Extended Essay - Mathematics

Creating a Model to Separate or Group Number Sets by their Cardinalities

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September 2009
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“The conquest of the actual infinite by means of set theory can be regarded as an extension of our scientific horizon whose importance is comparable to the importance of the Copernican system in astronomy and the theory of relativity and quantum mechanics in physics.”
- Abraham Fraenkel, “In Search of Infinity”

Word Count: 3982

Abstract

Have someone start counting the natural numbers, [1, 2, 3, 4…], and have another person begin counting the squares of these numbers [1, 4, 9, 16…]. If given an infinite amount of time, both persons would count exactly the same number of numbers or count the same cardinality. Even though the second person missed many natural numbers, one can say that they are as many perfect squares as there are their roots so the two sets must be equal.

Not limiting oneself to just the natural numbers or perfect squares, other number sets can be considered. However, before one considers examining other numbers sets, the general question, *"How can number sets be separated or grouped by their number of elements or cardinalities?"* must be answered, and this is the main focus of this investigation.

Through an analysis of some of Georg Cantor's techniques to enumerating number sets, including his famous diagonal argument, I will base his knowledge, other consulted sources as well as my own to generalize an answer to the aforementioned question.

At the end of the investigation I have coordinated a method to which any number set can be separated into four different groups with differing cardinalities. This will be represented in a flow chart which serves as a summary to all the unique examples of separating number sets. Following that will be brief explanations of the properties of each group and how one would come deduce which group a number set may fall into.

Word Count: 248

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1. Introduction

Number sets are generally grouped by their distinct characteristic properties. For example, the difference between the integers and the natural number set would primarily be absence of negative whole numbers and the number zero in the natural number set. However, another way to group number sets would be to group number sets by their number of elements. The set $A = \{1, 2, 3, 4\}$ could be grouped with the set $B = \{5, 6, 7, 8\}$ because they both have four elements or because they have the same *cardinality*. This is essay will look at methods in which number sets can be grouped by their cardinality and near the end of the investigation we will create a model for separating any number sets using a generalized approach.

2. Explaining Origins

Set theory and the cardinality of the infinite originate mainly from a German mathematician named Georg Cantor (1845-1918). Supported by fellow mathematician Richard Dedekind, Cantor first published his work on set theory in 1874 in Crelle's journal. Another paper was published in 1878 where Cantor further expanded on his theories of infinite number sets, defining equipotence of number sets of various dimensions. In his results, he proved that the set of real numbers deviated from other number sets and, in short, contained a "greater infinity" than other infinite sets. Some years later in 1892, Cantor presented his famous diagonal argument to a large audience during a meeting in Bremen (Grattan-Guinness, 2000).

3. Finite and Infinite Sets and their Cardinalities

3.1 Finite and Infinite Sets

A finite set is a set that contains a finite number of elements (Ex. The set of integers from 0 to 10 is a finite set, containing 11 elements). In a finite set, cardinality can be determined by counting the number of elements in the set. For example, in the set $A = \{1,2,3,4 \ldots 100\}$, there are exactly 100 elements in the set so we denote its cardinality as $card(A) = 100$. In these sets, the elements in a set do not play a crucial role in affecting the cardinality of the set. Even a large number like $10^{8^{10\times8}}$ holds no more value than the number 1 in sets; numbers in number sets are merely just placeholders. It is the characteristic properties of the numbers ordered within the set that are really important.

When one infinitely increases the number of elements in a set an infinite number of times, it is not possible to define cardinality with any natural number. These types of set are called infinite sets. To define the cardinality of such sets, one must derive it from concepts that are already well-known.
A property of these infinitely large collections of numbers is that they contain an infinite number of elements not bound by a largest element. A basic example would be the set of all natural numbers, an *infinite countable set*. However, for the purpose of this essay, a method should be constructed to prove it.

Assume that the natural number set was finite and call it $\mathbb{N} = \{1,2,3 \ldots n)$, and then have a set $A = \{x_1 \ldots x_n\}$ represent the basic finite set. It will be shown that there is no bijection from A to $\mathbb{N}$ and that there exists at least one element in $\mathbb{N}$ that cannot be mapped to A. Create an onto-function $f: A \to \mathbb{N}$ where $x_1, x_2, x_3, x_4, \ldots x_n \in \mathbb{N}$ where $f(x_1) = 1, f(x_2) = 2, \ldots f(x_n) = n$. Since the set $\{x_1, x_2, x_3, x_4, \ldots x_n\}$ is finite, there has to be a largest number (in terms of magnitude) and assume this number is x_y. Then we create the number $x_y + 1 > x_y$ and conclude $(x_y + 1) \notin \{x_1, x_2, x_3, x_4, \ldots x_n\}$. However we see that $x_y + 1 \in \mathbb{N}$ because it is the next natural number that succeeds x_y. We then see $f(x_y + 1) \in \{1,2,3 \ldots n\}$ since the set $\mathbb{N}$ assumes to contain all the natural numbers. Thus, $f(x_y + 1) = f(k) = k$ for some integer $k \in \{1,2,3 \ldots n\}$. However, we have previously stated that the natural number set is finite and there can only be a bijection between it and the finite set A. Since $k \notin A$, this is clearly a contradiction. The only possible conclusion is $card(\mathbb{N}) > card(A)$. This means that the natural number set is not finite but infinite (Faticoni, 2006). Although it is not possible to enumerate the infinite because it is completely abstract, we can say that anything not in the realm of being finite must be infinite. Using the same logic, it is possible to prove other number sets are infinite by using contradictions.

