V-Cycle Algorithm.

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1. Introduction

Given the differential equation,

$$\Delta v = 0 \ ; \quad x \in \Omega, \partial\Omega$$

The one-dimensional discretized Laplacian is,

$$-\frac{v_{j-1} - 2v_j + v_{j+1}}{2} = f_{i.j} = 0$$

When applied to a one dimensional grid defined on $x \, \epsilon \, [0,1]$ divided into $M = 8$ equidistant points, and with a Dirichlet boundary condition $f_{ij} = 0$ on $\partial\Omega$ yields the $7x7$ matrix equation,

$$\boldsymbol{A^h} \vec{v}_j^{\,h} = \vec{f}_j^{\,h} = 0 \ ; \qquad [1.1]$$

$$\boldsymbol{A^h} = \frac{1}{\Delta x^2}\begin{pmatrix} 2 & -1 & -0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 \end{pmatrix}$$

The eigenvalues and eigenvectors of matrix $\boldsymbol{A^h}$ are,

$$\lambda_j^h = \frac{1}{\Delta x^2}\left(2 - 2\omega \cos\frac{j\pi}{M}\right) \qquad \vec{w}_j = \begin{bmatrix} w_{j1} \\ w_{j2} \\ w_{j3} \\ w_{j4} \\ w_{j5} \\ w_{j6} \\ w_{j7} \end{bmatrix} = 2\begin{pmatrix} \sin\frac{j\pi}{M^h} \\ \sin\frac{2j\pi}{M^h} \\ \sin\frac{3j\pi}{M^h} \\ \sin\frac{4j\pi}{M^h} \\ \sin\frac{5j\pi}{M^h} \\ \sin\frac{6j\pi}{M^h} \\ \sin\frac{7j\pi}{M^h} \end{pmatrix}; \qquad M^h = 8 \qquad [1]$$

While no attempt to prove this formula will be made, a demonstration that it works is offered in appendix (see equation A.1). Also derived in the appendix (as part of the derivation of equation A.2 and A.5) the restriction and prolongation operators are,

$$\boldsymbol{I_h^{2h}} = \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix} \quad \rightarrow \quad \text{restriction} \qquad [1.2]$$

$$\boldsymbol{I_{2h}^h} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad \rightarrow \quad \text{prolongation} \qquad [1.3]$$

And the transformed matrix is,

$$I_h^{2h} A^h I_{2h}^h \;=\; A^{2h} = \frac{1}{(2\Delta x)^2}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \qquad [1.4]$$

Correspondingly, the eigenvalue of $\boldsymbol{A^{2h}}$ needs to be multiplied by $1/(2\Delta x)^2$ instead of $1/\Delta x^2$. The eigenvector has the same form albeit with $M^h = 8$ replaced with $M^{2h} = 4$.

In addition, as derived in the appendix (see the derivation of equations A.2 and A.5), we have the course grid interpolation matrices.

$$I_{2h}^{4h} = \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \qquad [1.5]$$

$$I_{4h}^{2h} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad [1.6]$$

And these gives for $\boldsymbol{A^{4h}}$ on the coarsest grid (G^{4h})

$$\begin{aligned} A^{4h} = I_{2h}^{4h} A^{2h} I_{4h}^{2h} \;&=\; \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} * \frac{1}{(2\Delta x)^2}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} * \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \\ &=\; \frac{1}{32\Delta x^2}\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} * \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \\ &=\; \frac{1}{8\Delta x^2} \qquad [1.7] \end{aligned}$$

As derived in the appendix (see equations A.2 - A.6) some other useful relations are,

$$I_h^{2h} \vec{w}_j^h \;=\; \cos^2\left(\frac{j\pi}{16}\right) \vec{w}_j^{2h} \qquad [A.2]$$

$$I_h^{2h} \vec{w}_{M-j}^h \;=\; -\sin^2\left(\frac{j\pi}{16}\right) \vec{w}_j^{2h} \qquad [A.3]$$

$$I_h^{2h} \vec{w}_{M/2}^h \;=\; 0 \qquad [A.4]$$

$$I_{2h}^h \vec{w}_j^{2h} \;=\; \cos^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_j^h - \sin^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_{M-j}^h \qquad [A.5]$$

$$I_h^{2h} \vec{r}_j^h \;=\; \vec{r}_j^{2h} \qquad [A.6]$$

2. Residual Correction for Weighted Jacobi scheme

For the Jacobi iteration method, the residual matrix $\boldsymbol{R}$ and the relation between its eigenvalues λ_j^R and that of the eigenvalues for matrix $\boldsymbol{A}$ (λ_j) are,

$$\boldsymbol{R = I - D^{-1}A} \quad \rightarrow \quad \boldsymbol{A \;=\; D(I - R)}$$

$$\lambda_j \;=\; d(1 - \lambda_j) \qquad [2.1]$$

Where $d = \frac{2}{\Delta x^2}$. The residual matrix can be rewritten as,

$$\boldsymbol{R \;=\; I - D^{-1}A \;=\; I - D^{-1}[L + D + U] \;=\; -D^{-1}[L + U]}$$

Therefore, the eigenvalues μ_j for the residual matrix obey the equation,

$$\boldsymbol{R}\vec{u}_j = \mu_j \vec{u}_j$$

$$-\frac{1}{d}\frac{1}{\Delta x^2}\begin{pmatrix} 0 & -1 & \dots & 0 \\ -1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \mu_j \begin{pmatrix} u_{j_1} \\ u_{j_2} \\ \vdots \\ u_{j_7} \end{pmatrix} \quad ; \quad on\ G^h$$

$$-\frac{1}{d}\frac{1}{(2\Delta x)^2}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = \mu_j \begin{pmatrix} u_{j_1} \\ u_{j_2} \\ u_{j_3} \end{pmatrix} \quad ; \quad on\ G^{2h}$$

Where μ_j is the eigenvector of the Jacobi residual matrix and d is the diagonal entry of matrix $\boldsymbol{A}$. Corresponding to $\boldsymbol{A^h}$ and $\boldsymbol{A^{2h}}$, the variable $d = \frac{2}{\Delta x^2}$ for the fine grid (G^h) iteration matrix ($\boldsymbol{R^h}$) and $d = \frac{2}{(2\Delta x)^2}$ for the course grid (G^{2h}) iteration matrix $\left(\boldsymbol{R^{2h}}\right)$. The same formula (see derivation of A.1 in the appendix) which was used to calculate the eigenvalues for matrix $\boldsymbol{A}$ can be used to find μ_j which is the same for both cases;

$$\mu_j^h = \frac{2}{d\Delta x^2}\cos\frac{j\pi}{M} = \cos\frac{j\pi}{M}$$

$$\mu_j^{2h} = \frac{2}{d(2\Delta x)^2}\cos\frac{j\pi}{M} = \cos\frac{j\pi}{M} \qquad [2.2]$$

