# Optimal Portfolio Simulation in Maple 14

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## Purpose

This Maple document intends to construct the Markowitz efficient frontier and examines how it behaves to certain variables in its construction.

## Generating the Basic Variables

We first clear all existing variables.

> *restart* :

Next in Maple, we use the slider and coding tools to experiment on the efficient frontier formed by three risky assets. The mean and volatility are defined using the sliders below. The units for the 2 variables will be written in terms of percentage gains.

**Asset 1** $(a_1)$**:**

$\mu_1 = .1$ 0.0 0.2 0.4 0.6 0.8 1.0 , $\sigma_1 = .4$ 0.0 0.05 0.1 0.15 0.2

**Asset 2** $(a_2)$**:**

$\mu_2 = .6$ 0.0 0.2 0.4 0.6 0.8 1.0 , $\sigma_2 = .13$ 0.0 0.05 0.1 0.15 0.2

**Asset 3** $(a_3)$**:**

$\mu_3 = .4$ 0.0 0.2 0.4 0.6 0.8 1.0 , $\sigma_3 =$ 0.5e-1 0.0 0.05 0.1 0.15 0.2

From here, we can generate the expected return vector and the covariance matrix. We assume for the sake of simplicity that $Cov(a_i, a_j) = k\sigma_i\sigma_j$ for $i \neq j$ and some $0 < k < 1$ chosen using a slider. We will also generate a risk-free borrowing or lending rate $r$ and a factor $B$ for an expected utility function to be used later in this document.

$k = .3$ 0.0 0.2 0.4 0.6 0.8 1.0

$r = .2$ 0.0 0.1 0.2 0.3

$B = 2.0$

0.0 1.0 2.0 3.0 4.0 5.0 6.0

Generate Vector of Expected Returns (R)

Generate Matrix of Covariances (S)

## Calculating the Efficient Frontier

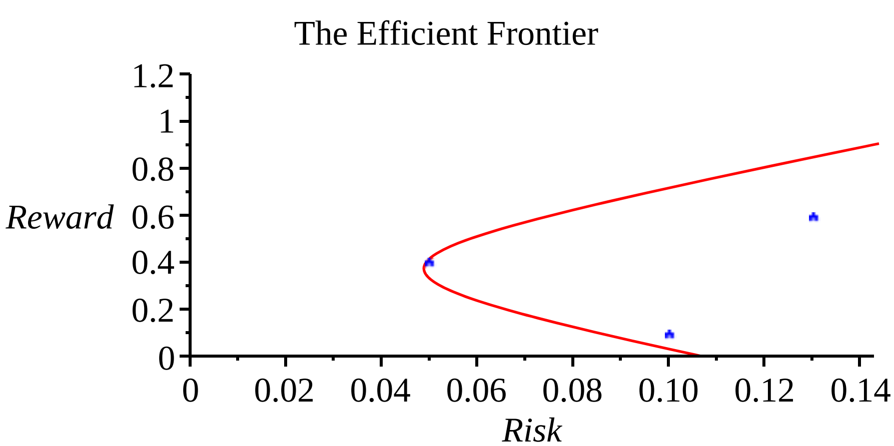
We know algebraically, and from literature published by Merton, that the the equation for the frontier in terms of $\sigma$, the portfolio volatility, is $\sigma = \sqrt{\frac{a\mu^2 - 2b\mu + c}{d}}$ where $\mu$ is an expected value of return using some combination of the three assets, $a = 1^t S^{-1} 1 > 0$, $b = 1^t S^{-1} R$, $b = R^t S^{-1} R$, and $D = ac - b^2 > 0$. Here, 1 denotes a unity vector of degree 3 defined by

```
> u3 := Vector(1..3, 1) :
```

We then generate an equation for $\sigma$ and plot the resulting equation using the code blocks below.

