# Wavefield Reconstruction Inversion: an example

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#### ABSTRACT

Full waveform inversion, or nonlinear least squares data-fitting driven by wave equation simulation, has proven its ability to extract detailed maps of earth structure from near-surface seismic observations. However, the least squares error function at the heart of this method tends to develop a high degree of nonconvexity, so that local optimization methods (the only numerical methods feasible for field-scale problems) tend to fail, unless provided with initial estimates of a quality not always available. A number of alternative optimization principles have been advanced that promise some degree of release from the multimodality of Full Waveform Inversion, amongst them Wavefield Reconstruction Inversion, the focus of this paper. Applied to a simple 1D acoustic transmission problem, both Full Waveform and Wavefield Reconstruction Inversion methods reduce to minimization of explicitly computable functions, in an asymptotic sense. The analysis presented here shows that Wavefield Reconstruction Inversion can be vulnerable to the same "cycle-skipping" failure mode as Full Waveform Inversion.

## INTRODUCTION

Full waveform inversion (FWI) is the current nomenclature in the seismology literature for data-fitting earth structure estimation driven by wavefield modeling. The earth properties to be estimated (material densities, stiffnesses, attenuation rates,...) form a vector c of spatially fields that appear as coefficients in systems of hyperbolic partial differential equations, modeling seismic wave propagation. The wavefields used in structure estimation ("imaging") are small motion disturbances of the earth's equilibrium state, so the equations of motion are typically linear(ized). Right-hand side vectors f in these systems model energy input that initiates waves (earthquakes, man-made sources such as explosives or mechanical vibrators). Data vectors d are simulated by sampling the solution fields at the locations of measurement devices (accelerometers, microphones,...) over appropriate time intervals. The relation between the energy source f and the simulated data is linear in f, but nonlinear in the coefficient vector c, so is naturally represented by a family of linear operators S[c] parametrized by the coefficient vector c.

The objective of FWI is to find c and f, given d, so that  $S[c]f \approx d$ . A typical method for enforcing this requirement is the minimization of an objective misfit measure (objective, for short), the most common choice being the square norm of a

Hilbert space in which the data is presumed to reside:

Given d, find c and f to minimize 
$$||d - S[c]f||^2$$
 (1)

in which  $\|\cdot\|$  is the norm in a suitable Hilbert space. This approach was first suggested in the 1980's ((Bamberger et al., 1979; Tarantola, 1984; Kolb et al., 1986; Crase et al., 1990), and many other papers since then). Usually some form of regularization is applied, to compensate for poorly determined aspects of c and/or f, as is explained in Tarantola's influential book (Tarantola, 2005). Also, f may be constrained in one way or another to embody characteristics of field energy sources, or even regarded as known (an example of this is given below).

Within a few years of its introduction into quantitative seismology, FWI was understood to suffer from a severe limitation. Because of the typical dimensions of earth models, and consequent cost of accurate computation of S, iterative local optimization provides the only feasible route to estimation of c via solution of the optimization problem 1. However the objective function of this optimization problem (the mean-square residual appearing in display 1) has many local minima in general, most having nothing to do with a usable estimate of earth structure. [An explicit example of this multi-modal behaviour appears below.] Reliable estimation of c via iterative local optimization requires that the initial estimate predict the correct arrival time of waves, as they appears in the data, within a wavelength at dominant frequencies (Gauthier et al., 1986; Virieux and Operto, 2009).

Despite this severe constraint, FWI has shown enough promise as a tool for both industrial and academic seismology that it is now a mainstream research topic, and to some extent a commercial product. Estimation of sufficiently accurate initial models via non-FWI methods is common practice, though what "sufficiently accurate" means may be difficult to discern (Plessix et al., 2010). The wavelength criterion mentioned in the last paragraph may be made less onerous by collection of relatively low-frequency data (Dellinger et al., 2016). Finally, many alternatives to straightforward least-squares data fitting have been suggested, some of which appear to exhibit less tendency to develop local minima than does the problem described in 1 (Symes, 2009).

This topic of this paper is one of these alternative approaches, Wavefield Reconstruction Inversion ("WRI"). WRI was introduced by van Leeuwen and Herrmann (2013), and further developed by van Leeuwen and Herrmann (2016); Wang et al. (2016) and other authors. It is based on the presumption that the correct source (right hand side in the equations of motion) q is known, and combines a penalty for data misfit with a penalty for failing to solve the equations of motion with the correct right hand side:

Given 
$$d$$
 and  $q$ , find  $c$  and  $f$  to minimize  $||d - S[c]f||^2 + \alpha^2 ||f - q||^2$ . (2)

Examples given by van Leeuwen and Herrmann (2013) and others suggest that the problem defined in display 2 is less likely to develop uninformative local minima if

 $\alpha$  is small. This might be so for several reasons: one is that when less weight is put on making the residual in the wave equation (f-q) small, f may be chosen to make the data residual d-S[c]f small instead. One aspect of failure to predict the arrival times of waves accurately is that small data residual is then difficult to achieve. By maintaining fidelity to the data, local minimization of the WRI objective might provide a spurious-minimum-free path to a satisfactory estimate of c.

