# A DETAILED PROOF ON THE IRRATIONALITY OF PI

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Abstract. This paper will detail the proof used by Ivan Niven, summarized here, to show that pi is an irrational number. The proof will mainly be targeted towards curious high school students and first year university students and will assume that the reader has sufficient knowledge in basic differential and integral calculus as well as some basic trigonometry. Some of the proofs have been intentionally left as an exercise for the reader to avoid taking up space.

**Claim:** $\pi$ is an irrational number.

**Proof:** We will proceed through a proof by contradiction. Suppose that $\pi$ is a positive rational number and is in the form $\pi = \frac{a}{b}$ for some $a, b \in \mathbb{N}$ (the set of positive integers) where $\pi$ is already in its reduced fraction form. We now define the following functions.

$$f(x) = \frac{x^n(a-bx)^n}{n!}$$

and

$$F(x) = f(x) + ... + (-1)^j f^{(2j)}(x) + ... + (-1)^n f^{(2n)}(x)$$

For some positive integer $n \in \mathbb{N}$ where $f^{(2j)}(x)$ denotes the $2j^{th}$ derivative of $f$. To proceed, we will first make some remarks about these functions.

**Remark 1.** $f(x)$ is a polynomial of degree $2n$ with integer coefficients, when $\frac{1}{n!}$ is factored out, for any $n \in \mathbb{N}$ by observation.

**Remark 2.** $f(x) = f(x - \pi)$.

**Proof:** This can be done through a simple evaluation of $f(x-\pi)$.

$$\begin{aligned} f(x-\pi) &= \frac{(x-\pi)^n(a-b(x-\pi))^n}{n!} \\ &= \frac{(bx-a)^n(a+(-bx-a))^n}{b^n n!} \\ &= \frac{b^n(bx-a)^n(-x)^n}{b^n n!} \\ &= \frac{x^n(a-bx)^n}{n!} \\ &= f(x) \end{aligned}$$

**Remark 3.1.** For values of $x \in (0,\pi)$, $f(x) > 0$.

**Proof:** Show as an exercise.

**Remark 3.2.** $0 \leq f(x) \leq \frac{\pi^n a^n}{n!}$, for $0 \leq x \leq \pi$ and any $n \in \mathbb{N}$.

**Proof:** We first note that $f(\pi) = f(\pi - \pi) = f(0) = 0$. With Remark 3.1., $0 \leq f(x)$ follows trivially. We now look for the local maxima of $f$ in the interval $(0,\pi)$. Through some simple calculations, we can derive $f'$ as

$$f'(x) = \frac{nx^{n-1}(a-bx)^{n-1}(a-2bx)}{n!}.$$

So $f'(x) = 0 \implies x = 0, \frac{\pi}{2}, \pi$. Discarding 0 and $\pi$, we have

$$\begin{aligned} f(\frac{\pi}{2}) &= \frac{(\frac{\pi}{2})^n(a-b(\frac{\pi}{2}))^n}{n!} \\ &= \frac{\pi^n a^n}{2^{2n} n!} \\ &\leq \frac{\pi^n a^n}{n!} \end{aligned}$$

**Remark 4.** For $0 \leq j < n$, $f^{(j)}(x)$ is zero when $x = 0$ and $x = \pi$ for any $n \in \mathbb{N}$.

**Proof:** We can proceed by induction. Suppose that for any $0 \leq m < n-1$, we have

$$f^{(m)}(x) = \frac{m!\binom{n}{m}x^{n-m}(a-bx)^{n-m}g_m(x)}{n!}$$

where $g_m(x)$ is some polynomial with the property of $g_m(0) = k_{0m}$ and $g_m(\pi) = k_{\pi m}$ for any $0 \leq m < n-1$ where $k_{0m}, k_{\pi m} \in \mathbb{Z}$ (the set of integers) and $g_m(x)$ only contains only integer coefficients. By observation, we can see that $f^{(m)}(0) = f^{(m)}(\pi) = 0$. In the base case $(m = 1)$, we have

$$f^{(1)}(x) = \frac{nx^{n-1}(a-bx)^{n-1}(a-2bx)}{n!} = \frac{nx^{n-1}(a-bx)^{n-1}g_1(x)}{n!}$$

from above, with $g_1(x) = (a-2bx)$, $g_1(0) = a$ and $g_1(\pi) = -a$. Now we will check if our hypothesis holds for $m+1$. We have

$$\begin{aligned} f^{(m+1)}(x) &= \frac{(m+1)!\binom{n}{m+1}x^{n-(m+1)}(a-bx)^{n-(m+1)}(a-2bx)g_m(x) + m!\binom{n}{m}x^{n-m}(a-bx)^{n-m}g_m'(x)}{n!} \\ &= \frac{(m+1)!\binom{n}{m+1}x^{n-(m+1)}(a-bx)^{n-(m+1)}\left[(a-2bx)g_m(x) + (m+1)(x)(a-bx)g_m'(x)\right]}{n!} \\ &= \frac{(m+1)!\binom{n}{m+1}x^{n-(m+1)}(a-bx)^{n-(m+1)}g_{m+1}(x)}{n!} \end{aligned}$$

By observation, we can see that $f^{(m+1)}(0) = f^{(m+1)}(\pi) = 0$.

