Asymptotic Inversion of the Variable Density Acoustic Model

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Outline

- Introduction: Born Modeling \ Inversion
- Approximate Born Inversion:
  1. Constant Density.
  2. Variable Density.
- Numerical Examples
- Concluding Remarks

Seismic Inversion

Relate the model space M to the data space D through the forward map F

$$\mathcal{F}: M \to D$$

Find the model $m \in M$ that fits best the observed datum $d \in D$.

Minimization of the L_2 misfit function:

$$J[m,d] = \frac{1}{2}||\mathcal{F}[m] - d||^2$$

Full Waveform Inversion (FWI).

Extends into any modeling physics, or data geometry.

Nonlinear, ill-posed problem.

Born Scattering Theory

- **Scalar wave equation (constant density):**

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p(t,\mathbf{x};\mathbf{x_s}) = f(t,\mathbf{x};\mathbf{x_s})$$

$$\mathcal{F}[v](t,\mathbf{x_r};\mathbf{x_s}) = p(t,\mathbf{x_r};\mathbf{x_s})$$

- **Linearization:**

$$v = v_0 + \delta v$$
$$p = p_0 + \delta p$$

$$\mathcal{F}[v_0] \cong \mathcal{F}[v_0] + F[v_0]\delta v$$

- **Born-type wave equation:**

$$\left(\frac{1}{v_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\delta p(t,\mathbf{x};\mathbf{x_s}) = \frac{2\delta v}{v_0^3}\frac{\partial^2}{\partial t^2}p_0(t,\mathbf{x};\mathbf{x_s})$$

$$F[v_0]\delta v(t,\mathbf{x_r};\mathbf{x_s}) = \delta p(t,\mathbf{x_r};\mathbf{x_s})$$

- **Linearized Misfit function:**

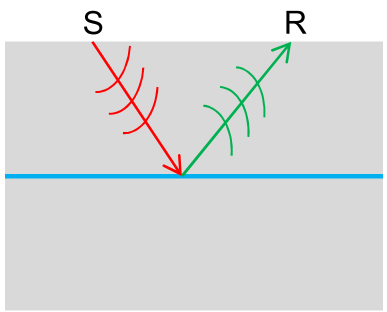
$$J_L[v_0,\delta v,\delta \mathrm{p}] = \tfrac{1}{2}\|F[v_0]\delta v - (\delta p - \mathcal{F}[v_0])\|^2$$

Born Scattering Theory

- **Forward Modeling / Adjoint Imaging (constant density):**

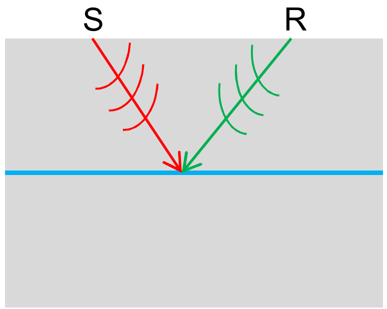
➢ Modeling:

$$(F[v_0]\delta v)(t, \mathbf{x_r}; \mathbf{x_s}) = \frac{\partial}{\partial t^2}\int d\mathbf{x}\,\frac{2\delta v(\mathbf{x})}{v_0^3(\mathbf{x})}\int \delta\tau\, G(t-\tau, \mathbf{x}; \mathbf{x_r})\, G(\tau, \mathbf{x}; \mathbf{x}_s)$$



➢ Imaging (migration):

$$(F^*[v_0]\delta p)(\mathbf{x}) = \frac{2}{v_0^3(\mathbf{x})}\int d\mathbf{x_r}\int d\mathbf{x}_s\int dt\,\frac{\partial^2}{\partial t^2}\delta p(t, \mathbf{x_r}; \mathbf{x}_s)\int \delta\tau\, G(t-\tau, \mathbf{x}; \mathbf{x_r})\, G(\tau, \mathbf{x}; \mathbf{x}_s)$$



Model Extension

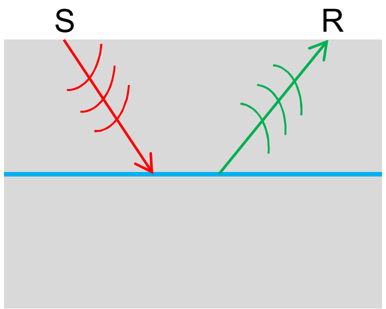
- **Subsurface offset extension:**

$$M \rightarrow \bar{M}$$

$$\mathcal{F}: \bar{M} \rightarrow D$$

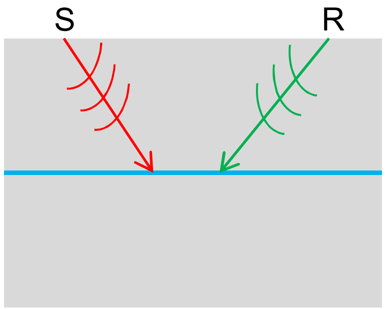
➢ Extended Modeling:

$$(\bar{F}[v_0]\delta\bar{v})(t, \mathbf{x_r}; \mathbf{x_s}) = \frac{\partial}{\partial t^2} \int d\mathbf{x} \int d\mathbf{h} \frac{2\delta\bar{v}(\mathbf{x},\mathbf{h})}{v_0^3(\mathbf{x})} \int \delta\tau G(t-\tau, \mathbf{x}+\mathbf{h}; \mathbf{x_r}) G(\tau, \mathbf{x}-\mathbf{h}; \mathbf{x}_s)$$



➢ Extended Imaging (migration):

$$(\bar{F}^*[v_0]\delta p)(\mathbf{x}, \mathbf{h}) = \frac{2}{v_0^3(\mathbf{x})} \int d\mathbf{x_r} \int d\mathbf{x}_s \int dt \frac{\partial^2}{\partial t^2} \delta p(t, \mathbf{x_r}; \mathbf{x}_s) \int \delta\tau G(t-\tau, \mathbf{x}+\mathbf{h}; \mathbf{x_r}) G(\tau, \mathbf{x}-\mathbf{h}; \mathbf{x}_s)$$



Approximate Born Inversion

- **Approximate Inverse Operator (constant density):**

Modify migration (adjoint) into an inverse operator.