The main result of this paper is that in one simple case, in which all of the necessary computations can be carried out by hand, this hope is not realized: minimization of the WRI objective is just as likely to be trapped in a spurious local minimizer as is the FWI objective. The context of this conclusion is a simple transmission inverse problem for the 1D acoustic wave system, which models pulse transmission along a 1D continuum from a source point to a receiver point. The pulses used in this thought experiment are short, so the main information content of the data is the time of transit from source to receiver. The predominant information about the material model, in this case reduced to the wave velocity (a scalar function of position), is just this transit time, so I constrain both the target wave velocity  $c_*$  generating the data and the trial wave velocity c to be constant, that is, independent of position along the 1D continuum. I introduce a family of inverse problems, depending on a parameter  $\lambda$  playing the role of wavelength. For sufficiently small  $\lambda$ , it is possible to show explicitly that the FWI problem possesses local minimizers far from the global minimizer at the target  $c_*$ , and that initiating a local iterative optimization, such as steepest descent or Newton's method, at a distance from  $c_*$  bounded below by a multiple of the "wavelength"  $\lambda$  will result in convergence to these spurious local minima. That is, FWI behaves in exactly the manner described in much of the literature on this topic. However, an analysis of WRI applied to the same context yields the same result: spurious local minima exist for sufficiently small  $\lambda$ , and will be found by local optimization unless the starting point is within  $O(\lambda)$  of the target. That is, WRI behaves in a manner qualitatively indistinguishable from FWI. In particular, its ability to allow good fit to data for small  $\alpha$  does not safeguard it from failure to converge globally to a "good" local minimum. In fact, the  $\alpha \to 0$  limit of the WRI objective is well-defined (after scaling by  $1/\alpha^2$ ) and also behaves in qualitatively the same way as the FWI objective. So the apparent ability to maintain better data fit via reduction of  $\alpha$  does not lead to global behaviour asymptotically more amenable to local optimization.

This paper begins with a description of the 1D inverse transmission problem and explicit computation of various components of the FWI approach, based on explicit solution of the 1D acoustic system presented in Appendix A. In the third section I introduce the  $\lambda$ -dependent family of problems, and establish the asymptotic properties of FWI as  $\lambda \to 0$ . The fourth section develops the algebraic structure of WRI, culminating in a remarkable identity revealing WRI to be equivalent to minimization of a weighted norm of the data residual, with a weight operator depending on the coefficient vector c. This identity has also been derived by van Leeuwen (2019), using a different argument. The result is quite general, applying to essentially any realization of WRI, and already shows that its behaviour must be closely related to that

of FWI. In the fifth section I return to 1D acoustics problem and the  $\lambda$ -dependent family of inverse problems, compute that weight operator explicitly, and deduce the global behaviour of WRI in this instance. The paper ends with a discussion of the relation of the analysis presented here to a previous analysis of optimization formulations of wave inversion problems based on parameter-dependent quadratic forms (Stolk and Symes, 2003), and a brief discussion of some other alternatives to FWI.

#### FWI FOR 1D ACOUSTICS

The example of FWI to be explored in this paper is one of the simplest possible, based on the 1D acoustics system connecting excess pressure p, particle velocity v, sonstitutive law defect ("source") f, density  $\rho$ , and wave velocity c:

$$\frac{\partial p}{\partial t} + \rho c^2 \frac{\partial v}{\partial z} = f$$

$$\rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial z} = 0$$

$$p, v = 0, t \ll 0.$$
(3)

The fields p, v, f are functions of spatial position  $z \in \mathbf{R}$  and time  $t \in \mathbf{R}$ , whereas  $c, \rho$  are functions of z alone, so that the system A-3 is autonomous.

The system A-3 has classical (smooth) solutions (p, v) when  $c, \rho$ , and f are smooth, and  $\log c$  and  $\log \rho$  are bounded on  $\mathbf{R}$ , as is well-established (Lax, 2006). In this paper, for reasons to be discussed below, c and  $\rho$  are constrained to be constant (z-independent) in which case solutions may be constructed by elementary methods (Appendix A).

In fact, I shall assume  $\rho > 0$  to be fixed for the remainder of this paper, that is, not updated in the inversion process. The wave velocity c will range over an interval: it is the parameter to be inverted. To be specific, choose  $c_{\text{max}} > c_{\text{min}} > 0$ , and require that c satisfy  $c_{\text{min}} \leq c \leq c_{\text{max}}$ .

Limit observations to the time interval [0, T], at the spatial ("receiver") location  $z_r$ . The modeling operator outputs the pressure trace  $p(z_r, t)$  over the time interval [0, T]:

$$S[c]f = p|_{\{z_r\} \times [0,T]} \tag{4}$$

The formulation of the inverse problem via least-squares requires a choice of Hilbert space structure for the domain and range of S[c]. A natural choice is

$$S[c]: L^2([z_{\min}, z_{\max}] \times \mathbf{R}) \to L^2[0, T]$$
 (5)

That is, the support of f will be assumed to lie in the strip  $[z_{\min}, z_{\max}] \times \mathbf{R}$ . As it will turn out, the choice of spatial interval  $[z_{\min}, z_{\max}]$  is arbitrary, so long as it has positive length.

For homogeneous (z-independent)  $\kappa$ , Appendix A provides an explicit expression:

$$S[c]f(t) = \frac{1}{2c} \int_{z_{\min}}^{z_{\max}} dz f\left(z, t - \frac{|z_r - z|}{c}\right)$$
 (6)

From this expression it is simple to see that S[c] is a bounded operator with the domain and range described in 5. The role of compact support in z in ensuring boundedness is also evident.

With this framework, the data is presumed to lie in the range space of S[c]:  $d \in L^2[0,T]$ . It is simple to verify from the identity 6 that S[c] is also surjective. That is, any data at all can be fit by S[c] with an appropriate choice of input. This observation is important in the development of WRI, to be explained below.

The version of FWI discussed here presumes that the source field corresponding to the data d is supported at a point  $z_s \in \mathbf{R}$ , and is known. The source position  $z_s$  must satisfy  $z_{\min} \leq z_s \leq z_{\max}$  and  $z_s \neq z_r$ , but is otherwise arbitrary.

