Chapter 5.

Branch-and-Bound

Foundations of Algorithms, 5^{th} Ed.

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Contents

The Branch-and-Bound Strategy

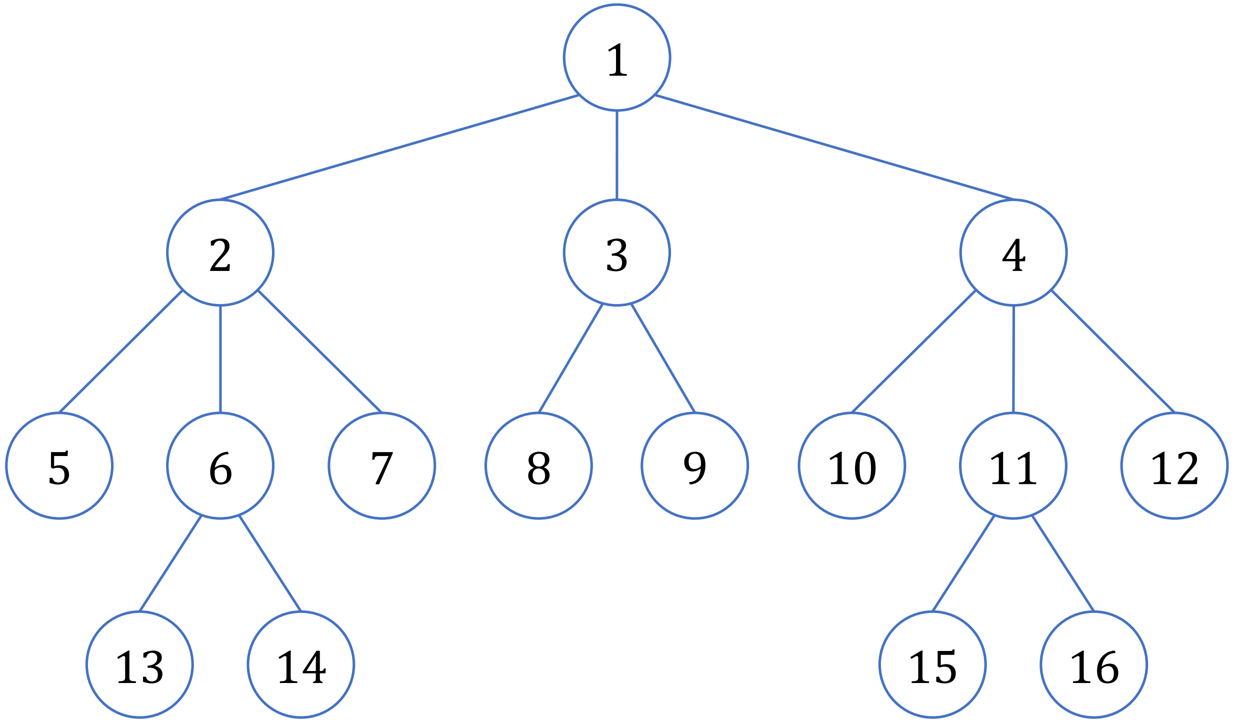
- The Branch-and-Bound algorithm
  - is an improvement on the backtracking algorithm.
  - The design strategy of the branch-and-bound algorithm:
    - uses a state space tree is used to solve a problem.
    - *does not limit* us to any particular *way of traversing* the tree
    - is used for *only for optimization problem*.
  - In backtracking, a depth-first-search is used to traverse a state space tree.
  - In branch-and-bound,
    - a ***breadth-first-search*** is used to traverse the tree,
    - and a ***best-first-search*** can be used, more efficiently.

The Branch-and-Bound Strategy

- The Breadth-First-Search:
  - We can implement the breadth-firs-search using a *queue*.

```
void breadth_first_search(tree T) {
    queue_of_node Q;
    node u, v;

    initialize(Q);
    v = root of T;
    visit v;
    enqueue(Q, v);
    while (!empty(Q)) {
        dequeue(Q, v);
        for (each child u of v) {
            visit u;
            enqueue(Q, u);
        }
    }
}
```

The Branch-and-Bound Strategy

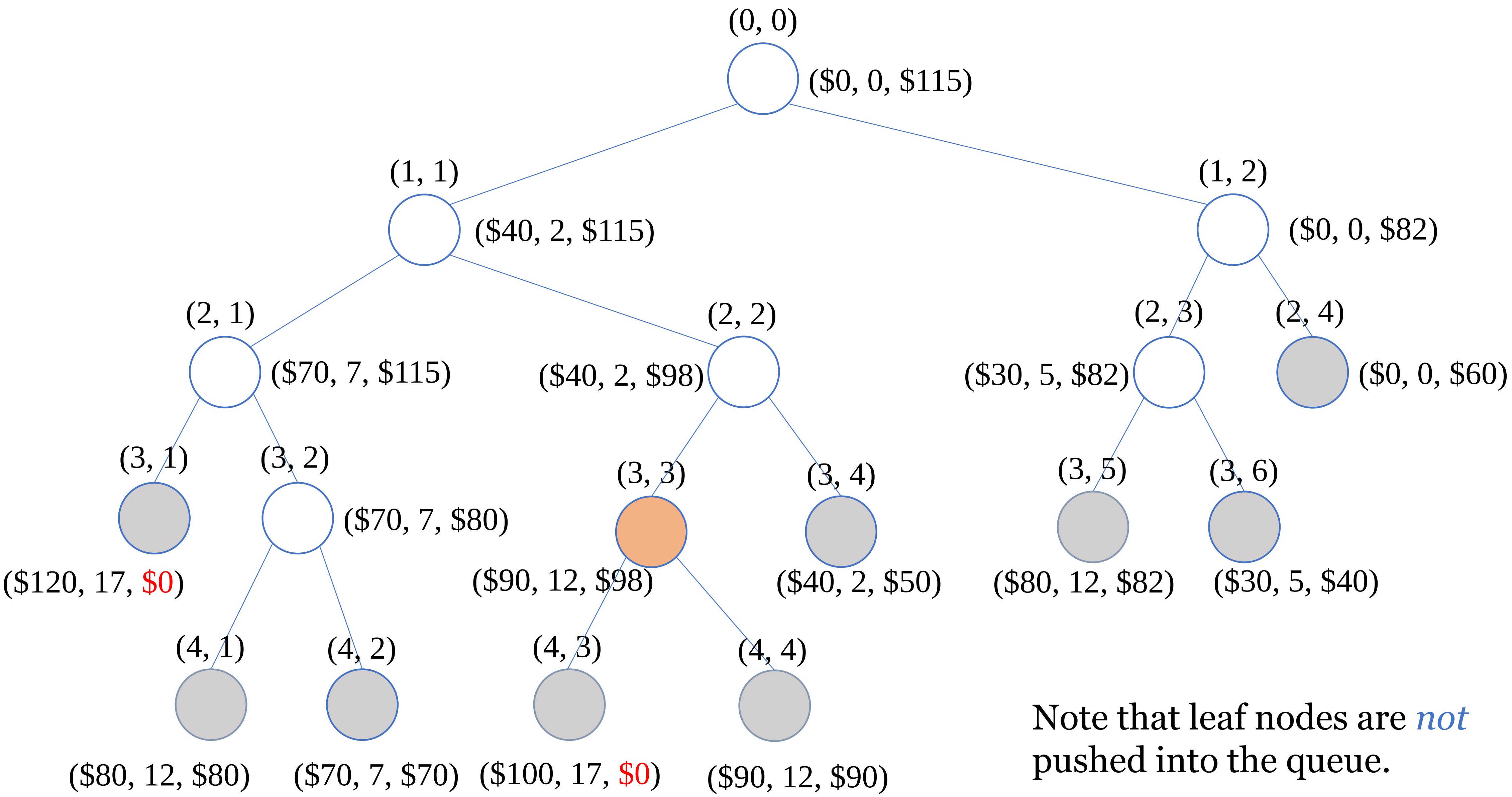


6.1 Branch-and-Bound for the 0-1 Knapsack Problem

- Solving the 0-1 Knapsack with the Branch-and-Bound Pruning:
  - The backtracking is also actually a branch-and-bound algorithm.
  - However, we can *compare* the *bounds* of *promising nodes*
    - and *visit* the children of the one with the *best bound*.
  - In this way, we often can *arrive* at an *optimal* solution
    - *faster than* we would by visiting in some predefined order (DFS).
  - This approach is called
    - ***best-first-search with branch-and-bound pruning***.
  - The implementation of this approach is a simple modification of
    - *breadth-first-search with branch-and-bound pruning*.

- Breadth-First-Search (BFS) with Branch-and-Bound Pruning:
  - An illustrative example:
    - $n = 4, W = 16$
    - $p_i = [40, 30, 50, 10]$
    - $w_i = [\ 2,\ \ 5, 10,\ \ 5]$
    - $\frac{p_i}{w_i} = [20,\ \ 6,\ \ 5,\ \ 2]$: already ordered in nondecreasing order.
  - We proceed exactly as we did using backtracking
    - except that we do a breadth-first-search instead of a depth-first-search.

6.1 Branch-and-Bound for the 0-1 Knapsack Problem



Note that leaf nodes are *not* pushed into the queue.