3.2 Countable Sets

Any set, finite or infinite is said to be countable if there exists an injective function between it and the set of natural numbers, the lowest of the infinite sets. These may include any finite or infinite subsets of the natural numbers or other infinite sets that can form a one to one function with the set of natural numbers.

So, for any finite set A, we can write $card(A) < card(\mathbb{N})$ or replace $card(\mathbb{N})$ with $\aleph_0$, the equivalent to the cardinality of the natural numbers. Symbols used to represent infinite sets, such as $\aleph_0$are called aleph numbers and these are used to symbolize cardinalities the infinite.

For any infinite countable set B we write, $card(B) \leq \aleph_0$ unless it is proven that there exists a bijection between the two sets. Generalizing for any finite set A and countably infinite set B, we make the relation:

$$card(A) < card(B) \leq \aleph_0$$

We can further generalize the relationship between infinite sets by stating that any two infinite sets, A and B where $A \subset B$, will have the following relationship, $card(A) \leq card(B)$and this will be true for all infinite sets, regardless of being countable or not. Expanding this further into sets, we can write:

$$card(\mathbb{N}) \leq card(\mathbb{Z}) \leq card(\mathbb{Q}) \leq card(\mathbb{R})$$

3.3 Uncountable Sets

A set that is uncountable is said to be infinite and not countable; it is a set where there is larger collection of elements than the set of natural numbers. Georg Cantor labeled the cardinality of these sets with the symbol c. While it is impossible to enumerate the cardinalities between an uncountable set and $\aleph_0$ to show that $\aleph_0 < c$, another possibility exists. Recall the property that any two sets are said to have the same cardinality if there exists a bijection between the two. Since an uncountable set should be considerably larger than $\aleph_0$, there should not be a bijection between the two nor should there be an injective function from an uncountable set to a countable set. One set that follows this property is the set of real numbers, and by comparing this set to the set of natural numbers we will see that there is at least one element in $\mathbb{R}$ that cannot be mapped to $\mathbb{N}$. To simplify the proof, we will only look at the set of real numbers between the intervals, $[0,1]$ which later on it will be proven to have the same number of elements of real numbers between any intervals. For the sake of contradiction, assume that we can have a one-to-one function, $f: \mathbb{N} \rightarrow [0,1]$. Then, map each natural number to a real number in the following way (Faticoni, 2006):

$$\begin{array}{llllllll} f(1) & = x_1 = & \boxed{.d_{11}} & d_{12} & d_{13} & d_{14} & d_{15} & \cdots \\ f(2) & = x_2 = & .d_{21} & \boxed{d_{22}} & d_{23} & d_{24} & d_{25} & \cdots \\ f(3) & = x_3 = & .d_{31} & d_{32} & \boxed{d_{33}} & d_{34} & d_{35} & \cdots \\ f(4) & = x_4 = & .d_{41} & d_{42} & d_{43} & \boxed{d_{44}} & d_{45} & \cdots \\ \vdots & & & \vdots & & & & \end{array}$$

This is simply creating the function $f(n) = x_n$ where $n \in \mathbb{N}$ and every x_n value is a unique real number comprised of an infinite decimal expansion. The digits that are in this expansion go by the form of d_{nm} where n is the natural number that maps to the number containing the digit and m is the placement of that digit. For example, the digit d_{97} is the 7th digit in the decimal expansion that maps to the natural number 9.

By looking at the digits where $m = n$, we can create a number that cannot be mapped to the natural number set. Define this number as having the digit expansion, $\boxed{.d_1 \; d_2 \; d_3 \; d_4 \; d_5 \; \cdots}$ and call this value x. Create the digits of x by the following property, $d_n \neq d_{nm}$. This makes $x \neq x_1, \; x_2, x_3$ … because $d_1 \neq d_{11}, \; d_2 \neq d_{22}, \; d_3 \neq d_{33}$ … and generalizing, we can say that x is unique from any number x_n due to differing values between d_n and d_{nm}. We have thus created a real number that cannot be mapped to a natural number proving a contradiction. This means that $card(\mathbb{R}) \neq card(\mathbb{N})$ and knowing that $\mathbb{N} \subset \mathbb{R}$, meaning $card(\mathbb{N}) \leq card(\mathbb{R})$, the only conclusion is that:

$$card(\mathbb{N}) < card(\mathbb{R})$$
$$\therefore \aleph_0 < c$$

3.4 Cardinality of the Continuum

Although, it was explained in the previous section that the real numbers were uncountable, it was not established what the cardinality of the set was, relative to $\aleph_0$. The symbol $\aleph_0$ symbolized the natural number set because it was the smallest of the infinite sets, but in the previous section, "*another infinity"* was discovered. Even though, the symbol c can be used to represent the cardinality, it is imperative to understand the relationship between $\aleph_0$ and c. Every real number can defined through an infinite decimal expansion where all the digits are natural numbers and all the digits in a real number can be bijected to the natural number set. This makes the cardinality of the digits in a real number equal to $\aleph_0$. It is also known that the real numbers can be arranged in any mathematical base, so for simplicity, assume the real numbers are arranged in binary. Thus, the number of ways of create the real numbers using 0 and 1 is $2^{\aleph_0}$ and therefore $2^{\aleph_0} = card(\mathbb{R}) = c$. It is possible to do this in base 10, but by using some cardinal arithmetic (Faticoni, 2006) the result is still the same ("Intuitive Argument," 2009):

$$10^{\aleph_0} = 2^{[(\frac{\ln 10}{\ln 2})\times\aleph_0]} = 2^{\aleph_0}$$

4. Other Number Sets

The following sections will determine how other number sets will be proven to have cardinality of $\aleph_0$ or $2^{\aleph_0}$ by looking at the characteristic property of each unique set or a property that separates sets besides their cardinalities. Characteristic properties can vary between sets and can be distinct from how collections of numbers are separated by cardinality. Consider the following sets:

$$A = \{The\ set\ of\ all\ non\ zero\ integers\ where -2 < x < 2\}$$

$$B = \{The\ of\ all\ roots\ to\ the\ equation\ x^2 - 1 = 0\}$$

Sets A and B contain different characteristic properties but what is noticeable is that both sets will have the elements $\{-1, 1\}$ and will have $card(A) = card(B) = 2$.

4.1 Integers

Integers are natural numbers that include the number zero and also their negative number counterparts. Since the integers are only comprised of only whole numbers they should be countable. We may then say $\mathbb{N} \subset \mathbb{Z}$ and intuitively argue:

$$card(\mathbb{N}) \leq card(\mathbb{Z})$$

To prove the equipotence between the integers and the natural numbers we may then begin to create a one-to-one function. First, divide the integers into two sets, the positive and the negative set of integers, $\mathbb{Z}^+$and $\mathbb{Z}^-$ while ignoring the number 0 in the set. Divide the natural number set into two sets, even and odd. Mapping the even set to positive integers and the odd set to the negative integers, we create the two functions $f: \mathbb{Z}^+ \to 2n$ and $f: \mathbb{Z}^- \to 2n - 1$ where $n \in \mathbb{N}$. Then, create the functions $f: \mathbb{N} \to 2n$ and $f: \mathbb{N} \to 2n - 1$. Putting it together in a diagram, it would look something like this:

$\mathbb{N}_{odd}$, $\mathbb{N}_{even}$	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	…	…
	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	
$\mathbb{N}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	…	…
	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	↓	
$\mathbb{Z}$	-1	1	-2	2	-3	3	-4	4	-5	5	-6	6	-7	7	-8	8	-9	9	…	…

So, including the number 0 that was left out,

$$card(\mathbb{Z}) = card(\mathbb{N}) + card(\mathbb{N}) + 1 = \aleph_0 + \aleph_0 + 1 = \aleph_0.$$

4.2 Rational Numbers and Cantor's Diagonal Argument

Rational numbers are in the form $\frac{m}{n}$ where $\{m, n \in \mathbb{Z} \; and \; n \neq 0\}$. Initially, we may consider that $card(\mathbb{N}) \neq card(\mathbb{Q})$ because $\mathbb{N}$ is so thinly distributed into $\mathbb{R}$ than $\mathbb{Q}$. This means that $\mathbb{Q}$ holds a higher degree of accuracy in approximating points than $\mathbb{N}$ in $\mathbb{R}$, making the set more dense. However, we cannot completely make this assumption unless it is formally proven that there is no bijection between the rational number set and the natural numbers. Georg Cantor proved that there is in fact an existing bijection using one of his most renowned proofs, Cantor's diagonal argument (Faticoni, 2006). Initially we start by addressing:

$$\because \mathbb{N} \subset \mathbb{Z}$$
$$card(\mathbb{N}) \leq card(\mathbb{Q})$$