Introduced by **Hou and Symes (2017)** :

$$\boxed{\bar{F}^{\dagger}(\mathbf{x},\mathbf{h}) = W_m^{-1}\bar{F}^{*}(\mathbf{x},\mathbf{h})W_d}$$

<u>model-side weight</u>

$$W_m^{-1} = -8v_0^4\partial_z$$

<u>data-side weight</u>

$$W_d = I_t(I_t\partial_{z_s})(I_t\partial_{z_r})$$

Variable Density Acoustics

- **Dynamical Laws of Linear Acoustics:**

$$\begin{cases} \text{Constitutive Law:} & \dfrac{1}{\kappa}\dfrac{\partial p}{\partial t} = -\nabla \cdot \mathbf{v} + f \\[2ex] \text{Momentum Balance:} & \rho \dfrac{\partial \mathbf{v}}{\partial t} = -\nabla p \end{cases}$$

- **Linearization (extended):**

$$\kappa = \kappa_0 + \delta\bar{\kappa} \qquad \mathbf{v} = \mathbf{v}_0 + \delta\bar{\mathbf{v}}$$
$$\rho = \rho_0 + \delta\bar{\rho} \qquad p = p_0 + \delta\bar{p}$$

- **Born-type wave equation:**

$$\begin{cases} \dfrac{1}{\kappa_0}\dfrac{\partial \delta\bar{p}}{\partial t} = -\nabla \cdot \delta\bar{\mathbf{v}} + \dfrac{1}{\kappa_0}\delta\bar{\kappa}\dfrac{1}{\kappa_0}\dfrac{\partial p_0}{\partial t} \\[2ex] \rho_0 \dfrac{\partial \boldsymbol{\delta}\bar{\mathbf{v}}}{\partial t} = -\nabla \delta\bar{p} - \delta\bar{\rho}\dfrac{\partial \mathbf{v}_0}{\partial t} \end{cases}$$

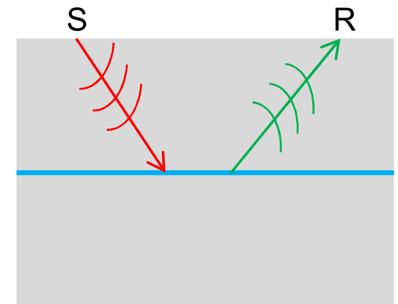
- **Born forward map:**

$$(\bar{F}[\kappa_0, \rho_0][\delta\bar{\kappa}, \delta\bar{\rho}])(t, \mathbf{x_r}; \mathbf{x_s}) = \delta\bar{p}(t, \mathbf{x_r}; \mathbf{x_s})$$

Variable Density Acoustics

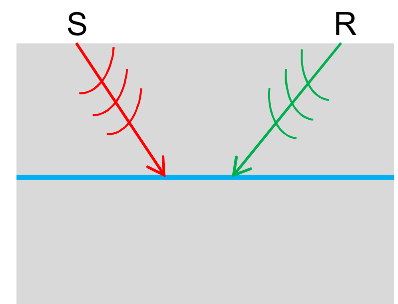
- Extended Modeling:

$$
\begin{aligned}
&(\bar{F}[\kappa_0,\rho_0][\delta\bar{\kappa},\delta\bar{\rho}])(t,\mathbf{x_r};\mathbf{x_s}) \\
&= \int d\mathbf{x}\int d\mathbf{h}\int \delta\tau\left(G(t-\tau,\mathbf{x}+\mathbf{h};\mathbf{x_r})\frac{1}{\kappa_{0+}}\delta\bar{\kappa}\frac{1}{\kappa_{0-}}\frac{\partial^3}{\partial t^3}G(\tau,\mathbf{x}-\mathbf{h};\mathbf{x}_s)\right. \\
&\qquad \left. + \nabla G(t-\tau,\mathbf{x}+\mathbf{h};\mathbf{x_r})\cdot\left(\frac{1}{\rho_{0+}}\delta\bar{\rho}\frac{1}{\rho_{0-}}\nabla\frac{\partial}{\partial t}G(\tau,\mathbf{x}-\mathbf{h};\mathbf{x}_s)\right)\right)
\end{aligned}
$$



- Extended Imaging (migration):

$$
(\bar{F}^*[\kappa_0,\rho_0]\delta p)(\mathbf{x},\mathbf{h})
= \begin{bmatrix}
\frac{1}{\kappa_{0+}}\frac{1}{\kappa_{0-}}\int d\mathbf{x_r}\int d\mathbf{x}_s\int dt\int \delta\tau G(t-\tau,\mathbf{x}+\mathbf{h};\mathbf{x_r})\frac{\partial^3}{\partial t^3}G(\tau,\mathbf{x}-\mathbf{h};\mathbf{x}_s)\delta p(t,\mathbf{x_r};\mathbf{x}_s) \\
\frac{1}{\rho_{0+}}\frac{1}{\rho_{0-}}\int d\mathbf{x_r}\int d\mathbf{x}_s\int dt\int \delta\tau\nabla G(t-\tau,\mathbf{x}+\mathbf{h};\mathbf{x_r})\nabla\frac{\partial}{\partial t}G(\tau,\mathbf{x}-\mathbf{h};\mathbf{x}_s)\delta p(t,\mathbf{x_r};\mathbf{x}_s)
\end{bmatrix}
$$



Asymptotic Analysis

- **The (Modified) Normal Operator:**

$$M = (I_t\bar{F}[\kappa_0,\rho_0])^{*}\partial_{z_s} I_t \partial_{z_r} I_t \bar{F}[\kappa_0,\rho_0]$$