6.1 Branch-and-Bound for the 0-1 Knapsack Problem

- A general algorithm for BFS with branch-and-bound pruning:

```
void breadth_first_branch_and_bound(state_space_tree T, int &best) {
    queue_of_node Q;
    node u, v;

    intialize(Q);
    v = root of T;
    enqueue(Q, v);
    best = value(v);
    while (!empty(Q)) {
        dequeue(Q, v);
        for (each child u of v) {
            if (value(u) is better than best)
                best = value(u);
            if (bound(u) is better than best)
                enqueue(Q, u);
        }
    }
}
```

6.1 Branch-and-Bound for the 0-1 Knapsack Problem

- Applying the strategy to the 0-1 Knapsack Problem:
  - Notice that two nodes (3, 1) and (4, 3) have bounds of \$0.
    - Unlike the promising function in backtracking,
      - *bound* function should return an integer.
  - Because we do not have the benefit of recursion,
    - we need to store all the information pertinent to a node at that node.

```
typedef struct node *node_pointer;
typedef struct node {
    int level;   // the node's level in the state space tree
    int weight;
    int profit;
} nodetype;

typedef queue<node_pointer> queue_of_node;
```

ALGORITHM 6.1: The BFS with Branch-and-Bound Pruning for the 0-1 Knapsack Problem

```
void knapsack5() {
    queue_of_node Q; node_pointer u, v;

    maxprofit = 0;
    Q.enqueue(create_node(0, 0, 0));
    while (!Q.empty()) {
        v = Q.dequeue();
        u = new_node(v->level + 1,
                     v->weight + w[v->level + 1],
                     v->profit + p[v->level + 1]);
        if (u->weight <= W && u->profit > maxprofit)
            maxprofit = u->profit;
        if (bound(u) > maxprofit)
            Q.enqueue(u);
        u = new_node(v->level + 1, v->weight, v->profit);
        if (bound(u) > maxprofit)
            Q.enqueue(u);
    }
}
```

6.1 Branch-and-Bound for the 0-1 Knapsack Problem

ALGORITHM 6.1: The BFS with Branch-and-Bound Pruning for the 0-1 Knapsack Problem

```
float bound(node_pointer u) {
    int j, k, totweight; float result;
    if (u->weight >= W)
        return 0;
    else {
        result = u->profit;
        j = u->level + 1;
        totweight = u->weight;
        while (j <= n && totweight + w[j] <= W) {
            totweight += w[j];
            result += p[j];
            j++;
        }
        k = j;
        if (k <= n)
            result += (W - totweight) * ((float)p[k] / w[k]);
        return result;
    }
}
```

6.1 Branch-and-Bound for the 0-1 Knapsack Problem

- ***Best-First-Search*** with Branch-and-Bound Pruning:
  - Note that the breadth-first-search strategy
    - has *no advantage over a depth-first-search*.
  - For example,
    - there are only 13 nodes in the state space tree of backtracking.
    - whereas there are 17 nodes in the state space tree produced by
      - the breadth-first-search with brand-and-bound pruning.
  - However, we can *improve* our search by using our bound
    - to do more than just determine whether a node is promising.
  - After visiting all the children of a given node,
    - *the node with the best bound* can be expanded *first*.

- $n = 4, W = 16$
- $p_i = [40, 30, 50, 10]$
- $w_i = [2, 5, 10, 5]$
- $\frac{p_i}{w_i} = [20, 6, 5, 2]$

$maxprofit = \$0$

$profit = \$0, weight = 0$

$bound = \$115$

Node (0, 0) is *inserted* into the PQ.

PQ

(0, 0) ($0, 0, $115)

6.1 Branch-and-Bound for the 0-1 Knapsack Problem

(0, 0)

($0, 0, $115)

$maxprofit = \$40$

1. Node (0, 0) is *removed* from the PQ.
   - $profit = \$0, weight = 0, bound = \$115$
   - $maxprofit = \$40$

2. Visit node (1, 1).
   - $profit = \$40, weight = 2$
   - $bound = \$115$
   - Insert (1, 1) into the PQ.

3. Visit node (1, 2).
   - $profit = \$0, weight = 0$
   - $bound = \$82$
   - Insert(1, 2) into the PQ.

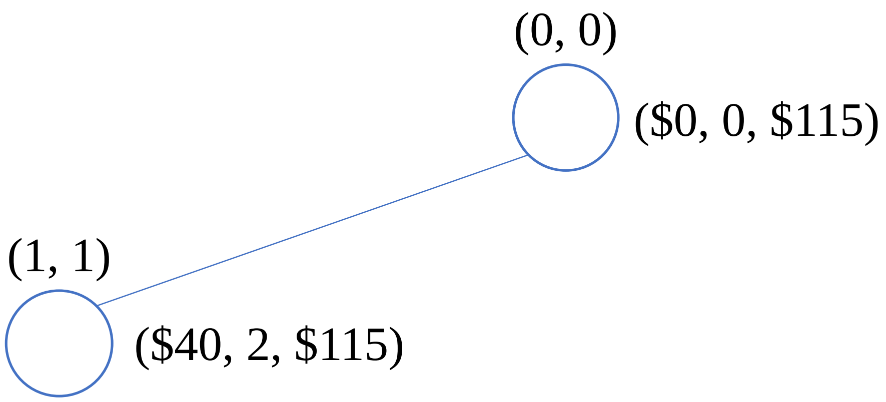
4. Determine *promising, unexpanded node* with the *greatest bound*.
   - Because node (1, 1) has the greatest bound,
     - we visit node (1, 1) which is removed from the PQ next.

PQ

(1, 1)
($40, 2, $115)

(1, 2)
($0, 0, $82)

6.1 Branch-and-Bound for the 0-1 Knapsack Problem



$maxprofit = \$70$

5. Node (1, 1) is *removed* from the PQ.
   - $profit = \$40, weight = 2, bound = \$115$
   - $maxprofit = \$70$

6. Visit node (2, 1).
   - $profit = \$70, weight = 7$
   - $bound = \$115$
   - Insert (2, 1) into the PQ.

7. Visit node (2, 2).
   - $profit = \$40, weight = 2$
   - $bound = \$98$
   - Insert (2, 2) into the PQ.

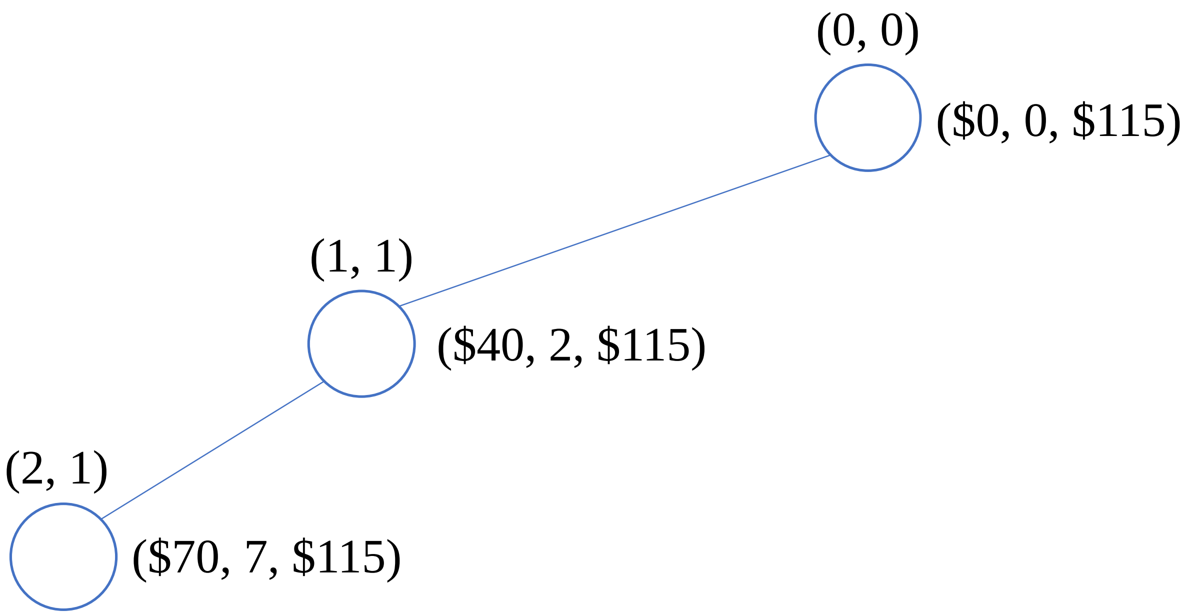
PQ

(2, 1)
($70, 7, $115)

(2, 2)
($40, 2, $98)

(1, 2)
($0, 0, $82)

6.1 Branch-and-Bound for the 0-1 Knapsack Problem



$maxprofit = \$70$

8. Node (2, 1) is *removed* from the PQ.
   - $profit = \$70, weight = 7, bound = \$115$

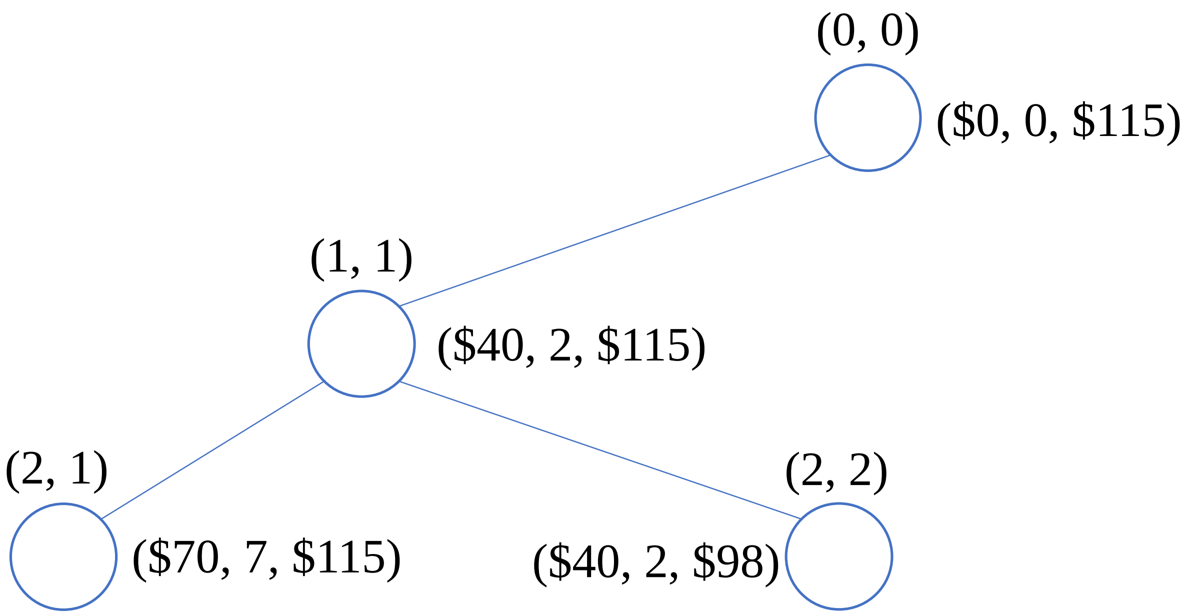
9. Visit node (3, 1).
   - $profit = \$120, weight = 17$
   - $bound = \$0$
   - Do NOT insert (3, 1) into the PQ.

