CORRELATED RANDOM WALKS

By Wayne Wignes

Introduction

The random walk is commonly used in financial engineering models as well as in any physical process involving Brownian motion, for which Langevin's equation is probably the most notable example. An uncorrelated random walk is a walk in which future jumps are in no way affected by what has already happened, and this is a *Markov process*. In contrast to this, the correlated walk - as the name implies - relates futre steps to jumps made in the past.

The paper reviewed is *Correlated Random Walks, Hyperbolic Systems, and Fokker-Planck Equations* by E. Zauderer [2]. Zauderer begins by showing that a correlated random walk results in a hyperbolic set of equations - a thing which is in contrast to the uncorrelated case which typically results in a parabolic set of equations. His primary aim in writing this paper however, is to demonstrate the effect which allowing for variable probability coefficients has on the resulting differential equation(s). Zauderer considers two cases for the probabilities shown in equation 3; $a(x), b(x)$ are both constants and $a(x)$ variable with $b(x) = 0$. In both cases Zauderer aims to reduce the coupled set of equations which result for the correlated walk (see equations 8 and 9) to a single diffusion-like equation. In order to acheive this it is found that the reversal rates $\lambda^{\pm}$ within the probabilities for the correlated walk (see equation 7) must take a specific form and furthermore that these rates must be taken to infinity in order for the resulting equations to reduce to a diffusion like equation.

It is found that taking the limit of the reversal rates to infinity for the two cases results in equations that are fundamentally different in their form. The case of constant probability coefficients results in the 'Itô type' differential qeuation while the case of variable coefficient $a(x)$ (and $b = 0$ results in the 'Stratonovich type' differential equation. These are classifications of *stochastic calculus*, in fact Itô and Stratonovich calculus are entirely different *kinds* of calculus, and each has their own distinctive properties that are of interest for a number of reasons in differing applications.

Though some of the Taylor expansions may be tedious, there are no esoteric theories being used throughout the paper - the author uses basic Taylor expansions to derive the final results. Following the authors derivations however offers no insights as to the final claim that the resulting differential equation is either Itô or Stratonovich type. These are concepts that are not typically required in a standard physics curriculum, and hence it is reasonable to assume many physics graduate students are not familiar with them.

In section I. a basic run through of Zauderer's results is made so as to clarify the problem at hand as well as to make the derivation of his results easier for the interested reader to work through. In section II the classification of pde's as hyperbolic, elliptic, or parabolic is briefly outlined so as to substantiate Zauderer's claim that the correlated random walk results in a hyperbolic system, as well as to give the reader a basic understanding of what is meant by these classifications. Section III introduces the basic elements of stochastic calculus. While one can hardly hope to expound upon the subject in the detail such a broad subject requires in a single paper, the aim is just to give the reader a feel for the fundamental differences between stochastic calculus and the standard calculus which physics typically deals with (The Langevin equation being a notable exeption). The general form of an Itô stochastic differential equations (or sde's) is then derived and we will see that it is essentially the *Fokker-plank* equation which Zauderer continuously refers to. Finally, using the Fokker-Plank equation, an example which connects the Itô *process* [which describes the evolution of sample paths[to the the deterministic evolution of probabilities [the Itô type sde] is offered via Langevin's equation which is well known to physicists.

Part I
Review Zauderer's paper

1 The Uncorrelated Walk

For the uncorrelated random walk jumps are independent of their past, which is to say it is a Markov process; what happens next has nothing to do with what has happened in the past. We make the following definitions:

$$\begin{aligned} v(x,t): &\quad \text{Probability that a particle is at position } x \text{ at time } t \\ p(x): &\quad \text{Probability that a particle jumps right} \\ q(x): &\quad \text{Probability that a particle jumps left} \qquad part \end{aligned} \tag{1}$$

The total probability that a particle is occupying position x at time t is then given by,

$$\begin{aligned} v(x,t+\Delta t) =& (1-p^(x)-q^(x))v(x,t) && \longrightarrow \quad P_1 : \text{Stationary particle; no jumps} \\ &+ p(x-\Delta x)v(x-\Delta x,t) && \longrightarrow \quad P_1 : \text{arrives at x by jumping right (from the left)} \\ &+ q(x+\Delta x)v(x+\Delta x,t) && \longrightarrow \quad P_1 : \text{arrives at } x \text{ by jumping left (from the right)} \end{aligned} \tag{2}$$

For reasons that the author leaves unstated it is assumed that the probabilities take the form

$$p(x) = \frac{1}{2}\left(a(x) + b(x)\frac{\Delta t}{\Delta x}\right), \qquad q(x) = \frac{1}{2}\left(a(x) - b(x)\frac{\Delta t}{\Delta x}\right) \tag{3}$$

The form of the second term $b\frac{\Delta t}{\Delta x}$ is likely related to the concept of probability flux which is of great importance to solving the Fokker-Plank equations numerically[5, 6]. Explitly, the probability flux is defined as,

$$J = \int \frac{\partial p}{\partial t} dx$$

which for the term in question would imply the probability flux of any integrated region is simply $b(x)$.

For small $\Delta x, \Delta t$ we can expand the above quantities as well as $v(x, t + \Delta t)$ in equation 2 and the results will be the diffusion equation if we require that as Δx and Δt go to zero we have $\frac{\Delta x^2}{\Delta t} \to D$; the diffusion constant. This in turn has the effect that $\Delta t \to 0$ *faster* than Δx. Consequently we have $lim(\frac{\Delta x}{\Delta t}) = c \to \infty$. As derived in the appendix (see equation A.1) this results in the following Itô type sde,

$$v_t + (bv)_x - \frac{1}{2}D(av)_{xx} = 0 \tag{4}$$

The above differential equation is readily compared to the Fokker-Plank equation,,

$$P_t + (AP)_x - (B^2P)_{xx} = 0 \qquad \longrightarrow \qquad \text{Itô type}$$

Which is the Itô type version of the Fokker-Plank equation as is elaborated on in section III. Here P represents a probability, A the *drift coefficient* and B the *diffusion coefficient*. Note that the diffusion coefficient when applied to the Fokker-Plank equation is defined in a more general sense in mathematical literature, hence we do not necessarily have the equivalence $D = B$.

2 The Correlated Walk

For the correlated random walk we presume that jumps are in some way related to the previous step (but independent of steps prior to this). We make the following definitions:

$$
\begin{aligned}
r(x,t):&\quad \text{Probability that a particle is at } x \text{ at time } t \text{ and came from the left (i.e. it jumped right)}\\
l(x,t):&\quad \text{Probability that a particle is at } x \text{ at time } t \text{ and came from the right (i.e. it jumped left)}\\
p^+(x):&\quad \text{Probability that right moving particle keeps going right (persistence)}\\
q^+(x):&\quad \text{Probability that right moving particle reverses}\\
p^-(x):&\quad \text{Probability that left moving particle keeps going left (persistence)}\\
q^-(x):&\quad \text{Probability that left moving particle reverses}
\end{aligned} \tag{5}
$$

So p denotes 'persistence' in direction of travel while 'q' denotes reversal of direction. Plus or minus simply indicates direction of travel; right or left, respectively. These give the following relation for the total probability that particles jump rightward (from the left) at time $t + \delta t$,

$$
\begin{aligned}
r(x,t+\Delta t) =& (1 - p^+(x) - q^+(x))r(x,t) \quad \longrightarrow \quad P_1 : \text{Stationary particle; no jumps}\\
&+ p^+(x-\Delta x)r(x-\Delta x,t) \quad \longrightarrow \quad P_1 : \text{Right moving particle continues}\\
&+ q^-(x-\Delta x)l(x-\Delta x,t) \quad \longrightarrow \quad P_1 : \text{left moving particle reverses}
\end{aligned} \tag{6}
$$

The situation is depicted in figure 1.

