Autocovariance and Innovations

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Wiener Process

The Wiener process is a random walk model which attempts to give mathematical form to the phenomena of brownian motion. As the name implies, a random walk is a set of successive steps over time which appear to be random in their aim or direction. Brownian motion is a phenomena which arises in a large number of natural processes, e.g. molecules moving in seemingly random directions as they are suspended in fluid. Random walk models are then one means of attempting to characterize Brownian motion. The Wiener process exploits the central limit theorem in order to develop a probability distribution which would characterize the steps of a random walker. The central limit theoreom (or law of large numbers), like Brownian motion, is a thing observed in numerous processes throughout nature, hence one appeal of the Wiener process is that it attempts to bring the two together.

We will not be concerned with rigorously constructing the Wiener process so much as we will be with using the relation described by equation 1 to derive some quantifiable variable that emulates the behavior of the Wiener process, i.e. a numerical expression which we can perform computations with.

Given ε that is distributed as a standard normal the *Wiener process* is characterized by the relation

$$\Delta w = w_i - w_{i-1} = \varepsilon_i \sqrt{\Delta t} = \varepsilon \sqrt{t_i - t_{i-1}} \tag{1}$$

For convenience the variable $w(t)$ is often denoted with a subscript as w_t. w_t is a stochastic variable meaning it has random components for which, given the state of w_t, the value of w_{t+1} cannot be predicted with certainty, but only probabilistically. The stochastic nature of w_t comes from the term ε_i which is a random variable with an expected mean of zero and variance of one, i.e. it is a standard normal variable denoted as $\varepsilon_i \sim N[0, 1]$. The term Δt is a deterministic term meaning there is nothing random about it; given an initial value of t_i we can predict the value of t_{i+1}.

First we consider the case of discrete time values then extend the model to the continuous case.

Because $\sqrt{\Delta t}$ is deterministic it is treated as a constant and we have $E[\Delta w_i] = \sqrt{\Delta t} E[\varepsilon_i] = 0$ and $Var[\Delta w_i] = \Delta t E[\varepsilon_i^2] = \Delta t$ where $E[\Delta w_i], Var[\Delta w_i]$ are the expectation and variance values, respectively.

Given a succession of time steps we have the final time as $t_n = \sum_i^n \Delta t = n\Delta t = n$ for discrete and uniformly spaced time steps (because $\Delta t = 1$ in this case). An equivalent representation is $\sum_i^n (t_i - t_{i-1}) = t_n - t_0 = \text{n}$ if we further assume that $t_0 = 0$. Noting that ε_i variables are independent of one another, for the expectation and variance we have $E[\Delta w_t] = 0$ and $Var[\Delta w_t] = \sum_{i=1}^n \Delta t = n\Delta t = t_n$.

The Wiener process requires that $w_0 = 0$. Ergo, if we let the variable $x_i = \Delta w = \varepsilon_i \sqrt{\Delta t}$ then we have $\sum_{i=1}^n x_i = \sum_{i=1}^n \Delta w_i = w_n$. We can exploit the latter equality $x_i = \varepsilon_i \sqrt{\Delta t}$ to develop a distribution for x_i. We know that $\varepsilon_i \sim N[0, 1]$ so, using the standard means of transforming arguments of a probability density function (pdf), we have for a transformation of variables $y = az$ that the pdf transforms as $f(y) = f(z)|J|$ where J is the Jacobian matrix of the transformation. Applying this to $x_i = \varepsilon_i \sqrt{\Delta t}$ we get,

$$f(x_i) = \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{1}{2}\frac{x_i}{\sqrt{\Delta t}}^2} dx_i$$

Which is to say that $x_i \sim N[0, \Delta t]$. Now it is generally true that a sum of normally distributed variables is normally distributed with additive mean and variance, i.e. given a set of variables $x_i \sim N[\mu_i, \sigma_i^2]$ we have $\sum_i^n x_i \sim N[\sum_i^n \mu_i, \sum_i^n \sigma_i^2]$. In our case we have $\sum_i^n x_i \sim N[0, n\Delta t] = N[0, t_n]$.