10. Visit node (3, 2).
    - $profit = \$70, weight = 7$
    - $bound = \$80$
    - Insert (3, 2) into the PQ.

PQ

(2, 2)
($40, 2, $98)
(1, 2)
($0, 0, $82)
(3, 2)
($70, 7, $80)

6.1 Branch-and-Bound for the 0-1 Knapsack Problem



$maxprofit = \$90$

11. Node (2, 2) is *removed* from the PQ.
   - $profit = \$40, weight = 2, bound = \$98$
   - $maxprofit = \$90$

12. Visit node (3, 3).
   - $profit = \$90, weight = 12$
   - $bound = \$98$
   - Insert (3, 3) into the PQ.

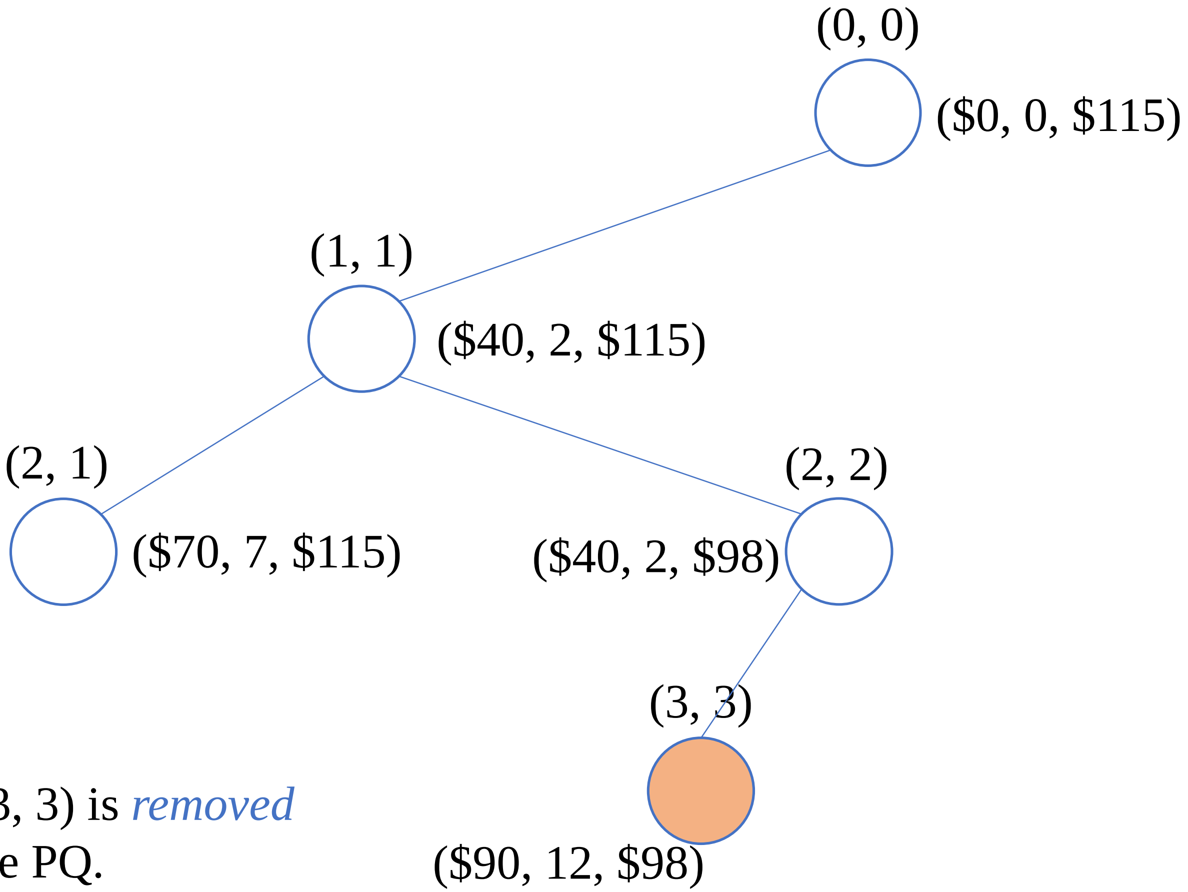
13. Visit node (3, 4).
   - $profit = \$74, weight = 2$
   - $bound = \$50$
   - Do NOT insert (3, 4) into the PQ.

PQ

(3, 3)
($90, 12, $98)
(1, 2)
($0, 0, $82)
(3, 2)
($70, 7, $80)

6.1 Branch-and-Bound for the 0-1 Knapsack Problem

$maxprofit = \$90$



14. Node (3, 3) is *removed* from the PQ.

15. Visit node (4, 1).
- \$100, 17, \$0
- Do NOT Insert (4, 1) into the PQ.

16. Visit node (4, 2).
- \$90, 12, \$90
- Do NOT insert (4, 2) into the PQ.

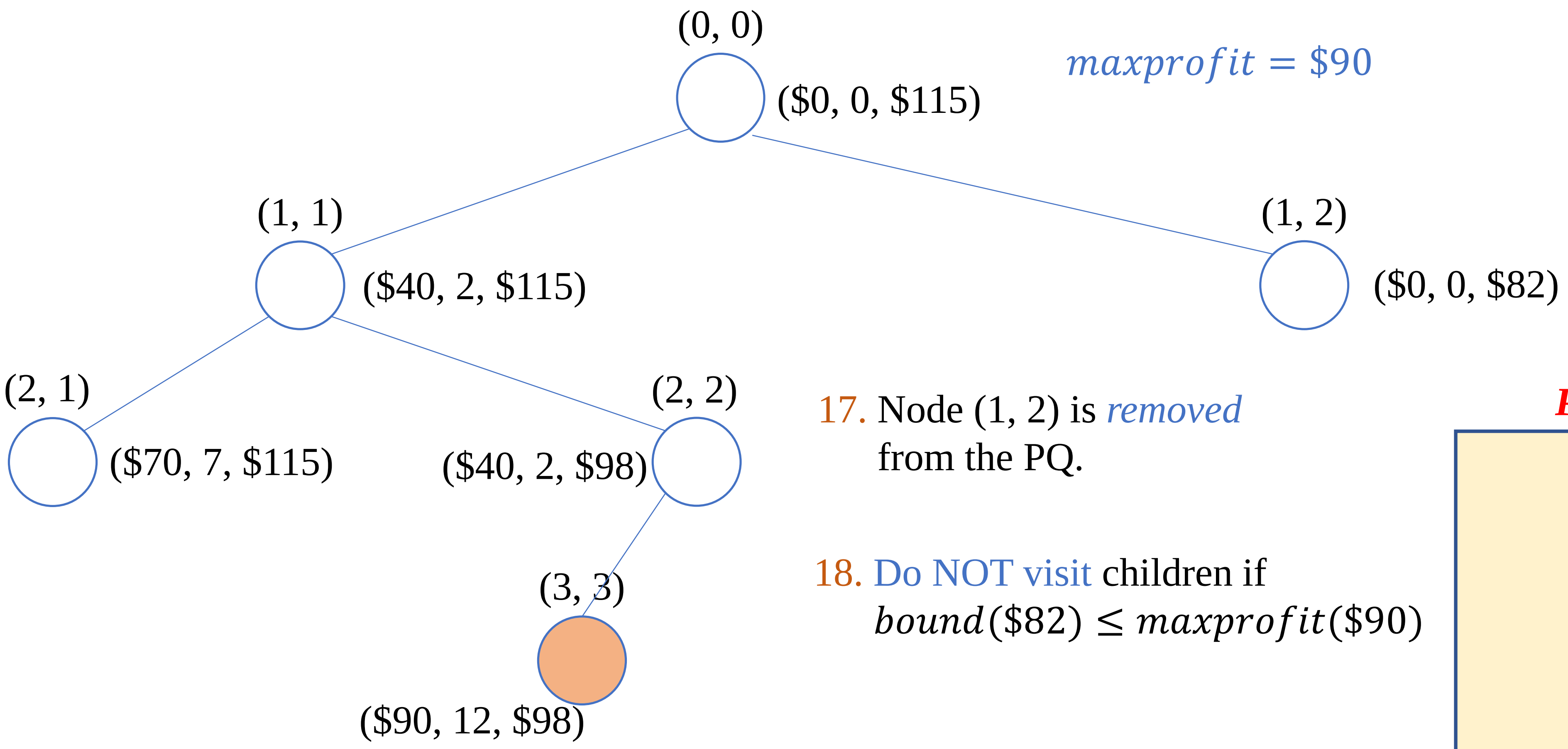
PQ

(1, 2)
($0, 0, $82)

(3, 2)
($70, 7, $80)

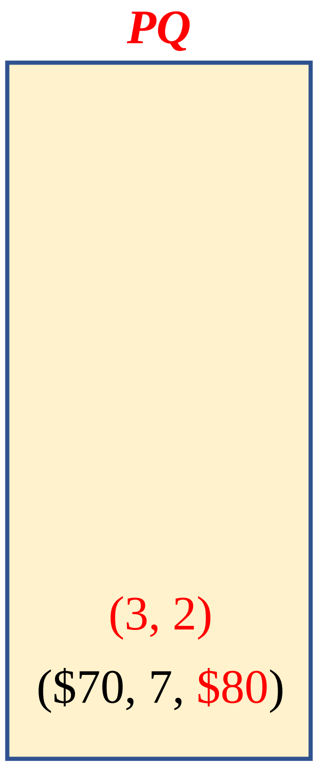
6.1 Branch-and-Bound for the 0-1 Knapsack Problem