This point source field depends on a function of time ("wavelet")  $w \in L^2(\mathbf{R})$ . Formally, the resulting acoustic field satisfies the system A-3 with  $f = w(t)\delta(z - z_s)$ . Inserting this expression in the explicit expression 6, obtain

$$S[c](w\delta(\cdot - z_s)) = \frac{1}{2c}w\left(t - \frac{|z_r - z_s|}{c}\right) = S_p[c]w(t)$$
(7)

Thus for this restricted class of source fields, the FWI objective can be redefined as

$$J_{\text{FWI}}[c, d.w] = \frac{1}{2} ||S_p[c]w - d||^2.$$
 (8)

 $S_p[c]$  is a bounded operator with domain  $L^2(\mathbf{R})$  and range  $L^2([0,T])$ .

To end this section, it is necessary to address an irritating technical point: the point source defined above is not a member of the domain of S[c], as it was defined in display 5, so the left-hand side of equation 7 does not actually make sense. The fix for this incompatibility actually elucidates the relation between S[c] and  $S_p[c]$ . Appendix B describes the construction of a family of bounded injective operators  $E[c]: L^2(\mathbf{R}) \to L^2([z_{\min}, z_{\max}] \times \mathbf{R})$  for which

$$S_p[c] = S[c] \circ E[c] \tag{9}$$

This relation exhibits S[c] as an extension of  $S_p[c]$  as described by Symes (2009). WRI for 1D acoustic transmission is based on S[c], and so is identified as an extended version of FWI.

The construction described in Appendix B requires that  $z_s \in (z_{\min}, z_{\max})$ , as mentioned above.

## GLOBAL ASYMPTOTICS OF 1D TRANSMISSION FWI

The FWI objective is well-known to exhibit non-convexity unless data frequency content is limited to a small range near 0 Hz, how small being determined by other scales and by the extent to which the initial wave velocity differs from the target.

To understand the non-convexity phenomenon and the relation of the various scales, it is advantageous to introduce a family of source wavelets, depending on a parameter  $\lambda$ , having dimensions of time and playing the role of wavelength:

$$w_{\lambda}(t) = \frac{1}{\sqrt{\lambda}} w_1 \left(\frac{t}{\lambda}\right). \tag{10}$$

In the definition 10, the "mother wavelet"  $w_1 \in C_0^{\infty}(0,1)$  has dimensionless argument, and the scaling is chosen so that  $||w_{\lambda}||_{L^2(\mathbf{R})}$  is independent of  $\lambda > 0$ .

Evidently there is no control of c at all in the data if the time interval of the observation, namely [0, T], is so short that no signal arrives within it. Accordingly, add to the other assumptions made so far the requirement that the transit time between source and receiver at the slowest permitted velocity is less than T:

$$\frac{|z_s - z_r|}{c_{\min}} < T \tag{11}$$

Choose a target wave velocity  $c_* \in [c_{\min}, c_{\max}]$ . Introduce a family of consistent data  $d_{\lambda}$ , generated by  $c_*$  and the wavelet family  $w_{\lambda}$ :

$$d_{\lambda} = S_p[c_*]w_{\lambda} \tag{12}$$

and a corresponding family of FWI objectives:

$$J_{\text{FWI}}[c, d_{\lambda}, w_{\lambda}] = \frac{1}{2} \|d_{\lambda} - S_p[c]w_{\lambda}\|^2$$

$$= \int_0^T dt \left| \frac{1}{2c_*} w_\lambda \left( t - \frac{|z_s - z_r|}{c_*} \right) - \frac{1}{2c} w_\lambda \left( t - \frac{|z_s - z_r|}{c} \right) \right|^2 \tag{13}$$

Note that supp  $w_{\lambda} \subset [0, \lambda]$ , so

$$\operatorname{supp} S_p[c]w_{\lambda} \subset \left[\frac{|z_s - z_r|}{c}, \lambda + \frac{|z_s - z_r|}{c}\right] \cap [0, T]$$
(14)

The transit time condition 11 implies that there exists  $\lambda_0 > 0$  so that for  $\lambda < \lambda_0$ ,

$$\lambda + \frac{|z_s - z_r|}{c} < T$$

for all admissible c. That is, for  $\lambda < \lambda_0$ , supp  $S_p[c]w_\lambda \subset (0,T)$  for  $c \in [c_{\min}, c_{\max}]$ , and

$$||S_p[c]w_\lambda||^2 = \frac{1}{4c^2} \int_{-\infty}^{\infty} dt \left| w_\lambda \left( t - \frac{|z_s - z_r|}{c} \right) \right|^2 = \frac{||w_1||^2}{4c^2}.$$
 (15)

Recall that the object of this study is the global behaviour of objective functions for velocity estimation: in this context, that means the behaviour for c far from  $c_*$ . Define

$$L = \frac{2c_{\text{max}}^2}{|z_s - z_r|}. (16)$$

Then if  $|c_* - c| > L\lambda$ ,

$$\left| \frac{|z_s - z_r|}{c} - \frac{|z_s - z_r|}{c_*} \right| = \frac{|c - c_*||z_s - z_r|}{cc_*}$$

$$\geq \frac{|c - c_*||z_s - z_r|}{c_{\max}^2} > L\lambda \frac{|z_s - z_r|}{c_{\max}^2} = 2\lambda.$$

That is, the infima of the supports of  $S_p[c]w_{\lambda}$ ,  $d_{\lambda} = S_p[c_*]w_{\lambda}$  are further apart than the lengths of these supports. In that case, necessarily

$$\operatorname{supp} S_p[c]w_{\lambda} \cap \operatorname{supp} S_p[c_*]w_{\lambda} = \emptyset.$$

Therefore  $S_p[c]w_\lambda$  and  $S_p[c_*]w_\lambda$  are orthogonal in  $L^2[0,T]$ , and

$$J_{\text{FWI}}[c, d_{\lambda}, w_{\lambda}] = \frac{1}{2} \left( \frac{1}{4c^2} + \frac{1}{4c_*^2} \right) \|w_1\|^2.$$
 (17)

Amongst other consequences, one immediately deduces from the expression 17 the non-convexity result:

**Theorem 1.** For L > 0 given by equation 16 and  $\lambda < \lambda_0$ , the minimizer of  $J_{\text{FWI}}[c, d_{\lambda}, w_{\lambda}]$  on the complement of  $[c_* - L\lambda, c_* + L\lambda]$  is  $c = c_{\text{max}}$ .

