1 The Model

1.1 Population and sampling parameters

- Sets of domains and strata indices in domains:

$\mathcal{D} := \{1, \ldots, D\}$	domain indices; D is the total number of domains,
$\mathcal{H}_d := \{1, \ldots, H_d\},\ d \in \mathcal{D}$	strata indices in domain d where H_d is the total number of strata in domain d,
$\mathcal{H} := \{(d, h) :\ d \in \mathcal{D},\ h \in \mathcal{H}_d\}$	domain-stratum indices.

- Population quantities:

$N_{d,h},\ (d,h) \in \mathcal{H}$	size of stratum h in domain d,
$S_{d,h},\ (d,h) \in \mathcal{H}$	standard deviation of a given study variable in stratum h in domain d,
$\tau_d,\ d \in \widetilde{\delta}(\mathcal{H})$	total in domain d, i.e. a sum of values of a given study variable for population elements in domain d,
$\kappa_d,\ d \in \widetilde{\delta}(\mathcal{H})$	priority weight for domain d,

$\rho_d := \tau_d \sqrt{\kappa_d},\ d \in \widetilde{\delta}(\mathcal{H}),$

where function $\widetilde{\delta}$ is as defined in (1.1a).

- Sampling parameters:

n	total sample size.

1.2 Model definition and properties

The model serves as an abstract representation of all the population and sampling quantities defined in Section 1.1.

Definition 1.1. *Let $I \subset \mathbb{N}_+^2$ and $J \subseteq I$. Then,*

$$\widetilde{\delta}(I) := \{d :\ \pi_1(i) = d,\ i \in I\}, \tag{1.1a}$$

$$\widetilde{\delta}_d(J) := \{h :\ \pi_1(i) = d,\ \pi_2(i) = h,\ i \in I\}, \qquad d \in \widetilde{\delta}(I), \tag{1.1b}$$

where

$$\pi_i(\boldsymbol{a}) := a_i, \qquad \boldsymbol{a} = (a_i,\, i \in I) \in \mathbb{R}^{|I|},\ i \in I. \tag{1.2}$$

Definition 1.2. *By a model, we understand a quintuple* $(\mathcal{H},\, \boldsymbol{N},\, \boldsymbol{S},\, \boldsymbol{\rho},\, n)$*, such that*

$$\mathcal{H} \subset \mathbb{N}_+^2,\ \ 0 < |\mathcal{H}| < \infty, \tag{1.3a}$$

$$\boldsymbol{N} = (N_{d,h},\ (d,h) \in \mathcal{H}) \in \mathbb{N}_+^{|\mathcal{H}|}, \tag{1.3b}$$

$$\boldsymbol{S} = (S_{d,h},\ (d,h) \in \mathcal{H}) \in \mathbb{R}_+^{|\mathcal{H}|}, \tag{1.3c}$$

$$\boldsymbol{\rho} = (\rho_d,\, d \in \widetilde{\delta}(\mathcal{H})) \in \mathbb{R}_+^{|\widetilde{\delta}(\mathcal{H})|}, \tag{1.3d}$$

$$n \in \Big(0,\, \sum_{(d,h)\in\mathcal{H}} N_{d,h}\Big], \tag{1.3e}$$

where $\widetilde{\delta}$ *is defined in* (1.1a).

Definition 1.3. *We define the following classes of models:*

$$\mathcal{P} := \big\{(\mathcal{H},\, \boldsymbol{N},\, \boldsymbol{S},\, \boldsymbol{\rho},\, n) :\ (1.3)\ \textit{hold}\,\big\}. \tag{1.4a}$$

Example of the model. Let

$$\begin{aligned}
&\mathcal{H} = \{(1,\,1),\,(1,\,2),\,(2,\,1),\,(2,\,2),\,(2,\,3)\},\\
&\boldsymbol{N} = (N_{1,1},\, N_{1,2},\, N_{2,1},\, N_{2,2},\, N_{2,3}) = (100,\, 200,\, 150,\, 40,\, 50),\\
&\boldsymbol{S} = (S_{1,1},\, S_{1,2},\, S_{2,1},\, S_{2,2},\, S_{2,3}) = (10,\, 2,\, 50,\, 30,\, 20),\\
&\boldsymbol{\rho} = (\rho_1,\, \rho_2) = (1.26,\, 2.32).\\
&n = 350
\end{aligned} \tag{1.5}$$

Then $(\mathcal{H},\, \boldsymbol{N},\, \boldsymbol{S},\, \boldsymbol{\rho},\, n) \in \mathcal{P}$.