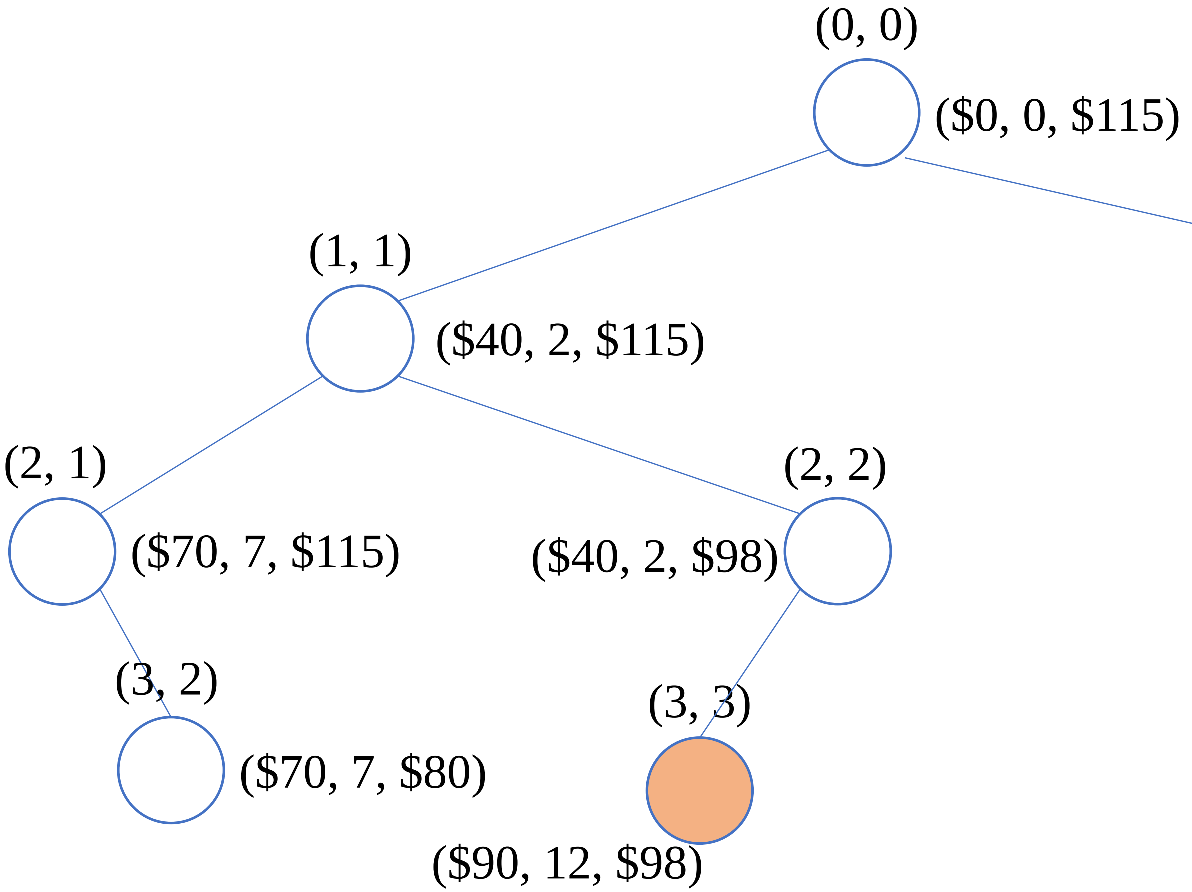
$maxprofit = \$90$



17. Node (1, 2) is *removed* from the PQ.

18. Do NOT visit children if $bound(\$82) \le maxprofit(\$90)$





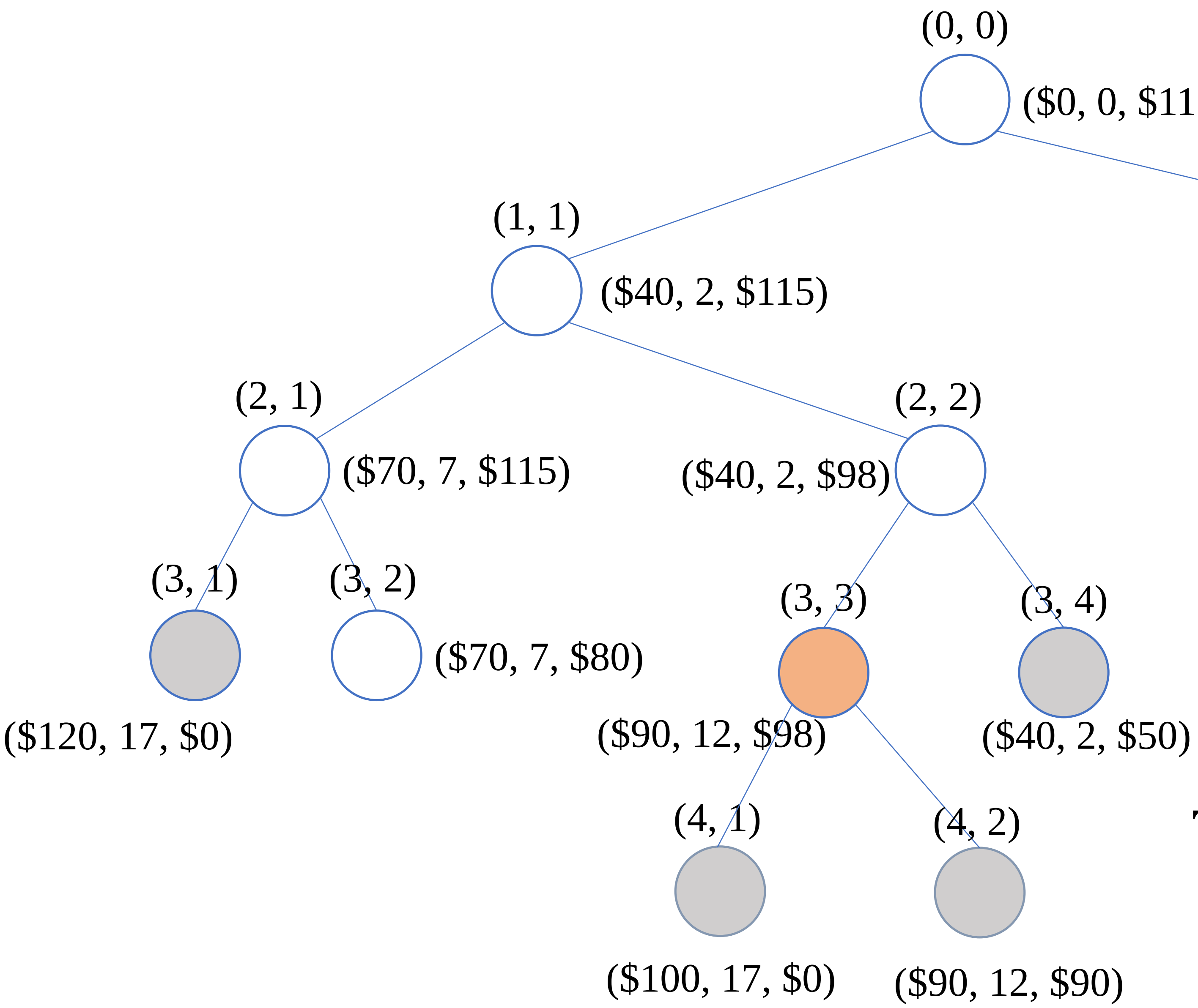
$maxprofit = \$90$

PQ

19. Node (3, 2) is *removed* from the PQ.

20. Do NOT visit children if $bound(\$80) \leq maxprofit(\$90)$

6.1 Branch-and-Bound for the 0-1 Knapsack Problem



Note that shaded nodes are *not* inserted into the priority queue, *but* visited to see if it is promising.

The number of nodes in state space tree:

- *depth-first*: 13 nodes
- breadth-first: 17 nodes
- *best-first*: 11 nodes.

- The implementation of Best-First-Search:
  - Note that there is no guarantee that
    - the node that appears to be best will actually lead to an optimal solution.
    - ex) (2, 1) is better than (2, 2), but (2, 2) leads to the optimal solution.
  - In general, best-first-search
    - still ends up creating *all of the state space tree* for some instances.
  - Best-first-search is a simple modification to breadth-first-search.
    - Instead of using a *queue*, use a *priority queue*.

- A general algorithm for the *best-first-search*:

```
void best_first_branch_and_bound(state_space_tree T, int &best) {
    priority_queue_of_node PQ;
    node u, v;

    initialize(PQ);
    v = root of T;
    best = value(v);
    insert(PQ, v);
    while (!empty(PQ)) {
        remove(PQ, v);
        if (bound(v) is better than best)
            for (each child u of v) {
                if (value(u) is better than best)
                    best = value(u);
                if (bound(u) is better than best)
                    insert(PQ, u);
            }
    }
}
```

6.1 Branch-and-Bound for the 0-1 Knapsack Problem

- Applying the strategy to the 0-1 Knapsack Problem:
  - Notice that a node is promising at the time when we insert it into the PQ.
    - However, it can be nonpromising when it is removed from the PQ.
  - So, compare its bound with $maxprofit$ after it is removed from the PQ.
  - Because we need the bound for a node
    - at insertion time, at removal time, and to order the nodes in the PQ,
    - we store the bound at the node.

```
typedef struct node *node_pointer;
typedef struct node {
    int level;    // the node's level in the state space tree
    int weight;
    int profit;
    float bound;
} nodetype;
```

ALGORITHM 6.2: The Best-First-Search with B&B Pruning for the 0-1 Knapsack Problem

```
void knapsack6() {
    priority_queue_of_node PQ; node_pointer u, v;
    maxprofit = 0;
    PQ.insert(create_node(0, 0, 0));
    while (!PQ.empty()) {
        v = PQ.remove();
        if (v->bound > maxprofit) {
            u = create_node(v->level + 1,
                            v->weight + w[v->level + 1],
                            v->profit + p[v->level + 1]);
            if (u->weight <= W && u->profit > maxprofit)
                maxprofit = u->profit;
            if (u->bound > maxprofit)
                PQ.insert(u);
            u = create_node(v->level + 1, v->weight, v->profit);
            if (u->bound > maxprofit)
                PQ.insert(u);
        }
    }
}
```

