Mapping Properties of Surface Source Extension

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ABSTRACT

Abstract goes here.

SYMMETRIC HYPERBOLIC SYSTEMS

If A_j is positive definite, then x_j can play the role of time, and the Cauchy problem with initial data $u_0 \in L^2(\mathbf{R}^n, \mathbf{R}^k)$ on x_j = constant has a unique weak solution $u \in L^2(\mathbf{R}^{n+1})^k$ with well-defined L^2 traces on any x_j level set (Lax, Taylor). Want to understand traces on other hypersurfaces, to begin with coordinate hypersurfaces, wlog $x_0 = 0$.

Notation:$x_0 = t$, $\partial_t = \partial_0$, $\mathbf{x} = (x_1, ..., x_n)$, $D = (\partial_1, ...\partial_n)$

$$K(t, \mathbf{x}, D) = \sum_{i=1}^{n} A_i(t, \mathbf{x})\partial_i,\; M(t, \mathbf{x}) = A_0(t, \mathbf{x}).$$

If M is positive definite, trace properties coverd by standard theory (above). Let $\Pi_0(t, \mathbf{x})$ = projection on null space V_0 of $M(t, \mathbf{x})$, $\Pi_1 = I - \Pi_0$ = projection on range V_1. Denote the rank of M by $l = \mathrm{dim} V_1$ Then

$$M\partial_t u + Ku = \Pi_1 M \partial_t \Pi_1 u + \sum_{\alpha,\beta=0,1} \Pi_\alpha K \Pi_\beta u + \text{ l.o.t.}$$

Since the ranges are orthogonal,

$$\begin{aligned} \Pi_1 M\partial_t \Pi_1 u + \Pi_1 K \Pi_1 u + \Pi_1 K \Pi_0 u &\in L^2_{\mathrm{loc}}(\mathbf{R}^{n+1}, V_1), && (1)\\ \Pi_0 K \Pi_0 u + \Pi_0 K \Pi_1 u &\in L^2_{\mathrm{loc}}(\mathbf{R}^{n+1}, V_0) && (2) \end{aligned}$$

Set $K_{\alpha,\beta} = \Pi_\alpha K \Pi_\beta$.

Main assumption: for all $t \in \mathbf{R}, (\mathbf{x}, \xi) \in T^*(\mathbf{R}^n) \setminus \{0\}$, $K_{0,0}(t, \mathbf{x}, \xi)$ is invertible on V_0.

Let $L_{00}(t, \mathbf{x}, \xi) \in C^\infty(\mathbf{R}, S^{-1}(\mathbf{R}^n))$ be inverse to $K_{0,0}(t, \mathbf{x}, \xi)$ on V_0: that is,

$$K_{0,0}(t, \mathbf{x}, \xi) L_{0,0}(t, \mathbf{x}, \xi) v = v = L_{0,0}(t, \mathbf{x}, \xi) K_{0,0}(t, \mathbf{x}, \xi) v$$

for all $v \in V_0$. Construct $L_{0,0}$ to be homogeneous of degree -1 outside of a neighborhood of $\xi = 0$, with V_1 an invariant subspace for $L_{0,0}(t, \mathbf{x}, \xi)$ on which it vanishes. It follows that $L_{0,0}$ is symmetric, and that

$$\Pi_0 u - L_{0,0}K_{0,0}\Pi_u = \Pi_0 u - L_{0,0}K_{0,0}u = \Pi_0 u + L_{0,0}K_{0,1}u \in H^1_{\mathrm{loc}}(\mathbf{R}^{n+1})^k$$

so

$$K_{1,0}u + K_{1,0}L_{0,0}K_{0,1}u \in L^2_{\mathrm{loc}}(\mathbf{R}^{n+1})^k$$

Define

$$\tilde{K} = K_{11} - K_{10}L_{00}K_{01}.$$

Then equation 1 can be rewritten

$$M\partial_t u_1 + \tilde{K}u_1 \in L^2_{\mathrm{loc}}(\mathbf{R}^{n+1}, V_1) \tag{3}$$