*Exercise:* Show that $g_{m+1}(0), g_{m+1}(\pi) \in \mathbb{Z}$ and $g_{m+1}(x)$ has only integer coefficients for all $0 \leq m \leq n-1$.

**Remark 5.** For $n \leq j$, $f^{(j)}(x)$ is integer at $x = 0$ and $x = \pi$ for any $n \in \mathbb{N}$.

**Proof:** We first notice by Remark 4 that

$$\begin{aligned}
f^{(n)}(x) &= \frac{d}{dx} f^{(n-1)}(x) \\
&= \frac{d}{dx} \frac{(n-1)!\binom{n}{n-1} x^{n-(n-1)}(a-bx)^{n-(n-1)} g_{n-1}(x)}{n!} \\
&= \frac{n!\binom{n}{n} x^{n-n}(a-bx)^{n-n}\left[(a-2bx)g_{n-1}(x) + nx(a-bx)g'_{n-1}(x)\right]}{n!} \\
&= (a-2bx)g_{n-1}(x) + nx(a-bx)g'_{n-1}(x) \\
&= g_n(x)
\end{aligned}$$

Thus, $f^{(n)}(0) = g_n(0) = k_{0n} \in \mathbb{Q}$ and $f^{(n)}(\pi) = g_n(\pi) = k_{\pi n} \in \mathbb{Q}$ since $g_n(x)$ is still an integer coefficient polynomial with integer values at $x = 0, \pi$. Now for $f^{(j)}(x)$ where $j > n$, we may note that

$$\begin{aligned}
f^{(j)}(x) &= \frac{d^{j-n}}{dx^{j-n}} f^{(n)}(x) \\
&= g_n^{(j-n)}(x).
\end{aligned}$$

Since $g_n(x)$ contains only integer coefficients and integer powers of $x$, then its derivatives do so as well. Thus, $f^{(j)}(x)$ must also be integer for any $j \geq n$.

**Remark 6.** Using Remarks 1,4 and 5, we can note that $F(0)$ and $F(\pi)$ are integer for any $n \in \mathbb{N}$.

**Remark 7.** $F'' + F = f$.

**Proof:** Note that

$$F'' = f^{(2)}(x) + \ldots + (-1)^j f^{(2j+2)}(x) + \ldots + (-1)^n f^{(2n+2)}(x)$$

but since $f$ is a polynomial of order $2n$, we have $f^{(2n+2)}(x) = 0$. Thus, when adding the two together, we get

$$\begin{aligned}
F'' + F &= f(x) + (1-1)f^{(2)}(x) + \ldots + (1-1)^j f^{(j)}(x) + \ldots + (1-1)^n f^{(n)}(x) \\
&= f(x)
\end{aligned}$$

**Remark 8.** $\frac{d}{dx}(F' \sin x - F \cos x) = f \sin x$

**Proof:** Show as an exercise.

Using the eighth remark, we can use integration to begin our proof by contradictions. Integrating $\frac{d}{dx}(F' \sin x - F \cos x)$ from 0 to $\pi$, we get

$$\begin{aligned}
\int_0^\pi \frac{d}{dx}(F' \sin x - F \cos x)\, dx &= (F' \sin x - F \cos x)\Big|_0^\pi \\
&= (F'(\pi) \sin \pi - F(\pi) \cos \pi) - (F'(0) \sin 0 - F(0) \cos 0) \\
&= F(0) + F(\pi)
\end{aligned}$$

which is clearly an integer from Remark 6. Now notice that

$$\int_0^\pi f(x)\sin x\,dx \le \int_0^\pi f(x)\,dx$$

and

$$\int_0^\pi f(x)\,dx \le M(\pi - 0) = M\pi$$

where $M$ is some upper bound of $f$ (you can try showing these as an exercise). From Remark 3, we have $f(x) \le \frac{\pi^n a^n}{n!}$ and so taking $n$ large enough, we can reduce $\frac{\pi^n a^n}{n!}$ small enough such that it is less than $\frac{1}{\pi}$(you can show that the sequence converges using the ratio test). Suppose choosing $N \in \mathbb{N}$ is sufficient enough to reduce our upper bound to less than $\frac{1}{\pi}$. Then defining

$$f(x) = \frac{x^N(a-bx)^N}{N!}$$

we still have $F(0) + F(\pi) \in \mathbb{Z}$ but now with $M < \frac{1}{\pi}$, we have

$$0 < \int_0^\pi f(x)\,dx \le M\pi < 1$$

since $f(x) > 0$ on the interval $x = (0, \pi)$ (you can show how this implies the integral is always non-zero and non-negative as an exercise). However, note that this makes our integral non-integer, which contradicts Remark 8.

Hence, our original assumption of pi being rational must be false and therefore pi must be irrational.

$\square$