That is, outside of a neighborhood of width proportional to a wavelength, minimization of  $J_{\text{FWI}}$  yields a local minimizer far from the target velocity  $c_*$  that generates the (noise-free) data.

For this 1D problem, a happy 1D accident occurs: a descent minimization starting at  $c_0 < c_*$  will at least initially proceed in the right direction. With sufficiently small steps, it is possible that an interation might land in the (small) domain of attraction around  $c_*$ . However neither this nor various other accidental advantages stemming from the very special form of this problem should be regarded as of any importance.

## WAVEFIELD RECONSTRUCTION INVERSION

This section will describe Wavefield Reconstruction Inversion (WRI) and develop some of its formal algebraic properties. The notation S[c] will represent a modeling operator based on wave dynamics of some sort, depending on a vector c of material parameters. The conclusions developed here in fact apply to WRI in any such setting. These conclusions will be applied to 1D acoustics in the following section, with c specialized to a scalar (wave velocity).

In the application below, the target source q will be a point source as in the previous section, however for the development of the basic properties of WRI, that is immaterial.

Note that in general  $\alpha$  is a dimensional parameter, having the same dimensions as S.

van Leeuwen and Herrmann (2013) posed this problem slightly differently: instead of the first order acoustics system (equation A-3 for the 1D case), they pose the wave dynamics in terms of the second order wave equation for the pressure wavefield p. From this viewpoint,  $\partial (f-q)/\partial t$  is the residual, that is, the difference between the image  $\partial f/\partial t$  of the 2nd order wave operator on p, and the assumed right-hand side  $\partial q/\partial t$ . So in this form, the second term penalizes the failure of p to solve the wave equation with the assumed source. The formulation presented here is equivalent, and was introduced by Wang et al. (2016).

van Leeuwen and Herrmann (2016) used the variable projection method (Golub and Pereyra, 2003), eliminating the source f in the inner step and updating the bulk modulus (or an equivalent quantity) in the outer step. That is, the problem 2 is equivalent to minimization of

$$J_{\text{WRI}}^{\alpha}[c, d, q] = \min_{f} \frac{1}{2} (\|d - S[c]f\|^2 + \alpha^2 \|f - q\|^2)$$

over c. This is the approach that I shall pursue here.

It will turn out to be convenient to define the residual with the target source q as r[c] = d - S[c]q, and set g = f - q. Then this definition can be rewritten as

$$J_{\text{WRI}}^{\alpha}[c, d, q] = \min_{g} \frac{1}{2} (\|r[c] - S[c]g\|^2 + \alpha^2 \|g\|^2)$$
 (18)

In any penalty method, control of the penalty parameter has a large influence on the speed of convergence. Aghamiry et al. (2019) use an augmented Lagrangian algorithm to minimize the influence of the penalty weight choice. Alternatively, one can use a version of the discrepancy principle to adjust  $\alpha$  dynamically (Fu and Symes, 2017), as the WRI problem has the necessary features described in that paper.

"Most" source fields f are non-radiating, that is, S[c]f = 0, and such sources contribute nothing to the data fit term in the definition of  $J_{\text{WRI}}^{\alpha}$ . If the domain and range of S were finite dimensional (which of course they are, after discretization), then the Fundamental Theorem of Linear Algebra identifies the null space of S[c] (the non-radiating sources) as the orthocomplement of the range of the transpose  $S[c]^T$  (Strang, 1993). In the infinite-dimensional setting of this paper, the Closed Range Theorem (Yosida, 1996) states that the same is true if the range of S[c] is closed. There are various ways to ensure this property, but the simplest is relevant here:

For all admissible models c, S[c] is surjective.

That is, em any data can be fit exactly using the extended model space (the domain of S). This property is a characteristic of of extended modeling methods (Symes, 2008): the ability to fit any data appears to be a essential for such methods to produce objectives without spurious local minima. It holds for the problems for which WRI has been advocated. For the simple model problem considered here, surjectivity follows from the explicit expression for S[c]f, as was noted in the discussion following equation 6.

Assuming that S[c] is surjective for any admissible c, the orthocomplement of the subspace of non-radiating sources is the range of the adjoint operator  $S[c]^T$ . In the definition 18, decompose  $g = S[c]^T e + n$ , in which e is the same type of object as d and S[c]n = 0 (that is, n is a non-radiating source), and note that the decomposition is orthogonal. Then

$$J_{\text{WRI}}^{\alpha}[c,d,q] = \min_{e,n} \frac{1}{2} (\|d - S[c](S[c]^{T}e + q)\|^{2} + \alpha^{2} (\|S[c]^{T}e\|^{2} + \|n\|^{2}))$$

$$= \min_{e} \frac{1}{2} (\|r[c] - S[c]S[c]^{T}e\|^{2} + \alpha^{2}\|S[c]^{T}e\|^{2}). \tag{19}$$

This reformulation has some computational advantages (Wang et al., 2016; Rizzuti et al., 2019), but also leads to a useful analytic transformation of the WRI problem. The minimizer on the RHS of equation 19 is the solution  $e = e_{\alpha}[c]$  of the normal equation

$$((S[c]S[c]^{T})^{2} + \alpha^{2}S[c](S[c]^{T})e = S[c]S[c]^{T}r[c]$$

whence

$$S[c]S[c]^{T}e_{\alpha}[c] = S[c]S[c]^{T}(S[c]S[c]^{T} + \alpha^{2}I)^{-1}r[c]$$