Definition 1.4. *Let functions* $\widetilde{\delta}$ *and* $\widetilde{\delta}_d$ *be as defined in* (1.1)*. We define the following model functions:*

$$\delta(p) := \widetilde{\delta}(\mathcal{H}), \qquad\qquad p \in \mathcal{P}, \tag{1.6a}$$

$$\delta_d(p) := \widetilde{\delta}_d(\mathcal{H}), \qquad d \in \delta(p), \qquad p \in \mathcal{P}, \tag{1.6b}$$

$$A_{d,h}(p) := \tfrac{N_{d,h}S_{d,h}}{\rho_d}, \qquad (d,h) \in \mathcal{H}, \qquad p \in \mathcal{P}, \tag{1.6c}$$

$$c_d(p) := \sum_{h\in\delta_d(p)} \tfrac{[A_{d,h}(p)]^2}{N_{d,h}}, \qquad d \in \delta(p), \qquad p \in \mathcal{P}, \tag{1.6d}$$

where $p = (\mathcal{H},\, \boldsymbol{N},\, \boldsymbol{S},\, \boldsymbol{\rho},\, n)$.

2 Allocation Problem

Problem 2.1 (FPDA)**.** *For a given* $p = (\mathcal{H},\, \boldsymbol{N},\, \boldsymbol{S},\, \boldsymbol{\rho},\, n) \in \mathcal{P}$*, the* FPDA($p$) *problem is defined as follows:*

$$\underset{(T,\,\boldsymbol{x})}{\text{minimize}} \quad T \tag{2.1}$$

$$\text{subject to} \quad \sum_{(d,h)\in\mathcal{H}} x_{d,h} - n = 0 \tag{2.2a}$$

$$\sum_{h\in\delta_d(p)} \frac{[A_{d,h}(p)]^2}{x_{d,h}} - c_d(p) - T = 0, \qquad d \in \delta(p) \tag{2.2b}$$

$$x_{d,h} - N_{d,h} \leq 0, \qquad (d,h) \in \mathcal{H}, \tag{2.2c}$$

where $(T,\, \boldsymbol{x}) = (T,\, (x_{d,h},\ (d,h) \in \mathcal{H})) \in \mathbb{R} \times \mathbb{R}_+^{|\mathcal{H}|}$ *is an optimization variable, and* δ, δ_d, $A_{d,h}$, c_d *are the functions as defined in* (1.6)*.*

3 Set functions and some population matrix

Definition 3.1. *Let $p = (\mathcal{H},\, \boldsymbol{N},\, \boldsymbol{S},\, \boldsymbol{\rho},\, n) \in \mathcal{P}$. Suppose that $\mathcal{A} \subsetneq \mathcal{H}$ and $\boldsymbol{v} = (v_d,\, d \in \widetilde{\delta}(\mathcal{A}^c)) \in \mathbb{R}_+^{|\widetilde{\delta}(\mathcal{A}^c)|}$. Then set functions s_d, for $d \in \widetilde{\delta}(\mathcal{A}^c)$, are defined as follows:*

$$s_d(\mathcal{A}, \boldsymbol{v} \,|\, p) := \frac{n - \sum_{(i,h)\in\mathcal{A}} N_{i,h}}{\sum_{(i,h)\in\mathcal{A}^c} v_i A_{i,h}(p)}\, v_d, \qquad d \in \widetilde{\delta}(\mathcal{A}^c), \tag{3.1}$$

where $\mathcal{A}^c := \mathcal{H} \setminus \mathcal{A}$.

Definition 3.2. *Let $p = (\mathcal{H},\, \boldsymbol{N},\, \boldsymbol{S},\, \boldsymbol{\rho},\, n) \in \mathcal{P}$. Suppose that $\mathcal{B} \subseteq \mathcal{H}$ is such that $\sum_{(d,h)\in\mathcal{B}^c} N_{d,h} < n$. Then matrix-valued function $\mathbf{D}$ is defined by*

$$\mathbf{D}(\mathcal{B} \,|\, p) := \frac{1}{n - \sum_{(d,h)\in\mathcal{B}^c} N_{d,h}} \boldsymbol{a}\, \boldsymbol{a}^\mathsf{T} - \mathrm{diag}(\boldsymbol{c}), \tag{3.2}$$

where $\mathcal{B}^c := \mathcal{H} \setminus \mathcal{B}$, and

$$\boldsymbol{a} = (a_d,\ d \in \widetilde{\delta}(\mathcal{B}))^\mathsf{T} := \Big(\sum_{h\in\widetilde{\delta}_d(\mathcal{B})} A_{d,h}(p),\ d \in \widetilde{\delta}(\mathcal{B}) \Big)^\mathsf{T}, \tag{3.3a}$$

$$\boldsymbol{c} = (c_d,\ d \in \widetilde{\delta}(\mathcal{B}))^\mathsf{T} := \Big(\sum_{h\in\widetilde{\delta}_d(\mathcal{B})} \tfrac{[A_{d,h}(p)]^2}{N_{d,h}},\ d \in \widetilde{\delta}(\mathcal{B}) \Big)^\mathsf{T}, \tag{3.3b}$$

Symbol T denotes matrix transpose (i.e. $\mathbf{a}$, $\mathbf{c}$ are column vectors) and $\mathrm{diag}(\boldsymbol{c})$ is a diagonal matrix with vector $\mathbf{c}$ on the diagonal.