- **Has the form of a Generalized Radon Transform (GRT) operator:**

$$Mu(\mathbf{x},\mathbf{h}) = \int d\mathbf{x_r} d\mathbf{x_s} d\mathbf{x}' d\mathbf{h}' A(\mathbf{x}_s,\mathbf{x}_r,\mathbf{x},\mathbf{h},\mathbf{x}',\mathbf{h}')\delta(\phi(\mathbf{x}_s,\mathbf{x}_r,\mathbf{x},\mathbf{h}) - \phi(\mathbf{x}_s,\mathbf{x}_r,\mathbf{x}',\mathbf{h}'))u(\mathbf{x}',\mathbf{h}')$$

in which:

$$\phi = T_s + T_r$$

$$A_0(\mathbf{x}_s,\mathbf{x}_r,\mathbf{x},\mathbf{h},\mathbf{x}',\mathbf{h}') = \pi^2 a_s a_r \frac{cos\boldsymbol{\theta}_\mathbf{s}}{v(\mathbf{x_s})}\frac{cos\boldsymbol{\theta}_{\boldsymbol{r}}}{v(\mathbf{x_r})} a'_s a'_r$$

and:

$$A(\mathbf{x}_s,\mathbf{x}_r,\mathbf{x},\mathbf{h},\mathbf{x}',\mathbf{h}') = \begin{bmatrix} A_0 \dfrac{1}{\sqrt{\kappa_{0+}\kappa_{0-}\kappa'_{0+}\kappa'_{0-}}} & A_0 \dfrac{\nabla T_s \nabla T_r}{\sqrt{\rho_{0+}\rho_{0-}\kappa'_{0+}\kappa'_{0-}}} \\ A_0 \dfrac{\nabla T_s \nabla T_r}{\sqrt{\rho_{0+}\rho_{0-}\kappa'_{0+}\kappa'_{0-}}} & A_0 \dfrac{\nabla T_s \nabla T_r \nabla T'_s \nabla T'_r}{\sqrt{\rho_{0+}\rho_{0-}\rho'_{0+}\rho'_{0-}}} \end{bmatrix}$$

Asymptotic Analysis

- **Asymptotic analysis of the normal operator reveals:**

$$M[\bar{r}_\kappa, \bar{r}_\rho]^T \cong \int dk_z dk_x dk_h \frac{e^{i(zk_z+xk_x+hk_h)}(q_s+q_r)^2}{256\pi^3(-ik_z)(s_+^2 q_s^2 + s_-^2 q_r^2 + (s_+^2+s_-^2)q_s q_r)v_+^2 v_-^2}[\bar{r}_\kappa, \bar{r}_\rho]^T [1, cos2\theta][1, cos2\theta]^T$$

where:

$$\bar{r}_\kappa = \frac{\delta\bar{\kappa}}{\sqrt{\kappa_{0+}\kappa_{0-}}} \qquad \bar{r}_\rho = \frac{\delta\bar{\rho}}{\sqrt{\rho_{0+}\rho_{0-}}}$$

$$q_s = \frac{\partial T_s}{\partial z} \qquad q_r = \frac{\partial T_r}{\partial z} \qquad s = \frac{1}{v}$$

- **The κ component:**

$$(M[\delta\bar{\kappa}, \delta\bar{\rho}\,]^T)_k \cong \int dk_z dk_x dk_h \frac{e^{i(zk_z+xk_x+hk_h)}}{256\pi^3\kappa_0^3\rho_0^{-1}(-ik_z)} PB(\delta\bar{\kappa} + (\kappa_0\rho_0^{-1})\delta\bar{\rho}\, cos2\theta)$$

in which θ is the scattering angle.

Approximate Born Inversion

- **The variable density acoustics Approximate Inverse:**

$$(I_t\bar{F}^*[\kappa_0,\rho_0])\partial_{z_s} I_t \partial_{z_r} I_t \bar{F}[\kappa_0,\rho_0][\delta\bar{\kappa},\delta\bar{\rho}] \\ \cong \int dk_z dk_x dk_h \frac{e^{i(zk_z+xk_x+hk_h)}}{256\pi^3\kappa_0^3\rho_0^{-1}(-ik_z)} PB(\delta\bar{\kappa}+(\kappa_0\rho_0^{-1})\delta\bar{\rho}\, cos2\theta)$$

Define the weights:

$$W_d = I_t(I_t\partial_{z_s})(I_t\partial_{z_r})$$

$$W_m^{-1} = 32\kappa_0^3\rho_0^{-1}\partial_z$$

The approximate Inverse:

$$\boxed{\bar{F}^\dagger\bar{F}\,[\delta\bar{\kappa},\delta\bar{\rho}] = W_m^{-1}\bar{F}^* W_d \bar{F}[\delta\bar{\kappa},\delta\bar{\rho}] \\ \cong 8\pi^3 \int dk_z dk_x dk_h e^{i(zk_z+xk_x+hk_h)} (\delta\bar{\kappa}+(\kappa_0\rho_0^{-1})\delta\bar{\rho} cos2\theta)}$$

- **Angle-dependent Reflectivity:**

$$\boxed{R(x,\theta) \cong \delta\bar{\kappa}+(\kappa_0\rho_0^{-1})\delta\bar{\rho} cos2\theta}$$