Since the null space of S[c] is orthogonal to the range of  $S[c]^T$  under the surjectivity assumption,  $S[c]S[c]^T$  is injective, whence

$$e_{\alpha}[c] = (S[c]S[c]^T + \alpha^2 I)^{-1}r[c]$$
 (20)

Consequently

$$J_{\text{WRI}}^{\alpha}[c,d,q] = \frac{1}{2} (\|r[c] - S[c]S[c]^{T}e_{\alpha}[c]\|^{2} + \alpha^{2}\|S[c]^{T}e_{\alpha}[c]\|^{2})$$

$$= \frac{1}{2} (\|r[c] - S[c]S[c]^{T}(S[c]S[c]^{T} + \alpha^{2}I)^{-1}r[c]\|^{2}$$

$$+ \langle (S[c]S[c]^{T} + \alpha^{2}I)^{-1}r[c], S[c]S[c]^{T}(S[c]S[c]^{T} + \alpha^{2}I)^{-1}r[c]\rangle)$$

$$= \frac{1}{2} (\|\alpha^{2}(S[c]S[c]^{T} + \alpha^{2}I)^{-1}r[c]\|^{2}$$

$$+ \alpha^{2} \langle (S[c]S[c]^{T} + \alpha^{2}I)^{-1}r[c], -\alpha^{2}(S[c]S[c]^{T} + \alpha^{2}I)^{-1}r[c]\rangle$$

$$+ \alpha^{2} \langle (S[c]S[c]^{T} + \alpha^{2}I)^{-1}r[c], r[c]\rangle)$$

$$= \frac{\alpha^2}{2} \langle (S[c]S[c]^T + \alpha^2 I)^{-1} r[c], r[c] \rangle$$

That is,

$$J_{\text{WRI}}^{\alpha}[c,d,q] = \frac{1}{2} \langle r[c], W_{\alpha}[c]r[c] \rangle \tag{21}$$

with

$$W_{\alpha}[c] = \frac{\alpha^2}{2} (S[c]S[c]^T + \alpha^2 I)^{-1}$$
 (22)

This remarkable identity shows that the WRI objective function is a weighted norm of the data residual r[c].

van Leeuwen (2019) gives a different derivation of an identity equivalent to equations 21, 22.

### GLOBAL ASYMPTOTICS OF 1D TRANSMISSION WRI

The preceding section provides the necessary ingredients for an assessment of the relation between WRI and FWI. While the conclusion reached below applies to many wave propagation settings, the 1D acoustic setting is particularly simple and yet illustrates clearly the nature of this relation.

The first task is to give an explicit expression for the operator  $S[c]S[c]^T$  appearing repeatedly in the expression 22. From the definition 6, it follows immediately that

$$S[c]^{T}e(z,t) = \begin{cases} \frac{1}{2c}e\left(t + \frac{|z_{r} - z|}{c}\right), z_{\min} \le z \le z_{\max};\\ 0, else. \end{cases}$$
 (23)

whence

$$S[c]S[c]^{T}e(t) = \frac{z_{\text{max}} - z_{\text{min}}}{4c^{2}}e(t),$$

that is,

$$S[c]S[c]^{T} = \frac{z_{\text{max}} - z_{\text{min}}}{4c^{2}}I$$
(24)

Thus the weight operator W[c] appearing in 21 takes the form

$$W_{\alpha}[c] = u(c)I,$$

$$u(c) = \frac{\alpha^2}{2} \left( \frac{z_{\text{max}} - z_{\text{min}}}{4c^2} + \alpha^2 \right)^{-1}.$$
 (25)

Next, suppose that  $q = w\delta(z - z_s)$ , that is, the target source is a point source, so that  $S[c]q = S_p[c]w$  in the notation used in the discussion of FWI. Thus 21 can be re-written as

$$J_{\text{WRI}}^{\alpha}[c, d, q] = u[c]J_{\text{FWI}}[c, d, w]$$
(26)

Recall the wavelength-dependent family of problems introduced in the derivation and statement of Theorem 1: target wave velocity  $c_*$ , wavelength parameter  $\lambda$ , parametrized family of wavelets  $w_{\lambda}$  and corresponding data  $d_{\lambda}$ .

Define

$$\beta = \frac{z_{\text{max}} - z_{\text{min}}}{c_*^2} - 4\alpha^2 \tag{27}$$

**Theorem 2.** For L as defined in 16, and  $\lambda < \lambda_0$ , the minimizer of  $J_{\text{WRI}}^{\alpha}[c, d_{\lambda}, w_{\lambda}\delta(\cdot - z_s)]$  on the complement of  $[c_* - L\lambda, c_* + L\lambda]$  is

- $c = c_{\text{max}}$  if  $\beta < 0$ ;
- $c = c_{\min}$  if  $\beta > 0$ ;
- any  $c < c_* L\lambda$  or  $> c_* + L\lambda$  if  $\beta = 0$ .

*Proof.* From 26, 25, and 17, if  $|c - c_*| > L\lambda$ ,

$$J_{\text{WRI}}^{\alpha}[c, d_{\lambda}, w_{\lambda}\delta(\cdot - z_{s})] = \frac{\alpha^{2}}{2} \left( \frac{z_{\text{max}} - z_{\text{min}}}{4c^{2}} + \alpha^{2} \right)^{-1} \frac{1}{2} \left( \frac{1}{4c^{2}} + \frac{1}{4c_{*}^{2}} \right) \|w_{1}\|^{2}$$
$$= \frac{\alpha^{2}}{4} \frac{1 + \frac{c^{2}}{c_{*}^{2}}}{z_{\text{max}} - z_{\text{min}} + 4c^{2}\alpha^{2}}$$

This linear fractional function of  $c^2$  is increasing, decreasing, or constant if  $\beta > 0$ ,  $\beta < 0$  or  $\beta = 0$ , respectively.