For the weighted Jacobi iteration method, we have,

$$\boldsymbol{R}^\omega = \boldsymbol{I} - \omega\boldsymbol{D}^{-1}\boldsymbol{A} = \boldsymbol{I} - \omega\boldsymbol{D}^{-1}[\boldsymbol{L} + \boldsymbol{D} + \boldsymbol{U}] = (1-\omega)\,\boldsymbol{I} + \omega\boldsymbol{R} \qquad [2.3]$$

From this the eigenvalues for the weighted Jacobi scheme can easily be inferred to be,

$$\begin{aligned} \mu_j^\omega = (1-\omega) + \omega\mu_j &= (1-\omega) - \frac{2}{d}\omega\cos\left(\frac{j\pi}{M}\right) \\ &= 1 - \omega\left(1 - \cos\left(\frac{j\pi}{M}\right)\right) \\ &= 1 - 2\omega\cos^2\left(\frac{j\pi}{M}\right) \end{aligned}$$

Rearranging equation [2.3] gives the relation between $\boldsymbol{A}$ and matrix $\boldsymbol{R}^\omega$, and by extension that of their eigenvalues,

$$\begin{aligned} \boldsymbol{A} = \frac{1}{\omega}\boldsymbol{D}(\boldsymbol{I} - \boldsymbol{R}^\omega) \quad \rightarrow \quad \lambda_j &= \frac{d}{\omega}\left(1 - \mu_j^\omega\right) \\ &= \frac{1}{\omega}\frac{2}{\Delta x^2}\left(1 - \left[1 - 2\omega\cos^2\left(\frac{j\pi}{M}\right)\right]\right) \\ &= \frac{4}{\Delta x^2}\cos^2\left(\frac{j\pi}{M}\right) \end{aligned} \qquad [2.4]$$

**Note: from here on out the eigenvalue μ_j^ω will be denoted as μ_j^h (fine grid) or μ_j^{2h} (course grid) with the understanding that weighted Jacobi eigenvalue is intended.*

3. Multigrid:

**Note to professor: I spent the majority of my time this last couple of weeks [painfully] trying to decipher J.W. Thomas' multigrid (section 10.10.4). As it turned out, this section was mostly unnecessary, but enlightening. I chose to leave the work done included in this report only because I spent a lot of time on it (even if it seems I simply coped his results). And also, the calculations are relevant up to the initial calculation of the residual on G^{2h}. Everything beyond this point has been highlighted which is to indicate it is not directly relevant to the V-cycle algorithm.*

We begin by guessing an initial solution of the form,

$$\vec{u}_0^h = \sum_{j=1}^{M^h-1} \vec{w}_j^h \qquad [3.0]$$

With $M^h = 8$ the vector $\vec{u}_0^h$ is a sum of eight eigenvectors (to matrix $\boldsymbol{A^h}$) $\vec{w}_j^h$. The following analysis will be broken into three cases; $j = 0, \dots, \frac{M^h}{2} - 1$, $j = \frac{M^h}{2}$, and $j = \frac{M^h}{2} + 1, \dots, M^h - 1$. Furthermore, because the vectors $\vec{w}_j^h$ are added to produce $\vec{u}_0^h$, we can break the problem into $M^h - 1 = 7$ problems to be solved with the weighted Jacob residual correction method;

$$\vec{u}_{0\,j}^h = \vec{w}_j^h \quad ; \quad k = 1, \dots, M^h - 1 = 7$$

<u>**Case: $\boldsymbol{j = 0, \dots, M^h/2 - 1}$**</u>

Let $\vec{v}_j^h$ denote the solution to [1.1]. With $\vec{v}_j^h = 0$ we have for the initial error,

$$\vec{e}_0^h = 0 - \vec{u}_{0_j}^h = -\vec{u}_{0_j}^h = -\vec{w}_j^h$$

Where $\vec{u}_{0_j}^h = \vec{w}_j^h$ as the initial approximation for the eigenvector which satisfies equation [1.1] the error after the n^{th} iteration is,

$$\vec{e}_n^h = 0 - \vec{u}_{n_j}^h = (\boldsymbol{R}^\omega)^n \vec{e}_0 = -(\boldsymbol{R}^\omega)^n \vec{w}_j^h = -\left(\mu_j^h\right)^n \vec{w}_j^h \qquad [3.1]$$

And the residual on the fine grid is,

$$\begin{aligned} \vec{r}_{n\,j}^{\,h} &= \vec{f}_{n\,j}^{\,h} - \boldsymbol{A^h}\vec{u}_{n_j}^h \\ &= 0 - \left(\mu_j^h\right)^n \boldsymbol{A^h}\vec{w}_j^h \\ &= -\left(\mu_j^h\right)^n \lambda_j^h \vec{w}_j^h \end{aligned} \qquad [3.2]$$

Using equation A.2 and A.6 together with the above relations gives,

$$\begin{aligned} \vec{r}_{n\,j}^{\,2h} &= \boldsymbol{I_h^{2h}} \vec{r}_{n\,j}^{\,h} \\ &= -\boldsymbol{I_h^{2h}}\left(\mu_j^h\right)^n \lambda_j^h \vec{w}_j^h = -\left(\mu_j^h\right)^n \lambda_j^h \cos^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_j^{2h} \end{aligned} \qquad [3.3]$$

Which implies that the solution vector $\vec{e}_{n\,j}^{\,2h}$ on the course grid may be written as the above quantity divided by the eigenvalue of matrix $\boldsymbol{A^{2h}}$,

$$\vec{e}_{n_j}^{\ 2h} = -\left(\mu_j^h\right)^n \frac{\lambda_j^h}{\lambda_j^{2h}} \cos^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_j^{2h} = c_{n_j} \vec{w}_j^{2h} \qquad [3.4.a]$$

Where,

$$\vec{c}_{n_j} = -\frac{\lambda_j^h}{\lambda_j^{2h}} \boldsymbol{I_h^{2h}} (\boldsymbol{R}^\omega)^n \vec{w}_j^h = -\left(\mu_j^h\right)^n \frac{\lambda_j^h}{\lambda_j^{2h}} \cos^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_j^{2h} \qquad [3.4.b]$$

From this we get the general relation

$$c_{(m+n)_j} = -\frac{\lambda_j^h}{\lambda_j^{2h}} \boldsymbol{I_h^{2h}} (\boldsymbol{R}^\omega)^{m+n} \vec{w}_j^h = \left(\mu_j^h\right)^m \left(\mu_j^h\right)^n c_{n_j} \qquad [3.5]$$

Using [3.1] then [A.5] the fine grid error becomes,

$$\vec{e}_{n_j}^h = \boldsymbol{I_{2h}^h} \vec{e}_j^{2h} = c_{n_j} \boldsymbol{I_{2h}^h} \vec{w}_j^{2h} = c_{n_j} \left[\cos^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_j^h - \sin^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_{M-j}^h \right]$$