Approximate Born Inversion

- **Approximate Inverse Operator (variable density):**

Similar construction as in Hou and Symes (2017) :

$$\boxed{\bar{F}^{\dagger}(\mathbf{x},\mathbf{h}) = W_m^{-1}\bar{F}^{*}(\mathbf{x},\mathbf{h})W_d}$$

model-side weight

$$W_m^{-1} = 32\kappa_0^3\rho_0^{-1}\partial_z$$

data-side weight

$$W_d = I_t(I_t\partial_{z_s})(I_t\partial_{z_r})$$

Net Result

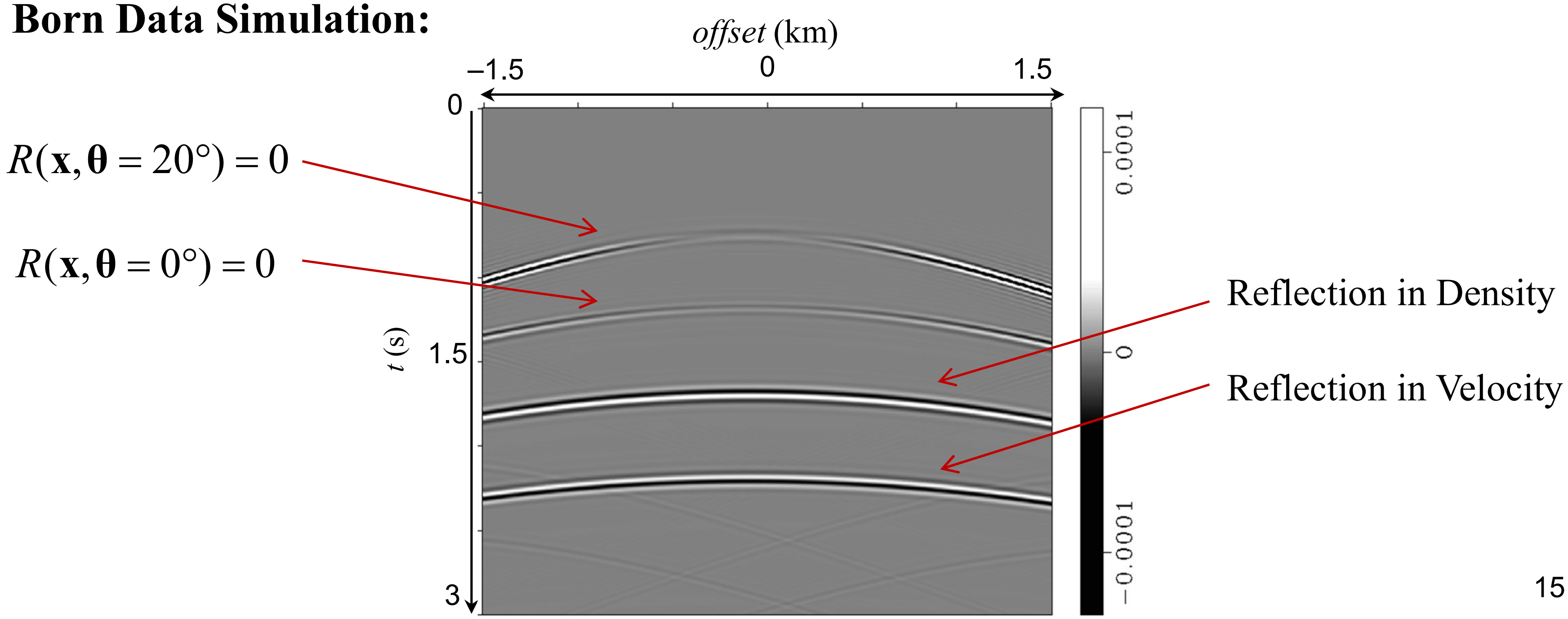
- Born data (or simulation) $(\bar{F}[\kappa_0,\rho_0][\delta\bar{\kappa},\delta\bar{\rho}])(t,\mathbf{x_r};\mathbf{x_s}) = d(t,\mathbf{x_r};\mathbf{x_s})$
- Data-side weight operator $W_d d(t,\mathbf{x_r};\mathbf{x_s})$
- Migration (subsurface offset extended) $\bar{F}^*[\kappa_0,\rho_0]W_d d(\mathbf{x},h)$
- Model-side weight operator $W_m^{-1}\bar{F}^*[\kappa_0,\rho_0]W_d d(\mathbf{x},h)$
- Angle-domain transformation $\mathcal{R}[W_m^{-1}\bar{F}^*[\kappa_0,\rho_0]W_d d(\mathbf{x},h)\ ](\mathbf{x},\theta)$
- AVA inversion $R(x,\theta) \cong \delta\bar{\kappa} + \left(\kappa_0\rho_0^{-1}\right)\delta\bar{\rho} cos2\theta$

Example #1

- **Layered Variable Density Media:**

Layer #	*Depth* (km)	*Velocity* (km/s)	*Density* (gr/cm^3)
1	0.750	2.000	2.300
2	1.250	2.300	1.963
3	1.750	1.963	2.300
4	2.250	1.963	2.000
5	3.000	2.200	2.000