In order to examine the continuous case, for reasons that will soon become apparent it is of use to employ the central limit theorem on $\sum_i^n x_i$;

$$\sum_i^n x_i \sim N[0, t_n] \qquad \longrightarrow \qquad \frac{1}{\sqrt{t_n}} \sum_i^n x_i = \frac{1}{\sqrt{n}} \sum_i^n x_i \sim N[0, 1]$$

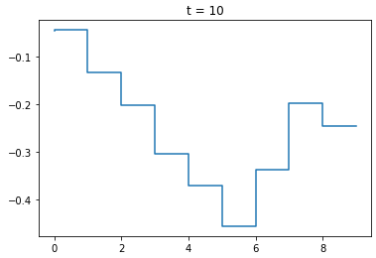
The standardized sum $\frac{1}{\sqrt{n}} \sum_i^n x_i$ is precisely the variable which we've set out to find as it emulates *in continuous* time the behavior of Brownian motion, least in the sense that, as will now be shown, with the appropriate scaling of time values it is approximately distributed as Δw_t is, which is to say it will be distributed just as $\sum_i^n x_i = \sum_i^n (w_i - w_{i-1}) \sim N[0, t_n]$ which only holds for discrete time values.

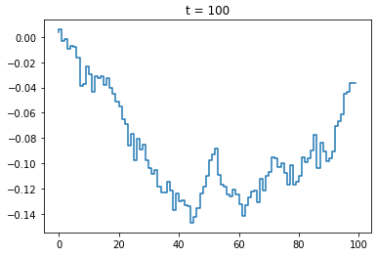
To make the conversion into [approximately] continuous time consider a re-scaling of the time interval $[0, n]$ to $[0, 1]$. Obviously this is be accomplished by dividing each unscaled time value $t_{unscaled} \in (i-1, i]$ by $t_n = n$. Notice the left hand inequality is not inclusive as this would cause an overlap with the interval $(i-2, i-1]$ which lies to the left of the interval in question. This is a thing we definitely want to avoid as it would violate the stipulation that Δw increments be independent of one another, else they would not properly emulate a truly random walk. If n is very large we can consider the scaled time value $t \approx i/n$ to be approximately continuous. Nevertheless, there will always be an infinite number of 'in-between' time values unaccounted for (hence, it is an approximation). It would be ideal if, instead of dividing by n we could divide the time interval by some arbitrary number which could be taken to infinity, but this would have little correlation to physical reality; the idea here is to invoke the central limit theorem for large sample sizes. For the interval in question we have,

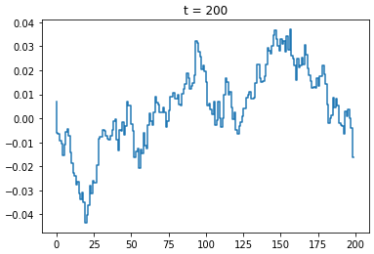
$$\frac{i-1}{n} < t \leq \frac{i}{n} \qquad i = 1, 2, ..., n$$

Where t is a truly continuous [scaled] time variable. By using the scaled time value $t = t_{unscaled}/n$ we are able to say that if $t \approx i/n$, and if n is large, then so too must i be large – a fact we will soon exploit.

As the figures below suggest, for low values of n (say $n < 100$) the approximation $t \approx \frac{i}{n}$ isn't all that good. But for larger values of n the approximation quickly begins to emulate brownian motion (or random walk).







Noting that in discrete time $t_i = i$ we have for the standardized partial sum up to interval i

$$\frac{1}{\sqrt{n}} S_i = \frac{1}{\sqrt{n}} \sum_{j=1}^{i} x_j$$

This is the quantifiable term which we use in algorithms to emulate the Wiener process. What follows is only to show it is approximately distributed as $N[0,\sqrt{t}]$ for all $t \approx t_i \ ; \ i = 1, 2, 3, ..., n$.