Note that for each $t \in \mathbf{R}$, $(\mathbf{x}, \xi) \in T^*(\mathbf{R}^n)$, both V_0 and V_1 are invariant subspaces of $\tilde{K}(t,\mathbf{x},\xi)$, and $V_0 \subset \mathrm{ker}\tilde{K}(t,\mathbf{x},\xi)$. Also, M is invertible when restricted to $V_1 = \mathrm{Rng}M = \mathrm{Rng}\Pi_1$, but not (necessarily) positive definite.

Restricted determinant: if V is an invariant subspace of A, then $\det_V A$ is the determinant of the matrix of A in any orthonormal basis of V. For $(t,\mathbf{x},\tau,\xi) \in T^*(\mathbf{R}^n)$, define the restricted characteristic polynomial

$$\mathcal{C}(t,\mathbf{x},\tau,\xi) = \det_{V_1}(M(t,\mathbf{x})\tau + \tilde{K}(t,\mathbf{x},\xi)) = 0$$

Since V_0 and V_1 are invariant subspaces of the matrix in parentheses, and V_0 is (contained in) its null space, there are at most $l = \mathrm{dim}V_1$ non-zero roots in τ for $\xi \neq 0$.

Choose a conic submanifold $\gamma \subset T^*(\mathbf{R}^n)$, so that for $|t| \leq \delta t, (\mathbf{x},\xi) \in \gamma$ there are exactly $m \leq l$ distinct real roots τ of $\mathcal{C}(t,\mathbf{x},\tau,\xi)$. Denote these by $\tau_i(t,\mathbf{x},\xi), i = 1,...,m$.

Choose a conic $\gamma_1 \subset T^*(\mathbf{R}^n)$ so that $\bar{\gamma}_1 \subset \gamma$, and $Q \in OPS^0(\mathbf{R}^n)$ with $ES(Q) \subset \gamma$, $Q(\mathbf{x},\xi) = 1$ for $(\mathbf{x},\xi) \in \gamma_1$.

Let $\mathcal{N}$ = normal bundle of $x_0 = 0$ and $\Gamma \subset T^*(\mathbf{R}^n)$ for which $\Gamma \cap \mathcal{N} = \emptyset$.

Assume that $u \in L^2(\mathbf{R}^{n+1}) \cap H^1_{\mathrm{loc}}(\Gamma \cap \Pi^{-1}_{\mathcal{N}}(\gamma_1^C))$.

For each $\tau_i, i = 1,...m$, choose $U_i(t,\mathbf{x},\xi) \in C^\infty(\mathbf{R} \times T^*(\mathbf{R}^n), V_1(t,\mathbf{x}))$, so that

$$\tau_i M(\mathbf{x})U_i(t,\mathbf{x},\xi) + \tilde{K}(t,\mathbf{x},\xi)U_i(t,\mathbf{x},\xi) = 0,\ i = 1,..,m.$$

$U_1,...,U_m$ span a subspace $\tilde{V}_1(t,\mathbf{x},\xi)$. Scale U_i so that $|U_i(t,\mathbf{x},\xi)| = 1$.

Note that $\tau_i(t,\mathbf{x},\xi), i = 1,...,m$ are homogeneous of order 1 in ξ and smooth in $\mathbf{x}$, $U_i(t,\mathbf{x},\xi), i = 1,...,m$ are homogeneous of order 0 in ξ, smooth in $\mathbf{x}$. So both are symbol-valued functions, of orders 1 and 0 respectively. Since the U's must be orthogonal, $U_i^T U_j = \delta_{ij}$.