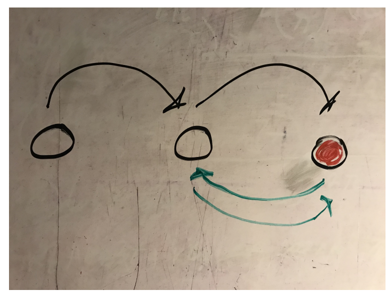

Figure 1: Three events are depicted which can account for a particle arriveing at state x (red dot) from the *left*. Red dot implies stationary - no jumps. One can imagine a similar scenario for particles arriving from the right.

Let λ^+ and λ^- denote the probability that a particle which is traveling left or right (minus or plus, respectively) suddenly reverses direction (reversal rates). Zauderer proposes the following forms of the persistence and reversal probabilities;

$$
\begin{aligned}
p^\pm(x) &= \hat{a}(x) \pm b(x)\frac{\Delta t}{\Delta x} - \lambda^\pm(x)\Delta t\\
q^\pm(x) &= \lambda^\pm(x)\Delta t
\end{aligned} \tag{7}
$$

One can easily see that as $\Delta t \to 0, q \to 0$ and $p \to a(x)$ and we are left with simple and unbiased (direction ally independent; there is no plus or minus present anymore) jump probabilities $p^\pm = \hat{a}(x)$ which - minus the hat symbol used on a - corresponds to the case of the uncorrelated random walk.Note that the hat symbol is not used on the coefficient b, presumably because this coefficient is either constant or zero in the two cases to be considered.

As derived in the appendix (see equations A.2.a and A.2.b) Taylor expanding all quantities which involve Δt in equation 6 then making use of the above probability expressions in the following set of coupled hyperbolic pde's,

$$r_t - [(b + c\hat{a})r]_x = -\lambda^+ r + \lambda^- l \tag{8}$$

$$l_t - [(b - \hat{a})r]_x = -\lambda^+ r - \lambda^- l \tag{9}$$

Take note that in there is a finite speed c in the above set of equations, and that as previously mentioned this is in contrast to the uncorrelated walk for which we required that as $\Delta t, \Delta x$ both approach zero we have finite value for the diffusion constant, i.e. $lim(\frac{\Delta x^2}{\Delta t}) \to D$ and this in turn had the implication that Δt goes to zero faster than δx, hence $lim(\frac{\Delta x}{\Delta t} \to \infty)$. But in the above set of equations there is no diffusion term - it is a first order set of equations which at no point required we make such a definition. In cases to follow we will impose conditions on c as necessary in order to achieve a result which is comparable to equation 4.

In what follows the basic steps which lead to equations 12 and 13 will be outlined, but derivation of these equations will not be shown in detail as it basically is just manipulating equations 8 and 9 - other than specifying the coefficients, there is no physics involved in what follows. Also note that an attempt has been made to make equations so far exactly correspond in number to those of Zauderer.

We seek solutions in terms of $l + r$ and $w = r - l$. It is not difficult to find that by manipulating equations 8 and 9 we get ,

$$v_t + (bv)_x + c(\hat{a}w)_x = 0 \tag{10}$$

$$w_t + (bw)_x + c(\hat{a}w)_x + c(\hat{a}v)_x = (\lambda^- - \lambda^+)v - (\lambda^- + \lambda^+)w \tag{11}$$

Equation 12 can be arrived at by three steps; 1) assume $\lambda^{\pm}, \hat{a}, b$ all constant; 2) [separately] take both the space and time derivatives eqn. 10 and add the results (call this equation 10^*); 3) differentiate of eqn. 11 with respect to x and use the resulting expression to substitute for all the w terms (there will be two of them) in 10^*. These steps produce the following result after some rearrangement,

$$v_{tt} + 2bv_{xt} + (b^2 - c^2\hat{a}^2)v_{xx} - (\lambda^+ - \lambda^-)c\hat{a}v_x + c\hat{a}(\lambda^- + \lambda^+)bv_x + (\lambda^+ + \lambda^-)v_t = 0 \tag{12}$$

To obtain eqn. 13 in Zauderer set $b = 0$, multiply 11 by ca and take the derivative with respect to x of the entire expression - call this 11^*. With $b = 0$, and noting that $\lambda^+ + \lambda^-$ is independent of time, using 10 to substitute for two terms in 11^* quickly results in the following differential equation,

$$v_{tt} - c^2\left[\hat{a}(\hat{a}v)_x\right]_x + (\lambda^+ + \lambda^-)v_t + \left[ca(\lambda^+ + \lambda^-)v\right]_x \tag{13}$$

Equations 11 and 12 are essentially the focus of the paper. Equation 11 represents the case of constant coefficients and equation 12 the case of $a(x), b = 0$. In both cases Zauderer's aim is to reduce these to a diffusion like equation. Zauderer does so under no apparent guiding rule other than to make the end result comparable with equation 4. In both cases one can begin to see that the key to achieving this will be to divide by $\lambda^+ + \lambda^-$ and place stipulations on the reversal rates $\lambda^{\pm}$ as well as any other terms identified along the way in an effort to make the result possess all the terms in equation 4. Of course, being as we wish to compare the correlated random walk to the uncorrelated walk, we will maintain the definitions that as $\Delta x \to 0$, $\Delta t \to 0$ we have $lim\frac{\Delta x}{\Delta t} = c$ (not presumed infinite unless explicitly stated) and $lim(\frac{\Delta x^2}{\Delta t}) \to D$.

Once equations 12 and 13 are in a form comparable to 4, it remains to compare the actual coefficients. The tool Zauderer uses to bridge the gap between the limits of the correlated walk and the uncorrelated walk is the concept of steady state. Any physical system, if given enough time, we should expect diffusion to dissipate as particles have a chance to spread out and attain an equilibrium. Hence in the limit of $t \to \infty$ we expect the time rate of change of probabilities to go to zero[1]. This would imply for 8 and 9 that $l_t = r_t = 0$.

We begin by defining the total probabilities which a particle has of moving right or left,

$$R(x) = \int_{-\infty}^{\infty} r(\mathbf{x}, t)d\mathbf{x}, \qquad L(x) = \int_{-\infty}^{\infty} l(\mathbf{x}, t)d\mathbf{x}$$

The difference between the above and the probability defined in equation 6 is that, whereas 6 represents the probability that a particle located at a specified position x arrived there by jumping rightward, the above equations represent the probabilities which a particle has of moving right or left - regardless of it's position.

At steady state, intuitively it seems reasonable that we consider the case of constant values for the reversal rates $\lambda^{\pm}$. While this intuitive assumption is by no means intended to be taken as a rigorous statement (one might for example imagine a pattern of assymetrical reversal rates which average out to zero upon being integrated over a small region), the important thing is that we remain consistent with this *assumption* any time we make use of the steady state

[1]As demonstrated in Part III of this report, at least for the case of Langevin's equations this can be related to the canonical ensemble

probabilities from here on out. Integrating equations 8 and 9 over all space, and utilizing the fact that both r and l have zero probability density at infinity gives,

$$\begin{aligned} R_t &= -\lambda^+ R + \lambda^- L \\ L_t &= \lambda^+ R - \lambda^- L \end{aligned}$$

Invoking the steady state condition ($t \to \infty$) gives $L_t = R_t = 0$ gives,

$$\begin{aligned} R = L\frac{\lambda^-}{\lambda^+} = (1-R)\frac{\lambda^-}{\lambda^+} \qquad &\longrightarrow \qquad R = \frac{\lambda^-}{\lambda^- + \lambda^+} \\ L = R\frac{\lambda^+}{\lambda^-} = (1-L)\frac{\lambda^+}{\lambda^-} \qquad &\longrightarrow \qquad L = \frac{\lambda^+}{\lambda^- + \lambda^+} \end{aligned} \tag{14}$$

Recall that p^+ and p^- represent the probabilities that a particle persists in moving right or left, respectively, and that q^+ and q^- the probabilities that a right or left moving particle reverses its direction, respectively. Putting these together with the steady state probabilities R and L we are now prepared to construct [steady state] probabilities of a particle jumping right p or going left q,

$$p = p^+ R + q^- L; \qquad q = p^- L + q^+ R \tag{3*}$$

Where the equation is numbered 3^* to indicated that these probabilities are going to take the place of those in equation 2, and they are to be used in equation 2 (the equation describing the time evolution of the *uncorrelated.* In contrast to $r(x,t)$ and $l(x,t)$ defined in equation 8, the above probabilities are independent of time. The values of the persistence and reversal probabilities $p^\pm$ and $q^\pm$ are still given by equation 7.

