Lax-Wendroff Method

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Given,

$$u_t + uu_x = 0\,; \qquad u(x,0) = e^{-16x^2}, \qquad u(-1,t) = 0$$

$$-1 < x < 1, \qquad t < 1 \qquad [1]$$

Exact solutions will not be pursued. Writing the derivatives out explicitly,

$$\frac{u(x,t) - u(x,t-\Delta t)}{\Delta t} + u\,\frac{u(x+\Delta x,t) - u(x-\Delta x,t)}{2\Delta x} = 0$$

Where backward difference was applied to the time derivative and central difference to the spatial. Taylor expanding the first terms to second order gives,

$$\frac{u - \left[u - \Delta t u_t + \frac{\Delta t^2}{2} u_{tt}\right]}{\Delta t} + u\,\frac{\left[u + \Delta x u_x + \frac{\Delta t^2}{2} u_{xx}\right] - \left[u - \Delta x u_x + \frac{\Delta x^2}{2} u_{xx}\right]}{2\Delta x} = 0$$

$$\frac{\Delta t u_t - \frac{\Delta t^2}{2} u_{tt}}{\Delta t} + u\,\frac{2\Delta x u_x}{2\Delta x} = 0$$

$$u_t - \frac{\Delta t}{2} u_{tt} \quad + \; uu_x = 0$$

$$u_t + uu_x = \tfrac{\Delta t}{2} u_{tt} \qquad [2]$$

Two applications of the given differential equation [1] give a substitution for u_{tt}

$$\begin{aligned} u_{tt} &= -u_t u_x - uu_{xt} \\ &= -u_t u_x - u(uu_x)_x \\ &= uu_x^2 + u(u_x^2 + uu_{xx})_x \\ &= 2uu_x^2 + u^2 u_{xx} \end{aligned}$$

Substituting this into [2] gives,

$$\boxed{u_t + uu_x = \tfrac{\Delta t}{2}(2uu_x^2 + u^2 u_{xx})}$$

Which is the Lax-Wendroff scheme for the Burger's equation[1]. Discretizing the above gives,

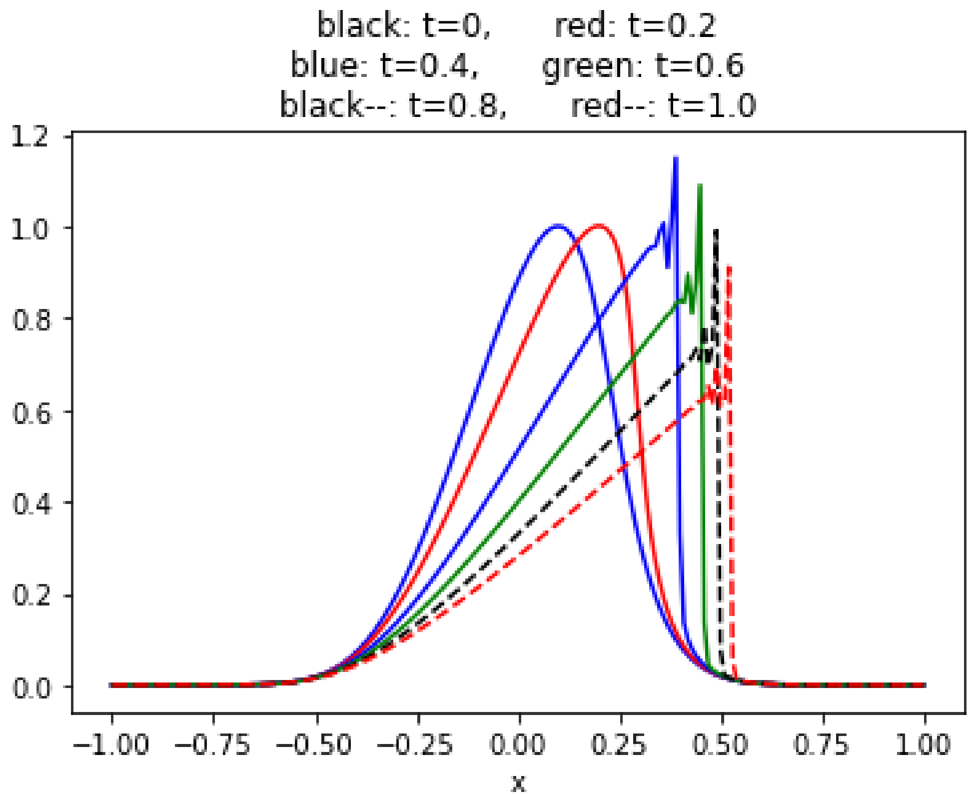
$$U_j^{k+1} = U_j^k - kU_j^k\left(\frac{U_{j+1}^k - U_{j-1}^k}{2h}\right) + \frac{k^2}{2}\left(U_j^k\left(\frac{U_{j+1}^k - U_{j-1}^k}{2h}\right)^2 + \left(U_j^k\right)^2\left(\frac{U_{j+1}^k - 2U_j^k + U_{j-1}^k}{h^2}\right)\right) \qquad [3]$$

Which is the equation which the code solves for. Due to the terms being squared I did not use a matrix method to solve this. Instead, for each time value k I went through and iterated the above code – nothing fancy. I believe this is just the Euler method, but the reference I am following refers to it as the 'Lax-Wendroff scheme'. Judging from other

[1] I am following *Numerical Solutions of Differential Equations* by Li, Qiao, and Tang.

applications in the same chapter of this reference, it seems it is possible to apply a method similar to ADI, but this will not be attempted.

Von-Neumann stability analysis does not work with this as the exponential terms do not all simplify to sins or cosines. The results are depicted below.



We see a shock wave develop at x = 0.5.