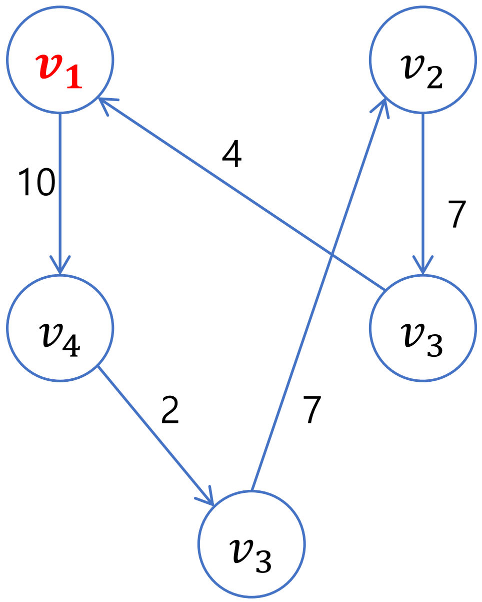
6.1 Branch-and-Bound for the 0-1 Knapsack Problem

```
typedef priority_queue<node_pointer, vector<node_pointer>, compare>
        priority_queue_of_node;

struct compare {
    bool operator()(node_pointer u, node_pointer v) {
        if (u->bound < v->bound)
            return true;
        return false;
    }
};

node_pointer create_node(int level, int weight, int profit) {
    node_pointer v = (node_pointer)malloc(sizeof(nodetype));
    v->level = level;
    v->weight = weight;
    v->profit = profit;
    v->bound = bound(v);
    return v;
}
```

6.2 The Traveling Salesperson Problem

- The Traveling Salesperson Problem Revisited:
  - The goal of this problem is to find an *optimal tour*, that is,
    - the *shortest path* in a directed graph that starts at a given vertex,
    - visits each vertex *exactly once*, and ends up back at the starting vertex.

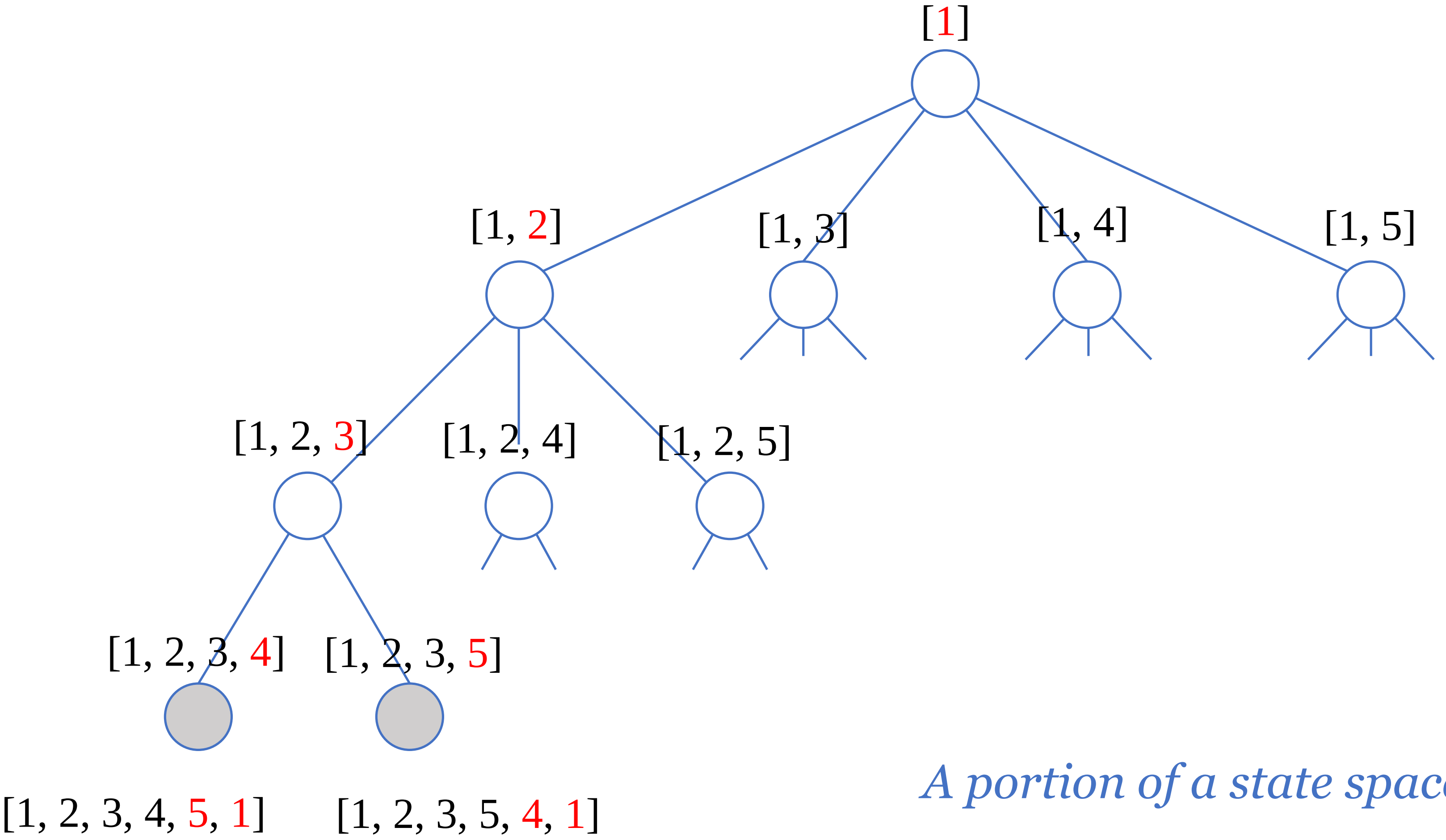
W	1	2	3	4	5
1	0	14	4	10	20
2	14	0	7	8	7
3	4	5	0	7	16
4	11	7	9	0	2
5	18	7	17	4	0



an optimal tour starting at v_1.
$(length = 30)$

- The *State Space Tree* for the TSP:
  - An obvious state space tree is one in which
    - each vertex other than the starting vertex
      - is tried as the *first vertex*, after the starting one, *at level* 1.
    - each vertex other than the *starting vertex* and the *first vertex*
      - is tried as the *second vertex at level* 2, and so on.
  - Then, a node represents
    - a *path* chosen up to that node starting from the root node.
  - We stop expanding the tree
    - when there are $n - 1$ vertices in the path stored at a node
    - because, at that time, the nth vertex is *uniquely* determined.

6.2 The Traveling Salesperson Problem



A portion of a state space tree when $n = 5$.

6.2 The Traveling Salesperson Problem

- Determining a ***bound*** for each node:
  - Determine a *lower bound* on the length of any tour
    - that can be obtained by expanding beyond a given node.
  - Then, the node is *promising*
    - only if its bound is *less* than the *current minimum tour length*.
  - We can obtain a *bound for a node*:
  - In any tour, the *length* of the *edge taken* when leaving a vertex
    - must be *at least as great as* the *length* of the *shortest edge*
      - emanating from the vertex.

- **Determining a *bound* for each node:**
  - Let *cost* be the length of the edge taken.
  - A *lower bound* on the *cost of leaving* vertex v_1 is given by
    - the *minimum* of all the nonzero entries in *row* 1 of the adjacency matrix.

W	1	2	3	4	5
1	0	14	4	10	20
2	14	0	7	8	7
3	4	5	0	7	16
4	11	7	9	0	2
5	18	7	17	4	0

- the *lower bound* leaving v_1: 4

W	1	2	3	4	5
1	0	14	4	10	20
2	14	0	7	8	7
3	4	5	0	7	16
4	11	7	9	0	2
5	18	7	17	4	0

- $v_2: 7$
- $v_3: 4$
- $v_2: 2$
- $v_2: 4$

- Determining a ***bound*** for each node:
  - Because a tour must leave every vertex exactly once,
    - a lower bound on the length of a tour is the sum of these *minimums*.

W	1	2	3	4	5
1	0	14	4	10	20
2	14	0	7	8	7
3	4	5	0	7	16
4	11	7	9	0	2
5	18	7	17	4	0

- $v_1 : 4$
- $v_2 : 7$
- $v_3 : 4$
- $v_2 : 2$
- $v_2 : 4$

- A *lower bound* on the length of a *tour* is
  - $4 + 7 + 4 + 2 + 4 = 21.$
- Note that this is not to say that
  - there is a tour with this length.
- Rather, it says that
  - there can be *no tour* with a *shorter length*.

- **Determining a *bound* for each node:**
  - Suppose we have visited the node $[1, 2]$.
    - The cost of getting to v_2 is the weight on the edge from v_1 to v_2.
  - Therefore, any tour obtained by expanding beyond this node,
    - has the following lower bounds on the *cost* of leaving v_2.

[1, 2]

W	1	2	3	4	5
1	0	14	4	10	20
2	14	0	7	8	7
3	4	5	0	7	16
4	11	7	9	0	2
5	18	7	17	4	0

- $v_1: 14$
- $v_2: 7$
- $v_3: 4$
- $v_2: 2$
- $v_2: 4$

- Do not include the edge to v_1:
  - v_2 cannot return to v_1.
- Do not include the edge to v_2:
  - we already have been at v_2.

- A *lower bound* obtained by expanding $[1, 2]$:
  - $14 + 7 + 4 + 2 + 4 = 31.$

6.2 The Traveling Salesperson Problem

- Determining a ***bound*** for each node:
  - Suppose we have visited the node $[1, 2, 3]$.

[1, 2, 3]

W	1	2	3	4	5
1	0	14	4	10	20
2	14	0	7	8	7
3	4	5	0	7	16
4	11	7	9	0	2
5	18	7	17	4	0

- $v_1: 14$
- $v_2: 7$
- $v_3: 7$
- $v_2: 2$
- $v_2: 4$

- Do not include the edge to v_1:
  - v_3 cannot return to v_1.
- Do not include the edge to v_2 and v_3:
  - we already have been at v_2 and v_3.