Before preceding, for clarity we review the general approach.

- The limits on 12 and 13 will be performed and stipulations will be made on certain variables in whatever way is required to make them take the form of a diffusion equation. Nothing more will be done with these equations once this is done aside from classifying its *form* as Itô or Stratonovich type.

Secondly, we need to develop a comparison between whatever coefficients remain after the aforementioned limits are taken. This is accomplished by the following steps,

- The persistent and reversal probabilities in equation 7 and the steady probabilities in 14 will be combined to give a specific value for the probabilities just defined in equation 3^*.
- We insert the probabilities defined in 3^* into equation 2 and Taylor expand all quantities which involved either Δx or δt. This step is algebra intensive so it will be carried out in the appendix.
- We formulated equivalences between the correlated variable $\hat{a}$ and its uncorrelated equivalent. in the uncorrelated random walk. Note that the actual math of this part may be tucked into the Taylor expansions carried out in the appendix.

Finally note that we are tempted to argue that the steady probabilities constructed in equation 3^* are in fact uncorrelated, i.e. that imposing steady state conditions decoupled the probabilities in 8 and 9. At the time of writing this report this statement seems premature. Before making such a claim we will go through the limits, identify constraints, *then* make this claim. Note that the key to this - and somewhat central to what Zauderer aims to demonstrate - is that when the persistent probabilities go to zero we recover an equation of the Itô type which is comparable to the uncorrelated case (equation 4). The answer to this may turn out to be trivial, but it is worth pointing out that simply setting $r_t = l_t = 0$ in 8 and 9 does *not* decouple the system.

2.1 Limiting case for a,b,$\lambda^{\pm}$= constant (eqn. 12)

Dividing 12 by $\lambda^+ + \lambda^-$ gives,

$$\frac{v_{tt} + 2bv_{xt} + (b^2 - (\hat{a}c)^2)v_{xx}}{\lambda^+ + \lambda^-} - \frac{\lambda^+ - \lambda^-}{\lambda^+ + \lambda^-}c\hat{a}v_x + bv_x + v_t = 0 \tag{15}$$

Comparing the above sde to equation 4 we see that in general we don't want the terms which are proportional to v_{xx}, v_x or v_t to disappear, but we do want terms such as v_{tt} and v_{xt} to dissapear. We have some flexibility here in that c can take on any value. We can exploit this fact, but the complication is that c appears in two terms - once as a first order term and once as a second order. If we require that the reversal rates take the general form $\lambda^{\pm} = \lambda + \sqrt{\lambda}h$ where h is some constant, then we find that, upon taking the limit $\lambda \to \infty$ the unwanted terms will go to zero while the desired terms remain if we exploit the variability of c and allow this term to go to infinity as well. Explicitly, upon substituting $\lambda^{\pm} = \lambda + \sqrt{\lambda}h$ into the above and letting $\lambda \to \infty$ we have,

$$\begin{aligned} lim_{\lambda\to\infty}\Big\{\frac{v_{tt} + 2bv_{xt} + (b^2 - (\hat{a}c)^2)v_{xx}}{2\lambda} + \frac{\sqrt{\lambda}h}{\lambda}c\hat{a}v_x + bv_x + v_t\Big\} &= 0 \\ \frac{-(\hat{a}c)^2v_{xx}}{2\lambda} + \frac{\sqrt{\lambda}h}{\lambda}c\hat{a}v_x + bv_x + v_t &= 0 \\ -\frac{1}{2}\Big(\frac{c^2}{\lambda}\Big)\hat{a}^2v_{xx} + \sqrt{\frac{c^2}{\lambda}}\hat{a}v_x + bv_x + v_t &= 0 \end{aligned} \tag{16}$$

If we require that $\hat{D} = \frac{c^2}{\lambda}$ as λ, c both go to infinity. We then have an equation which can be compared to equation 4.

$$v_t + bv_x + \sqrt{\hat{D}}h\hat{a}v_x - \frac{1}{2}\hat{D}\hat{a}v_{xx} = 0 \qquad \longrightarrow \qquad \text{Itô type} \tag{17}$$

Now we wish to develop an expression for the steady state probabilities in 3* so that we might substituted these expressions into equation 3, Taylor expand, and compare so as to identify some equivalence between $\hat{a}$ and a. With $b = 0$ and $\lambda^{\pm} = \lambda + \sqrt{\lambda}h$ putting equation 7 into 14 yields the following expressions for the steady probability of jumping right or left (this part is not shown by Zauderer),

$$p = \frac{1}{2}\big(\hat{a} + b\frac{\Delta t}{\Delta x}\big)\big(1 + \frac{h}{\sqrt{\lambda}}\big), \qquad q = \frac{1}{2}\big(\hat{a} + b\frac{\Delta t}{\Delta x}\big)\big(1 - \frac{h}{\sqrt{\lambda}}\big),$$

As demonstrated in equation A.3 of the appendix, inserting this into equation 2, Taylor expanding, and taking the limits $\lambda \to \infty$ and $c \to \infty$ gives

$$v_t - \frac{h}{\sqrt{\lambda}}bv_x - \sqrt{\hat{D}}hav_x + \frac{1}{2}\hat{D}\hat{a}v_{xx} = 0 \tag{18}$$

if we define $lim(\lambda\Delta t) = \hat{a}$ (see derivation in the appendix - there obviously a couple of sign errors when compared to equation 17). This definition implies the following equivalence;

$$\hat{D} = lim\Big(\frac{c^2}{\lambda}\Big) = lim\Big(\frac{(\Delta x/\Delta t)^2}{\lambda}\Big) = lim\Big(\frac{\Delta x^2}{\Delta t}\Big)lim\Big(\frac{1}{\Delta t\lambda}\Big) = D\frac{1}{\hat{a}}$$

Obviously equations 17 and 18 are comparable - minus a couple of sign errors which entered somewhere in the Taylor expansions in the appendix they are equivalent. But 17 was derived *without* presuming a steady state. We did however have to impose the condition that both the reversal rates and the speed go to infinity. This says that the steady state probabilities of the correlated walk

Finally we note that the persistent probabilities $p^{\pm}$ tend to zero as can be seen by writing them in terms of $c = lim(\Delta x\ \Delta t)$;

$$p^{\pm} = \hat{a} \pm \frac{1}{c}b - \lambda\Delta t \qquad \longrightarrow \qquad \hat{a} \pm 0 - \hat{a} = 0 \tag{19}$$

Zauderer compares this limiting behavior to that of the persistent probabilities in 7 ; the effect of imposing the condition $lim(\lambda\Delta t) = \hat{a}$ is that now it is the persistent probabilities $p^{\pm}$ which tend to zero rather than the reversal probabilities $q^{\pm}$ which would got to zero as $\Delta t \to \infty$ had we not imposed this limiting condition. This [I think] is saying that in steady state diffusion *can* occur, but only by particles continuously reversing back and forth between the same two states - they don't travel anywhere beyond this.

As is in line with earlier claims that imposing steady state conditions is equivalent to decoupling the probabilities in 8 and 9 so that we are left with an uncorrelated walk, we've seen that equation 17 is of the Itô type. To see that it is indeed comparable to 4, we identify the coefficient A in the Fokker-Plank equation of the Itô type (see unlabeled equation just below 4) as $A = b + \sqrt{\hat{D}}h\hat{a}$ and $B = \sqrt{2\hat{D}\hat{a}}$.