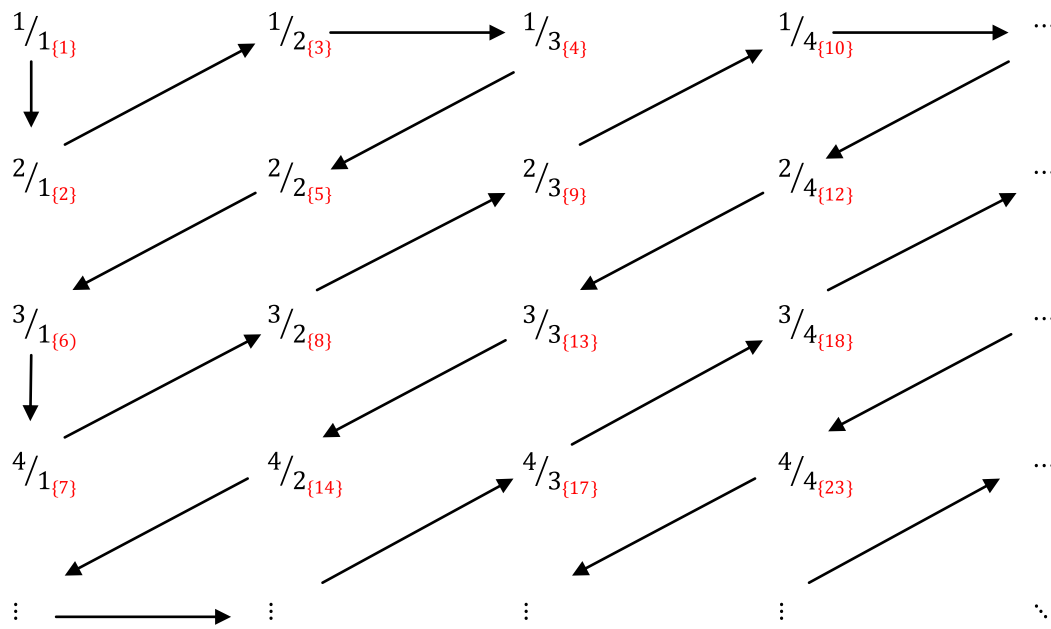
If it can be proven that there is a bijection between the positive rationals and the natural numbers, then there is also a bijection between the set of all rationals and the natural numbers using similar logic from the Integers section:

$$card(\{2, 4, 6, 8\}) = card(\mathbb{N}) = card(\mathbb{Q}^+)$$
$$card(\{1, 3, 5, 7\}) = card(\mathbb{N}) = card(\mathbb{Q}^-)$$
$$\therefore card(\mathbb{Q}) = \aleph_0 + \aleph_0 = \aleph_0$$

For simplicity, we will only examine the positive rationals. We may start off by listing all the rationals in a grid in the following way:

$$\begin{array}{ccccc} {}^1/_1 & {}^1/_2 & {}^1/_3 & {}^1/_4 & \cdots \\ {}^2/_1 & {}^2/_2 & {}^2/_3 & {}^2/_4 & \cdots \\ {}^3/_1 & {}^3/_2 & {}^3/_3 & {}^3/_4 & \cdots \\ {}^4/_1 & {}^4/_2 & {}^4/_3 & {}^4/_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

To prove equipotence between the two sets we will create an undulating line across the rational number grid and assign each rational number that it passes through as a number or point in the line as follows:



Then, we will improve this distribution by changing each assigned number to an even number since this grid only looks at the positive rationals. To further improve the design, rational numbers that are not completely reduced to their lowest denominator such as $\frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \cdots = \frac{n}{n}$ will be removed to avoid repeating rationals. Hence, we have completed the proof that the positive rational numbers are one to one with the even natural numbers and as well that that the negative rational numbers are one to one with the odd natural numbers.