- A *lower bound* obtained by expanding $[1, 2, 3]$:
  - $14 + 7 + 7 + 2 + 4 = 34.$

6.2 The Traveling Salesperson Problem

- Determining a ***bound*** for each node:
  - We will use the following data type in the algorithm.
    - The field *path* contains the partial tour stored at the node.
  - We need two functions *length* and *bound*.
    - *length* returns the length of the tour's *path*,
    - *bound* returns the bound for a node using the considerations discussed.

```
typedef vector<int> ordered_set;

typedef struct node *node_pointer;
typedef struct node {
    int level;
    ordered_set path;
    int bound;
} nodetype;
```

6.2 The Traveling Salesperson Problem

```
int length(ordered_set path) {
    vector<int>::iterator it;
    int len = 0;
    for (it = path.begin(); it != path.end(); it++)
        if (it != path.end() - 1)
            len += W[*it][*(it+1)];
    return len;
}
```

W	1	2	3	4	5
1	0	14	4	10	20
2	14	0	7	8	7
3	4	5	0	7	16
4	11	7	9	0	2
5	18	7	17	4	0

[1, 2, 3]

$$1 \xrightarrow{14} 2 \xrightarrow{7} 3$$

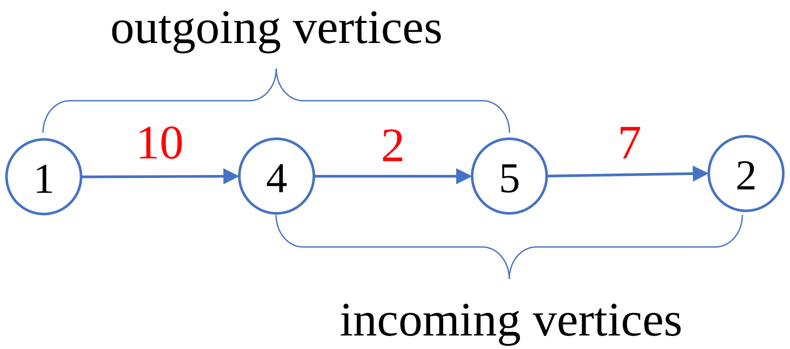
$$length([1,2,3]) = W[1][2] + W[2][3] = 21$$

```
int bound(node_pointer v) {
    // start from the length of path
    int lower = length(v->path);
    for (int i = 1; i <= n; i++) {
        if (hasOutgoing(i, v->path)) continue;
        int min = INF;
        for (int j = 1; j <= n; j++) {
            // Do not include self-loop
            if (i == j) continue;
            // Do not include an edge to which i cannot return
            if (j == 1 && i == v->path[v->path.size() - 1]) continue;
            // Do not include edges already in the path
            if (hasIncoming(j, v->path)) continue;
            // A lower bound (minimum) on the cost of leaving i
            if (min > W[i][j]) min = W[i][j];
        }
        lower += min;
    }
    return lower;
}
```

6.2 The Traveling Salesperson Problem

```
bool hasOutgoing(int v, ordered_set path) {
    vector<int>::iterator it;
    for (it = path.begin(); it != path.end() - 1; it++)
        if (*it == v) return true;
    return false;
}

bool hasIncoming(int v, ordered_set path) {
    vector<int>::iterator it;
    for (it = path.begin() + 1; it != path.end(); it++)
        if (*it == v) return true;
    return false;
}
```

[1, 4, 5, 2]



6.2 The Traveling Salesperson Problem

W	1	2	3	4	5
1	0	14	4	10	20
2	14	0	7	8	7
3	4	5	0	7	16
4	11	7	9	0	2
5	18	7	17	4	0

[1, 3, 2] $length([1,3,2]) = 4 + 5 = 9$

$bound([1,3,2]) = length[1,3,2] + \min(v_2) + \min(v_4) + \min(v_5)$

$= 9 + 9 + 16 + 18 = 22$

[1, 3, 4] $length([1,3,4]) = 4 + 7 = 11$

$bound([1,3,4]) = length[1,3,4] + \min(v_2) + \min(v_4) + \min(v_5)$

$= 11 + 11 + 18 + 20 = 27$

[1, 4, 5] $length([1,4,5]) = 10 + 2 = 12$

$bound([1,4,5]) = length[1,4,5] + \min(v_2) + \min(v_3) + \min(v_5)$

$= 12 + 12 + 19 + 23 = 30$

6.2 The Traveling Salesperson Problem

- Best-First-Search with Branch-and-Bound Pruning:
  - Initialize the value of the best solution to infinity
    - since there is no candidate solution at the root node.
  - *Candidate solutions* exist *only at leaves* in the state space tree.
  - We need not compute bounds for leaves
    - because the algorithm is written so as not to expand beyond leaves.

6.2 The Traveling Salesperson Problem

- $n = 5$

W	1	2	3	4	5
1	0	14	4	10	20
2	14	0	7	8	7
3	4	5	0	7	16
4	11	7	9	0	2
5	18	7	17	4	0

$minlength = \infty$

$opttour = [\quad]$

$bound = 21$

Node [1] is *inserted* into the PQ.

PQ

[1]

$(bound = 21)$

[1]

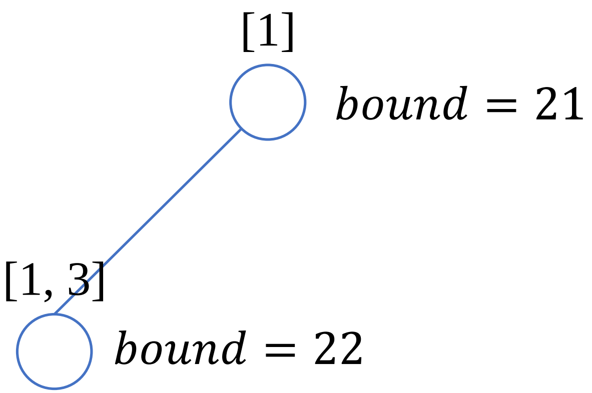
$bound = 21$

PQ

$[1,3]$
$(bound = 22)$
[1,4]
$(bound = 30)$
[1,2]
$(bound = 31)$
[1,5]
$(bound = 42)$

1. Node [1] is *removed* from the PQ. (the root)
   a. Bound is 21 and $minlength = \infty$
   b. Node [1,2] is inserted into the PQ. $bound = 31$.
   c. Node [1,3] is inserted into the PQ. $bound = 22$.
   d. Node [1,4] is inserted into the PQ. $bound = 30$.
   e. Node [1,5] is inserted into the PQ. $bound = 42$.

6.2 The Traveling Salesperson Problem

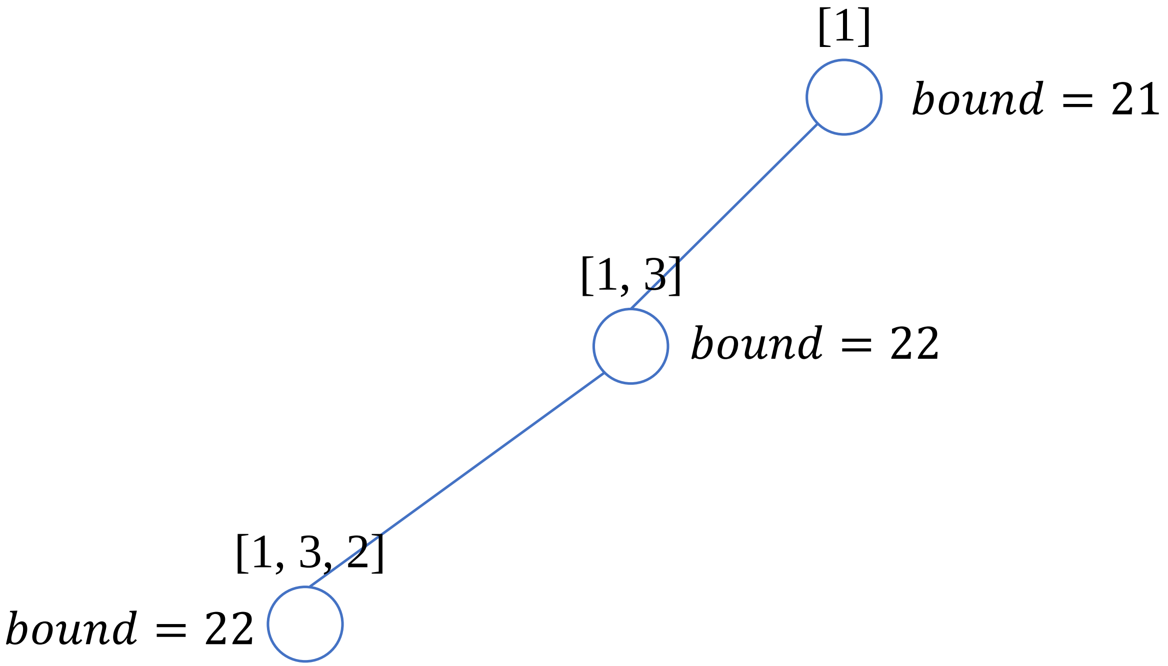


2. Node [1,3] is *removed* from the PQ.
   a. Bound is 22 and $minlength = \infty$
   b. Node [1,3,2] is inserted into the PQ. $bound = 22$.
   c. Node [1,3,4] is inserted into the PQ. $bound = 27$.
   d. Node [1,3,5] is inserted into the PQ. $bound = 39$.