In other words,  $J_{\text{WRI}}^{\alpha}$  has local minima far from the target velocity  $c_*$ , in the same way as does  $J_{\text{FWI}}$ . One of the local minima will be the result of a local optimization almost surely, unless the initial estimate of c is "within a wavelength" of the target velocity.

Note that L is independent of  $\alpha$  (definition 16), and for small enough  $\alpha$ ,  $\beta > 0$  (definition 27). Conclude that the region  $\{c \in [c_{\min}, c_{\max}] : |c - c_*| > L\lambda$  is independent of  $\alpha$ , and the minimizer of  $J_{\text{WRI}}[c, d_{\lambda}, w_{\lambda}\delta(\cdot - z_s)]$  in this region (away from  $c_*$ ) is  $c = c_{\min}$  for small enough  $\alpha$ . Therefore taking  $\alpha$  small does not change the multimodal nature of  $J_{\text{WRI}}^{\alpha}$ : there remain multiple far-apart local minima, no matter how small  $\alpha$  may be.

#### DISCUSSION

Theorems 1 and 2 call out the chief conclusions of this work: that at least for the 1D acoustic transmission inverse problem, both  $J_{\text{FWI}}[c, d, w]$  and  $J_{\text{WRI}}^{\alpha}[c, d, q]$  exhibit local minima (in c) far from the global minimum for consistent data, the domain of attraction of the global minimizer can be arbitrarily small, and these properties persist

as  $\alpha \to 0$ . These are striking conclusions, but the 1D acoustic transmission problem is very special and lacks fidelity to field practice. I shall show how these approaches to solving this special problem share properties with a much larger family of inversion methods. The theory developed to explain these properties suggests methods that may not suffer the non-convexity of FWI and WRI, and in fact do not in several cases that I will mention.

First, a consequence of the results proven here:  $J_{\text{FWI}}[c, d, w]$  and  $J_{\text{WRI}}[c, d, q]$  are not smooth as joint functions of model (c) and data (d) vectors. If they were, their derivatives would be bounded uniformly over bounded sets in c, d, but the two main results show that this is not the case. As  $\lambda \to 0$ ,  $d = d_{\lambda}$  varies within a ball  $B \subset L^2([0,T])$  of radius  $||d_1||$  (since the  $||d_{\lambda}||$  is independent of  $\lambda$ ), but the value of either objective changes from a positive value (bounded away from zero independently of  $\lambda$ ) to zero over an interval of c of length  $O(\lambda)$ . Therefore the derivatives with respect to c of both objective functions are not bounded over  $[c_{\min}, c_{\max}] \times B$ .

While lack of smoothness is not in itself the most important property established in the preceding sections, it is a necessary condition for stable and reliable parameter recovery via local optimization. Moreover, necessary conditions for smoothness are known for a much wider class of quadratic form objectives for inverse problems.

These results concern optimization problems of the genral form

Given d, find c to extremize 
$$J[c, d] = \langle G[c]d, A[c]G[c]d \rangle$$
 (28)

In this prescription, d is a data vector, as in the examples above, c is a vector of material parameters to be estimated, G[c] is a c-dependent family of operators having the same nature as the simulation operators S[c] and  $S_p[c]$  figuring in the preceding discussion, and A[c] is an operator-valued function of c, on the range of G[c].

Stolk and Symes (2003) assume that the operator-valided function G[c] is of a class typical of modeling operators for wave equation inverse problems, or their inverses or adjoints. The precise characterization of the so-called microlocally elliptic Fourier Integral Operators is quite technical (Duistermaat, 1996). Roughly speaking, such operators map high-frequency localized wave packets to other such packets with well-defined changes of position and direction of oscillation. If the dependence of such an operator G[c] on c is of sufficiently full rank, in the sense that destination packets can be shifted in any direction by changing c appropriately, along with a couple of other technical assumptions, one can conclude that J as defined in 28 is smooth in c and d jointly if and only if the operator A[c] is a pseudodifferential operator - again, a class of operators whose precise definition is quite technical. However these operators also have a rough characterization: they do not change the location of oscillatory wave packets or alter their direction of oscillation, but only scale such packets by smooth functions, to good approximation.

With a bit of fiddling, the FWI problem 8 for 1D acoustic transmission inversion can be rewritten in the form 28. Note that  $S_p[c]$  is invertible (more specifically, has a right inverse; if [0,T] were extended to  $(-\infty,\infty)$  it would have a left inverse too).

From the defintion 12 of the data family  $d_{\lambda}$ , one sees that  $w_{\lambda} = S_p[c_*]^{-1}d_{\lambda}$ . Therefore

$$J_{\text{FWI}}[c, d_{\lambda}, w_{\lambda}] = \frac{1}{2} \| (I - S_p[c] S_p[c_*]^{-1}) d_{\lambda} \|^2$$

$$= \frac{1}{2}(\|d_{\lambda}\|^{2} + \|S_{p}[c]S_{p}[c_{*}]^{-1}d_{\lambda}\|^{2}) + \langle d_{\lambda}, S_{p}[c]S_{p}[c_{*}]^{-1}d_{\lambda}\rangle$$

For the operator family  $S_p[c]$  defined above, it is easy to see that the second term is smooth in c, and the first is constant. The third can be rewritten as

$$\langle d_{\lambda}, S_p[c]S_p[c_*]^{-1}d_{\lambda}\rangle = \langle S[c]^T d_{\lambda}, (S_p[c]^T S_p[c_*])^{-1}S_p[c]^T d\rangle$$

which has the form 28 with the choices  $G[c] = S_p[c]^T$ ,  $A[c] = (S_p[c]^T S_p[c_*])^{-1}$ . A similar manipulation exhibits  $J_{WRI}$  as the sum of harmless terms and a quadratic form of the form 28.