2.2 Limiting case for b = 0 (eqn. 13)

For the case of $\hat{a}(x)$ variable and $b = 0$ dividing equation 13 by $\lambda^+ + \lambda^-$ and once again substituting $\lambda^\pm = \lambda(x) + \sqrt{\lambda(x)}h$ (notice this time $\lambda(x)$ is a variable function not a constant) then taking the limit of both $c, \lambda(x) \to \infty$ in order to achieve a diffusion like form of the equation gives,

$$v_t + \sqrt{\hat{D}}h(\hat{a}v)_x - \frac{1}{2}\hat{D}(\hat{a}(\hat{a}v)_x)_x = 0 \qquad \longrightarrow \qquad \text{Stratonovich type sde} \tag{20}$$

Where once again we've defined $\hat{D} = lim(c^2/\lambda(x))$ in order to get a diffusion like (but not Itô type) equation. And once again the limit implies both λ and c are going to infinity. $\lambda^\pm$ were given the same form as in the derivation of equation 16. Really not much changed as far the the method of derivation goes, only here we have a very different outcome; instead of an sde of the It$\hat{0}$ type we have an sde of the Stratonovich type.

We now construct the steady probabilities by substituting 7 and 14 into equation 3*, only this time we have $\hat{a}(x)$ and $b = 0$. The results for the right and left jump probabilities are easily found to be,

$$p = \frac{1}{2}\hat{a}\big(1 + \frac{h}{\sqrt{\lambda}}\big), \qquad q = \frac{1}{2}\hat{a}\big(1 - \frac{h}{\sqrt{\lambda}}\big), \tag{21}$$

As is derived in equation A.4 of the appendix, inserting the above probabilities into equation 2, Taylor expanding all values involving Δt or Δx then taking the limits λ and c go to infinity as we demanded in the case of the correlated walk gives the following Itô type sde,

$$v_t + \sqrt{\frac{D}{\tilde{a}}}h(\hat{a}v)_x - \frac{1}{2}D(\hat{a}v)_{xx} = 0 \qquad \longrightarrow \qquad \text{Itô type sde} \tag{22}$$

Where the variable $\tilde{a}$ (Notice the tilde instead of the hat symbol) is a limiting constant defined by $\tilde{a} = lim(\Delta t\lambda(x))$. Here we needed to differentiate between the two because $\hat{a}(x)$ is in this case a variable function. Notice that - in contrast to the case of constant coefficients - the limiting forms of the correlated walk (equation 20) and the limiting form of the steady state case produced equations of fundamentally different form; equation 20 is Stratonovich type while 22 is Itô type.

Zauderer points out that the cause of this discrepancy lies in the persistent probabilities which do not tend to zero for the case of variable coefficients. Explicitly, with $b = 0$ the persistent probabilities are,

$$p^\pm = \hat{a}(x) - \lambda(x)\Delta t \qquad \longrightarrow \qquad \hat{a}(x) - \tilde{a} \neq 0$$

which do not go to zero no matter the choice of $\tilde{a}$ because $\hat{a}(x)$ is variable while $\tilde{a}$ is a constant. However, once again we've found that imposing steady state conditions did produce an Itô type sde comparable to equation 4. Accordingly, we claim that to invoke steady - at least for the cases considered in which we were allowed to impose the conditions $\lambda, c \to \infty$ - seems to be equivalent to decoupling the system.

In the final paragraphs of his paper Zauderer shows that there is an exception to the dilemma of variable coefficients producing a Stratonovich type sde instead of an Itô type sde, namely the case of equal reversal rates, i.e. $\lambda^- = \lambda^+$. In this case dividing equation 11 by $\lambda^- + \lambda^+ = 2\lambda$, letting $\lambda \to \infty$, and requiring that the term $(aw)_x$ remain bounded gives the following expression,

$$w = -\frac{c(\hat{a}v)_x}{2\lambda}$$

Now substituting this into equation 10 we get,

$$w = v_t + (bv)_x - \frac{1}{2}\left(\left(\frac{c^2\hat{a}}{\lambda}\right)(\hat{a}v)_x\right)_x \tag{23}$$

Equation 23 requires that we identify

$$\hat{D} = \lim_{c,\lambda\to\infty}\left(\frac{c^2\hat{a}}{\lambda}\right) \quad = \quad \lim_{\Delta t\to 0,\lambda\to\infty}\left(\frac{\Delta x^2}{\Delta t}\frac{\hat{a}(x)}{\lambda\Delta t}\right) \quad = \quad D\lim_{\Delta t\to 0,\lambda\to\infty}\left(\frac{\hat{a}(x)}{\lambda\Delta t}\right)$$

So in this case - in order to obtain a diffusion like equation - we require that $lim(\lambda\Delta t) \to \hat{a}(x)$ - a *variable* quantity. With $\hat{D}$ defined as such taking the limit of 23 gives,

$$w = v_t + (bv)_x - \frac{1}{2}\hat{D}(\hat{a}v)_{xx} \qquad \longrightarrow \qquad \text{It}\hat{o}\text{ type sde} \tag{24}$$

As Zauderer emphasizes in the last paragraphs of his paper and important consequence of requiring $lim(\lambda\Delta t) \to \hat{a}(x)$ is that $\hat{a}(x)$ is now variable and the persistent probabilities for the case of $b(x) = 0$ now vanish in the limit;

$$p^{\pm} = \hat{a}(x) - \lambda(x)\Delta t \qquad \longrightarrow \qquad \hat{a}(x) - \hat{a}(x) = 0 \tag{25}$$

And so we see that, as Zauderer claims, the " importance of having the persistence probabilities tend to zero...".

Part II

Hyperbolic pde's

Zauderer claims equations 8,9 represent a system of hyperbolic equations. Tyically differential equations are classified as hyperbolic, elliptic, or parabolic according to their second order coefficients. In 8.9 howevr we have a linear set of equations. In the case of linear equations the method of characteristics and directional derivatives are utilized to classify systems as hyperbolic. I was unsuccessful in applying this theory to equations 8,9. For an idea of the method see [4]. However, no approximations were made when going from the coupled set of equations 8,9 to the single second order equation 12, hence the theory of linear second order classifications can serve as an alternative method of classification.

Linear second order differential equations of the form,

$$Af_{xx} + Bf_{xy} + Cf_{yy} + Du_x + Eu_y + Fu = 0 \tag{26}$$

are classified according to the discriminant of the second order terms [**?**] ; $\Delta = B^2 - AC$. To get a *rough* feel for how this term arises consider a transformation to coordinates $\zeta(x,y), \chi(x,y)$. In terms of these new variables a function $f(x,y)$ will transform as,

$$A^* f_{\chi\chi} + B^*_{\chi\zeta} + C^* f_{\zeta\zeta} + L* = 0 \tag{27}$$

Where L^* represents the lower order differential terms and the transformed coefficients and partial derivatives are as follows,

$$\begin{aligned}
A^* &= A\chi_x^2 + B\chi_x\chi_y + C\chi_y^2 \\
B^* &= 2A\chi_x\zeta_x + B(\chi_x\zeta_y + \chi_y\zeta_x) + 2C\chi_y\zeta_y \\
C^* &= A\zeta_x^2 + B\zeta_x\zeta_y + C\zeta_y^2 \\
\\
f_x &= f_\zeta\zeta_x + f_\chi\chi_x \\
f_y &= f_\zeta\zeta_y + f_\chi\chi_y \\
\\
f_{xx} &= f_{\zeta\zeta}\zeta_x^2 + 2f_{\chi\zeta}\chi_x\zeta_x + f_{\chi\chi}\chi_x^2 \\
f_{yy} &= f_{\zeta\zeta}\zeta_y^2 + 2f_{\chi\zeta}\chi_y\zeta_y + f_{\chi\chi}\chi_y^2 \\
f_{xy} &= f_{\zeta\zeta}\zeta_x\zeta_y + f_{\chi\zeta}\left(\chi_x\zeta_y + \chi_y\zeta_x\right) + f_{\chi\chi}\chi_x\chi_y
\end{aligned} \tag{28}$$

