Analysis

The discretization of given equation is as follows,

$$u_t = u_{xx} + F(u) \; ; \qquad u(x,0) = \begin{cases} 0.1 & -1 \le x \le 1 \\ 0 & otherwise \end{cases}, \qquad u_x = 0 \text{ as } x \to \pm\infty$$

Where $F(u) = u(1-u)\left(u - \frac{1}{4}\right) \quad or \quad u(1-u)$. Seeking a travelling wave solution of the form $u(x,t) = f(x-ct)$ this equation transforms as,

$$-cf' = f'' + F(f)$$

Which gives the system of equations,

$$f' = w$$

$$w' = -cw - F(u)$$

The Jacobian matrix of this transform is,

$$J = \begin{pmatrix} \frac{\partial f'}{\partial f} & \frac{\partial f'}{\partial w} \\ \frac{\partial w'}{\partial f} & \frac{\partial w'}{\partial w} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\partial F(f)}{\partial f} & -c \end{pmatrix}$$

The eigenvalues are found as follows,

$$\begin{vmatrix} -\lambda & 1 \\ \frac{\partial F(u)}{\partial f} & -c-\lambda \end{vmatrix} = 0$$

$$\lambda(c+\lambda) - \frac{\partial F(u)}{\partial f} - 0$$

$$\left(\lambda + \frac{c}{2}\right)^2 = \frac{c^2}{4} + \frac{\partial F(u)}{\partial f}$$

$$\lambda = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - \frac{\partial F(u)}{\partial f}}$$

$\underline{\boldsymbol{F(u) = u(1-u)\left(u - \frac{1}{4}\right)}}$:

For $F(u) = u(1-u)\left(u - \frac{1}{4}\right)$ we have,

$$-\frac{\partial F(u)}{\partial f} = \frac{\partial}{\partial f}\left\{(f^2 - f)\left(f - \frac{1}{4}\right)\right\}$$

$$= \frac{\partial}{\partial f}\left\{f^3 - \frac{1}{4}f^2 - f^2 + \frac{1}{4}f\right\}$$

$$= \frac{\partial}{\partial f}\left\{f^3 - \frac{5}{4}f^2 + \frac{1}{4}f\right\}$$

And the eigenvalue becomes,

$$\lambda = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + 3f^2 - \frac{5}{2}f + \frac{1}{4}}$$

For $|x| > 1$ and at $t = 0$ we have $u(x,0) = f(x) = 0$ this is,

$$\lambda = -\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2 + 1} \qquad \rightarrow \qquad \text{unconditionally stable} \quad [1]$$

For $-1 \leq x \leq 1$ and at $t = 0$ we have $u(x,0) = f(x) = \frac{1}{10}$ this is,

$$\lambda = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + \left(\frac{3}{100} - \frac{5}{20} + \frac{1}{4}\right)} = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + \frac{3}{100}} \qquad \rightarrow \qquad \text{unconditionally stable} \quad [2]$$

$F(u) = u(1-u)$:

For $F(u) = u(1-u)$ we have,

$$-\frac{\partial F(u)}{\partial f} = \frac{\partial}{\partial f}\{(f^2 - f)\} = \{2f - 1\}$$

And the eigenvalue becomes,

$$\lambda = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + 2f - 1}$$

For $|x| > 1$ and at $t = 0$ we have $u(x,0) = f(x) = 0$ this is,

$$\lambda = -\frac{c}{2} \pm \frac{1}{2}\sqrt{c^2 - 4} \qquad \rightarrow \qquad \text{stable if } c \geq 2 \qquad [3]$$

For $-1 \leq x \leq 1$ and at $t = 0$ we have $u(x,0) = f(x) = \frac{1}{10}$ this is,

$$\lambda = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} + \frac{2}{10} - 1} = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - \frac{8}{10}}$$

$$= -\frac{c}{2} \pm \frac{1}{2}\sqrt{\frac{5c^2 - 16}{5}} \qquad \rightarrow \qquad \text{stable if } c \geq \frac{4}{\sqrt{5}} \approx 1.788 \qquad [4]$$

Equations [1],[2],[3], and [4] suggest that Fishers equation is conditionally stable, but when the nonlinear term is multiplied by a factor of $\left(u - \frac{1}{4}\right)$ it becomes unconditionally stable.

