}_{m-1})$. If we denote $\approx{c}_k=W_1^{-1}(\tilde{c}_k)$, $k=1,\cdots,m-1$, and $\approx{c}_0=W_1^{-1}[W_1(c)+\|\lambda_1\|_{b_1}]$, we get

$$\approx{c}_k=W_{k+1}^{-1}[W_{k+1}(\approx{c}_{k-1})+\|\lambda_{k+1}\|_{b_1}].$$

Obviously,

$$\approx{c}_1=W_2^{-1}[W_2(\approx{c}_0)+\|\lambda_2\|_{b_1}]$$

$$=W_2^{-1}[W_2(W_1^{-1}[W_1(c)+\|\lambda_1\|_{b_1}])+\|\lambda_2\|_{b_1}]=c_2,\cdots,$$

$$\approx{c}_{m-1}=W_m^{-1}[W_m(\approx{c}_{m-2})+\|\lambda_m\|_{b_1}]$$

$$=W_m^{-1}[W_m(c_{m-1})+\|\lambda_m\|_{b_1}]=c_m.$$

Hence,

$$z(t)\leqslant W_{m+1}^{-1}[W_{m+1}(c_m)+\int_{t_0}^{t}\lambda_{m+1}(s)ds].$$

Therefore, we have proved the result for $n=m+1$. Note that $u(t)\leqslant z(t)$, so

$$u(t)\leqslant W_{m+1}^{-1}1[W_{m+1}(c_m)+\int_{t_0}^{t}\lambda_{m+1}(s)ds].$$

The proof has been completed.

Remark 1 When $n=1$, we get Theorem (7) in [6].

Remark 2 When $b_i\equiv0$ or $d_i\equiv0$, $i=1,\cdots,n$,, we get Theorem 1 in [5].

Remark 3 When

$$u(t)\leqslant c+\sum_{i=1}^{n}\int_{t_0}^{t}a_i(s)w_i(u(s))ds+\sum_{i=1}^{m}\int_{t_0}^{t}b_i(s)\int_{t_0}^{3}d_i(T)w_i(u(T))dTds$$

we can change it into (5) by adding the zero term.

References

[1] R. Bellman. Asymptotic series for the solutioin of linear differential-difference equation[J].Rendiconti del circolo Matematica Di Palermo.1958,(7):1~9

[2] H.E. Gollwritzer. A note on a functional inequality[J].Proc. Amer. Math. Soc. 1969,(23):642~647.

[3] B. G. Pachpatte. On some generalization of Bellman's Jemma[J]. J. Math. Anal. Appl,1975,(51):141~150.

[4] B. G. Pachpatte. A note on Gronwall-Bellman's inequality [J].J. Math. Anal. Appl.1973,(44): 758~762.

[5] M. Pinto. Integral Inequalities of Bihari—Type and Applications[J].Funkcialaj Ekvacioj,1990,(33):387~403.

[6] Sung Kyu Choi et.. Lipschitz and exponential asymptotic stablity for nonlinear functional differential systems[J].Dynamic Systems and Applications},1997,(6):397~410.

[7] Bihari,I. A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations [J].Acta Math. Acad. Sci. Hungar,1956,(7): 81~94.

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Bihari 积分不等式的推广

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摘 要:Bihari 不等式在微分方程中有十分重要的作用。本文作者把Bihari 不等式推广到含 n 个非线性项的积分不等式,并且用归纳法加以证明。所得结论包括了 M.Pinto 和Sung Kyu Choi 等的结论。最后考虑了更一般的情形。

关键词:Gronwall—Bellman 不等式; Bihari 不等式