One immediately deduces from the expressions for A^*, B^* and C^* that a clever choice of χ and ζ will cause the coefficients A^*, B^* to vanish. Completing the square and solving for the *ratio* χ_x/χ_y in A^* for example yields,

$$\frac{\chi_x}{\chi_y} = \frac{B}{2} \pm \frac{1}{2}\sqrt{B - 4AC} \tag{29}$$

And this will have certain implications for the solution. The three generalization which mathematicians use to classify such pde's are that the discriminant be greater, equal to, or less than zero, and these cases correspond to hyperbolic, parabolic, and elliptic pde's, respectively. In our case equation 12 in Part I has a discriminant of,

$$4b^2 - 4(1)(b^2 - (ab)^2) \quad = \quad (ab)^2 \quad > \quad 0 \tag{30}$$

That is, assuming that a, b are both greater than zero. Again, a, b represent the drift and diffusion coefficients, respectively. If a process is not drifting then [on average] it is not moving and nothing of measurable importance is happening (because it is the average value experiments typically measure). To diffuse means for the particles to move from a region of high concentration to lower concentration. If a process is no longer diffusing then entropy has been maximized and it has obtained equilibrium and there is no driving force behind it, hence it makes intuitive sense that if a is zero then b is necessarily zero. In section ... rigorous mathematical proof of this statement will be obtained through the concept of first and second moments in deriving the Chapman-Kolmogorov. Hence, a, b are non-zero, therefore the discriminant is greater than zero and equation12 represents a hyperbolic pde.

Part III
Stochastic Calculus

While it would be an appreciable digression to attempt to expound upon the entire theory of stochastic calculus, these are concepts which the average first or second year physics graduate student is unfamiliar with. Consequently it is unlikely such readers will have any clue what the author is talking about when he says 'Stratonovich type sde' or 'Itô type sde'. Here a basic ouline of the fundamental concepts of stochastic are breifly expounded upon as a prerequisite to derivation of the Chapman-kolmogorov equation which will allow us to understand these terms.

1 Stochastic Integrals

part A stochastic process is not an entirely rigorously defined thing. In general it is a set of probabilities which have been ordered in time. In terms of differential equations it is a process whose' differential has both a deterministic portion (the coefficient of dt) and another stochastic or random part (this will be the coefficient of $dw(t)$). Note that $w(t)$ is often represented as simply w_t, and we will adopt the latter convention in this paper. wE can write the differential of a stochastic process as,

$$dX_t = a(X_t, w_t, t)dt + b(X_t, w_t, t)dw_t \qquad \longrightarrow \qquad \text{Itô differential} \tag{1}$$

For clarity, dX_t is the differential stochastic *process* while x_t is the deterministic variable and w_t is the stochastic variable. The coefficients can be functions of the variables X_t, w_t, t. Note that the capitalization ofX_t is intentional as it represents a random variable rather than a deterministic one. For a variable to be random means it has an equal probability of being anywhere within its own associated probability distribution. This is not to say that the probability density function (or pdf) is everywhere the same - the magnitude of the distribution can change -, but just that we assume particles to be equally spread out under this curve which must be of finite value and go to zero at infinity. Each variable - stochastic or deterministic - has an associated pdf which will be denoted as $pdf_{X_t} = P(x_t, t)$. Note that the small x_t in the pdf is intentional and implies we are dealing with the explicitly form of x_t rather than simply referring to a distribution which we reserve for big X_t. Given the pdf of X_t, we say X_t is distributed according to the specific form of the pdf, e.g. Gaussian distribution, binomial distribution, etc. The expectation and variance values are,

$$\mu = E[X_t] = \int_{-\infty}^{\infty} x_t P(x_t) \qquad , \qquad \sigma^2 \quad = E[X_t^2] - (E[X_t])^2$$

When dealing with multiple variables X_t, Y_t (or when a single variable is dependent on multiple variables) we have a *joint pdf* $P(X_t, Y_t)$ which in the case of independence between X_t, Y_t can be separated; $P(X_t, Y_t) = P(X_t)P(Y_t)$. When dealing with stochastic processes it is actually more common to see probabilities listed as $P(X_t) = P(X, t)$, and so we will write probabilities as such but maintain the subscript on other quantities.

The two most widely used forms are Itô calculus and Stratonovich calculus. Whereas the standard calculus is defined with use of the Riemann integral, these alternative forms of calculus are defined with the Riemann–Stieltjes integral.

In discrete form the sum which defines Itô calculus is,

$$\int_0^t x_t dw_t = \sum_{i=1} x_{t_i}(w_{t_i+1} - w_{t_i}) \qquad \longrightarrow \qquad \text{Itô sum} \tag{2}$$

While the sum which defines Stratonivich calculus is,

$$\sum_{i=1}^{n-1} \frac{1}{2}(y_{t_i+1} + y_{t_i})(x_{t_i+1} - x_{t_i}) \qquad \longrightarrow \qquad \text{Stratonovich sum} \tag{3}$$

Upon taking the limit of infinitesimal partitions these become Riemann–Stieltjes integrals. The difference between Itô and Stratonovich calculus is how they chose X_t. Given an interval $\Delta t = t_{i+1} - t_i$ Itô chose $X_t = X_{t_i}$ to lie at the beginning of the interval while Stratonovich chose to average X_t over the time interval, so $X_t = \frac{X_{t_{i+1}} + X_{t_i}}{2}$ in

Stratonovich calculus.

The difference in choice has somewhat dramatic implications; in Itô calculus X_t is independent of the time interval because it is defined precisely at the beginning of the partition and we can always define the time interval to be the limit which excludes that particular point. This has the effect of decoupling X_t from $\Delta w_{t_i} = w_{t_i+1} - w_{t_i}$. While it would be a digression to go too far into this, one brief example may be useful to get a feel for the differences between the two forms. X_t being independent of W_t implies the joint pdf and by extension the expected values can be factored, i.e.

$$\mu = E[X_t W_t] = E[X_t]E[W_t] \tag{4}$$

W_t is taken to be normally distributed with $\mu = 0$ and $\sigma^2 = t$, i.e. X_t $N[0, \sqrt{t}]$ which is to say that it has a probability distribution of the following form,

$$P(w,t) = \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{x^2}{2t}} \tag{5}$$

The reasoning behind the assumption that w_t has a normal distribution with$\mu = 0$ and $\sigma^2 = t$ relates to the central limit theorem and Lindeberg's condtion, both of which are discussed in some detail in the appendix (see equations ...). Therefore, anytime we see $E[W_t]$ we don't even have to do the math, we just know because it has been given to us that $E[W_t] = \mu = 0$ which implies the expected value in equation 6 goes to zero.

Equally important is the variance of the differential dW_t which implies that $E[dW_t^2] = d\sigma^2 = dt$. Taking another look at equation 1 one begins to suspect this result will prove to be very useful in dealing with Itô differentials.

Equation 1 is the Itô differential which is associated with the summation in 4. For the differential which corresponds to the Stratonovich sum (equation 5) a *quadratic variation* on this sum leads to the following differential form[1],

$$y_t \partial x_s = y_t dx_t + \frac{1}{2} d(y_t) d(x_t) \qquad \longrightarrow \qquad \text{Stratonovich differential} \tag{6}$$

A significant difference between Stratonovich calculus and Itô calculus is that in Itô calculus the process W_t is a *Martingale* which mathematically speaking is to say that, given time variables s, t such that $t > s$ we have, $E[W_t|F_s] = W_s$, where F_s is a 'Filtration' (basically just an *ordered* set of possible outcomes). Essentially this is saying that being given information about the past in no way alters what you would expect in the future - it didn't improve your ability to predict the future. This property makes Itô calculus very important in financial engineering. The trade-off is that Itô calculus does not follow similar rules to standard calculus. In contrast, Stratonovich calculus does obeye similar rules to that of standard calculus, but the stochastic variable is not a martingale in Stratonovich calculus.