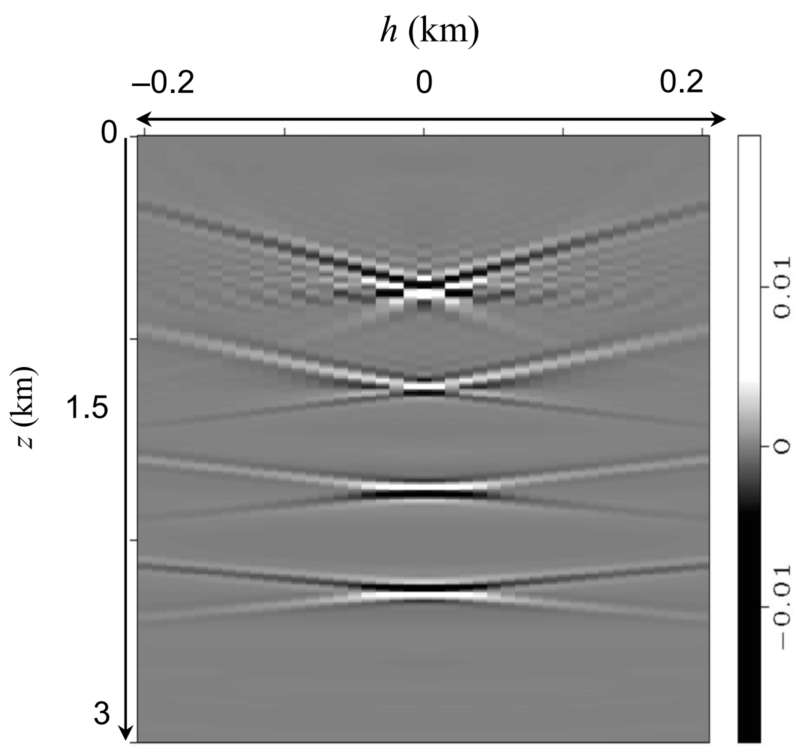
- **Born Data Simulation:**



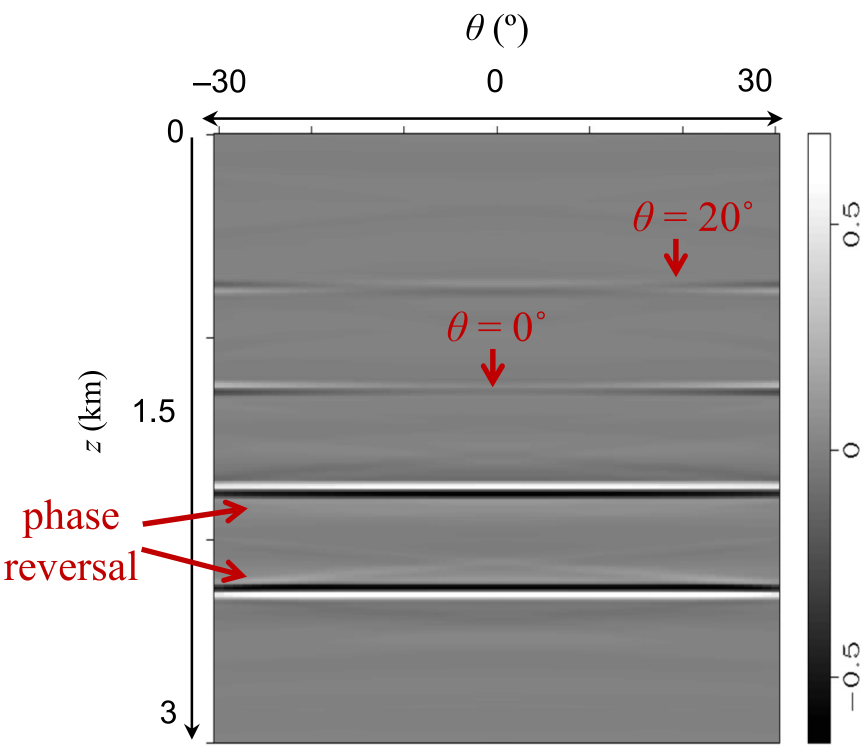
Example #1

- **Approximate Inverse:**

subsurface offset gather



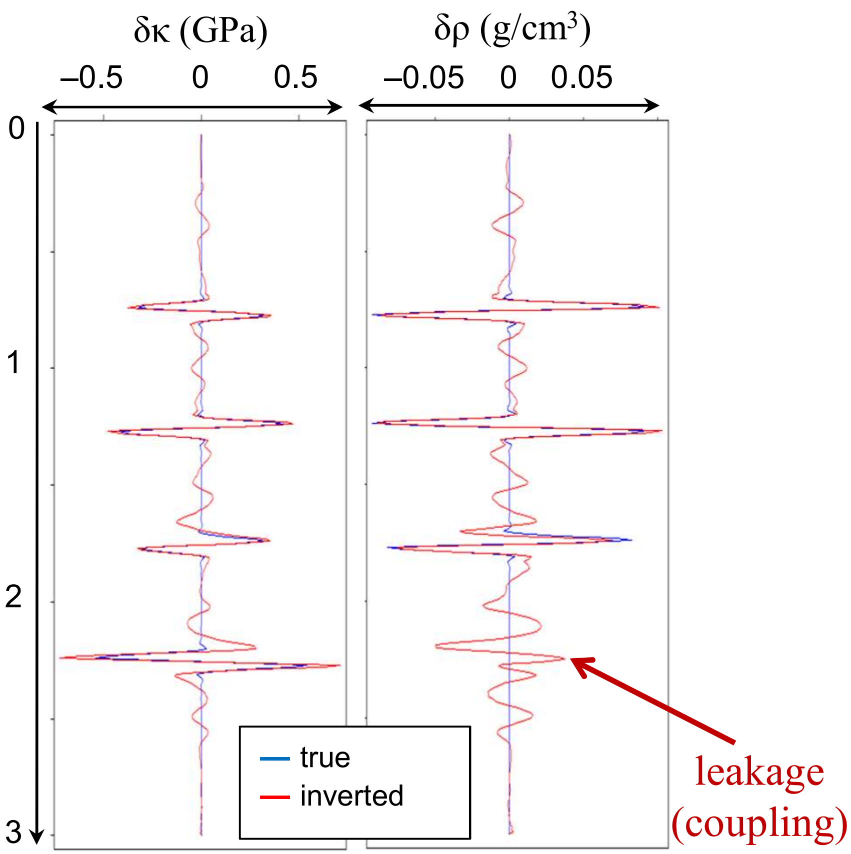
scattering-angle gather



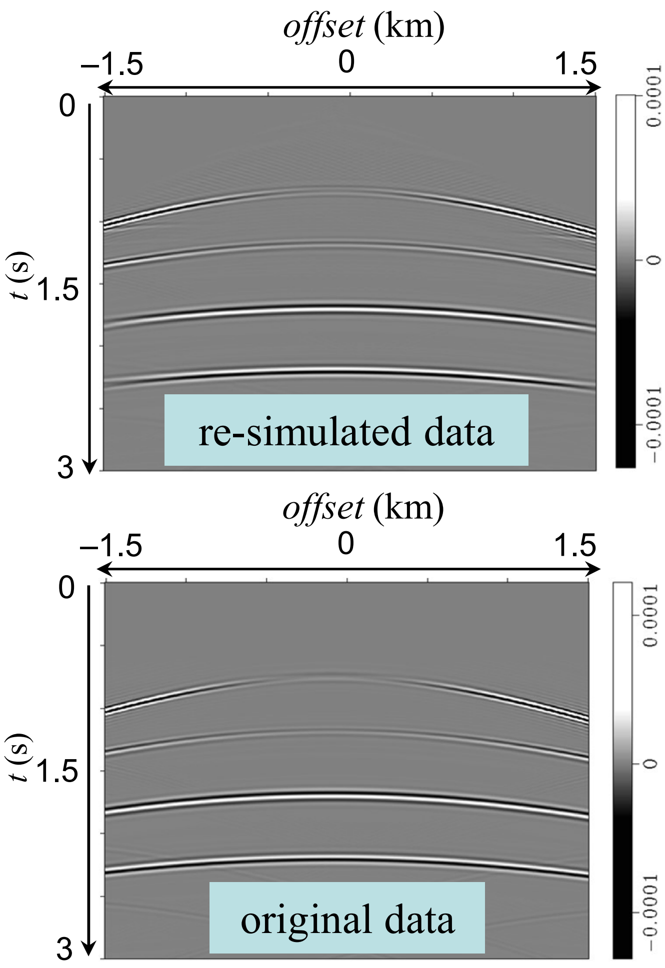
Example #1

- **AVA Inversion:**