PQ

[1,3,2]
$(bound = 22)$
[1,3,4]
$(bound = 27)$
[1,4]
$(bound = 30)$
[1,2]
$(bound = 31)$
[1,3,5]
$(bound = 39)$
[1,5]
$(bound = 42)$

6.2 The Traveling Salesperson Problem

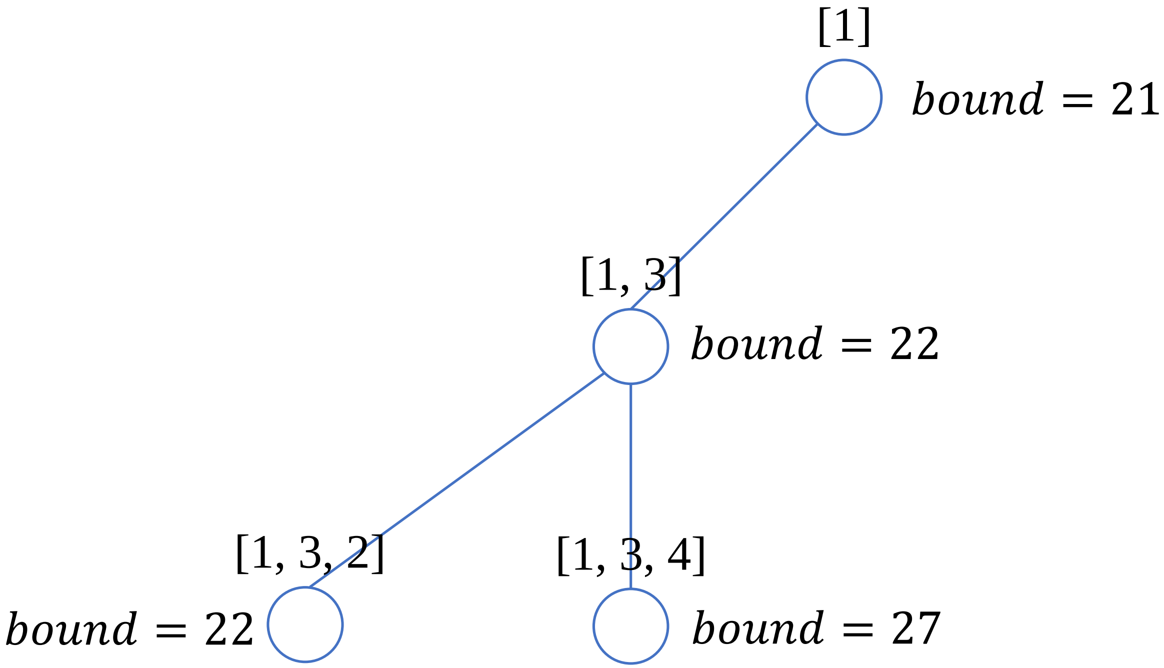


PQ

[1,3,4]
$(bound = 27)$
[1,4]
$(bound = 30)$
[1,2]
$(bound = 31)$
[1,3,5]
$(bound = 39)$
[1,5]
$(bound = 42)$

3. Node [1,3,2] is *removed* from the PQ.
   a. Bound is 22 and $minlength = \infty$
   b. Node [1,3,2,4] is a leaf node. $length = minlength = 37.$ $opttour = [1,3,2,4,5,1].$
   c. Node [1,3,2,5] is a leaf node. $length = minlength = 31.$ $opttour = [1,3,2,5,4,1].$

6.2 The Traveling Salesperson Problem



3. Node [1,3,4] is *removed* from the PQ.
   a. Bound is 27 and $minlength = 31$
   b. Node [1,3,4,2] is a leaf node. $length = 43$. $minlength = 31.$
   c. Node [1,3,4,5] is a leaf node. $length = 34$. $minlength = 31.$

PQ

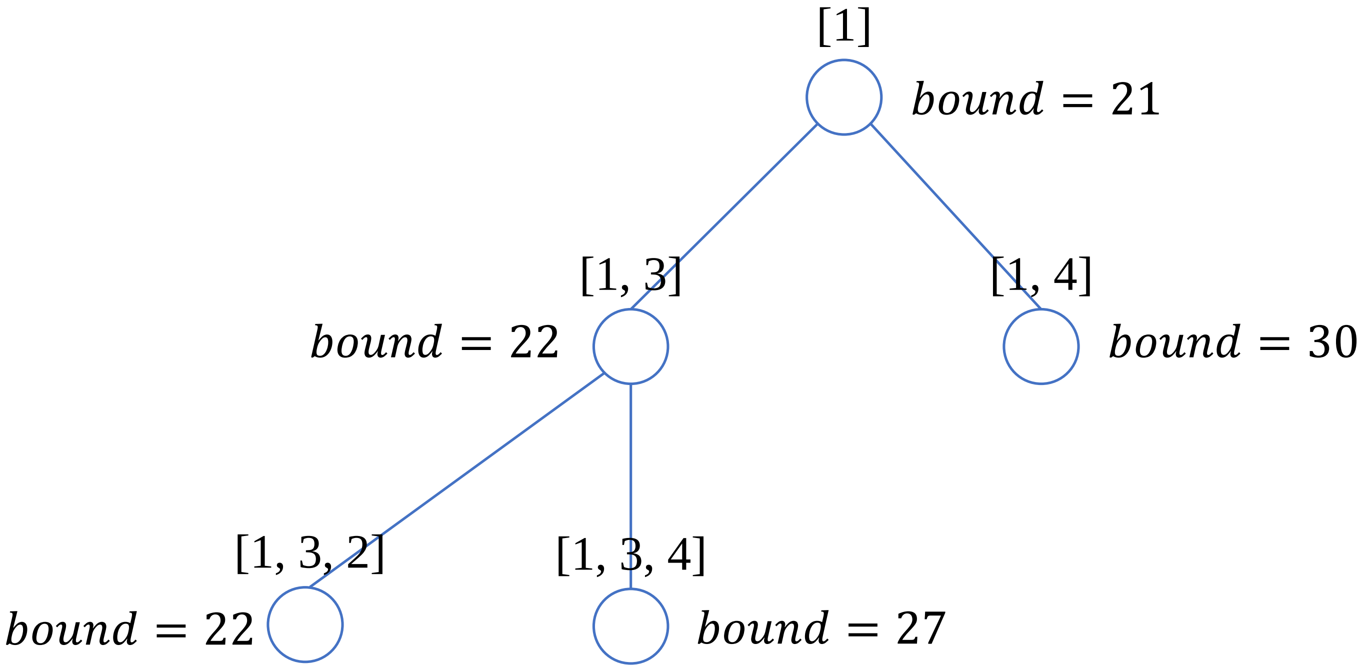
[1,4]
($bound = 30$)

[1,2]
($bound = 31$)

[1,3,5]
($bound = 39$)

[1,5]
($bound = 42$)

6.2 The Traveling Salesperson Problem

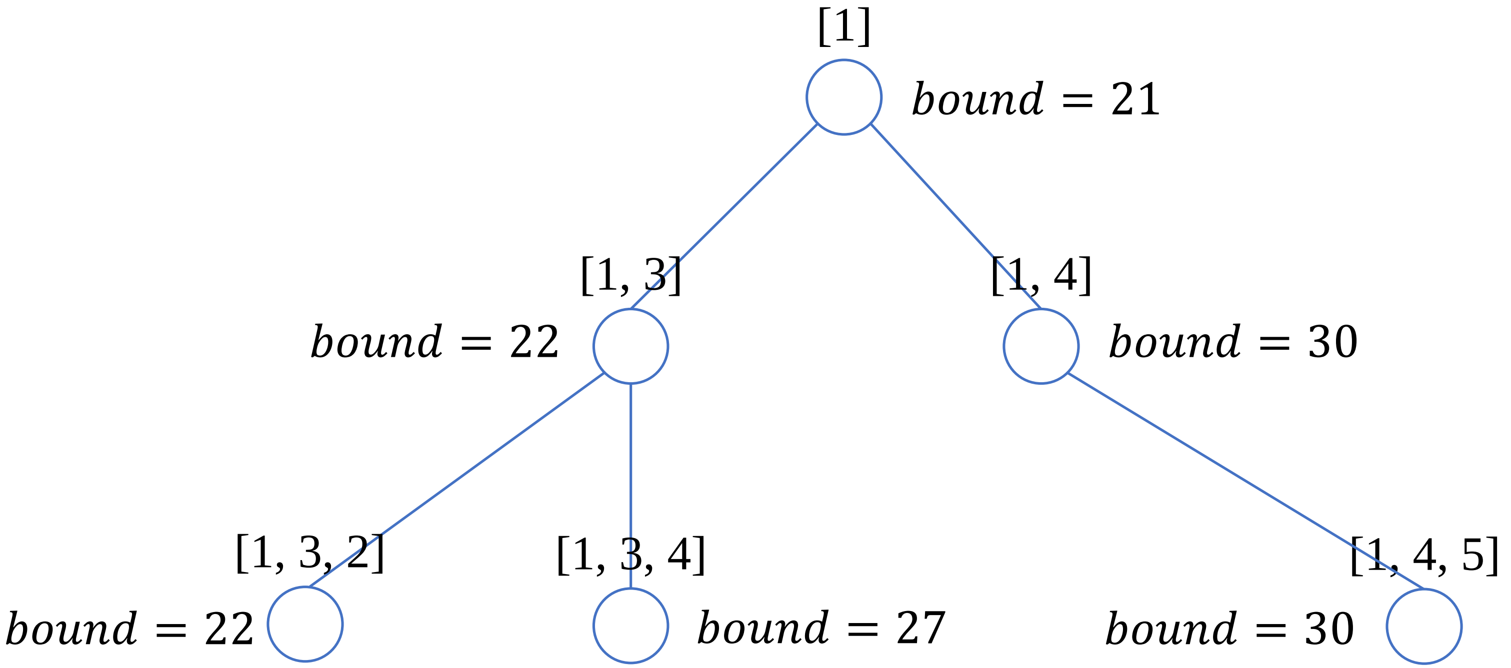


3. Node [1,4] is *removed* from the PQ.
   a. Bound is 30 and $minlength = 31$.
   b. Node [1,4,2] is NOT inserted into the PQ. $bound = 45 > minlenth = 31$.
   c. Node [1,4,3] is NOT inserted into the PQ. $bound = 38 > minlength = 31$.
   d. Node [1,4,5] is inserted into the PQ. $bound = 30$.