Generate Equation for sigma

Plot Markowitz Frontier



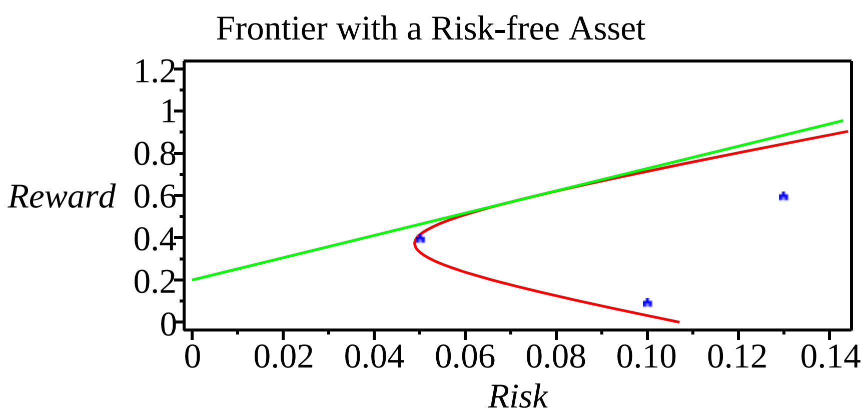

Here, the blue asterisks are where the three assets lie, inside the frontier.

## Portfolio of Risk-free and Risky Assets

This time, we add a risk free asset with interest rate *r* (defined above) to our portfolio and using portfolio optimization theory, we find the optimal portfolio of these 3 assets which will provide the greatest return per unit of risk. This known to be the point $(\sigma_m, \mu_m)$ where the line $(\mu - \mu_m) = slope \cdot (\sigma - \sigma_m)$ for some $slope > 0$ lies tangential to the frontier. This line is known as the capital allocation line (CAL) and the portfolio associated with that point is the called the market portfolio. Algebraically, we can calculate the weights of the assets (in numerical order) in the market portfolio using the equation $w_m = \frac{S^{-1}(R - r1)}{b - ra}$ which implies that $(\sigma_m, \mu_m) = (w_m^t S w_m, w_m{}^t R)$. Since we know that the point $(0, r)$ lies on the CAL, then we have the equation $\mu = \left(\frac{\mu_m - r}{\sigma_m}\right)\sigma + r$ representing the CAL.

We generate a sample plot below.

Plot New Frontier




## Separation Principle

The CAL represents the possible combinations of a portfolio consisting of a risk free asset, and the three assets in fixed proportions, such that the return on reward per unit risk (or Sharpe ratio) is maximized. Where a particular investor stands on the CAL, however, is dependent on his or her risk preferences. These preferences are generally encoded in a suitable utlity function and the problem of maximizing an investor's expected utility for some proportion of risky and risk-free assets came to be know as Merton's portfolio problem.

For simplicity, we will only assume a quadratic utility function of the form $U(W) = W - \frac{B}{2}W^2$ where $W$ is the random variable that corresponds to the realized growth $1 + r_A$ of a particular portfolio $A$. Here, $B$ determines the risk preference of an investor. That is, higher values of $B$ indicate a risk seeking behaviour while the opposite holds for lower values of $B$. We first note that $U'(W) = 1 - BW$ and to preserve monotonicity of $U$, we should ensure that $B$ is chosen with the condition that $B < \frac{1}{W}$. Next, since we want to optimizean investor's expected utility, $E[U(A)] = \mu_A - \frac{B}{2}(\mu_A^2 + \sigma_A^2)$, on the CAL we note that the utility is maximized when the indifference curve of the optimal utility are tangental to the CAL.
This means that we can use the method of Lagrange multipliers to maximize $E[U(A)]$ subject to the

condition $\left(\frac{\mu_m - r}{\sigma_m}\right)\sigma + r - \mu = 0.$

## Interactive Optimal Portfolio Plot with Investor Preferences

The following plot uses the variables defined in the second section of this document to generate the efficient frontier and the optimal investor portfolio.

Refresh Plot