With this we have for one iteration on the fine grid,

$$\vec{u}_{(n+1)_j}^h = \vec{u}_{n_j}^h + \vec{e}_{n_j}^{\ h}$$

$$= \left(\mu_j^h\right)^n \vec{w}_j^h + c_{n_j} \left[\cos^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_j^h - \sin^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_{M-j}^h \right]$$

$$= \left[\left(\mu_j^h\right)^n + c_{n_j} \cos^2\left(\frac{j\pi}{2M^h}\right) \right] \vec{w}_j^h - c_{n_j} \sin^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_{M-j}^h \qquad [3.6]$$

Now let $m = n + n'$ where n' is the number of ,,,???? [unresolved]???.... For $\vec{u}_{(m+1)_j}^h$ we have,

$$\vec{u}_{(m+1)_j}^h = \vec{u}_{m_j}^h + \vec{e}_{m_j}^{\ h}$$

$$= \left(\mu_j^h\right)^m \vec{w}_j^h + \boldsymbol{I_{2h}^h} \vec{u}_{m_j}^{\ 2h}$$

$$= \left(\mu_j^h\right)^{n+n'} \vec{w}_j^h + c_{m_j} \boldsymbol{I_{2h}^h} \vec{w}_{m_j}^{\ 2h}$$

$$= \left(\mu_j^h\right)^{n+n'} \vec{w}_j^h + c_{m_j} \left[\cos^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_{m_j}^h - \sin^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_{m_{M-j}}^h \right]$$

Utilizing [3.5] this becomes,

$$\vec{u}_{(m+1)_j}^h = \left(\mu_j^h\right)^{n+n'} \vec{w}_j^h + c_{m_j} \left[\cos^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_{m_j}^h - \sin^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_{m_{M-j}}^h \right]$$

$$= \left(\mu_j^h\right)^{n'} \left[\left(\mu_j^h\right)^n + c_{n_j} \cos^2\left(\frac{j\pi}{2M^h}\right) \right] \vec{w}_j^h - c_{n_j} \left(\mu_{M-j}^h\right)^{n'} \sin^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_{M-j}^h \qquad [3.7]$$

<u>Case: $j = M/2$:</u>

With the initial guess $\vec{u}_{0_{M^h/2}}^h = \vec{w}_{M^h/2}^{2h}$, and as in [3.1] we have,

$$\vec{u}_{n_{M^h/2}}^h = \left(\mu_{M^h/2}^h\right)^n \vec{w}_{M^h/2}^h \qquad [3.8]$$

The residual vector is then,

$$\vec{r}^h_{M^h/2} = \vec{f}_{M^h/2_j}{}^h - \boldsymbol{A^h}\vec{u}^h_{n_{M^h/2}} = 0 - \left(\mu^h_{M^h/2}\right)^n \lambda^h_{M^h/2}\vec{w}^h_{M^h/2}$$

The above result along with [A.6] and [A.4] gives,

$$\vec{r}_{M^h/2_j}{}^{2h} = \boldsymbol{I^h_{2h}}\vec{r}_{M^h/2_j}{}^h = -\left(\mu^h_{M^h/2}\right)^n \lambda^h_{M/2}\boldsymbol{I^h_{2h}}\vec{w}^h_{M^h/2} = 0 \qquad [3.9]$$

Following the same reasoning that led to [3.3] and [3.4], the error $\vec{e}^h_{M/2}$ and coefficient $c_{M/2}$ on the fine grid are easily seen to be,

$$\vec{e}_{n_{M^h/2}}{}^h = 0$$

$$c_{n_{M^h/2}} = 0$$

And this gives for the iterative solution $\vec{u}^h_{(n+1)_j}$,

$$\vec{u}^h_{(n+1)_{M^h/2}} = \vec{u}^h_{n_{M^h/2}} + \vec{e}_{n_{M^h/2}}{}^h = \vec{u}^h_{n_{M^h/2}} \qquad [3.10]$$

$$\vec{u}^h_{(m+1)_{M^h/2}} = \vec{u}^h_{m_{M^h/2}} + \vec{e}_{m_{M^h/2}}{}^h = \left(\mu^h_{M^h/2}\right)^m \left(\mu^h_{M^h/2}\right)^n \vec{w}^h_{M^h/2} \qquad [3.11]$$

***Case*: $\boldsymbol{j = M^h/2 + 1, \dots, M^h - 1}$**

As before we have

$$\vec{u}^h_{0_j} = \vec{w}^h_j \;;\quad j = \frac{M^h}{2} \dots . M^h - 1$$

$$\vec{u}^h_{n_j} = \left(\mu^h_j\right)^n \vec{w}^h_j \qquad [3.12]$$

$$\vec{r}_{n_j}{}^h = -\left(\mu^h_j\right)^n \lambda^h_j \vec{w}^h_j$$

Using [A.2] we get for the residual vector on the course grid

$$\vec{r}_{n_j}{}^{2h} = \boldsymbol{I^{2h}_h}\vec{r}_{n_j}{}^h = -\left(\mu^h_j\right)^n \lambda^h_j \boldsymbol{I^{2h}_h}\vec{w}^h_j = -\left(\mu^h_j\right)^n \lambda^h_j \cos^2\left(\frac{j\pi}{2M}\right) \vec{w}^{2h}_j \qquad [3.13]$$

As before the equation $\boldsymbol{A^{2h}}\vec{e}_{n_j}{}^{2h} = \vec{r}_{n_j}{}^{2h}$ gives $\vec{e}_{n_j}{}^{2h}$ as the above quantity divided by the eigenvalue of matrix $\boldsymbol{A^{2h}}$

$$\vec{e}_{n_j}{}^{2h} = -\left(\mu^h_j\right)^n \frac{\lambda^h_j}{\lambda^{2h}_{M^h-j}} \cos^2\left(\frac{j\pi}{16}\right)\vec{w}^{2h}_j = -\,c_{n_{M^h-j}}\vec{w}^{2h}_j$$

Where

$$c_{n_{M^h-j}} = \left(\mu^h_j\right)^n \frac{\lambda^h_j}{\lambda^{2h}_{M-j}} \cos^2\left(\frac{j\pi}{16}\right)$$

Note that this time the coefficient does not include the negative sign. Using A.5 the error on the fine grid is,

$$\vec{e}^h_{n_j} = \boldsymbol{I^h_{2h}}\vec{e}_{n_j}{}^{2h} = -c_{n_{M^h-j}}\boldsymbol{I^h_{2h}}\vec{w}^h_j$$

$$= c_{n_{M^h-j}}\left[\sin^2\left(\frac{j\pi}{2M^h}\right)\vec{w}^h_{M^h-j} - \cos^2\left(\frac{j\pi}{2M^h}\right)\vec{w}^h_j\right] \qquad [3.14]$$