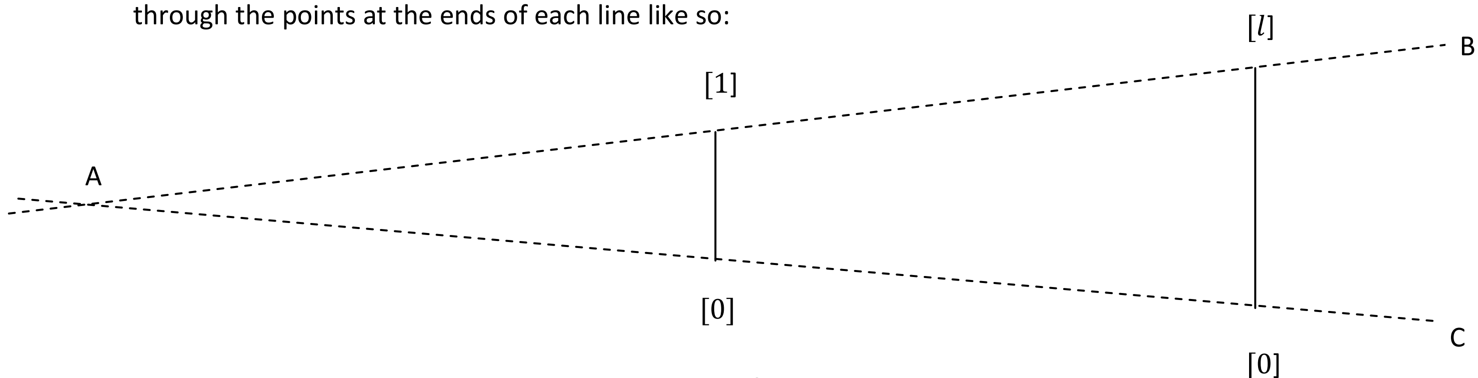
4.3 Irrational Numbers

Irrational numbers are real numbers that are neither fractions nor repeating decimals and are represented by the symbol $\mathbb{Q}'$. In fact, the real number set is actually composed of only rational and irrational numbers, so $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$. Using this, we may prove the uncountability of the irrational number set. Assume that that $\mathbb{Q}'$ is countable. We may establish that there is an infinite collection of elements that belong to $\mathbb{Q}'$ and this is the set of all the roots of prime numbers (Faticoni, 2006). Since there are as many primes as natural numbers, using Euclid's proof of infinite primes (See Appendix), we should conclude $card(\mathbb{Q}^{'}) = card(\mathbb{N}) = \aleph_0$. It is also known that the union of two disjoint infinite countable sets is also infinitely countable. Thus, we see $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^{'} = card(\mathbb{Q}) + card(\mathbb{Q}^{'}) = \aleph_0$. However, $\mathbb{R} \neq \aleph_0$, clearly a contradiction. From this the only conclusion is that $\mathbb{Q}^{'}$ must be uncountable since union of an uncountable set and a countable set can only be uncountable.

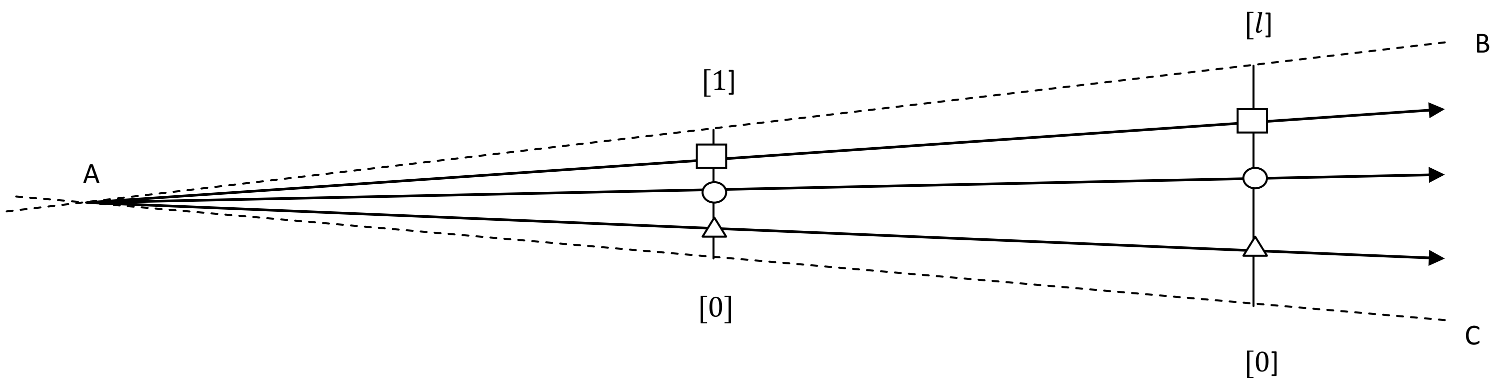
5. Is Cardinality Outside Spatial Dimensions?

When referring to cardinal numbers such as $\aleph_0$ or $2^{\aleph_0}$ it is important to know that it is impossible to bind these representations of cardinality to any value. Just like ∞, these symbols are merely abstractions. To demonstrate, consider the following statement, “All lines have the same number of points, regardless of their lengths.” To prove this is simple, construct two lines, one of length 1 and one of length $l \neq 1$ and assume one end of any of the two lines has value 0 and the other has the value equivalent to its length.

Each point on the line segment with length 1 can be written as 0, 1, or any infinite decimal expansion $(0.d_1\ d_2\ d_3\ d_4\ d_5\ \cdots)$. Looking back to section 3.3, this is exactly all the elements of the real numbers between 0 and 1. To prove that this line has the same amount of points as the line with length l, create a point A where two lines, AB and AC can be drawn from A such that each line only passes through the points at the ends of each line like so:



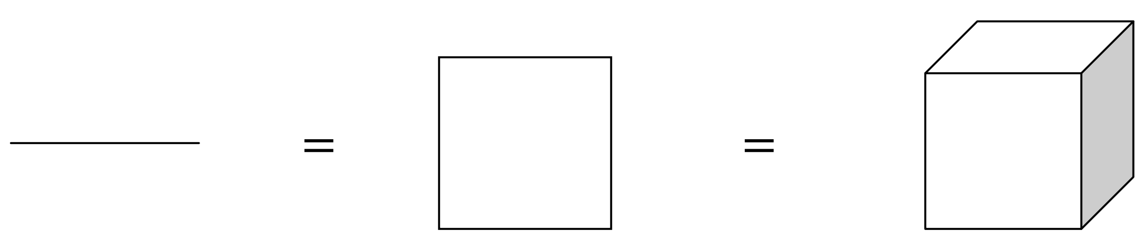
To prove equipotence it must be proven that there is a bijection between the two lines, but in the above diagram this is already possible. Starting from A, we can draw any line within the boundaries of B and C to map any two elements between the two sets together:



If we continue to infinitely map all of these points, we will have created a bijective function from $[0,1]$ and $[0, l]$, proving the initial statement. What this also means is that the rules of cardinality are different when compared to the rules of magnitude. For example, using the concept that any two lines have the same number of points, so should the number of real numbers between any intervals, meaning $card(0,1) = card(-1,1) = card(-100000,100000) = card(a, b)$ for any numbers a and b and even $card(a, b) = card(-\infty, \infty)$. This clearly is contradictory to how we judge lengths by magnitude.

