Exact Solution:

Given,

$$u_t = u_{xx} + u_{yy} + u$$

Making the substitution $u = we^t$ gives,

$$w_t = w_{xx} + w_{yy}$$

In spherical coordinates we have

$$w_t = \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^3 w}{\partial r^3} + \frac{1}{r^2}\frac{\partial^3 w}{\partial \theta^3}$$

The exact solution is beyond the scope of this assignment. However, we still need to develop a spatial boundary condtion ($w(t = 0)$ which satisfies the above equation. Given an initial velocity $w_t = v_0$ and proposing a radially symmetric solution $w(x, y) = R(r)$ gives,

$$v_0 = \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^3 w}{\partial r^3} = \frac{1}{r}\frac{\partial w}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$$

$$rv_0 = \frac{\partial w}{\partial r}\left(r\frac{\partial w}{\partial r}\right)$$

$$\frac{1}{2}r^2 v_0 + \mathrm{c}_1 = r\frac{\partial w}{\partial r}$$

$$\frac{1}{r}r^2 v_0 + \mathrm{c}_1 \ln(r) + c_2 = w$$

The natural log blows up at the origin which is included in this problem, so we require $\mathrm{c}_1 = 0$. And we require that $w(R) = 0$ so this give the value of c_2 such that we are left with,

$$w(r) = \frac{1}{r}(r^2 - R^2) \quad \rightarrow \quad \text{Initial condition} \qquad [1]$$

Computational Solution (Douglas-Rachford ADI):

Calling the differential operators $\nabla_{\mathrm{xx}} = A$ and $\nabla_{\mathrm{yy}} = B$ then using the backward Euler method gives,

$$\frac{U^{n+1} - U^n}{k} = (A + B)U^{n+1} + U^{n+1} \qquad [2]$$

Or in terms of spherical coordinates

$$w_t = \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^3 w}{\partial r^3} + \frac{1}{r^2}\frac{\partial^3 w}{\partial \theta^3}$$

$$\frac{U^{n+1} - U^n}{k} = \left(\frac{1}{r_i}\frac{U_{i+1,j} - U_{i,j}}{h_R} + \frac{U_{i-1,j} - 2\,U_{i,j} + U_{i+1,j}}{h_R}\right) + \frac{1}{r_i^2}\left(\frac{U_{i,j-1} - 2\,U_{i,j} + U_{i,j+1}}{h_R}\right) + U_{i,j}$$

$$= \left(\frac{U_{i-1,j} - \left(2 + \frac{h_R}{r_i}\right)U_{i,j} + \left(1 + \frac{h_R}{r_i}\right)U_{i+1,j}}{h_R}\right) + \left(\frac{U_{i,j-1} - 2\,U_{i,j} + U_{i,j+1}}{r_i^2 h_R}\right) + U_{i,j}$$

Factoring out $1/r_i^2$ so as to avoid singularities,

$$\frac{U^{n+1} - U^n}{k} = \frac{1}{r_i^2}\left\{\left(\frac{U_{i-1,j}r_i^2 - \left(2r_i^2 + r_i h_R\right)U_{i,j} + \left(r_i^2 - r_i h_R\right)U_{i+1,j}}{h_R}\right) + \left(\frac{U_{i,j-1} - 2\,U_{i,j} + U_{i,j+1}}{h_R}\right)\right\} + U_{i,j}$$

$$= \frac{1}{r_i^2}\{\boldsymbol{R}U^{n+1} + \boldsymbol{T}U^{n+1}\} + U^{n+1}$$

$$= \frac{1}{r_i^2}\{(\boldsymbol{R} + \boldsymbol{T})U^{n+1}\} + U^{n+1} \qquad [2]$$

Where $\boldsymbol{T}$ stands for 'theta'. The proposed Douglas Rachford ADI scheme is just like last time, only now the definition of the matrices have changed,

$$\left(1 - \frac{k}{r_i^2}\boldsymbol{R}\right)U^{n^*} = \left(1 + \frac{k}{r_i^2}\boldsymbol{T}\right)U^n$$

$$\left(1 - \frac{k}{r_i^2}\boldsymbol{T}\right)U^{n+1} = U^{n^*} - \frac{k}{r_i^2}\boldsymbol{T}U^{n^*}$$

$$\left(r_i^2 - k\boldsymbol{R}\right)U^{n^*} = \left(r_i^2 + k\boldsymbol{T}\right)U^n$$

$$\left(r_i^2 - k\boldsymbol{T}\right)U^{n+1} = r_i^2 U^{n^*} - k\boldsymbol{T}U^{n^*} \qquad [2]$$