Again with $\vec{f}_{n_j}{}^h = 0$ we get for the error on the fine grid,

$$\vec{u}^h_{(n+1)_j} = \vec{u}^h_{n_j} + \vec{e}_{n_j}^{\ h}$$

$$= \left(\mu^h_j\right)^n \vec{w}^h_j + c_{n_{M^h-j}} \left[\sin^2\left(\frac{j\pi}{2M^h}\right)\vec{w}^h_{M^h-j} - \cos^2\left(\frac{j\pi}{2M^h}\right)\vec{w}^h_j\right]$$

$$= \left[\left(\mu^h_j\right)^n - c_{n_{M^h-j}} \cos^2\left(\frac{j\pi}{2M^h}\right)\right]\vec{w}^h_j + c_{n_{M^h-j}} \sin^2\left(\frac{j\pi}{2M^h}\right)\vec{w}^h_{M^h-j} \qquad [3.15]$$

For $\vec{u}^h_{(m+1)_j}$ we have,

$$\vec{u}^h_{(m+1)_j} = \vec{u}^h_{m_j} + \vec{e}_{m_j}^{\ h}$$

$$= \left(\mu^h_j\right)^{n+n'} \vec{w}^h_j + c_{m_{M^h-j}} \left[\sin^2\left(\frac{j\pi}{2M^h}\right)\vec{w}^h_{M^h-j} - \cos^2\left(\frac{j\pi}{2M^h}\right)\vec{w}^h_j\right]$$

Using [3.5] this becomes,

$$\vec{u}^h_{(m+1)_j} = \vec{u}^h_{m_j} + \vec{e}_{m_j}^{\ h}$$

$$= \left(\mu^h_j\right)^{n+n'} \vec{w}^h_j + c_{n_{M^h-j}} \left[\left(\mu^h_{M^h-j}\right)^{n'} \sin^2\left(\frac{j\pi}{2M^h}\right)\vec{w}^h_{M^h-j} - \left(\mu^h_j\right)^{n'} \cos^2\left(\frac{j\pi}{2M^h}\right)\vec{w}^h_j\right]$$

$$= \left(\mu^h_j\right)^{n'} \left[\left(\mu^h_j\right)^n - c_{n_{M^h-j}} \cos^2\left(\frac{j\pi}{2M^h}\right)\vec{w}^h_j\right]\vec{w}^h_j + c_{n_{M^h-j}} \left(\mu^h_{M^h-j}\right)^{n'} \sin^2\left(\frac{j\pi}{2M^h}\right)\vec{w}^h_{M^h-j} \qquad [3.16]$$

Final solution(s)

In accordance with [3.1], the final solution(s) are,

$$\vec{u}^h_{n+1} = \sum_{j=1}^{M^h/2-1} \vec{u}^h_{(n+1)_j} + \vec{u}^h_{(n+1)_{M^h/2}} + \sum_{j=M^h/2+1}^{M^h-1} \vec{u}^h_{(n+1)_j} \qquad [3.17]$$

In the case of the simple course grid correction scheme (***not the V-cycle***) $\vec{u}^h_{(n+1)_j}$ in the first, second, and third term is given by [3.6], [3.10], and [3.15], respectively.

$$\vec{u}^h_{m+1} = \sum_{j=1}^{M^h/2-1} \vec{u}^h_{(m+1)_j} + \vec{u}^h_{(m+1)_{M^h/2}} + \sum_{j=M^h/2+1}^{M^h-1} \vec{u}^h_{(m+1)_j} \qquad [3.18]$$

Equations [3.17] and [3.18] are the equations to be coded for. Again, *highlighted portions of the above analysis are redundant to the v-cycle algorithms*. However, the derived equations for $\vec{u}_{n_j}^{\ 2h}$ (equations [3.1], [3.8], and [3.12]) will be used to calculate the final correction(s) to $\vec{u}_{n_j}^{\ h}$. After n initial relaxations on the fine grid [followed by some grid transfers and recalculation of error terms resulting in $\vec{e}_{m_j}^{\ h}$], then an additional m relaxations on the fine grid after course grid corrections have been made, these equations are used to calculate,

$$\vec{u}_{n_j}^{\ h} = \vec{u}_{n_j}^{\ h} + \vec{e}_{m_j}^{\ h} = \left(\mu^h_j\right)^n \vec{w}^h_j + \boldsymbol{I}^{\boldsymbol{h}}_{\boldsymbol{2h}} \vec{e}_{m_j}^{\ 2h}$$

Also $\vec{r}_{n_j}^{\ 2h}$ (equations [3.3], [3.9], and [3.13]) can and will still be used to solve the equation,

$$\boldsymbol{A}^{\boldsymbol{2h}} \vec{e}_{n_j}^{\ 2h} = \vec{r}_{n_j}^{\ 2h}$$

And this is where the v-cycle algorithm begins.

4. About the Code

The first step in the V-cycle is to make an initial guess $\vec{u}_{0_j}^h$ (a linear combination of eigenvectors of matrix $\boldsymbol{A^h}$) then relax n times using the weighted Jacobi (or some other scheme) so as to give an approximate solution $\vec{u}_{n_j}^{\ h}$ to the equation,

1) $\boldsymbol{A^h}\vec{u}_{n_j}^{\ h} = \vec{f}_{n_j}^{\ h}$ → compute n times on G^h with initial guess $\vec{u}_{0_j}^h$. Results in $\vec{u}_{n_j}^h$.

[coding unnecessary – see next equation]

2) $\vec{u}_{n_j}^{\ h} = \left(\mu_j^h\right)^n \vec{w}_j^h$ → invoke pre-defined function [see lines and 40-42 and 182]

Given that $\vec{f}_{n_j}^{\ h} = 0$ on the fine grid this step can be bypassed as equation [3.1] gives $\vec{u}_{n_j}^{\ h}$. The next step is to calculate the fine grid error,

3) $\vec{r}_{n_j}^{\ h} = \vec{f}_{n_j}^{\ h} - \boldsymbol{A^h}\vec{u}_{n_j}^{\ h}$ → compute [see lines 43-46 and 186]

Once again, with $\vec{f}_{n_j}^{\ h} = 0$ and with $\vec{u}_{0_j}^h$ a linear combination of eigenvectors of matrix $\boldsymbol{A^h}$ we are spared the computation for this step as equation [3.3] gives,

4) $\vec{r}_{n_j}^{\ 2h} = \boldsymbol{I_h^{2h}}\vec{r}_{n_j}^{\ h} = -\left(\mu_j^h\right)^n \lambda_j^h \cos^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_j^{2h}$ → compute [see lines 48-53 and 187]

