Waveform Inversion via Source Extension

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Workshop in honor of Jean Virieux
June 2025

Agenda

Inversion

Extension

Triplication

FWI: adjust model parameters to make modeled data close to observed data

u = modeled wavefield in space-time, m = model parameters, R = sampling operator, $L[m]$ = wave operator, s = energy source, wave equation

$$L[m]u = s$$

Predicted data: $Ru = RL[m]^{-1}s = F[m]s$, $F[m] = R : L[m]^{-1}$ = modeling operator

Least squares ("FWI"): minimize

$$J_{\mathrm{FWI}}[m] = \sum (F[m]s - d)^2$$

$\sum$ = sum over all sources, receivers, times

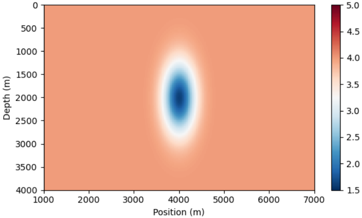
How to do it?

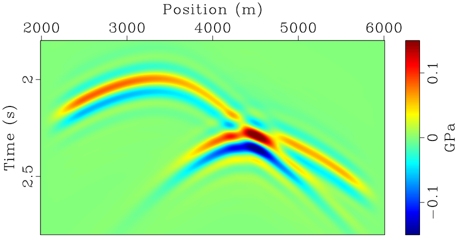
- Modeling (computing u etc.): many methods - e.g. time-domain staggered grid finite differernce (Virieux, 84)

- Iterative descent update using ∇J_{FWI} - efficient computation via *adjoint state method* (Chavent-Lemmonier 74,...), first multi-D demo by Gauthier-Virieux-Tarantola 86

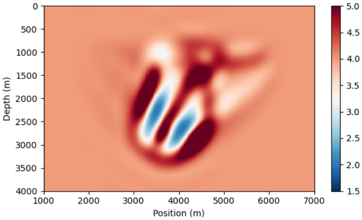
So what happens?

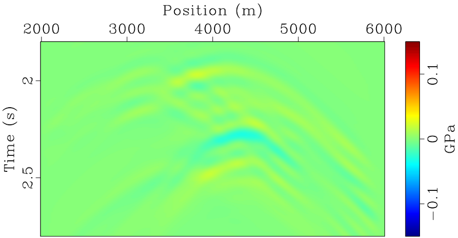
A simple example: acoustic lens with isotropic point source @ $\mathbf{x}_s$:
$f = w(t)\delta(\mathbf{x} - \mathbf{x}_s)$, $\mathbf{x}_s$ = source position





Left: target bulk modulus κ, source at $x_s = 3500$ m, $z_s = 3000$ m, receivers at $x_r = 2000 - 6000$ m, $z_r = 1000$ m. Right: synthetic data, source pulse = [1.0, 2.0, 7.5, 12.5] Hz trapzoidal bandpass filter]





Left: target model; Right: FWI estimate of $m =$ bulk modulus: initial model $\kappa_0 \equiv 4.0$ GPa, 450 steps steepest descent with smoothing preconditioner. Right: data residual $(Ru - d)$ - energy (norm) $\approx 25\%$ of data norm.

Failure mechanism: "cycle skipping", first documented in Gauthier-Virieux-Tarantola 86 - half-wavelength criterion, (Virieux-Operto 09)

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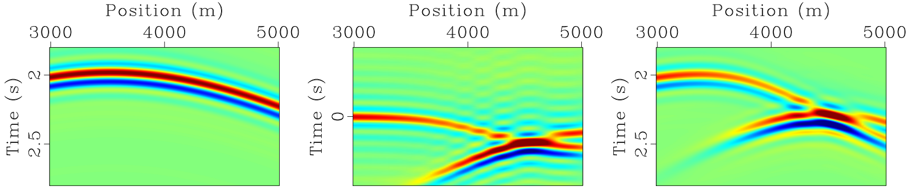
Many proposed cures for cycle skipping:

- acquire lower frequency data (Dellinger et al SEG 2016,...)
- combine with traveltime tomography (Luo-Schuster 91, Prieux-...-Virieux 13)
- use a different objective, e.g. transport metric (Yang-Engquist 18, Métivier-...-Virieux 18)
- use a different model - *enlarge search space, easier data fit* ("model extension")

Many extension methods - Operto et al. 23, Farshad-Chauris 23, Huang et al 19, S. 08. Common theme: maintain data fit while suppressing additional degrees of freedom

Example: Adaptive Waveform Inversion (AWI, Warner-Guasch 14): acoustic model, *known* isotropic point source $s = w(t)\delta(\mathbf{x} - \mathbf{x}_s)$,

Idea: if events in predicted data $F[m]s$ are in wrong place, use a filter f to put them in the right place: extended modeling op $\bar{F}[m, f] = f * F[m]$.



Left: Predicted data $F[m_0]s$, $m_0 =$ homogenous model; Center: adaptive filter f;
Right: adapted data $f * F[m_0]s \approx d =$ lens data

$f = d$ deconvolved (trace-by-trace) by $F[m]s$ ("easy"), so can make $\bar{F}[m,f]s$ fit data *for any m* (common feature of extension methods)

but $\bar{F}[m,f] \neq F[m]$ unless $f(\mathbf{x}_s, \mathbf{x}_r, t) = \delta(t)$ - to test, find an operator vanishing on $\delta(t)$ ("annihilator" - another common feature), update m to reduce its output

$t\delta(t) = 0$, so minimize

$$J_{\mathrm{AWI}}[m] = \sum_{\mathbf{x}_s,\mathbf{x}_r} \frac{\sum_t |t f[m](\mathbf{x}_s, \mathbf{x}_r, t)|^2}{\sum_t |f[m](\mathbf{x}_s, \mathbf{x}_r, t)|^2}$$

Good News: suppose

- transmission data dominated by single arrivals - unique travel time τ between source and receiver,
- $d \approx F[m_*]s$
- RMS wavelength of $w \rightarrow 0$.

Then

$$J_{\mathrm{AWI}}[m] \rightarrow \sum_{\mathbf{x}_s,\mathbf{x}_r} |\tau[m] - \tau[m_*]|^2$$

For single-arrival data, AWI is asymptotic to travel time tomography $\Rightarrow$ no cycle-skipping

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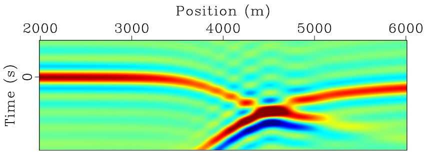
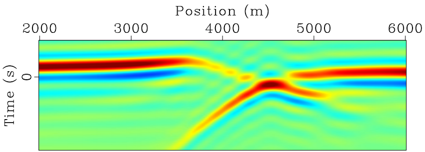
Conclusion does not apply to acoustic lens example - multiple arrivals.

Asymptotic analysis, numerical examples $\Rightarrow$

Significant energy in d at multiple arrival times $\Rightarrow$ approach based on source-receiver extension (e. g. AWI) behaves like FWI (S. 94, Plessix et al. 00, Huang et al. 17, Yong et al. 23.)





Adaptive filters for lens data. Left: $f[m]$, m=initial model. Right: $f[m]$, m updated to reduce J_{MSWI} (similar to J_{AWI}).

Remedy: localize inversion around individual events

- localize in time - Yong-Brossier-Métivier-Virieux 23: Local Adaptive Waveform Inversion (LAWI) via Gabor filter
- localize in phase space - surface source extension (SSE) - Huang et al. 19

SSE extended source: $\bar{s} = \bar{w}(\mathbf{x}, t; \mathbf{x}_s)\delta_{\mathcal{S}}(\mathbf{x})$, $\mathcal{S}=$ surface containing source locations - *spread source energy over a surface*, acts as antennae, controls full 2D phase spectrum