model perturbations

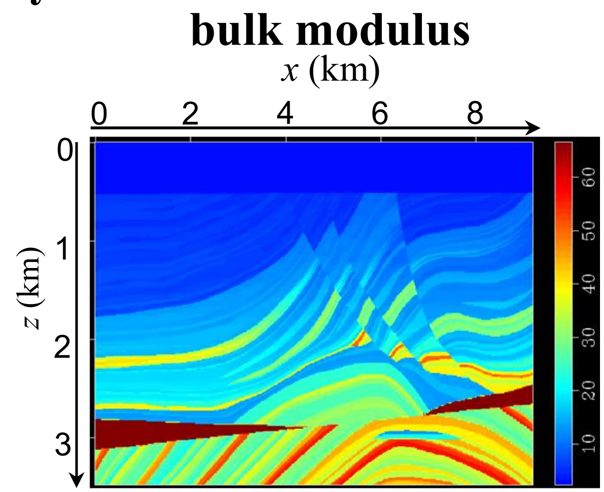


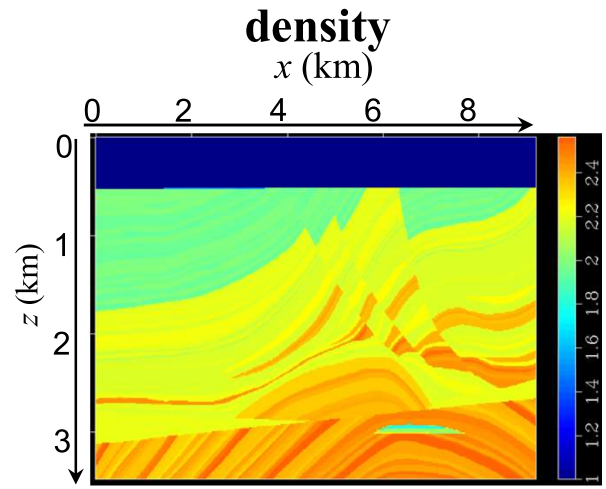
Born data re-simulation



Example #2

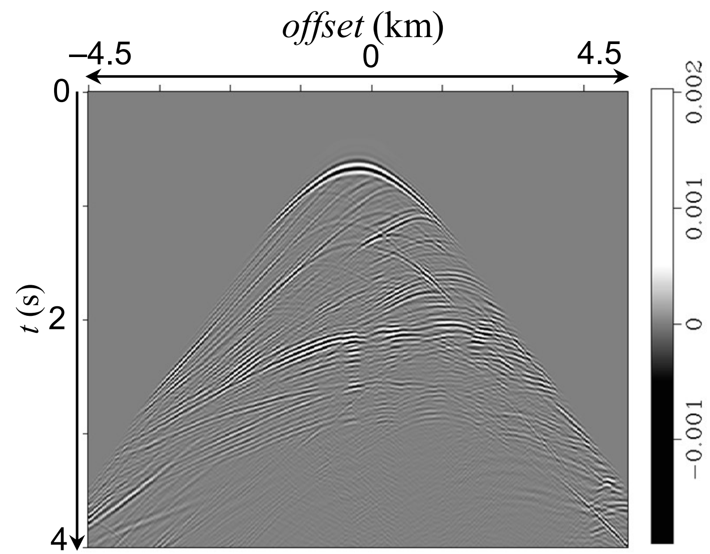
- **Marmousi Variable Density Model:**





- **Born data simulation:**

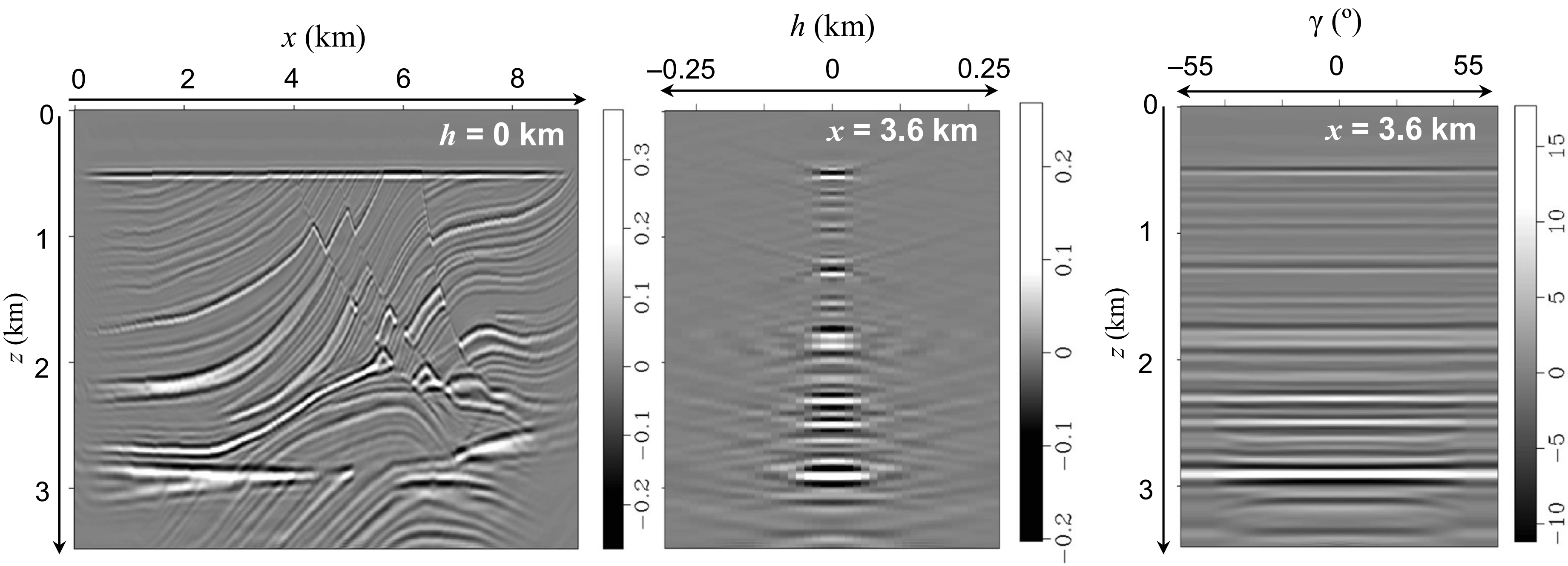
Shot gather at x = 4.56km :