The first real computational step is to make the initial guess $\vec{e}_{0_j}^h = \vec{0}$ (or any guess) then utilize the weighted Jacobi scheme to calculate[1] $\vec{e}_{n_j}^{\ 2h}$

5) $\boldsymbol{A^{2h}}\vec{e}_{n_j}^{\ 2h} = \vec{f}_{n_j}^{\ 2h}$ → relax n times on G^{2h} with initial guess $\vec{e}_{0_j}^h = 0$. Results in $\vec{e}_{n_j}^h$

[see line 137-147 and 192]

We then use the computed value of $\vec{e}_{n_j}^{\ 2h}$ to calculate $\vec{r}_{n_j}^{\ 2h}$

6) $\vec{r}_{n_j}^{\ 2h} = \vec{f}_{n_j}^{\ 2h} - \boldsymbol{A^{2h}}\vec{e}_{n_j}^{\ 2h}$ → compute [see lines 159-163 and 193]

We can then use $\vec{r}_{n_j}^{\ 2h}$ to compute the residual $\vec{r}_{n_j}^{\ 4h}$ [the residual on G^{4h}] by applying the restriction operator to $\vec{r}_{n_j}^{\ 2h}$

7) $\vec{r}_{n_j}^{\ 4h} = \boldsymbol{I_{2h}^{4h}}\vec{r}_{n_j}^{\ 2h}$ → compute [see line 199]

Which in turn can be used to calculate $\vec{e}_{n_j}^{\ 4h}$. From [1.7] $\boldsymbol{A^{4h}}$ is a single valued matrix so no special computation beyond simple division is needed to solve the equation

8) $\boldsymbol{A^{4h}}\vec{e}_{n_j}^{\ 4h} = \vec{f}_{n_j}^{\ 4h}$ → solve [see lines 202-203]

Having solved for $\vec{e}_{n_j}^{\ 4h}$, we can prolongate this error to $\vec{e}_{n_j}^{\ 2h}$ then use this to correct the previous calculation of the error on G^{2h}.

---

[1] The same value of weighting factor ω is here used for relaxations that solve for $\vec{e}_n$ and $\vec{u}_n$. To the authors knowledge there is no issue with utilizing the weighted Jacobi method to calculate the error. Admittedly, this warrants mathematical justification which hasn't been pursued in this paper.

$$\textbf{9)}\ \vec{e}_{n_j}^{\ 2h} = \vec{e}_{n_j}^{\ 2h} + \boldsymbol{I_{4h}^{2h}}\vec{e}_{n_j}^{\ 4h} \quad \rightarrow \quad \text{correct [see line 206]}$$

We now use $\vec{e}_{n_j}^{\ 2h}$ as an initial guess then perform m relaxations on G^{2h} which returns $\vec{e}_{m_j}^{h}$ which is an improved calculation of the error[2].

$$\textbf{10)}\ \boldsymbol{A^{2h}}\vec{e}_{m_j}^{2h} = \vec{r}_{m_j}^{\ 2h} \rightarrow \text{relax } m \text{ times on } G^{2h} \text{ with initial guess } \vec{e}_{n_j}^{2h}\text{. Results in } \vec{e}_{m_j}^{2h}$$

[see lines 130-140 and 210-211]

By prolongating $\vec{e}_{m_j}^{2h}$ to the fine grid, $\vec{e}_{m_j}^{2h}$ can be used to improve the original calculation of $\vec{u}_{n_j}^{h}$,

$$\textbf{11)}\ \vec{u}_{n_j}^{h} = \vec{u}_{n_j}^{h} + \vec{e}_{m_j}^{\ h} = \left(\mu_j^h\right)^n \vec{w}_j^h + \boldsymbol{I_{2h}^{h}}\vec{e}_{m_j}^{\ 2h} \quad \rightarrow \quad \text{correct } \vec{u}_{n_j}^{h} \text{ [see lines 218-222]}$$

Using this improved calculation of $\vec{u}_{n_j}^{h}$ as an initial guess, m relaxations on the fine grid are performed yielding the final approximation $\vec{u}_{m_j}^{\ h}$,

$$\textbf{12)}\ \boldsymbol{A^{h}}\vec{u}_{m_j}^{\ h} = \vec{f}_{m_j}^{\ h} \rightarrow \text{relax } m \text{ times on } G^{2h} \text{ with initial guess } \vec{u}_{n_j}^{h}\text{. Results in } \vec{u}_{m_j}^{h}$$

[see lines 130-140 and 225-226]

The total final approximations to $\vec{u}_{m}^{h}$ is then given by equation [3.18],

$$\textbf{13)}\ \vec{u}_{m+1}^{h} = \sum_{j=1}^{M^h/2-1} \vec{u}_{(m+1)_j}^{h} + \vec{u}_{(m+1)_{M^h/2}}^{h} + \sum_{j=M^h/2+1}^{M^h-1} \vec{u}_{(m+1)_j}^{h}$$

[See line 270]

The code operates by defining two primary functions; RELAX and V_cycle.

As the name implies 'RELAX' performs n weighted Jacobi relaxations. RELAX accepts argument M^h or M^{2h} which in turn are used define Matrix $\boldsymbol{A^h}$ or $\boldsymbol{A^{2h}}$**.** n (an input to RELAX) weighted Jacobi relaxations are then performed. RELAX then returns the vector $\vec{u}_{n_j}$ on the grid specified by the input M. A number of other vectors are also returned, and these are,

- Uk - returns vector $\vec{u}_{n_j}^{\ h}$ or $\vec{u}_{n_j}^{\ 2h}$
- rk - returns $\vec{r}_{n_j}^{\ h}$ or $\vec{r}_{n_j}^{\ 2h}$ after k=n iterations
- R - returns a vector whose elements are the norm $\left\|\vec{r}_{k_j}\right\|; k = 1, \ldots, n$
- DIFF - returns a vector whose elements are the norm of $\left\|\vec{u}_{k+1_j} - \vec{u}_{k_j}\right\|; k = 1, \ldots, n$
- N - returns a vector whose elements are the indices' $k = 1, \ldots, n$

---

[2] See note made on line 208 of code. It is uncertain whether step #10 should be calculated using the value of $\vec{r}_{n_j}^{\ 2h}$ from step #4 or step #4; res2_n or r2h;

$$step\ \#4)\quad \vec{r}_{n_j}^{\ 2h} = \boldsymbol{I_h^{2h}}\vec{r}_{n_j}^{\ h} = -\left(\mu_j^h\right)^n \lambda_j^h \cos^2\left(\frac{j\pi}{2M^h}\right) \vec{w}_j^{2h} \quad \rightarrow \quad \text{'r2h 'in code}$$

$$step\ \#6)\quad \vec{r}_{n_j}^{\ 2h} = \vec{f}_{n_j}^{\ 2h} - \boldsymbol{A^{2h}}\vec{e}_{n_j}^{\ 2h} \quad \rightarrow \quad \text{'res2\_n' in code}$$

Both seem to produce equivalent results, so the most updated value from step #6 was ultimately chosen.