Given the rough understanding of the results of Stolk and Symes (2003) sketched above, one would conclude that neither  $J_{\text{FWI}}$  nor  $J_{\text{WRI}}$  are likely to be smooth jointly in c and d. Indeed, apart from scale,  $S_p[c]$  is composition with a shift (translation) by  $(z_{\text{max}} - z_{\text{min}})c^{-1}$ , so  $A[c] = (S_p[c]^T S_p[c_*])^{-1}$  is composition with a shift by  $(z_{\text{max}} - z_{\text{min}})(c^{-1} - c_*^{-1})$ . Thus application of A[c] does not leave the position of a wave packet fixed, unless  $c = c_*$  - and indeed A[c] is not a pseudodifferential operator unless  $c = c_*$ . On the other hand, G[c] is a shift operator, the simplest prototype of an elliptic Fourier Integral Operator. Therefore the conclusion, derived directly from Theorem 1, that  $J_{\text{FWI}}$  is not smooth jointly in c and d would also appear to follow from the main result of (Stolk and Symes, 2003). This conclusion can be made precise by proper attention to detail, and the same is true of  $J_{\text{WRI}}$ .

In the context of the 1D acoustic transmission problem as formulated here, the question immediately arises: do quadratic forms 28 exist that are smooth jointly in c and d, and whose global minimizer is the correct velocity  $c = c_*$ ? An affirmative answer is provided for precisely this example problem in (Symes, 2019). The operator  $G[c] = S_p[c]^T$  is precisely the same as appeared in the reformulation of  $J_{\text{FWI}}$ . Ignoring a c-dependent multiplier,

$$A[c]u(t) = tu(t).$$

Applied to the wavelength-dependent family of data  $d_{\lambda}$  and source wavelets  $w_{\lambda}$  used repeatedly throughout this paper, A[c] yields a vanishing result as  $\lambda \to 0$  for the correct velocity  $c = c_*$  and a stably non-zero result otherwise. For these choices, it can be established that

$$\frac{d}{dc}J[c, d_{\lambda}] \begin{cases} > 0 \text{ if } c < c_* + O(\lambda), \\ < 0 \text{ if } c > c_* + O(\lambda) \end{cases}$$

That is, all local minima of  $J[c, d_{\lambda}]$  lie within  $O(\lambda)$  ("a wavelength") of the target velocity  $c_*$ . Not surprisingly the wavelength parameter also regulates the accuracy of the inversion.

The reader is directed to (Huang et al., 2019) for an extensive discussion of other similar source extension methods for various wave inversion problems, and for references to earlier work on this topic.

## CONCLUSION

The tendency of iterative FWI to become trapped in uninformative local minima has been much discussed and still drives a substantial worldwide research program, almost 35 years after the phenomenon was first identified. WRI is amongst the many remedies proposed for this pathology, and numerical experiments have appeared to suggest that it may succeed. The example investigated in this report is simple enough to allow for rigorous mathematical conclusions regarding the behaviour of both FWI and WRI. The complete explanation for the behaviour of FWI is no surprise. As it turns out, the same conclusion may be reached for WRI: in this example at least, iterative minimization of the WRI objective it is no more likely to produce a useful estimate of wave velocity than is FWI, and for the same reason - indeed, the two are very closely linked (equations 21, 22).

While these specific conclusions are of course tied to the extremely simple homogeneous acoustic 1D transmission inverse problem studied here, the relations 21, 22 are straightforward algebraic properties of WRI and appear to link it closely to FWI in any wave propagation setting. As explained in the discussion section, even mere smoothness of a quadratic form objective function in both the model and data parameters may impose restrictions on the operators involved in the construction of the form. These restrictions are generally not met by any version of FWI. An examination of other versions of WRI from this point of view may prove informative.

## APPENDIX A

## 1D RADIATION PROBLEM

Begin with the 1D acoustics point source system.

$$\frac{\partial p}{\partial t} + \rho c^2 \frac{\partial v}{\partial z} = w(t)\delta(z - z_s)$$

$$\rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial z} = 0$$

$$p, v = 0, t \ll 0.$$
(A-1)

Since the right hand side is singular, so is the solution, so it must be a solution in the weak sense. It follows from the weak solution conditions that the pressure is continuous at  $z = z_s$ , whence v must have a discontinuity.

In  $z \neq z_s$ , the right hand side vanishes, so the solution must be locally a combination of plane waves; causality implies that

$$p(z,t) = a\left(t - \frac{|z - z_s|}{c}\right), v(z,t) = \operatorname{sgn}(z - z_s)b\left(t - \frac{|z - z_s|}{c}\right)$$

From the second dynamical equation (Newton's law) it follows that  $b = a/(\rho c)$ . The

singularity on the LHS of the first dynamical equation (constitutive law) is

$$\rho c^2[v]_{z=z_s}\delta(z-z_s) = 2\rho c^2b\delta(z-z_s) = 2ca\delta(z-z_s).$$

This must in turn equal the RHS of the constitutive law, whence a = w/(2c). Thus

$$p(z,t) = \frac{1}{2c}w\left(t - \frac{|z - z_s|}{c}\right)$$

$$v(z,t) = \operatorname{sgn}(z - z_s)\frac{1}{2\rho c^2}w\left(t - \frac{|z - z_s|}{c}\right)$$
(A-2)

This result (computation of the Green's function for the acoustic system) permits an explicit expression for the system with a space-time source:

$$\frac{\partial p}{\partial t} + \rho c^2 \frac{\partial v}{\partial z} = f(z, t)$$

$$\rho \frac{\partial v}{\partial t} + \frac{\partial p}{\partial z} = 0$$

$$p, v = 0, t \ll 0.$$
(A-3)

Since

$$f(z,t) = \int dz_1 f(z_1,t) \delta(z-z_1)$$

obtain

$$p(z,t) = \frac{1}{2c} \int dz_1 f\left(z_1, t - \frac{|z - z_1|}{c}\right) \tag{A-4}$$

$$v(z,t) = \frac{1}{2\rho c^2} \int dz_1 \operatorname{sgn}(z - z_1) f\left(z_1, t - \frac{|z - z_1|}{c}\right)$$
 (A-5)

#### APPENDIX B

## EQUIVALENCE OF POINT AND NON-POINT SOURCES

As noted in the text, the point source  $w(t)\delta(z-z_s)$  is not a member of the domain of the simulation operator S[c], as it is not square-integrable. The object of this appendix is to construct a square-integrable right-hand side in the system A-3 for which the pressure field p is the same as that of the weak solution to the point source problem A-1 constructed in the last section, near the receiver point  $z=z_r$ , and to exhibit this square-integrable replacement for the point source as the image of the point source wavelet under a bounded extension map, as in equation 9.