Example #2

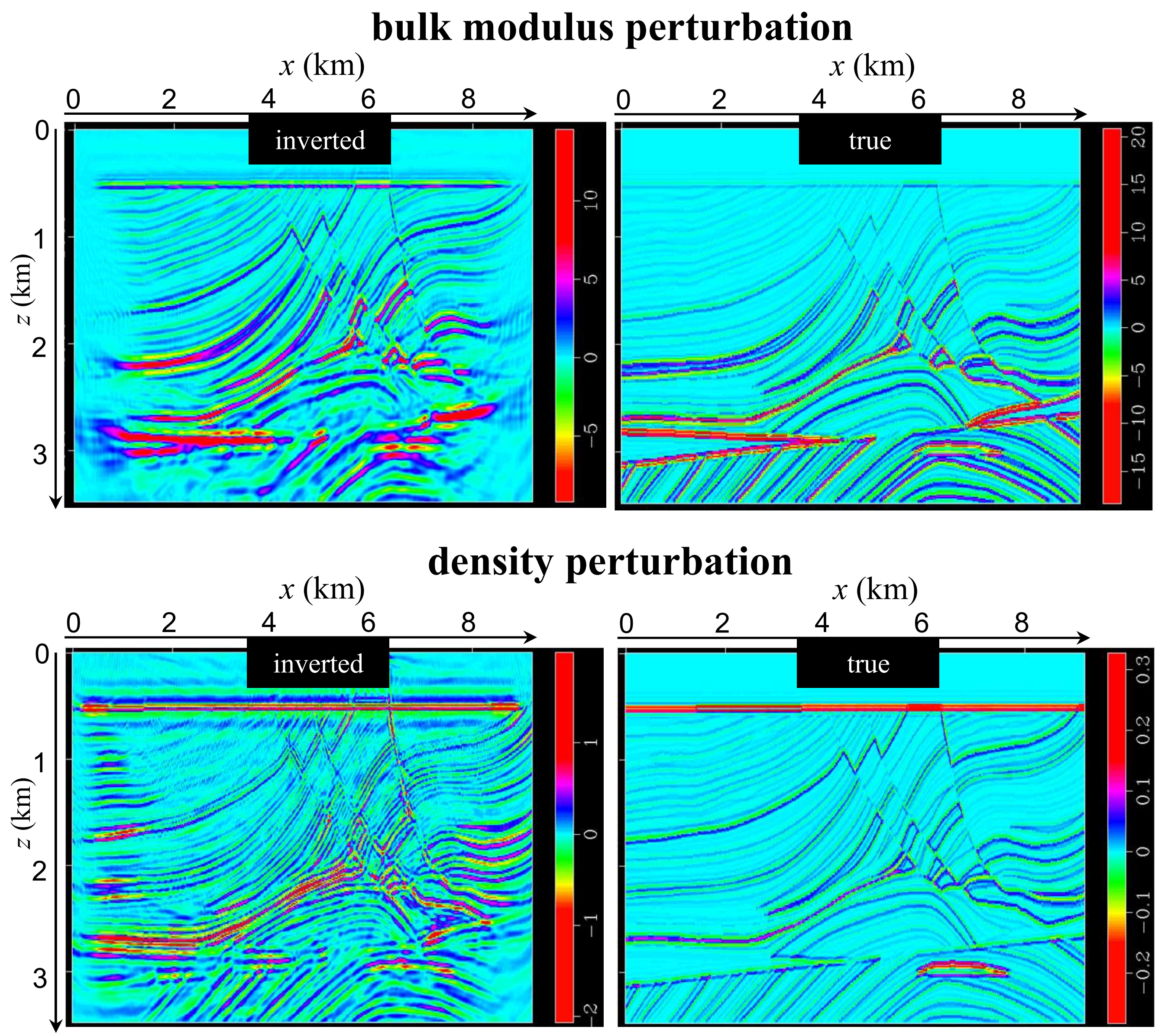
- **Approximate Inverse:**

subsurface offset extended image | **scattering-angle gather**



Example #2

- **AVA Inversion:**

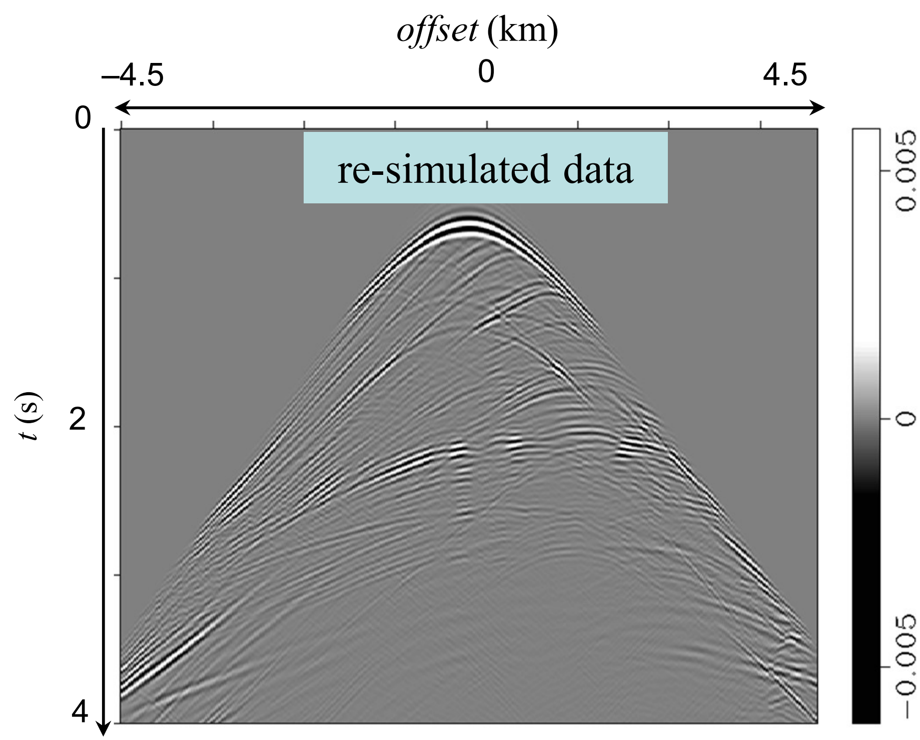


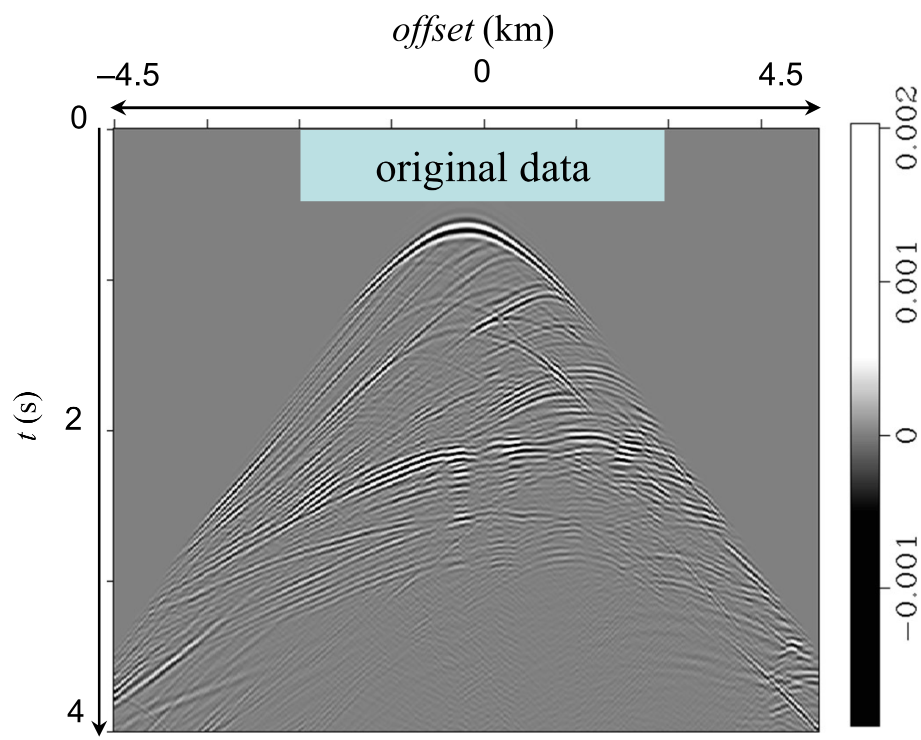
Example #2

- **AVA Inversion:**

  Born data re-simulation

Shot gather at x = 4.56km :





Concluding Remarks

- Approximate Born Inversion = weighted variation of RTM (subsurface offset extended)
- Explicit data and model side weights, independent of ray-theory properties.
- Post-migration angle-transformation naturally exposes the angle-dependent reflectivity.
- Simultaneous inversion of model perturbations: $\delta\kappa$, $\delta\rho$
- May be extended similarly to Elastic Media (pure and converted modes).
- Main Limitations:

  1) asymptotic approximation (high-frequency, short-scale)

  2) data bandwidth

  3) image is assumed to be well focused (optimal velocity)

  3) coupling between inverted parameters
- The Approximate Inverse application may accelerate the convergence of LSM (SEG 2018, Tue, SPMI2, **“Accelerated Acoustic LSM”** by Dafni & Symes).

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Thank you!