‘V_cycle’ goes through all of the aforementioned steps except the final summation (equation [3.18]) which is performed through a series of loops at the end of the code. The outputs are as follows;

uh_m - returns vector ${\vec{u}_{m}}_{j}^{h}$

n_plot - returns ${\vec{r}_{n}}_{j}^{h}$ after $k = n$ iterations

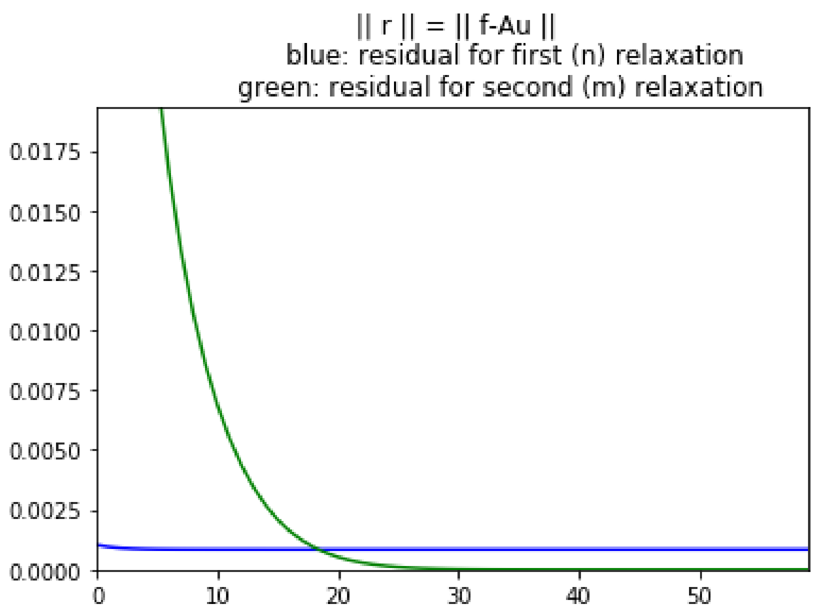
norm1_n - returns a vector whose elements are the norm $\left\| {\vec{r}_{k}}_{j} \right\|; k = 1, \dots, m$

m_plot - returns a vector whose elements are the norm of $\left\| {\vec{u}_{k+1}}_{j} - {\vec{u}_{k}}_{j} \right\|; k = 1, \dots, m$

norm1_m - returns a vector whose elements are the indices’ $k = 1, \dots, n$

obviously these vectors are in fact calculated via RELAX (V_cycle invoked RELAX), but their values change according to the inputs $M, \vec{r}_n, \vec{e}_n$.

By the same reasoning which justified the linear superposition of solutions in [3.0], the vectors whose elements are $\left\| {\vec{r}_{k}}_{j} \right\|; k = 1, \dots, n, m$ can be calculated for $j = 1, \dots, 7$ then added together (linear superposition). Lines [236-246] of the code sum the vector norms over the index $j = 1,2,3$. The norm for $j = 1$ (a practically negligible value) is then calculated. Lines 257-266 of the code sum the vector norms over the index $j = 5,6,7$. the resulting three vectors are then added to produce two vectors of length n and m, respectively [see lines 271-272]. These vectors show the value of $\left\| {\vec{r}_{k}}_{j} \right\|$ as the iterations go from $k = n, m$.



The blue line which represents the initial n relaxations on the fine grid shows a hardly perceptible relaxation before leveling off (as is typical for a weighted Jacobi relaxation scheme). The flat nature of the curve likely stems from the fact that equation [3.3] was an exact calculation.

The green line represents the final m relaxations on the fine grid. While the vector norm values start much higher than the initial relaxation scheme, they quickly level off to a value which is *below* the value for which the first relaxation scheme leveled off. This was the entire purpose of the V-cycle algorithm.

Appendix

Given a tridiagonal matrix of the form

$$\begin{pmatrix} b & c & 0 & 0 & \dots \\ a & b & c & 0 & \dots \\ 0 & a & b & c & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The eigenvectors and eigenvalues are,

$$\lambda_j \; = b + 2c\sqrt{\frac{a}{c}}\cos\frac{j\pi}{M} \qquad\qquad \vec{w}_j \; = 2\begin{pmatrix} w_{j_{k-1}} \\ w_{j_k} \\ w_{j_{k+1}} \\ \vdots \end{pmatrix}; \qquad w_{j_k} = \; \sin\frac{jk\pi}{M}$$

A proof of this will not be offered. Instead it will simply be demonstrated that if λ_j is an eigenvalue and $\vec{w}_j$ and eigenvector of the matrix $\boldsymbol{A}$, then the formula simply works as demonstrated below.

$$\boldsymbol{A}\vec{w}_j = \; \lambda_j\vec{w}_j$$

$$\frac{1}{\Delta x^2}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} \sin\frac{j\pi}{M_{2h}} \\ \sin\frac{2j\pi}{M_{2h}} \\ \sin\frac{3j\pi}{M_{2h}} \end{pmatrix} = \frac{1}{\Delta x^2}\left(2 - 2\cos\frac{j\pi}{M_{2h}}\right)\begin{pmatrix} \sin\frac{j\pi}{M_{2h}} \\ \sin\frac{2j\pi}{M_{2h}} \\ \sin\frac{3j\pi}{M_{2h}} \end{pmatrix}$$

$$\begin{pmatrix} 2\sin\frac{j\pi}{M^{2h}} - \; \sin\frac{2j\pi}{M^{2h}} \\ -\sin\frac{j\pi}{M^{2h}} + \; 2\sin\frac{2j\pi}{M^{2h}} - \; \sin\frac{3j\pi}{M^{2h}} \\ -\sin\frac{2j\pi}{M^{2h}} + 2\sin\frac{3j\pi}{M^{2h}} \end{pmatrix} = 4\begin{pmatrix} \sin\frac{j\pi}{M^{2h}} - \; \cos\frac{j\pi}{M^{2h}}\sin\frac{j\pi}{M^{2h}} \\ \sin\frac{2j\pi}{M^{2h}} - \; \cos\frac{j\pi}{M_{2h}}\sin\frac{2j\pi}{M^{2h}} \\ \sin\frac{3j\pi}{M^{2h}} - \; \cos\frac{j\pi}{M^{2h}}\sin\frac{3j\pi}{M^{2h}} \end{pmatrix}$$

$$= 4\begin{pmatrix} \sin\frac{j\pi}{M^{2h}} - \; \frac{1}{2}\left\{\sin\frac{2j\pi}{M^{2h}}\right\} \\ \sin\frac{2j\pi}{M^{2h}} - \; \frac{1}{2}\left\{\sin\frac{3j\pi}{M^{2h}} - \; \sin\frac{-j\pi}{M^{2h}}\right\} \\ \sin\frac{3j\pi}{M^{2h}} - \; \frac{1}{2}\left\{\sin\frac{4j\pi}{M^{2h}} - \; \sin\frac{-2j\pi}{M^{2h}}\right\} \end{pmatrix}$$

$$= \begin{pmatrix} 4\sin\frac{j\pi}{M^{2h}} - \; 2\sin\frac{2j\pi}{M^{2h}} \\ -2\sin\frac{j\pi}{M^{2h}} + \; 4\sin\frac{2j\pi}{M^{2h}} - \; 2\sin\frac{3j\pi}{M^{2h}} \\ -2\sin\frac{2j\pi}{M^{2h}} + 4\sin\frac{3j\pi}{M^{2h}} \end{pmatrix}$$