2 Stochastic Differential Equations (sde's)

Let $f(x_t)$ be a twice differentiable function. using the Itô differential a Taylor expansion gives,

$$df = f^{'} dx_t + \frac{1}{2} f^{''} (dx_t)^2 + ...$$

Where f' represents the standard derivative $df(x)/dx$. All higher order terms in the expansion vanish as will soon become apparent. Just for convenience, let $a(x_t, w_t, t), b(x_t, w_t, t)$ both be constants and note that the following expansion will work out the exact same regardless if they are variable as *no derivatives of these coefficients will be taken*, rather the Itô differential will be substituted into the above expansion and these coefficients are simply part of the Itô differential. Explicitly, we will make the following substitutions for the differential operators,

$$\begin{aligned} dx_t &= adt + bdw_t \\ (dx_t)^2 &= a^2 dt^2 + 2abdtdw_t + b^2 (dw_t)^2 = 0 + 0 + b^2 dt \end{aligned} \tag{7}$$

Where for the second order differential substitution we used the relations $dt = dtdw_t = 0$ and $(dw_t)^2 = dt$. These same relations can be used to show that higher order terms in the expansion vanish. Substituting these into the Taylor expansion yields,

$$
\begin{aligned}
df &= f^{'}\{adt + bdw_t\} + \frac{1}{2}b^2 f^{''}dw_t + ... \\
&= \left\{ f^{'}a + \frac{1}{2}b^2 f^{''} \right\} dt + bdw_t
\end{aligned} \tag{8}
$$

Dividing by dt and taking the [conditional] expectation value of the above gives,

$$
\begin{aligned}
E\left\{ \frac{df(x)}{dt} | x_0, t_0 \right\} &= \int_{-\infty}^{\infty} dx \frac{df}{dt} \rho(x,t|x_0,t_0) \\
&= \int_{-\infty}^{\infty} dx \left\{ f^{'}(x)a + \frac{1}{2} f^{''}(x) \right\} \rho(x,t|x_0,t_0)
\end{aligned} \tag{9}
$$

Now let $a = a(x), b = b(x)$. Integrating by parts and noting that the probability density must vanish at infinity we are left with,

$$
\int_{-\infty}^{\infty} dx f(x) \frac{d}{dt} \rho(x,t|x_0,t_0) \quad = \quad -\int_{-\infty}^{\infty} f(x) \frac{\partial}{\partial x}[a(x)\rho(x,t|x_0,t_0)] + \frac{1}{2}\int_{-\infty}^{\infty} f(x) \frac{\partial^2}{\partial x^2}[b^2(x)\rho(x,t|x_0,t_0)] \tag{10}
$$

Taking away the integral and dividing by $f(x)$ we are left with the Itô sde,

$$
\boxed{\frac{d}{dt}\rho(x,t|x_0,t_0) = -\frac{\partial}{\partial x}[a(x)\rho(x,t|x_0,t_0)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[b^2(x)\rho(x,t|x_0,t_0)] \qquad \longrightarrow \qquad \text{Itô form}} \tag{11}
$$

Comparing this relation to the Itô differential we have just developed a very simple and powerful relation between stochastic differentials and the Kolmogorov equation. We write this differential again with all dependance on the variables x_t, t, w_t restored just to clarify the comparison as well as the fact that it applies to the drift and diffusion coefficients a, b regardless of their dependence on these variables.

$$
\boxed{dx_t = a(x_t,t,w_t)dt + b(x_t,t,w_t)dw_t} \tag{12}
$$

3 Langevin's Equation

To brink the concepts discussed thus far together, it seems fitting to close with an example. In terms of applications in the physical sciences, Langevin's equation is probably the most commonly used model that is applied to physical phenomena that demonstrate stochastic fluctuations. The langevin equation reads,

$$
\frac{dx}{dt} = a(x_t) + b\frac{dw_t}{dt} \qquad ; \qquad a(x_t) = -\frac{D}{kT}\frac{\partial V(x)}{\partial x}, \qquad b = \sqrt{2D} \tag{13}
$$

Where $dw_t dt$ is the derivative of the stochastic variable and is known as *white noise*.The parameter D is the well known Diffusion coefficient in physics. It is curious to note there is slight discrepancy between this diffusion coefficient and B. It is a trivial matter to generate sample paths satisfying the langevin equation; multiplying by dt and discretizing the above Langevin's equation gives,

$$
dx_t = a(x_t)dt + bdw_t \tag{14}
$$

$$
x(t+\Delta t) = x(t) + a(x_t)\Delta t + b\sqrt{\Delta t}N[0,1] \tag{15}
$$

Where the factor $dw_t = \sqrt{\Delta t}N[0,1]$ comes from the fact that $w(t)$ is normally distributed with standard deviation $\sqrt{\Delta t}$. This is briefly discussed in the appendix (see 'Lindeberg's condition'). In terms of coding in Python a random number generator was used in place of the expression $N[0,1]$. The results of simulating various sample paths are depicted in figure 3 below.

To show a detailed calculation would risk overloading the reader with details that are questionable in their pertinance, but suffice it to say that - armed with the tools of Itô calculus discussed thus far - the variance can easily be read off from equation 16 to be as follows,

$$Var[x_t^2] = \sigma^2 = b^2 t = 2Dt \tag{16}$$

Which is in agreement with well known results in standard physics textbooks.

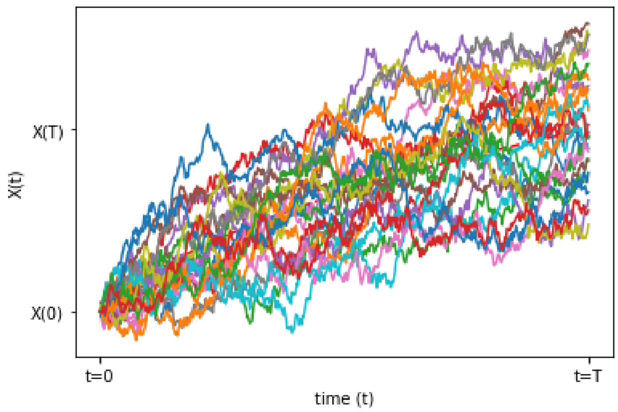

Figure 2: Sample paths for Langevin's equation with parameters $D = 100, K = 1, t = 1, B = \sqrt{2D}$

The above algorithm for simulating sample paths, while very easy to implement, does not obey the principle of detailed balance[6]. According to the relation between equation 13 and 14 we can immediately deduce the corresponding Fokker-Plank equation which allows us to solve for the probability in lieu of being able to solve for deterministic sample paths (x_t).

$$\frac{dP(x)}{dt} = D\frac{\partial}{\partial y}\left[\frac{1}{kT}V'(y)P(y)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[P(y)\right] \tag{17}$$

Equation 19 cannot be solved analytically for arbitrary potentials. Finite volume methods are often applied in biological applications[5, 6] and these give some very useful and [computationally speaking] very simple algorithms for modeling protein dynamics, e.g. one can imagine a molecule traversing a one dimensional polymer chain.

We close noting an interesting connection between the above Fokker-Plank equation for the Langevin equation and the the probability within the canonical ensemble for which - given that temperature is constant as the canonical ensemble requires - we have,

$$P_i^{eq} \propto e^{V_i/kT} \tag{18}$$

In fact, aside from a couple of constants, this *is* the equilibrium (or steady state) solution to equation 28 (see equation A.5 in reference [6]).

.