To add to this, we can also say that cardinality is *outside spatial dimensions* or is not affected by how many planes that are holding what is being bijected to.

This means, in terms of cardinality of the number of points:



To prove this, create a bijective function from $\mathbb{R}$ to $\mathbb{R} \times \mathbb{R}$ where $\mathbb{R}$ would represent a one-dimensional line and $\mathbb{R} \times \mathbb{R}$ a two dimensional square. Any point in $\mathbb{R} \times \mathbb{R}$ can be written as an ordered pair (x, y) and points on $\mathbb{R}$ can be written as a single real number n. For any point on $\mathbb{R} \times \mathbb{R}$ the values of x and y are:

$$x = x_0 .\ x_1\ x_2\ x_3 \ldots$$
$$y = y_0 .\ y_1\ y_2\ y_3 \ldots$$
$$where\ \{x_n,\ y_n \in \mathbb{N}\}$$

With this, we can then create the function by mapping it to n where the value of n uses all the values of x_n and y_n:

$$n = x_0\ y_0 .\ x_1\ y_1\ x_2\ y_2\ x_3\ y_3 \ldots$$

A similar method can be used to prove equivalency of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ using ordered triplets and in general $\mathbb{R}^n$ where $n \in \mathbb{N}$. Replacing $\mathbb{R}$ with $\mathbb{N}$ or any subset of $\mathbb{R}$ and using the same proof will also reinforce the notion that cardinality is not merely untouched by dimensional boundaries, but it is *completely outside it*.

6. Creating the Model for Grouping Number Sets

Using all the various proofs and models that were examined, a model can be created to establish how exactly any number set can be grouped with a certain cardinal number. For simplicity, I have decided to represent this model in a flow chart. Following the flow chart is a summary and conclusion of some key steps, the various endpoints and examples of methods that can be used to carry out the steps. These methods are taken from important extracts of sections that were previously covered,

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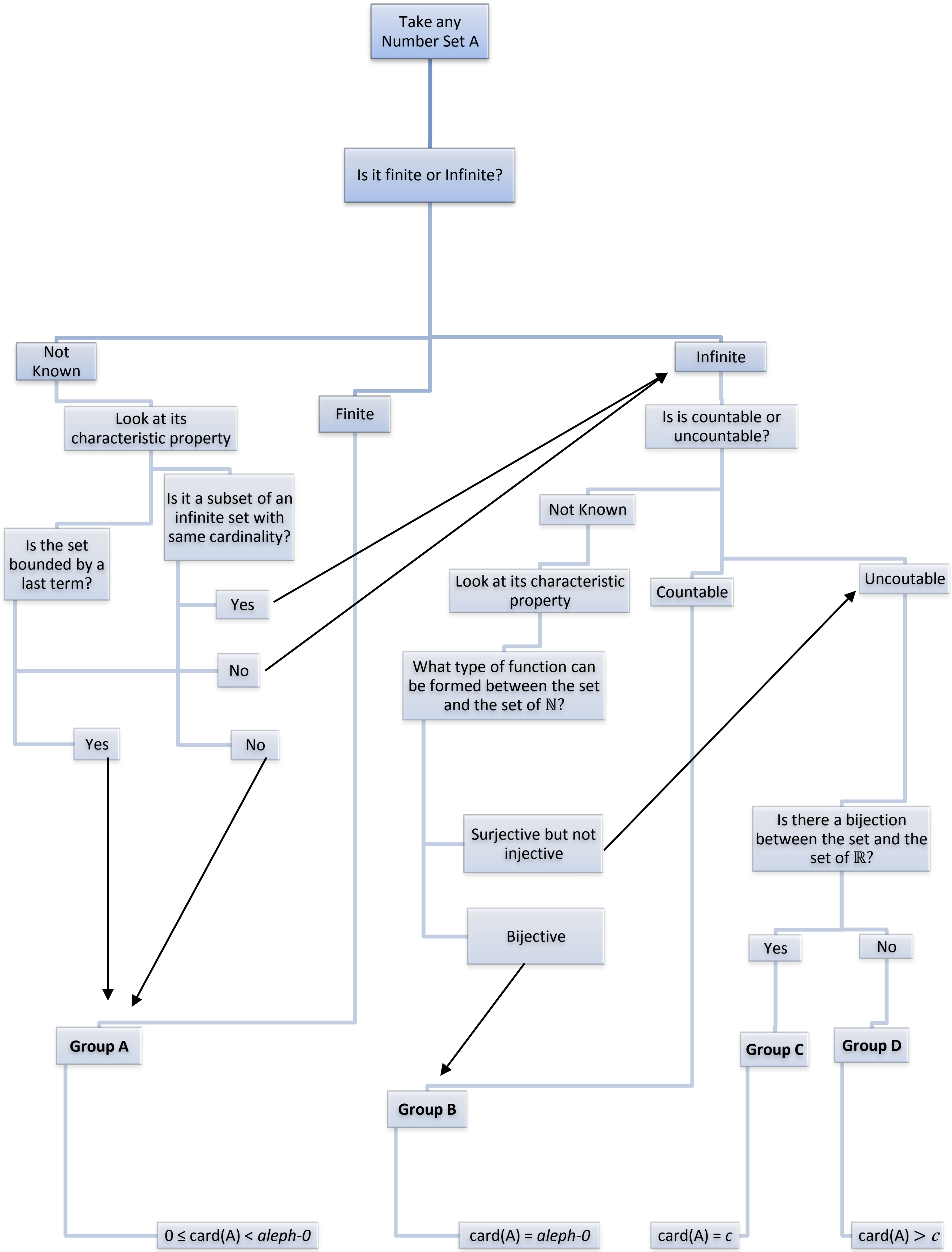