Which verifies the claim that the eigenvectors and eigenvalues for matrix $\boldsymbol{A}$ are,

$$\boxed{\vec{w}_j \; = \; 2\sin\frac{jk\pi}{M^h} \; ; k = 1, \dots, 7 \; , \qquad \lambda_j \; = \; 2\left(1 - \cos\frac{j\pi}{M^h}\right) \qquad\qquad [A.1]}$$

Where in the second to last line the sine addition formulas were utilized and $\sin\frac{4j\pi}{M_{2h}} = 0$ because $M_{2h} = 4$. By extension of the above mathematics, the eigenvalue equation for the fine grid matrix works out in a similar fashion.

Some other useful relations are proved beginning with the formula,

$$I_h^{2h}\vec{w}_j^h \;=\; \cos^2\frac{j\pi}{2M^h}\vec{w}_j^{2h} \qquad [A.2]$$

Proof: the entries of vector $\vec{w}_j^{2h}$ are $w_{jk}^{2h} = \sin\frac{jk\pi}{M^{2h}} = \sin\frac{jk\pi}{4}$, which gives,

$$\begin{aligned}
\cos^2\frac{j\pi}{16}w_{jk}^{2h} &= \frac{1}{2}\left(1+\cos\frac{j\pi}{8}\right)\sin\frac{jk\pi}{4} \\
&= \frac{1}{2}\left(\sin\frac{k\pi}{4}j + \frac{1}{2}\left\{\sin\left(\frac{2k+1}{8}j\pi\right) + \sin\left(\frac{2k-1}{8}j\pi\right)\right\}\right) \\
&= \frac{1}{4}\left\{\sin\left(\frac{2k-1}{M^h}j\pi\right) + 2\sin\left(\frac{2k\pi}{M^h}j\right) + \sin\left(\frac{2k+1}{M^h}j\pi\right)\right\} \\
&= \frac{1}{4}\left\{\sin\left(\frac{2k-1}{2M^{2h}}j\pi\right) + 2\sin\left(\frac{2k\pi}{2M^{2h}}j\right) + \sin\left(\frac{2k+1}{2M^{2h}}j\pi\right)\right\} \;;\qquad j = 1,2,3 \quad k = 1,2,3 \\
&= I_h^{2h}\vec{w}_j^h
\end{aligned}$$

As an example, for $j = 1$ we have,

$$I_h^{2h}w_{1k}^h \;=\; I_h^{2h}\begin{pmatrix} w_{11}^h \\ w_{12}^h \\ w_{13}^h \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \sin\left(\frac{\pi}{8}\right) + 2\sin\left(\frac{2\pi}{8}\right) + \sin\left(\frac{3\pi}{8}\right) \\ \sin\left(\frac{3\pi}{8}\right) + 2\sin\left(\frac{4\pi}{8}\right) + \sin\left(\frac{5\pi}{8}\right) \\ \sin\left(\frac{5\pi}{8}\right) + 2\sin\left(\frac{6\pi}{8}\right) + \sin\left(\frac{7\pi}{8}\right) \end{pmatrix}$$

Which suggests the matrix of the form,

$$\boldsymbol{I_h^{2h}} \;=\; \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

Changing w_{jk}^{2h} to w_{jk}^{4h} in the above derivation results in the same equation but with M^{4h} in the denominator and is restricted to just $k = 1$, hence for the course grid transfer we have,

$$\boldsymbol{I_h^{2h}} \;=\; \frac{1}{4}\begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$$

Substituting $j = M^{2h} - j = 4 - j$ into A.2 gives,

$$\begin{aligned}
I_h^{2h}\vec{w}_{M-j}^h = &\quad -\sin^2\left(\frac{j\pi}{2M^h}\right)\vec{w}_j^{2h} \\
&= \frac{1}{4}\left\{\sin\left(\frac{2k-1}{8}[4-j\,]\pi\right) + 2\sin\left(\frac{2k\pi}{8}[4-j\,]\right) + \sin\left(\frac{2k+1}{8}[4-j\,]\pi\right)\right\} \\
&= \frac{1}{4}\left\{\sin\left(\frac{2k-1}{2}\pi - \frac{2k-1}{8}j\pi\right) + 2\sin\left(-\frac{2k\pi j}{8}\right) + \sin\left(\frac{2k+1}{2}\pi - \frac{2k+1}{8}j\pi\right)\right\} \\
&= \frac{1}{4}\left\{\sin\left(\frac{2k-1}{8}j\pi\right) - 2\sin\left(\frac{2k\pi j}{8}\right) + \sin\left(\frac{2k+1}{8}j\pi\right)\right\} \\
&= -\frac{1}{2}\sin\left(\frac{k\pi j}{4}\right) + \frac{1}{2} * \frac{1}{2}\left[\sin\left(\frac{2k-1}{8}j\pi\right) + \sin\left(\frac{2k+1}{8}j\pi\right)\right] \\
&= -\frac{1}{2}\sin\left(\frac{k\pi j}{4}\right) + \frac{1}{2}\left[\cos\left(\frac{j\pi}{8}\right)\,\sin\left(\frac{jk\pi}{4}\right)\right]
\end{aligned}$$

$$= -\frac{1}{2}\left(1-\cos\left(\frac{j\pi}{2M^{2h}}\right)\right)\sin\left(\frac{k\pi j}{M^{2h}}\right) \qquad ; \qquad k = 1,2,3$$

$$= -\sin^2\left(\tfrac{j\pi}{4M^{2h}}\right)\vec{w}_j^{2h}$$

$$= -\sin^2\left(\tfrac{j\pi}{2M^{h}}\right)\vec{w}_j^{2h}$$

$$\boxed{I_h^{2h}\vec{w}_{M-j}^{h} = -\sin^2\left(\tfrac{j\pi}{2M^{h}}\right)\vec{w}_j^{2h} \qquad [\text{A.3}]}$$