Computation: Forward Euler

The discretization of given equation is as follows,

$$u_t = u_{xx} + F(u)$$

$$\frac{U_m^{n+1} - U_m^n}{k} = \frac{U_{m-1}^n - 2U_m^n + U_{m+1}^n}{h^2} + kF(U_m^n)$$

$$U_m^{n+1} = U_m^n + r(U_{m-1}^n - 2U_m^n + U_{m+1}^n) + kF(U_m^n)$$

$$= \boldsymbol{A} * \underline{U}^n + kF(U_m^n) \qquad [4]$$

Noting that the algorithm begins at the left most point ($x = -Mh$), and that the corresponding index for this is actually $m = 0$, we can apply the boundary condition that the first spatial derivative vanish at $x = -Mh \rightarrow \infty$ with use of the central difference formula which gives for the boundary value $u_x = 0$ for $x \rightarrow \infty$,

$$\frac{U_1^n - U_{-1}^n}{2k} = 0 \quad \rightarrow \quad U_2^n = U_0^n$$

Substituting this into the initial special element ($m = 0$) gives,

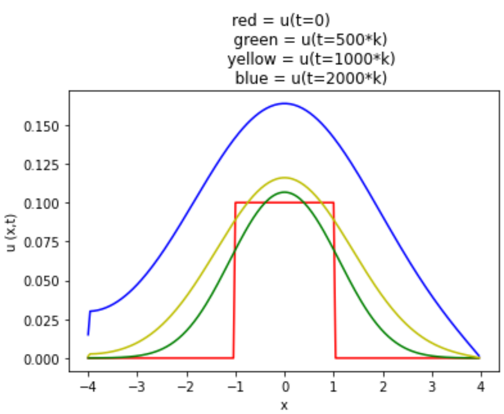
$$\begin{aligned} U_0^{n+1} &= U_0^n + r(U_{-1}^n - 2U_0^n + U_1^n) + kF(U_0^n) \\ &= U_0^n + r(-2U_0^n + 2U_1^n) + kF(U_0^n) \\ &= (1-2r)U_0^n + 2r2U_1^n + kF(U_0^n) \end{aligned}$$

Accordingly, in the algorithm the second line of the matrix $\boldsymbol{A}$ is adjusted (see lines 64-65). The term $kF(U_0^n)$ is accounted for in the time-iteration loop itself (see lines 90-95).

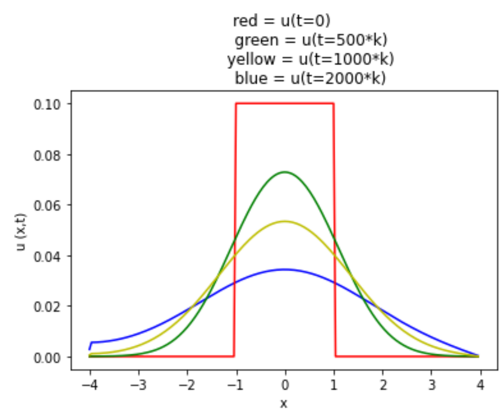
The spatial initial condition(s) are that $\bar{U}^0 = [0.1, 0.1, \dots, 0.1]$ for $|x| \le 1$ and $\bar{U}^0 = [0, 0, \dots, 0]$ for $|x| > 0$. These values are assigned via a function called 'bndry' (see lines 28-32).

The results for increasing values of n are shown below,

$\boldsymbol{F(u) = u(1-u)}$:



<u>$\boldsymbol{F(u) = u(1-u)\left(u - \frac{1}{4}\right)}$:</u>



We see that multiplying the nonlinear term in Fisher's equation by a factor of $\left(u - \frac{1}{4}\right)$ causes a dramatic difference; instead of growing with time, the solution decays with time.

The parameters used for this computation are as follows,

$$r = \frac{k}{h^2} \le \frac{2+\gamma k}{4} \quad \rightarrow \quad k \le \frac{1}{2\left(\frac{1}{h^2} - \frac{\gamma}{4}\right)} = \frac{\gamma h^2}{2 - \frac{h^2 \gamma}{2}} \;; \qquad 0 \le \gamma \le 1$$

$$h = \frac{L}{M} = \frac{1}{100}$$