7. Conclusion

What groups are different number sets placed into based on cardinality?

Taking in any number set into consideration, it can be separated into four recognizable groups that differentiate in cardinality. These are the finite set (A), the countably infinite set (B), the cardinality of the continuum (C) and the cardinality beyond the continuum (D). To deduce which group a number set would fall under, one must first consider several properties of the set first.

Finite Set (Group A):

Any sets that cannot form a bijection to any infinite set or only form an injection and not a surjection fall under this category. These sets are bound by a first and last term and are not necessarily well-ordered. The cardinality of these set will range anywhere from 0 – which is the empty set – to any natural number that is not an abstraction such as infinity. Sets that are in this group will form an injective function but not a surjective function to the natural number set.

Countably Infinite Set (Group B):

Sets that fall under this category always share the same cardinality of $\aleph_0$ and are the smallest of the infinite sets. Any subset that derives from a set from this category and that is also infinite is also a member of this group. A set that is almost synonymous with this group is the natural number set. Any other sets that fall under this group will always form a bijective function to the set of $\mathbb{N}$.

Cardinality of the Continuum (Group C):

A set that falls under this category always has cardinality $2^{\aleph_0}$. These sets are called uncountable sets and are significantly larger than countably infinite sets. Infinite subsets that derive from sets that go under this category do not necessarily share the cardinality. The name of this set is associated with the real number set because continuum is sometimes referred to the real number line and as the name implies, the real numbers also falls under this group. Any collection of elements that forms a bijective function with the set $\mathbb{R}$ falls under this category.

Cardinality beyond the continuum (Group D):

Although it was not explicitly explained in this essay, one cannot discount the existence of sets that far exceed the real number set. These collections of elements will have the property of forming a surjective but not injective function with the real number set.

How does one determine the group in which a number set falls into?

If it cannot be determined which groups a set falls under, several other properties have to be looked into, such as:

Countability:

Sets can be either finite or infinite and methods to determine this include, but are not limited to:

-Proof by contradiction (Section 3.1)
-What function can the set form with the set $\mathbb{N}$ (Surjective, Injective, Bijective)
-Proof/Disproof of a last term (Section 3.1)
-Proof/Disproof of being a subset of an infinite set with equal cardinality

If a set is finite, it falls under Group A but if it is infinite, more factors must be taken into consideration such as:

The Comparisons to the set $\mathbb{N}$:

The set $\mathbb{N}$ is very commonly associated with sets of cardinality $\aleph_0$ so the simplest way to know if a set falls under Group B would be to form a bijection to $\mathbb{N}$. Methods to doing this include:

-Proof/Disproof that the set is an infinite subset of $\mathbb{N}$
-Proof/Disproof that the set bijects to other know sets in Group B
-*Algorithms: Think of the set $\mathbb{N}$ as a long line with points of equal distance spread on the line. By using intuitive thinking, one can create an algorithm that is unique to the set being examined

such that each element can correspond to a point on the set. The "line" itself that is being mapped to the examined set does not necessarily have to be straight as seen in Section 4.2.
-Spatial Dimensions: It is noteworthy that the set $\mathbb{N} \times \mathbb{N}$ has the same cardinality as $\mathbb{N}$ or $\mathbb{N}$ multiplied to $\mathbb{N}$ any number of times. This property applies to any infinite set as well (Section 5)

If the set bijects to the set $\mathbb{N}$, then it falls under Group B, if the set is larger than the set $\mathbb{N}$, further investigation is needed. Similar to the last section, one would consider:

The Comparisons to the set $\mathbb{R}$:

A set that bijects to the set $\mathbb{R}$ will go under Group C and has cardinality $2^{\aleph_0}$. Methods to finding this bijection as well other more indirect methods include:

-Proof/Disproof that the set bijects to other know sets in Group C
-Algorithms: Same concept as in previous section
-Proof/Disproof that the union of the set and a countable set is equal to the real number set or another uncountable set (Section 4.3) either intuitively or proof by contradiction

If the set has cardinality of the set $\mathbb{R}$, it will go under Group C, and all other sets that exceed this cardinality go under Group D.