One step in this construction involves building a constitutive defect (pressure) source that is equivalent to a force (velocity) source, in the sense of generating the same solution outside of the source support. This construction is presented here in

the context of the 1D acoustic problem, but is a special case of a much more general construction of considerable interest in its own right (Burridge and Knopoff, 1964).

Let  $\epsilon$  be any positive number  $<|z_r-z_s|$ . Denote by  $(p,\mathbf{v})$  the (weak) solution A-2 of the point source problem constructed in the last section. Pick  $\phi \in C_0^{\infty}(\mathbf{R})$  so that  $\phi = 1$  if  $|z - z_s| \le \epsilon/2$  and  $\phi(z) = 0$  if  $|z - z_s| \ge \epsilon$ . Set  $p_0 = p(1 - \phi)$ ,  $v_0 = v(1 - \phi)$ . Then

$$\frac{\partial p_0}{\partial t} + \rho c^2 \frac{\partial v_0}{\partial z} = f_0$$

$$\rho \frac{\partial v_0}{\partial t} + \frac{\partial p_0}{\partial z} = g_0$$
(B-1)

in which

$$f_0(z,t) = -\rho c^2 v(z,t) \frac{\partial}{\partial z} (1 - \phi(z))$$

$$g_0(z,t) = -p(z,t) \frac{\partial}{\partial z} (1 - \phi(z))$$
(B-2)

vanish near  $z = z_s$ . If  $w \in L^2(\mathbf{R})$  and vanishes for large negative t (as it must, for the system A-1 to be compatible), then from expressions A-2 the distributions p, v are locally square-integrable in  $\{z : |z - z_s| \ge \epsilon/2\} \times \mathbf{R}$  and vanish for large negative t, whence the same is true of  $f_0, g_0$ .

Assume for the moment that  $w \in C_0^{\infty}(\mathbf{R})$ , so that p, v are smooth away from  $z = z_s$  and  $p_0, v_0, f_0, g_0$  are smooth. Then  $p_0$  is also the solution of the second-order initial value problem

$$\frac{1}{\rho c^2} \frac{\partial^2 p_0}{\partial t^2} - \frac{1}{\rho} \frac{\partial^2 p_0}{\partial z^2} = F$$

$$p = 0, t \ll 0$$
(B-3)

with the right-hand side F given by

$$F = \frac{1}{\rho c^2} \frac{\partial f_0}{\partial t} - \frac{1}{\rho} \frac{\partial g_0}{\partial z}$$
 (B-4)

Define f by

$$f(z,t) = \rho c^2 \int_{-\infty}^t ds \, F(z,t) = f_0(z,t) - c^2 \int_{-\infty}^t ds \, \frac{\partial g_0}{\partial z}(z,s)$$
 (B-5)

Then setting

$$p_1 = p_0,$$

$$v_1 = \frac{1}{\rho} \int_{-\infty}^t \frac{\partial p_0}{\partial z}$$

it follows from B-3 and B-5 that  $p_1, v_1$  solves A-3 with f as given above. Since  $p_1 = p_0$ , and  $p_0 = p$  in a neighborhood of  $z = z_r$ , it follows that

$$S[c]f = S_p[w], (B-6)$$

that is, that using RHS f in A-3 produces the same pressure field near  $z = z_r$  as does the point source in A-1. Also

$$\begin{split} f(z,t) &= -\rho c^2 v(z,t) \frac{\partial}{\partial z} (1-\phi(z)) - c^2 \int_{-\infty}^t ds \, \frac{\partial}{\partial z} \left( -p(z,s) \frac{\partial}{\partial z} (1-\phi(z)) \right) \\ &= -c^2 \left( \int_{-\infty}^t ds \, \left( \rho \frac{\partial v}{\partial t} - \frac{\partial p}{\partial z} \right) (z,s) \frac{\partial}{\partial z} (1-\phi(z)) - p(z,s) \frac{\partial^2}{\partial z^2} (1-\phi(z)) \right) \\ &= -2\rho c^2 v(z,t) \frac{\partial}{\partial z} (1-\phi(z)) + c^2 \frac{\partial^2}{\partial z^2} (1-\phi(z)) \int_{-\infty}^t ds \, p(z,s) \end{split}$$

using the second equation (momentum balance) in the system A-1. Use A-2 to replace p, v by explicit expressons in w:

$$=-\mathrm{sgn}(z-z_s)w\left(t-\frac{|z-z_s|}{c}\right)\frac{\partial}{\partial z}(1-\phi(z))+\frac{\partial^2}{\partial z^2}(1-\phi(z))\frac{c}{2}\left(\int_{-\infty}^t w\right)\left(t-\frac{|z-z_s|}{c}\right)$$

$$= E[c]w(z,t) \tag{B-7}$$

whence the image of w under E[c] is square-integrable, E[c] extends to a bounded operator  $L^2(\mathbf{R}) \to L^2([z_{\min}, z_{\max}] \times \mathbf{R})$ , and from equation B-6

$$S[c] \circ E[c] = S_p[c] \tag{B-8}$$

as asserted in equation 9.

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