If $t \approx \frac{i}{n}$ then make the substitution $n = \frac{i}{t}$ which gives $\frac{\sqrt{t}}{\sqrt{i}} \sum_{j=1}^{i} x_j$. As previously noted we have reason to say that i is large, so we'd be justified in making the central limit approximation which tells us that $\frac{1}{\sqrt{i}} \sum_{j=1}^{i} x_j \sim N[0,1]$. If we set $z = \frac{1}{\sqrt{i}} \sum_{j=1}^{i} x_j$ then we are left with $S_i \approx \sqrt{t}z$. As in the one dimensional case we perform a change of variables and find the Jacobian for the transformation. The results are

$$f(\sqrt{t}z) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}\frac{z}{t}^2} dz \qquad ; \quad z = \frac{1}{\sqrt{i}} \sum_{j=1}^{i} x_j$$

Note that the above has been derived under the presumption that $t_0 = 0$. For arbitary initial times $t_0 = s$ we'd get the well-known result,

$$f(\sqrt{t}z) = \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{1}{2}\frac{z}{t-s}^2} dz$$

We see that the quantity $\sqrt{t}z$ has the same distribution as $w_t = \sum_{i=1}^{t} (w_i - w_{i-1})$ in approximately continuous time.

To summarize our results we have the following relations

$$\sum_{i}^{n} x_i = \sum_{i}^{n} (w_i - w_{i-1}) \sim N[0,n] \qquad \textit{unscaled time; large } n.$$

$$\frac{1}{\sqrt{i}} \sum_{j=1}^{i} x_j \sim N[0,1] \qquad \textit{unscaled time; } i \textit{ large; } i \leq n$$

$$\frac{\sqrt{t}}{\sqrt{i}} \sum_{j=1}^{i} x_j \sim N[0,t] \qquad \textit{scaled time; large } n.$$

To emulate the Δw process we use,

$$\frac{1}{\sqrt{n}} S_i = \frac{1}{\sqrt{n}} \sum_{j=1}^{i} x_j$$

where $x_j = \Delta w \sim N[0, \Delta t] = N[0,1]$ where the last equality follows from the fact that $\Delta t = 1$ in the unscaled time-frame. For plotting/ computational purposes there are two ways to go about generating w_t. The first is to use a normally distributed random number generator (e.g. $np.random.normal(\mu, \sigma, t = i)$ in python) for the variables x_i and plot the cumulative sum of these variables then divide the resulting partial sum values at every step by $\sqrt{n}$. The second and equivalent way is to recognize that the above summation reduces to,

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{i} x_j = \frac{1}{\sqrt{n}} \sum_{j=1}^{i} (w_i - w_{i-1}) = \frac{w_i - w_0}{\sqrt{n}} = \frac{w_i}{\sqrt{n}}$$

Setting $i = 1$ we get $x_1 = w_1$ so w_1 is simply x_1 – a standard normally distributed random variable. For $i = 2$ we have $x_1 + x_2 = w_1 + x_2 = w_2$. Iterating the process we get the standard random walk model;

$$w_i = w_{i-1} + x_i$$

Conclusion

The Wiener process is just one way to model a random walk, albeit a powerful and widely used method. Both Brownian motion and the law of large numbers (the central limit theorem) are things observed everywhere in nature, so it is not a random walk model without justifiable cause for its form. There are a number of useful identities for the Wiener process which have not been stated here, but we have develop a quantifiable expression which can be used to emulate brownian motion.

References

G.S. Maddala and In-Moo Kim, *Unit Roots, Cointegration, and Structural Change*, Cambridge University Press,1999.

Patrick Billingsley, *Probability and Measure, Second Edition*, John Wiley and Sons, 1986.