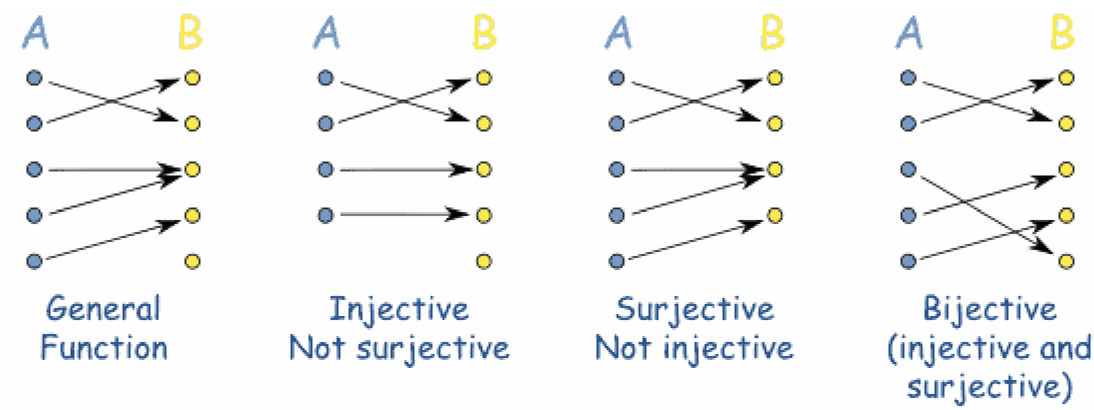
Closing Remarks

This essay should not be viewed as a formal writing into how to categorize number sets but should be viewed more as a guideline to understanding and interpreting how mathematicians determine the cardinality of math sets. Although the proofs that were presented here were amazingly elegant and simplistic, they should not deter people from exploring other potentially intuitive proofs. In the appendix I have prepared an example of how the algebraic number set was proven to be countably infinite using the guidelines above but again, one should not be limited to other people's interpretations. As Georg Cantor once said,

"The essence of mathematics lies in its freedom."

APPENDIX

Surjective, Injective, and Bijective Functions
(*Injective, Surjective and Bijective*," n.d.)



Some Basic Cardinal Arithmetic (Faticoni, 2006)
1. $\aleph_0 \times [Any\ Real\ Number] = \aleph_0$
2. $\aleph_0 + [Any\ Real\ Number] = \aleph_0$
3. $\aleph_0$ $cannot\ be\ negative\ nor\ can\ it\ be\ divisible\ by\ any\ real\ numbers$

Euclid's proof of Infinite Primes (Caldwell, n.d.)

Assume that there are a finite number of primes. Let this be the set of $A = \{p_1, p_2, p_3 \ldots p_n\}$ where p_n is the largest prime in the set. Let P be the product of all the primes. Now consider the number $P + 1$. If $P + 1$ is a new prime, then it is outside the set of A and thus all the primes are not contained within the set.

However if it is not prime, then it must contain some prime factors. Let p be a prime factor that can divide $P + 1$. We see that $p \notin A$ since dividing p with any member of the list A will result in remainder 1. Thus, p is new prime created from $P + 1$ and since p is not contained within a finite set of primes, there must be an infinite number of primes.

Cardinality of the Algebraic Numbers (Vilenkin, 1995)

Algebraic numbers are roots of any equation with the form:

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0$$

Where $a_0, a_1, a_2 \ldots a_n \in \mathbb{Z}$ and the roots of these equations can be approximated using the set of integers and several algebraic operations where the number are not necessarily rational. Such numbers could include

$$\sqrt[17]{\sqrt{156} + \sqrt[3]{123}}$$

However, even with the large range of algebraic numbers that can be produced through mathematical equations, they are still countable and only match with the natural numbers in cardinality.

We know that for every algebraic equation with a highest degree n will only have at most n roots. Thus, we may set up a table of every algebraic equation that lists the roots of one equation in the first row, the list of distinct roots in the second equation in the second that are not in the first equation, the list of distinct roots in the third equation in the third row that are not in the second equation and so on and so forth. So in each row there are a finite number of distinct roots so the cardinality of the set now depends on the number of rows or equations that support the algebraic numbers.

Recall that algebraic numbers are roots of equations that contain integer coefficients. Since there are a countably infinite amount of integers and a finite number of ways to arrange them as coefficients into a set, there are also a countably infinite number of equations that support algebraic numbers. This is also the representation of the number of rows in the table that was previously mentioned. Thus, the cardinality of the algebraic numbers is countably infinite which is equal to the integers which in turn are equal to the natural numbers.

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