For any m, determine $\bar{w} = \bar{w}[m]$ so that $F\bar{s} \approx d$. Update m to drive $\bar{w}(\mathbf{x}, t; \mathbf{x}_s) \rightarrow w(t)\delta(\mathbf{x} - \mathbf{x}_s)$ by minimizing

$$J_{\mathrm{SSE}}[m] = \sum_{\mathbf{x}_s} \sum_{\mathbf{x},t} ||(\mathbf{x} - \mathbf{x}_s)|\bar{w}[m](\mathbf{x}, t; \mathbf{x}_s)|^2$$

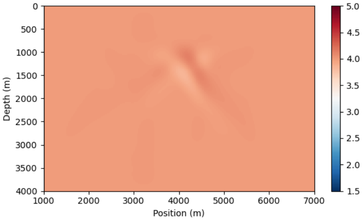
Penalty method: minimize

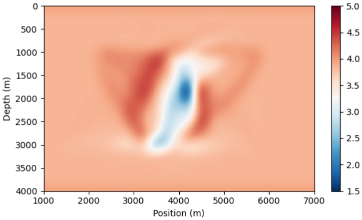
$$J_\alpha[m,\bar{w}] = \frac{1}{2}\sum |F[m]\bar{w} - d|^2 + \alpha^2 ||\mathbf{x} - \mathbf{x}_s|\bar{w}|^2$$

over m

- time-domain FD, gradient via adjoint state
- Variable Projection:
  - update $\bar{w}$ using Conjugate Gradient Iteration (inner loop)
  - update m using steepest descent, line search (outer loop) plus smoothing regularization
- update α using Discrepancy Principle (keep data residual $\in (e_-, e_+)$) - can always choose $\bar{w}$ to fit data $\Rightarrow$ can start with $\alpha = 0$ (Fu & S. 17)

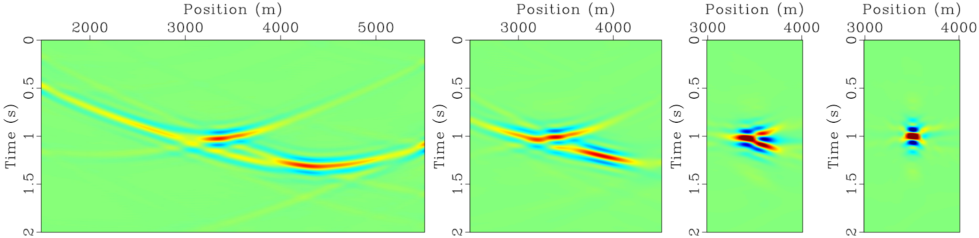
SSE for 1 shot lens data, source surface $\mathcal{S} = \{z = 3000m\}$



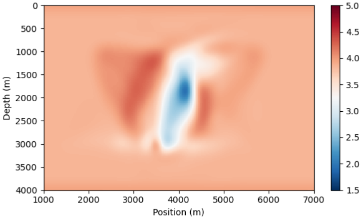


Start at $\kappa_0 = 4.0$ GPa, use $e_- = 0.05\|d\|, e_+ = 0.1\|d\|$.
Left: Iteration 1 $(\alpha = 2.9e - 6)$
Right: Iteration 41 $(\alpha = 1.99e - 5)$

$m = [\kappa]$ converges $\leftrightarrow$ α increases, extended source $\bar{f}$ (initially spread over $z = z_s = 3000$ m) focuses at $x = x_s = 3500$ m.



Left to right: $\bar{f}$ at Iterations 1 ($\alpha = 2.9e-6$), 9 ($\alpha = 5.2e-6$ m) 19 ($\alpha = 1.32e-5$) and 41 ($\alpha = 1.99e-5$).



Fig/residcovmestfwicgw41wind.pd

Left: "Jump to $\alpha = \infty$": bulk modulus estimated by FWI, 200 steepest descent steps starting with iteration 41 of SSE., final RMS residual reduction $=$ 0.05; Right: FWI data residual

Why it works - a hint:

provided that

- $d = F[m_*]s$,
- RMS wavelength of w (hence d) $\rightarrow 0$,

$$\nabla\nabla J_{\mathrm{SSE}}[m_*] \rightarrow \sum_{\mathbf{x},\mathbf{p}}$$