Substituting $j = M^{2h}$ into A.2 give

$$\begin{aligned}
I_h^{2h}w_{M^{2h},k}^{h} &= \frac{1}{4}\left\{\sin\left(\frac{2k-1}{2M^{2h}}M^{2h}\pi\right)+2\sin\left(\frac{\pi k}{M^{2h}}M^{2h}\right)+\ \sin\left(\frac{2k+1}{2M^{2h}}M^{2h}\pi\right)\right\}\\
&= \frac{1}{4}\left\{\sin\left(\frac{2k-1}{2}\pi\right)+\ \sin\left(\frac{2k+1}{2}\pi\right)\right\}\\
&= \frac{1}{2}\left\{\sin(k\pi)\cos\left(\frac{\pi}{2}\right)\right\}\\
&= 0
\end{aligned}$$

$$\boxed{I_h^{2h}\vec{w}_{M/2}^{h} = 0 \qquad [A.4]}$$

In the following case the index $k = 1,2,\dots,M^h - 1 = 7$. For an arbitrary k we have

$$\begin{aligned}
&\cos^2\left(\frac{j\pi}{2M^h}\right)w_{j,k}^{h} - \sin^2\left(\frac{j\pi}{2M^h}\right)w_{M^h-j,k}^{h}\\
&\quad= \cos^2\left(\frac{j\pi}{2M^h}\right)\sin\left(\frac{kj\pi}{M^h}\right) - \sin^2\left(\frac{j\pi}{2M^h}\right)\sin\left(\frac{k[M^h-j]\pi}{M^h}\right)\\
&\quad= \frac{1}{2}\left(1+\cos\left(\frac{j\pi}{M^h}\right)\right)\sin\left(\frac{kj\pi}{M^h}\right) - \frac{1}{2}\left(1-\cos\left(\frac{j\pi}{M^h}\right)\right)\sin\left(k\pi-\frac{kj\pi}{M^h}\right)
\end{aligned}$$

For even values of k we have $\sin\left(k\pi - \frac{kj\pi}{M^h}\right) = -\sin\left(\frac{kj\pi}{M^h}\right)$ and this becomes,

$$\left[\frac{1}{2}\left(1+\cos\left(\frac{j\pi}{M^h}\right)\right)+\frac{1}{2}\left(1-\cos\left(\frac{j\pi}{M^h}\right)\right)\right]\sin\left(\frac{kj\pi}{M^h}\right) \quad = \quad \sin\left(\frac{kj\pi}{M^h}\right)$$

For odd values of k we have $\sin\left(k\pi - \frac{kj\pi}{M^h}\right) = \sin\left(\frac{kj\pi}{M^h}\right)$ and we have,

$$\begin{aligned}
\frac{1}{2}\left(1+\cos\left(\frac{j\pi}{M^h}\right)\right)-\frac{1}{2}\left(1-\cos\left(\frac{j\pi}{M^h}\right)\right)\sin\left(\frac{kj\pi}{M^h}\right) &= \cos\left(\frac{j\pi}{M^h}\right)\sin\left(\frac{kj\pi}{M^h}\right)\\
&= \frac{1}{2}\left(\sin\left(\frac{k-1}{M^h}j\pi\right)+\ \sin\left(\frac{k+1}{M^h}j\pi\right)\right)
\end{aligned}$$

For $k = 1,2,3 \dots,7$ the following matrix equation is produced:

$$\boxed{\cos^2\left(\frac{j\pi}{2M^h}\right)\vec{w}_j^h - \sin^2\left(\frac{j\pi}{2M^h}\right)\vec{w}_{M-j}^h \quad = \quad \frac{1}{2}\begin{pmatrix}1&0&0\\2&0&0\\1&1&0\\0&2&0\\0&1&1\\0&0&2\\0&0&1\end{pmatrix}\begin{pmatrix}\sin\frac{j\pi}{M^{2h}}\\ \sin\frac{2j\pi}{M^{2h}}\\ \sin\frac{3j\pi}{M^{2h}}\end{pmatrix} \quad = \quad \boldsymbol{I_{2h}^{h}}\vec{w}_j^{2h} \quad ; M^{2h} = 4 \qquad [A.5]}$$

Changing M^h to M^{2h} on the course grid we have for the index $k = 1,2,3,$

$$\cos^2\left(\frac{j\pi}{2M^{2h}}\right)w_{j,k}^h - \sin^2\left(\frac{j\pi}{2M^{2h}}\right)w_{M^h-j,k}^h = \begin{cases} = \sin\left(\frac{kj\pi}{M^{2h}}\right) & ; \qquad k_{even} \\ = \frac{1}{2}\left(\sin\left(\frac{k-1}{M^{2h}}j\pi\right) + \sin\left(\frac{k+1}{M^{2h}}j\pi\right)\right); & \qquad k_{odd} \end{cases}$$

And the following matrix equation is produced:

$$\cos^2\left(\frac{j\pi}{2M^{2h}}\right)\vec{w}_j^h - \sin^2\left(\frac{j\pi}{2M^{2h}}\right)\vec{w}_{M-j}^h \quad = \quad \frac{1}{2}\begin{pmatrix}1\\2\\1\end{pmatrix}\sin\frac{j\pi}{M^{4h}} \quad = \quad \boldsymbol{I_{4h}^{2h}}\vec{w}_j^{4h}$$

The left-hand quantity is not what is of interest here. This was simply a means of deducing what the appropriate form of the matrix $\boldsymbol{I_{4h}^{2h}}$,

$$\boxed{\boldsymbol{I_{4h}^{2h}} = \frac{1}{2}\begin{pmatrix}1\\2\\1\end{pmatrix}}$$

Using the identity $\boldsymbol{I_h^{2h}}\boldsymbol{A^h}\boldsymbol{I_{2h}^h} = \boldsymbol{A^{2h}}$ and $\vec{e}_j^h = \boldsymbol{I_{2h}^h}\vec{e}_j^{2h}$ we have,

$$\boldsymbol{A^{2h}}\vec{e}_j^{2h} = \vec{r}_j^{2h}$$

$$\left(\boldsymbol{I_h^{2h}}\boldsymbol{A^h}\boldsymbol{I_{2h}^h}\right)\vec{e}_j^{2h} = \vec{r}_j^{2h}$$

$$\boldsymbol{I_h^{2h}}\boldsymbol{A^h}\left(\boldsymbol{I_{2h}^h}\vec{e}_j^{2h}\right) = \vec{r}_j^{2h}$$

$$\boldsymbol{I_h^{2h}}\boldsymbol{A^h}\vec{e}_j^h = \vec{r}_j^{2h}$$

$$\boxed{\boldsymbol{I_h^{2h}}\vec{r}_j^h = \vec{r}_j^{2h} \qquad [A.6]}$$

References

Thomas, J. W. (1995). Section 10.5 and 10.10. In *Numerical partial differential equations*. New York: Springer.