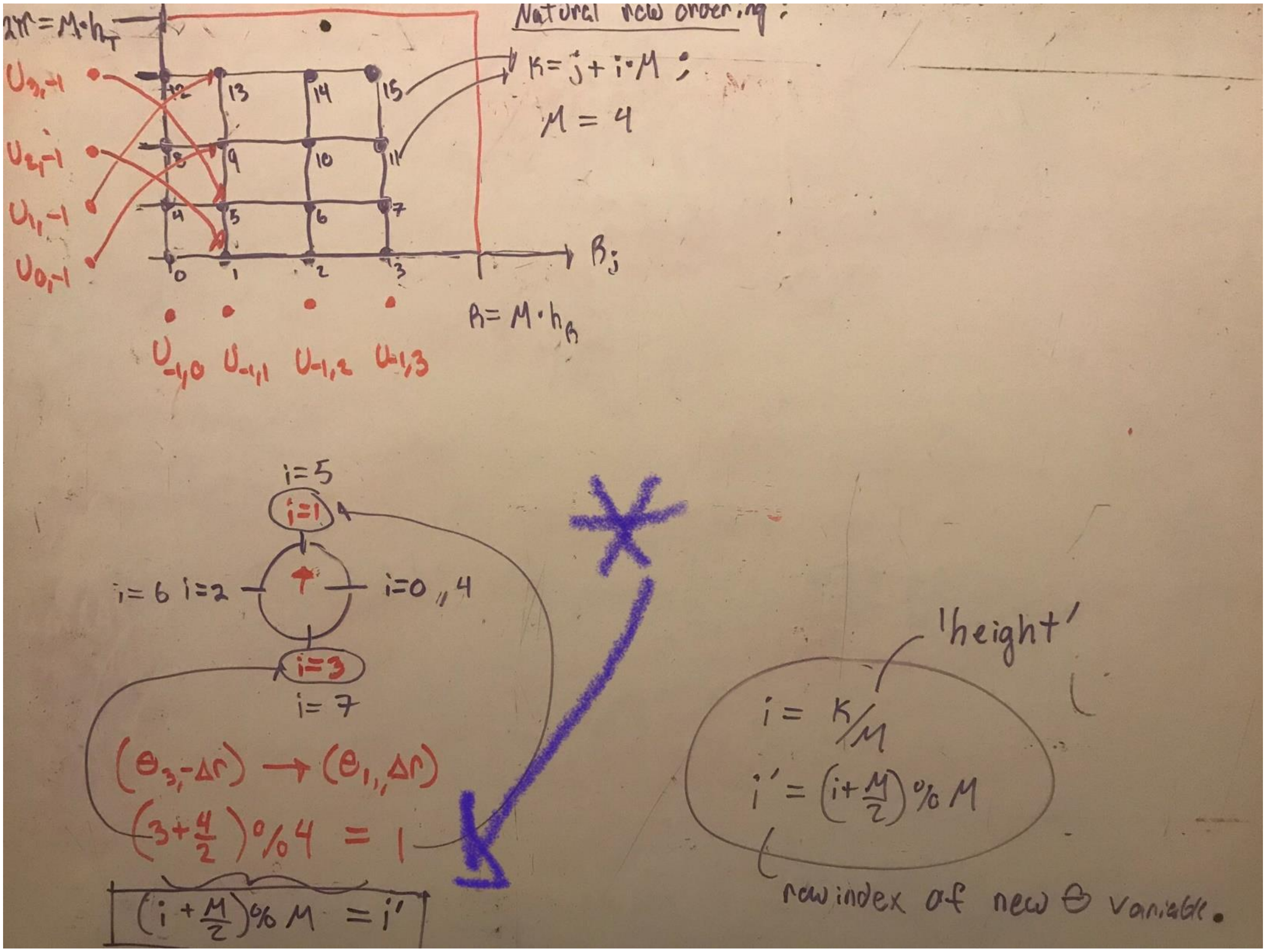
Some Notes About the Code:

The methodology adapted for this problem was to solve the problem in polar coordinates. Two primary difficulties were encountered

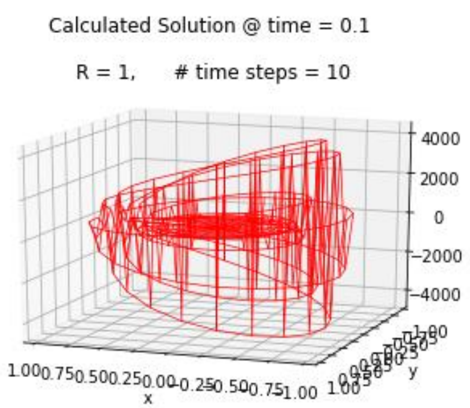
1) The factors of $1/r_i$, $1/r_i^2$ in the definitions of $\boldsymbol{R}, \boldsymbol{T}$ cause the algorithm to blow up. This was temporarily addressed by adjusting the vector $\vec{r}$ such that $\vec{r}[0] = \frac{h_R}{3} \approx 0.1$
2) Drawing the grid in the figure below, 'ghost points' arise on the left column and bottom row. These points (depicted in red in the figure) represent negative r, θ values. To deal with them, in lines 126-131 of the code the following identifications are made;

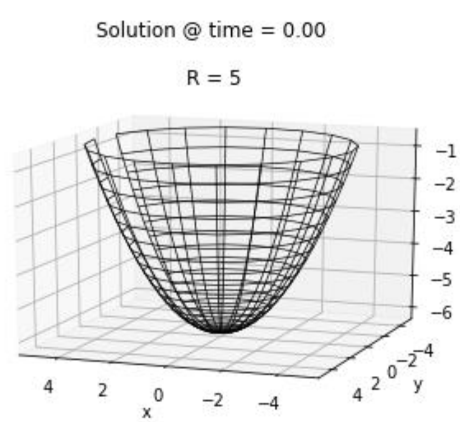
$i_g = j_A/M$	→	identify row index of ghost point (subscribt 'g' denotes that it is a subscript on the grid and subscript A denotes it is an index of matrix A)
$I_g = \left(i_g + \frac{M}{2}\right)\%M$	→	identify corresponding point that is within the grid and shifted by π, i.e. $\theta_{I_g} = \theta_{i_g} + \pi$

The figure below helps clarify what is happening;



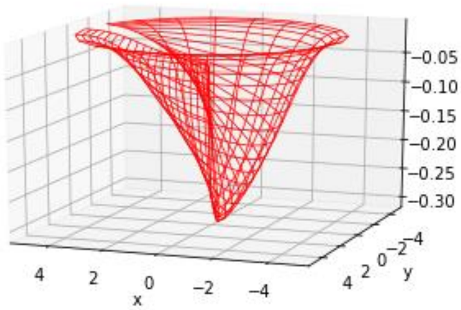
The results for a grid of $M = 20$, $N = 10$ time stps (with time step value $k = 1/1000$) and for various values of R are as follows;





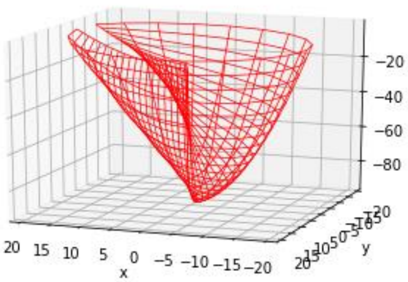
Calculated Solution @ time = 0.1

R = 5,   # time steps = 10



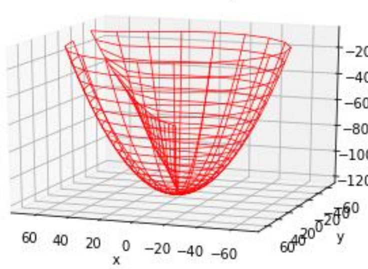
Calculated Solution @ time = 0.1

R = 20,   # time steps = 10



Calculated Solution @ time = 0.1

R = 70,   # time steps = 10



Calculated Solution @ time = 0.1

R = 100,   # time steps = 10