APPENDIX

Taylor Expansions

Uncorrelated Case with $(\boldsymbol{general}\ [\boldsymbol{not\ steady\ state}!] - \boldsymbol{no\ stipulations\ on\ coefficients})$

For the uncorrelated random walk we will require that $\lim_{\Delta t, \Delta x \to 0} \frac{(\Delta x)^2}{\Delta t} = D$. From equation 3 we have,

$$p(x) = \tfrac{1}{2} a(x) + \tfrac{1}{2} b(x) \frac{\Delta t}{\Delta x}$$

$$q(x) = \frac{1}{2} a(x) - \frac{1}{2} b(x) \frac{\Delta t}{\Delta x}$$

From equation 2 we have,

$$v(x, t - \Delta t) \quad = \quad \big(1 - p(x) - q(x)\big)\, v(x,t) \ + \ p(x - \Delta x)\, v(x - \Delta x, t) \ + \ q(x + \Delta x)\, v(x + \Delta x, t)$$

Taylor expanding the relevant quantities gives,

$$\begin{aligned}
(v - \Delta t v_t + \mathcal{O}(\Delta t^2)\ ) \ &= \ (1 - a)\, v \\
&+ \ \left\{\frac{1}{2}\left(a - \Delta x a_x + \frac{1}{2}\Delta x^2 a_{xx}\right) + \frac{1}{2}\frac{\Delta t}{\Delta x}\left(b - \Delta x b_x + \frac{1}{2}\Delta x^2 b_{xx}\right)\right\}\left(v - \Delta x v_x + \frac{1}{2}\Delta x^2 v_{xx}\right) \\
&+ \ \left\{\frac{1}{2}\left(a + \Delta x a_x + \frac{1}{2}\Delta x^2 a_{xx}\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left(b + \Delta x b_x + \frac{1}{2}\Delta x^2 b_{xx}\right)\right\}\left(v + \Delta x v_x + \frac{1}{2}\Delta x^2 v_{xx}\right)
\end{aligned}$$

$$\begin{aligned}
\Delta t v_t \quad &= \quad -a\, v \ + \ \left\{\left(a + \tfrac{1}{2}\Delta x^2 a_{xx}\right) - \tfrac{\Delta t}{\Delta x}\Delta x b_x\right\} v \\
&+ \ \left\{\Delta x a_x - \frac{\Delta t}{\Delta x}\left(b + \frac{1}{2}\Delta x^2 b_{xx}\right)\right\}\Delta x v_x \\
&+ \ \left\{\left(a + \frac{1}{2}\Delta x^2 a_{xx}\right) - \frac{\Delta t}{\Delta x}\Delta x b_x\right\}\frac{1}{2}\Delta x^2 v_{xx}
\end{aligned}$$

$$\begin{aligned}
v_t \quad &= \quad \left\{\tfrac{1}{2}\tfrac{\Delta x^2}{\Delta t} a_{xx} - b_x\right\} v \\
&+ \ \left\{\frac{\Delta x^2}{\Delta t} a_x v_x - b v_x\right\} \\
&+ \ \left\{\frac{1}{2}\frac{\Delta x^2}{\Delta t} v_{xx} a - \frac{1}{2}\Delta x^2 v_{xx} b_x\right\}
\end{aligned}$$

$$\begin{aligned}
v_t \quad &= \quad \left\{\tfrac{1}{2}\tfrac{\Delta x^2}{\Delta t}(a_{xx} + 2 a_x v_x + v_{xx} a) - (b_x\, v + v_x b)\right\} \\
&= \quad \left\{\tfrac{1}{2}\tfrac{\Delta x^2}{\Delta t}(va)_{xx} - (bv)_x\right\}
\end{aligned}$$

$$\boxed{v_t \xrightarrow[\Delta t, \Delta x \to 0]{} \tfrac{D}{2}(av)_{xx} - (ba)_x} \quad \rightarrow \qquad \text{[A.1]}$$

This corresponds to equation 4.

Correlated case (equation 8 , Zauderer eqn. 8,9):

Equation (6) is written again for convenience

$$r(x,t+\Delta t) \;=\; \big(1-p^{+}(x)-q^{+}(x)\big)\, r(x,t) \;\rightarrow\quad P_1 \text{ (stationary particle jumps from left)}$$
$$+\; p^{+}(x-\Delta x)\, r(x-\Delta x,t) \;\rightarrow\quad P_2 \text{ (right moving particle continues [from left])}$$
$$+\; q^{-}(x-\Delta x)\, l(x-\Delta x,t) \;\rightarrow\quad P_3 \text{ (left moving particle reverses)}$$

The case of particles jumping from the left (or moving right) will be treated in detail, and similar results will then be inferred for the case of particles jumping from the right (or moving left).

Taylor expanding variables w.r.t. both time and space then suppressing the notion such that [for example] $r(x,t) = r$,

$$r(x,t+\Delta t) \;\approx\; r \;+\; r_t\Delta t \;+\; \mathcal{O}(\Delta t^2)$$
$$r(x-\Delta x,t) \;\approx\; r \;-\; r_x\Delta x \;+\; \tfrac{1}{2}r_{xx}(\Delta x)^2$$
$$l(x-\Delta x,t) \;\approx\; l \;-\; l_x\Delta x \;+\; \tfrac{1}{2}l_{xx}(\Delta x)^2$$
$$p^{+}(x-\Delta x) \;\approx\; p^{+} \;-\; p_x^{+}\Delta x \;+\; \tfrac{1}{2}p_{xx}^{+}(\Delta x)^2$$
$$q^{-}(x-\Delta x) \;\approx\; q^{-} \;-\; q_x^{-}\Delta x \;+\; \tfrac{1}{2}q_{xx}^{-}(\Delta x)^2$$

Where second order time terms were set to zero because we are going to take the limits $\Delta t \to 0$ and $\Delta x \to 0$ keeping only the ratio $\lim\limits_{\Delta t,\Delta x\to 0}\frac{\Delta x}{\Delta t} = c$ constant (note that requiring $\lim\limits_{\Delta t,\Delta x\to 0}\frac{(\Delta x)^2}{\Delta t} = D$ is not specified in the derivation of the correlated case, but this will be required in the next derivation). Higher order terms that we see have no hope of satisfying these ratios and will go to zero will likewise be set to zero along the way in this somewhat tedious derivation. Inserting these values into equation 6 gives,

$$r \;+\; r_t\Delta t \;\approx\; (1-p^{+}-q^{+})\, r$$
$$+\left\{p^{+} - p_x^{+}\Delta x + \tfrac{1}{2}p_{xx}^{+}(\Delta x)^2\right\}\left\{r - r_x\Delta x + \tfrac{1}{2}r_{xx}(\Delta x)^2\right\}$$
$$+\left\{q^{-} - q_x^{-}\Delta x + \tfrac{1}{2}q_{xx}^{-}(\Delta x)^2\right\}\left\{l - l_x\Delta x + \tfrac{1}{2}l_{xx}(\Delta x)^2\right\}$$

$$r \;+\; r_t\Delta t \;=\; (1-p^{+}-q^{+})\, r$$
$$+\; r\left\{p^{+} - p_x^{+}\Delta x + \tfrac{1}{2}p_{xx}^{+}(\Delta x)^2\right\} - r_x\Delta x\{p^{+} - p_x^{+}\Delta x + \cdots\} + \tfrac{1}{2}r_{xx}(\Delta x)^2\{p^{+} - \ldots\}$$
$$+\; l\left\{q^{-} - q_x^{-}\Delta x + \tfrac{1}{2}q_{xx}^{-}(\Delta x)^2\right\} - l_x\Delta x\,\{q^{-} - q_x^{-}\Delta x + \cdots\} + \tfrac{1}{2}l_{xx}(\Delta x)\{q^{-} - \ldots\}$$

$$r_t\Delta t \;=\; \{q^{-}l - q^{+}r\} - \{(p_x^{+}r + p^{+}r_x) \;+\; (q_x^{-}l + q^{-}l_x)\}\Delta x$$
$$=\; \{q^{-}l - q^{+}r\} - \{p^{+}r \;+\; q^{-}l\}_x\Delta x$$

Now using the proposed probabilities in equation 3 gives,

$$r_t\Delta t \;\approx\; \Delta t\{l\lambda^{-} - r\lambda^{+}\} - \left\{\left(a + b\frac{\Delta t}{\Delta x} - \lambda^{+}\Delta t\right) r \;+\; \Delta t\, l\,\lambda^{-}\right\}_x \Delta x$$

$$\boxed{r_t \xrightarrow[\Delta t,\Delta x\to 0]{} l\lambda^{-} - r\lambda^{+} - ([ac+b]r)_x} \qquad \text{[A.2.a]}$$

And for the particles jumping from the right (or jumping left) a similar treatment yields,

$$l_t \approx r\lambda^+ - l\lambda^- - ([b-ac]l)_x \qquad \text{[A.2.b]}$$

These equation correspond to 8,9.