PQ

[1,4,5]
$(bound = 30)$
[1,2]
$(bound = 31)$
[1,3,5]
$(bound = 39)$
[1,5]
$(bound = 42)$

6.2 The Traveling Salesperson Problem



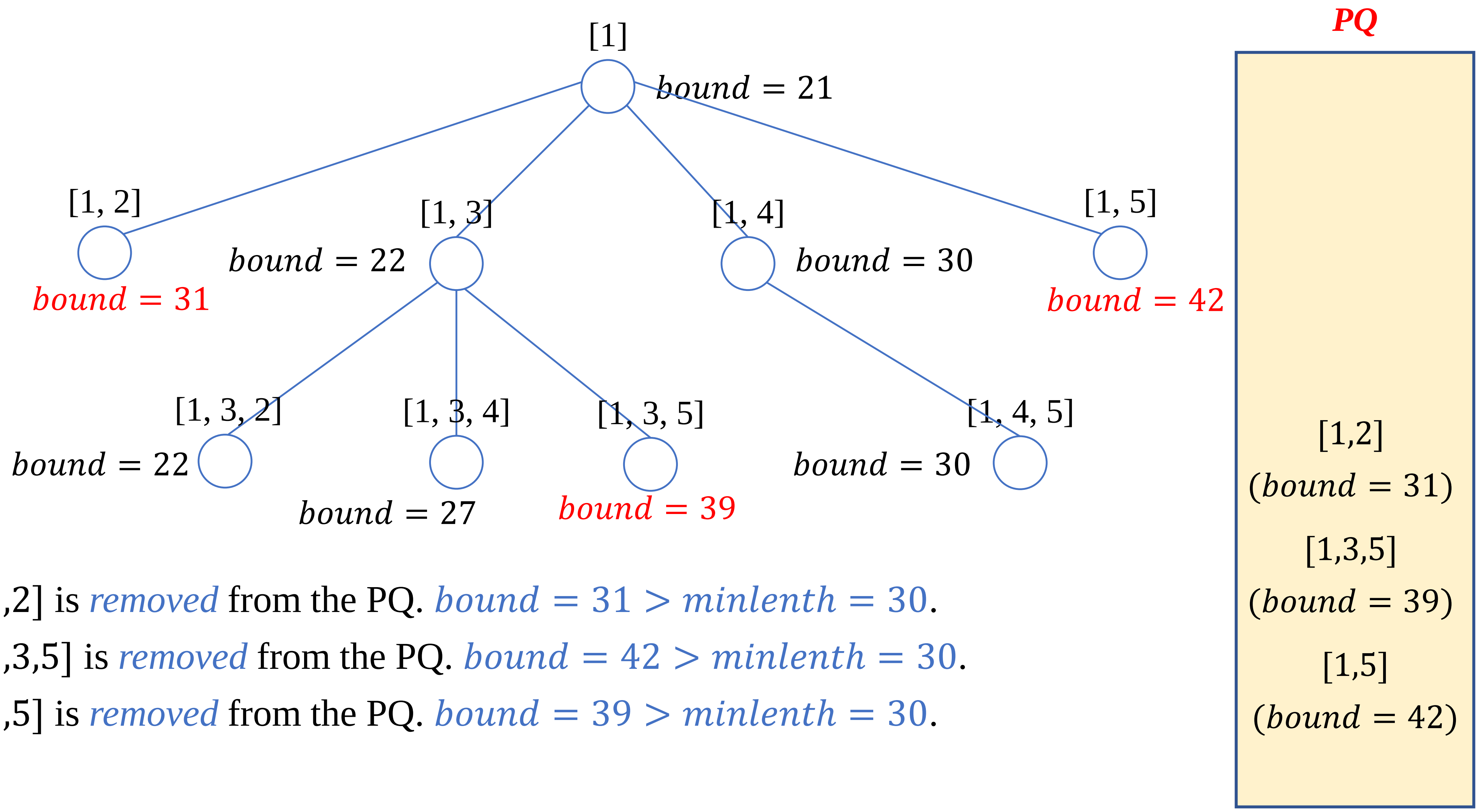
PQ

[1,3,5]
$(bound = 39)$

[1,5]
$(bound = 42)$

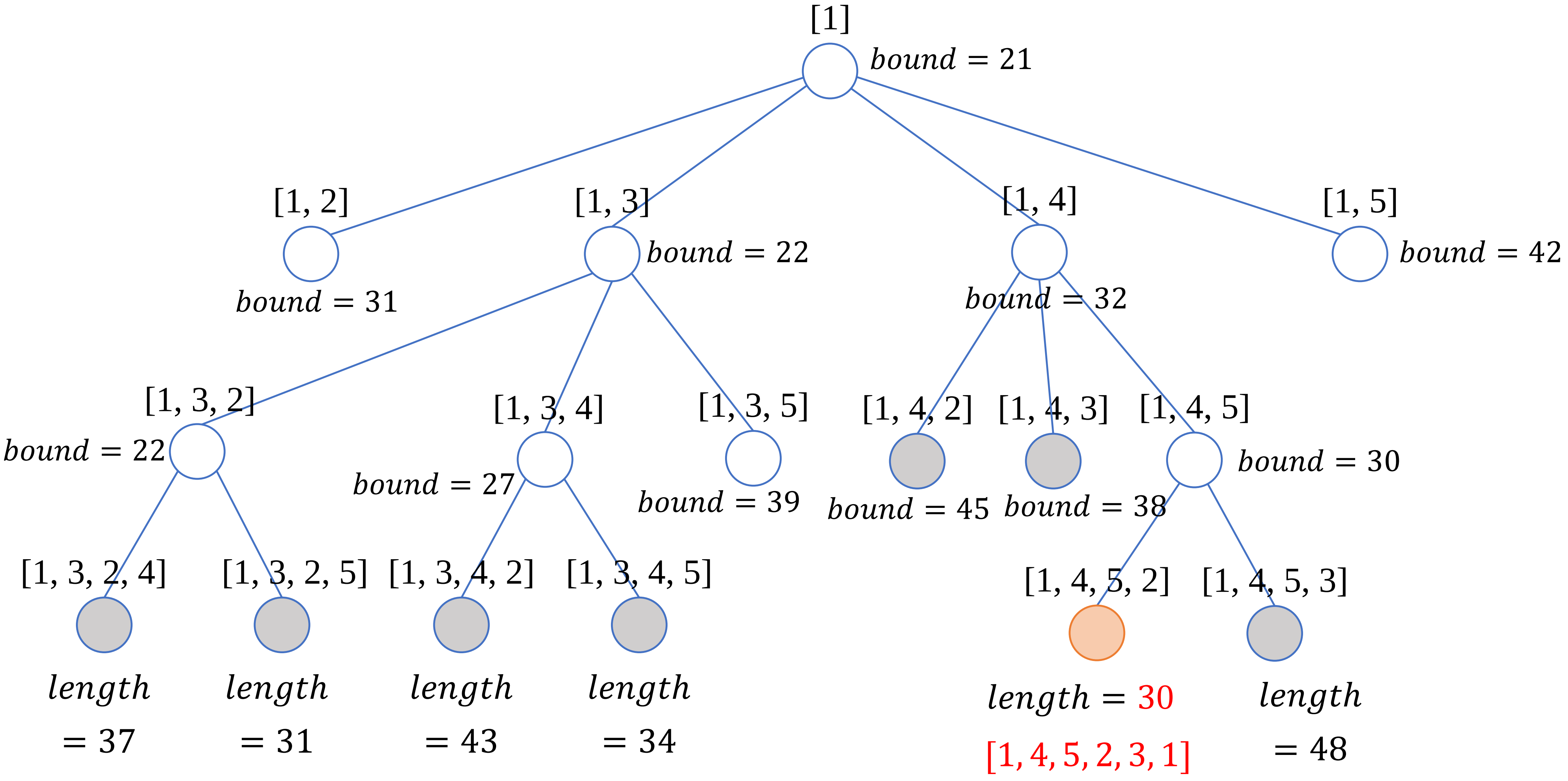
3. Node [1,4,5] is *removed* from the PQ.
   a. Bound is 30 and $minlength = 31$
   b. Node [1,4,5,2] is a leaf node. $length = 30$. $minlength = 30$. $opttour = [1,4,5,2,3,1]$.
   c. Node [1,4,5,3] is a leaf node. $length = 48$. $minlength = 30$.

6.2 The Traveling Salesperson Problem



3. Node [1,2] is *removed* from the PQ. $bound = 31 > minlenth = 30$.
4. Node [1,3,5] is *removed* from the PQ. $bound = 42 > minlenth = 30$.
5. Node [1,5] is *removed* from the PQ. $bound = 39 > minlenth = 30$.

6.2 The Traveling Salesperson Problem



ALGORITHM 6.3: The Best-First-Search with Branch-and-Bound Pruning for the TSP.

```
void travel2(ordered_set &opttour, int &minlength) {
    priority_queue_of_node PQ;
    node_pointer u, v;

    minlength = INF;
    v.level = 0; v.path = [1]; v.bound = bound(v);
    PQ.insert(v);
    while (!PQ.empty()) {
        v = PQ.remove();
        if (v->bound < minlength) {

            // ......

        }
    }
}
```

```
for (int i = 2; i <= n; i++) {
    // for all i such that i is not in v.path
    if (isIn(i, v->path)) continue;
    u = create_node(v->level + 1, v->path);
    u->path.push_back(i);
    if (u->level == n - 2) {
        // put the only vertex not in v.path
        u->path.push_back(remaining_vertex(u->path));
        u->path.push_back(1); // make first vertex last one
        if (length(u->path) < minlength) {
            minlength = length(u->path);;
            copy(u->path, opttour);
        }
    }
    else {
        u->bound = bound(u);
        if (u->bound < minlength)
            PQ.push(u);
    }
}
```

6.2 The Traveling Salesperson Problem

- The Efficiency for the TSP:
  - Note that there are *17 nodes* visited in the state space tree ,
    - whereas the entire state space tree has *41 nodes*.
    - $1 + 4 + 4 \times 3 + 4 \times 3 \times 2 = 41.$
  - When two or more bounding functions are available,
    - we can compute bounds using all available functions
  - However, remember that our goal is not to visit as few nodes as possible,
    - but rather to maximize the overall efficiency of the algorithm.
  - Final remark on the TSP is ***approximation algorithms***:
    - They are not guaranteed to yield optimal solution,
    - but rather yield solutions that reasonably close to optimal.

Any Questions?