Definition 3.3. *Let $p = (\mathcal{H},\, \boldsymbol{N},\, \boldsymbol{S},\, \boldsymbol{\rho},\, n) \in \mathcal{P}$. Suppose that $\mathcal{B} \subseteq \mathcal{H}$ is such that $\sum_{(d,h)\in\mathcal{B}^c} N_{d,h} < n$, and $\widetilde{\delta}(\mathcal{B}) = \{d_1,\, \ldots,\, d_D\}$, where $D := |\widetilde{\delta}(\mathcal{B})|$. Then the $Eigen$ operator is defined as follows:*

$$Eigen(\mathcal{B} \,|\, p) := (\lambda,\, \boldsymbol{v}), \tag{3.4}$$

where $(\lambda,\, \boldsymbol{v})$ is the unique eigenpair for matrix $\mathbf{D}(\mathcal{B} \,|\, p)$ such that $\boldsymbol{v} = (v_{d_1},\, \ldots,\, v_{d_D}) \in \mathbb{R}_+^D$ is a unit vector.

To illustrate the functioning of the $Eigen$ operator, consider $p \in \mathcal{P}$ with $\mathcal{B} \subseteq \mathcal{H}$, such that $\delta(p) = \{1,\, 2,\, 3\}$ and $\widetilde{\delta}(\mathcal{B}) = \{2,\, 3\}$. Suppose that $(\lambda,\, (w_1,\, w_2))$ is an eigenpair for the matrix $\mathbf{D}(\mathcal{B} \,|\, p)$. Then, $Eigen(\mathcal{B} \,|\, p) = (\lambda,\, \boldsymbol{v})$, where $\boldsymbol{v} = (v_2,\, v_3) = (w_1,\, w_2)$. This example demonstrates that $\widetilde{\delta}(\mathcal{B}) \subseteq \delta(p)$, and hence, the dimension of the vector $\boldsymbol{v}$ may be less than or equal to $|\delta(p)|$.

4 RFIXPREC Algorithm

Algorithm 1 *RFIXPREC*

Input: $p = (\mathcal{H},\, \boldsymbol{N},\, \boldsymbol{S},\, \boldsymbol{\rho},\, n) \in \mathcal{P}, \mathcal{U} \subseteq \mathcal{H},\, \mathcal{J} \subseteq \delta(p).$ ▷ see (1.4a) and (1.6a)

Require: $(\sum_{(d,h)\in\mathcal{U}} N_{d,h} < n$ or $\mathcal{U} = \mathcal{H})$ and $(\widetilde{\delta}(\mathcal{U}) \not\subset \mathcal{J}$ if $\mathcal{U} \neq \emptyset)$ and $\mathcal{J} \neq \emptyset$. ▷ see (1.1)

function RFIXPREC($p, \mathcal{U},\, \mathcal{J}$)
    $j \in \mathcal{J}$ ▷ chosen arbitrarily
    $\mathcal{H}_j \leftarrow \{(d,h) \in \mathcal{H} : d = j\}$
    do
        if $|\mathcal{J}| = 1$ **then**
            $(T,\, \boldsymbol{x}) \leftarrow$ FIXPRECACT($p, \mathcal{U}$) ▷ see the next page
        else
            $(T,\, \boldsymbol{x}) \leftarrow$ RFIXPREC($p, \mathcal{U},\, \mathcal{J} \setminus \{j\}$)
        end if
        $\mathcal{U}_j \leftarrow \big\{(d,h) \in \mathcal{H}_j :\; x_{d,h} \geq N_{d,h}\big\}$ ▷ $\boldsymbol{x} = \big(x_{d,h},\; (d,h) \in \mathcal{H}\big)$
        if $\mathcal{U}_j \neq \emptyset$ **then**
            $\mathcal{U} \leftarrow \mathcal{U} \cup \mathcal{U}_j$
            $\mathcal{H}_j \leftarrow \mathcal{H}_j \setminus \mathcal{U}_j$
        end if
    while $\mathcal{U}_j \neq \emptyset$
    return $(T,\, \boldsymbol{x})$
end function

Algorithm 2 *FIXPRECACT*

Input: $p = (\mathcal{H},\, \boldsymbol{N},\, \boldsymbol{S},\, \boldsymbol{\rho},\, n) \in \mathcal{P}$, $\mathcal{U} \subseteq \mathcal{H}$. ▷ see (1.4a)

Require: $\sum_{(d,h)\in\mathcal{U}} N_{d,h} < n$ or $\mathcal{U} = \mathcal{H}$.

function FIXPRECACT(p, $\mathcal{U}$)

  if $\mathcal{U} = \mathcal{H}$ **then**

    $T \leftarrow 0$

  else

    $(\lambda,\, \boldsymbol{v}) \leftarrow Eigen(\mathcal{H} \setminus \mathcal{U} \,|\, p)$ ▷ see Definition 3.3 of $Eigen$

    $T \leftarrow \lambda$

  end if

  return $(T,\, \boldsymbol{x})$ where $x_{d,h} = \begin{cases} N_{d,h}, & (d,h) \in \mathcal{U} \\ s_d(\mathcal{U},\, \boldsymbol{v} \,|\, p)\, A_{d,h}(p), & (d,h) \in \mathcal{H} \setminus \mathcal{U} \end{cases}$ ▷ see Definition 3.1 of s_d

end function