[Steady State] Uncorrelated Case with a, b, λ = constant

For $b \neq 0$ the steady state probabilities should take the form (not demonstrated by Zauderer),

$$p(x) = \left(\hat{a} + b\frac{\Delta t}{\Delta x}\right)\left(\frac{1}{2} + \frac{h}{2\sqrt{\lambda}}\right)$$

$$q(x) = \left(\hat{a} + b\frac{\Delta t}{\Delta x}\right)\left(\frac{1}{2} - \frac{h}{2\sqrt{\lambda}}\right)$$

For a, b, λ constant the above probabilities need not be expanded, so expanding equation 2,

$$v(x, t-\Delta t) \quad = \quad \big(1 - p(x) - q(x)\big)\, v(x,t) \quad + \quad p(x-\Delta x)\, v(x-\Delta x, t) \quad + \quad q(x+\Delta x)\, v(x+\Delta x, t)$$

gives,

$$\begin{aligned}
\left(v - \Delta t v_t + \mathcal{O}(\Delta t^2)\ \right) &= \left(1 - \left(\hat{a} + b\frac{\Delta t}{\Delta x}\right)\right) v \\
&\quad + \left(\hat{a} + b\frac{\Delta t}{\Delta x}\right)\left(\frac{1}{2} + \frac{h}{2\sqrt{\lambda}}\right)\left(v - \Delta x v_x + \frac{1}{2}\Delta x^2 v_{xx}\right) \\
&\quad + \left(\hat{a} + b\frac{\Delta t}{\Delta x}\right)\left(\frac{1}{2} - \frac{h}{2\sqrt{\lambda}}\right)\left(v + \Delta x v_x + \frac{1}{2}\Delta x^2 v_{xx}\right) \\
-\Delta t v_t &= \left(\hat{a} + b\frac{\Delta t}{\Delta x}\right) v \quad + \quad \left(\hat{a} + b\frac{\Delta t}{\Delta x}\right)\left(v - \frac{h}{\sqrt{\lambda}}\Delta x v_x + \frac{1}{2}\Delta x^2 v_{xx}\right) \\
&= \quad 0 \quad + \quad \left(\hat{a} + b\frac{\Delta t}{\Delta x}\right)\left(-\frac{h}{\sqrt{\lambda}}\Delta x v_x + \frac{1}{2}\Delta x^2 v_{xx}\right)
\end{aligned}$$

Dividing both sides by Δt gives,

$$-v_t \quad = \quad \hat{a}\left(-\frac{h}{\sqrt{\lambda}}\frac{\Delta x}{\Delta t} v_x + \frac{1}{2}\frac{\Delta x^2}{\Delta t} v_{xx}\right) \quad + \quad b\frac{1}{\Delta x}\left(-\frac{h}{\sqrt{\lambda}}\Delta x v_x + \frac{1}{2}\Delta x^2 v_{xx}\right)$$

Letting $\Delta t, \Delta x \to 0$

$$-v_t \quad = \quad -\frac{h}{\sqrt{\lambda}} c\hat{a} v_x + \frac{1}{2} D\hat{a} v_{xx} \quad - \frac{h}{\sqrt{\lambda}} b v_x$$

$$v_t \quad = \quad \sqrt{\widehat{D}} h\hat{a} v_x \quad - \frac{1}{2}\widehat{D}\hat{a}^2 v_{xx} \quad + \frac{h}{\sqrt{\lambda}} b v_x \qquad \text{[A.3]}$$

Where Zauderer has defined the limiting quantities $\frac{c^2}{\lambda} \to \widehat{D}$ and $\hat{a} \to \lambda\Delta t$. This corresponds to equation 16.

[Steady State] Uncorrelated Case with $b = 0$

$$\lim_{\Delta t, \Delta x \to 0} \frac{\Delta x}{\Delta t} = c \text{ and } \lim_{\Delta t, \Delta x \to 0} \frac{(\Delta x)^2}{\Delta t} = D$$

Where we require that $b = 0$ the steady state probabilities take the form,

$$p(x) = \frac{1}{2}\hat{a}(x) + \frac{1}{2}\frac{h}{\sqrt{\lambda}}\hat{a}(x)$$

$$q(x) = \frac{1}{2}\hat{a}(x) - \frac{1}{2}\frac{h}{\sqrt{\lambda}}\hat{a}(x)$$

Expanding,

$$v(x, t - \Delta t) \quad = \quad \big(1 - p(x) - q(x)\big)\, v(x,t) \quad + \quad p(x - \Delta x)\, v(x - \Delta x, t) \quad + \quad q(x - \Delta x)\, v(x - \Delta x, t)$$

Gives,

$$\begin{aligned}
(v - \Delta t v_t + \mathcal{O}(\Delta t^2)\ ) \quad &= \quad (1 - a)\, v \\
&\quad + \left\{\frac{1}{2}\left(a - \Delta x a_x + \frac{1}{2}\Delta x^2 a_{xx}\right) - \frac{h}{2\sqrt{\lambda}}\left(a - \Delta x a_x + \frac{1}{2}\Delta x^2 a_{xx}\right)\right\} \left(v - \Delta x v_x + \frac{1}{2}\Delta x^2 v_{xx}\right) \\
&\quad + \left\{\frac{1}{2}\left(a - \Delta x a_x + \frac{1}{2}\Delta x^2 a_{xx}\right) - \frac{h}{2\sqrt{\lambda}}\left(a - \Delta x a_x + \frac{1}{2}\Delta x^2 a_{xx}\right)\right\} \left(v - \Delta x v_x + \frac{1}{2}\Delta x^2 v_{xx}\right)
\end{aligned}$$

$$\begin{aligned}
\Delta t v_t \quad &= \quad -a\, v \quad + \quad \left\{\left(a + \frac{1}{2}\Delta x^2 a_{xx}\right) - \frac{h}{\sqrt{\lambda}}\Delta x a_x\right\} v \\
&\quad + \left\{\left(a + \frac{1}{2}\Delta x^2 a_{xx}\right) + \frac{h}{\sqrt{\lambda}}\Delta x a_x\right\} \frac{1}{2}\Delta x^2 v_{xx} \\
&\quad + \left\{\Delta x a_x - \frac{h}{2\sqrt{\lambda}}\left(a + \frac{1}{2}\Delta x^2 a_{xx}\right)\right\} \Delta x v_x
\end{aligned}$$

$$v_t \quad = \quad -\frac{h\Delta x}{\Delta t\sqrt{\lambda}}(a_x v + v_x a) - \frac{\Delta x^2}{2\Delta t}(a_{xx} + 2a_x v_x + a v_{xx})$$

$$v_t \quad = \quad -h\sqrt{\frac{\Delta x^2/\Delta t}{\Delta t \lambda}}(a_x v + v_x a) - \frac{\Delta x^2}{2\Delta t}(a_{xx} + 2a_x v_x + a v_{xx})$$

$$\boxed{v_t \quad \xrightarrow[\Delta t, \Delta x \to 0]{} \quad h\sqrt{\frac{D}{\tilde{a}}}(va)_x - \frac{D}{2}(va)_{xx} \quad \to \quad Ito\ type} \qquad \text{[A.4]}$$

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