CMOR 350 Final Exam

Spring 2024

Instructor: Dr. Eylem Tekin

Instructions:

Please read the following carefully before starting the exam.

1. Exam duration: 4 hrs
2. Please submit your exam as a **single pdf file**. If you have any supporting files such as computer codes, please email them to me separately.
3. Late submissions will not be graded.
4. You are allowed to use class notes and course textbook.
5. You *may not* use any other resources, in particular, the internet other than accessing the textbook.
6. **You are expected to work alone during the exam**. You are not allowed to collaborate with or provide/receive help to/from any other person during the exam by any means. Any academic misconduct will be handled based on the Rice Honor Code (http://honor.rice.edu/honor-system-handbook/).
7. To receive full credit, your writing must be legible. You will not get credit if I cannot read your answer.
8. Show all your work. Correct answers without any development and/or explanation will not receive any credit.
9. You may not share this final exam with anyone either now or in the future.
10. **Write your name and sign the pledge below.**

NAME: ——————————————

On my honor, I have neither given nor received any unauthorized aid on this exam.

SIGNATURE: ————————————

1) Suppose that a patient in a hospital is categorized into one of the three conditions at the beginning of each day: good, fair or critical. At the beginning of the next day, the patient will either still be in the hospital and be in good, fair or critical condition or will be discharged in one of the two conditions: improved or unimproved. The transition probabilities for the situation are as follows:

$$\begin{array}{l} \\ Good \\ Fair \\ Critical \end{array} \begin{array}{c} \begin{array}{ccc} Good & Fair & Critical \end{array} \\ \begin{pmatrix} 0.65 & 0.20 & 0.05 \\ 0.50 & 0.30 & 0.12 \\ 0.51 & 0.25 & 0.20 \end{pmatrix} \end{array}$$

$$\begin{array}{l} \\ Good \\ Fair \\ Critical \end{array} \begin{array}{c} \begin{array}{cc} Improved & Unimproved \end{array} \\ \begin{pmatrix} 0.08 & 0.02 \\ 0.05 & 0.03 \\ 0.02 & 0.02 \end{pmatrix} \end{array}$$

For example, a patient who begins the day in fair condition has a 50% chance of being in good condition, 30% chance of being in fair condition, 12% chance of being in critical condition the next day, and a 5% chance of being discharged from the hospital the next day in improved condition and a 3% chance of being discharged the next day in unimproved condition.

a) (10 pts) Consider a patient who enters the hospital in fair condition. On average, how many days does this patient spend in the hospital?

b) (10 pts) There were 300 patients in good condition, 500 patients in fair condition, 200 patients in critical condition in the hospital this morning. Tomorrow morning the following admissions will be made: 50 good condition patients, 40 fair condition patients, and 30 critical condition patients. What is your prediction for tomorrow's morning hospital census? That is, how many patients will there be in each group tomorrow morning?

c) (10 pts) What fraction of patients who enter the hospital in fair condition will leave the hospital in improved condition?

2) College Park, a small college town, has two ambulances. Ambulance 1 is based at the college campus, and ambulance 2 is based downtown.

If a request for an ambulance comes from the college, the college-based ambulance is sent if it is available. Otherwise, the downtown-based ambulance is sent, if available. If no ambulance is available, the call is assumed to be lost.

If a request for an ambulance comes from anywhere else in the town, the downtown-based ambulance is sent, if available. Otherwise, the college-based ambulance is sent, if available. If no ambulance is available, the call is considered to be lost.

An average of 3 calls per hour are received from the college, and an average of 4 calls per hour are received from the rest of the town. For both call types, the times in between calls are independent and identically distributed exponential random variables. The average time (exponentially distributed) it takes an ambulance to service a call (respond to a call and be ready to respond to another call at the base) is shown in the table below.

Ambulance base	Ambulance goes to	
	College	Noncollege
College	4 minutes	7 minutes
Downtown	5 minutes	4 minutes

For example, if the ambulance is college-based and serves a noncollege call, the service time is exponentially distributed with 7 minutes.

a) (5 pts) Given 5 calls have arrived in an hour, what is the probability that there were 2 calls received from college and 3 calls were received from rest of the town?

b) (10 pts) Model this system as a continuous-time Markov chain. Define the states and give the rate matrix.

c) (10 pts) Find the limiting distribution.

d) (5 pts) What fraction of time is the downtown ambulance busy in the long-run?

e) (5 pts) **This is a bonus question for 5 extra points.** Assume that half of service time is the time required for an ambulance to go from its base to a patient. What are the average waiting times for an ambulance by a college student and a town person? Who waits longer on average?

3) A machine produces items one at a time, the production times being independent and identically distributed exponential random variables with mean 1 hour. The produced items are stored in a warehouse of capacity 5. When the warehouse is full the machine is turned off, and it is turned on again when the warehouse has space for at least one item. Demands for the items arrive according to a Poisson process with rate 20 per day. Any demand that cannot be satisfied is lost.

Define the state of the system at time t as $X(t)$= the number of items in the warehouse at time t.

a) (6 pts) Model $X(t)$ as a birth and death process. That is, draw the rate diagram and identify the birth and death rates.

b) (3 pts) Based on your rate diagram in part a, in standard nomenclature, what type of a queueing system is this?

   Answer parts c and d using the formulas for the queueing system that you identified in part b.

c) (4 pts) Compute the fraction of time that the machine is turned off in the long-run.

d) (4 pts) What is the rate at which items enter the warehouse?

e) Now consider the same problem but with an infinite capacity warehouse. Suppose in this case demand rate is 12 per hour.

   – (5 pts) What is the largest mean production time so that at most 10% of the demands are lost?

   – (3 pts) What is the expected number of items in the warehouse under this production rate?

4) We want to efficiently operate a machine that can be in any one of 3 states, denoted $1, 2, 3$. State 1 corresponds to a machine in perfect condition. The cost of operating the machine in state i for one time period is given by $g(i)$, $i = 1, 2, 3$. In particular, $g(1) = \$3$, $g(2) = \$5$, $g(3) = \$15$. The actions at the start of each period are to (a) let the machine operate one more period in the state it currently is, or (b) replace the machine with a new one (state 1) at a cost $R = \$20$. Once replaced, the machine is guaranteed to stay in state 1 for one period. If the machine is in state i at the beginning of a period, it makes a transition into state j with probability p_{ij} at the beginning of next period. The transition probabilities are given by the following matrix:

$$P = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0.0 & 0.4 & 0.6 \\ 0.0 & 0.0 & 1 \end{bmatrix}.$$

Suppose that the objective is to minimize the total expected cost of operating the machine for T periods. The terminal cost at the end of the planning horizon of T periods is $\$2i$ for i=1,2,3.

a) (8 pts) Formulate this problem as a finite horizon Markov decision process. Define the states and value functions clearly, and present the optimality equations.

b) (7 pts) Find the optimal operating policy that minimizes the total expected cost over two periods ($T = 2$) using the backward induction algorithm. Assume that the machine is in state 3 at the start of the first period. You will find the optimal actions at the beginning of first and second periods. Clearly state what the optimal policy is, and give the minimum total expected cost over two periods. Perform your calculations manually showing each step of the backward induction algorithm. Do not use a computer code.