Suppose first that $m = l$, that is, that $\tilde{V}_1 = V_1$. Then $\{U_i(t,\mathbf{x},\xi) : i = 1,..,l\}$ is a basis for $V_1(t,\mathbf{x})$ for each $(t,\mathbf{x},\xi \in \Gamma$. Set $u_{1,i}(t,\mathbf{x}) = U_i(t,\mathbf{x},D_x)^T u_1$. Then for any

$w \in L^2(\mathbf{R}^{n+1}, V_1)$

$$\sum_{i=1}^{l} U_i(t,\mathbf{x},D)U_i(t,\mathbf{x},D)^T w(t,\mathbf{x}) \approx$$

$$\int d\xi\, e^{i\mathbf{x}\cdot\xi} U_i(t,\mathbf{x},\xi)U_i(t,\mathbf{x},\xi)^T \hat{w}(t,\xi) \approx$$

$$\int d\xi\, e^{i\mathbf{x}\cdot\xi}$$

$$\tilde{K}(t,\mathbf{x},D)u_{1,i} = \tilde{K}(t,\mathbf{x},D)U_i(t,\mathbf{x},D)^T$$

$$M(\mathbf{x})\partial_t u_{1,i}(t,\mathbf{x}) = -\tilde{K}(t,\mathbf{x},D)$$

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So $V_0 = \mathrm{span}(\{e2\})$, $V_1 = \mathrm{span}(\{e1, e3\}), \Pi_0 = \mathrm{diag}(0,1,0)$

$$K(t,\mathbf{x},\xi) = A_1(\mathbf{x})\xi_1 + A_2(\mathbf{x})\xi_2 = \begin{pmatrix} \frac{1}{\kappa(\mathbf{x})}\omega & \xi_x & 0 \\ \xi_x & \rho(\mathbf{x})\omega & 0 \\ 0 & 0 & \rho(\mathbf{x})\omega \end{pmatrix}$$

whence $K_{0,0} = \mathrm{diag}(0, \rho(\mathbf{x})\omega, 0)$.

$$K_{1,1}(t,\mathbf{x},\xi) = \begin{pmatrix} \frac{1}{\kappa(\mathbf{x})}\omega & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho(\mathbf{x})\omega \end{pmatrix},$$

$$K_{0,1}(t,\mathbf{x},\xi) = \begin{pmatrix} 0 & \xi_x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $K_{1,0} = K_{0,1}^T$.

Choose $\omega_0 > 0$, $\phi \in C^\infty(\mathbf{R})$ so that $\phi(\omega) = 1$ for $\omega > 2\omega_0$, $= 0$ for $\omega < \omega_0$. Set

$$L_{0,0} = \mathrm{diag}\left(0, \frac{\phi(|\omega|)}{\rho(\mathbf{x})\omega}, 0\right)$$

Then $L_{0,0}K_{0,0} = I$ restricted to V_0 and away from zero frequency, and

$$\tilde{K} = K_{1,1} - K_{1,0}L_{0,0}K_{0,1} = \begin{pmatrix} \frac{\omega}{\kappa} - \frac{\xi_x^2\phi(|\omega|)}{\rho\omega} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho\omega \end{pmatrix}.$$

Since $V_1 = \mathrm{span}(e_1, e_3)$, the determinant of the restricted characteristic polynomial is obtained by dropping the second row and column:

$$\det \begin{pmatrix} \frac{\omega}{\kappa} - \frac{\xi_x^2 \phi(|\omega|)}{\rho\omega} & \xi_z \\ \xi_z & \rho\omega \end{pmatrix}.$$

$$= \frac{\rho\omega^2}{\kappa} - \xi_x^2 \phi(|\omega|) - \xi_z^2$$
$$= \frac{\rho\omega^2}{\kappa} - \xi_x^2 - \xi_z^2 \mod S^{-\infty}$$

as expected. So a maximal choice for Γ would be

$$\Gamma = \{(z, (t, x), (\omega, \xi_x)) : c(z, x)|\xi_x| < |\omega|\}.$$

REFERENCES