Example: The Limit of Finite Approximations to an Area

Find the limiting value of lower sum approximations to the area of the region below the graph of y = 1 - x² and above the interval [0,1] on the x-axis using equal width rectangles whose widths approach zero and whose number approaches infinity.

We compute a lower sum approximation using n rectangles of equal width ∆x = (1 - 0)/n, and then we see what happens as n → ∞. We start by subdividing [0,1] into n equal width subintervals

Each subinterval has width 1/n, The function y = 1 - x² is decreasing on [0,1], and its smallest value in a subinterval occurs at the subinterval’s right endpoint. So a lower sum is constructed with rectangles whose height over the subinterval [(k-1)/n,k/n] is f(k/n) = 1-(k/n) ², giving the sum

We write this in sigma notation and simplify,

We have obtained an expression for the lower sum that holds for any n. Taking the limit of this expression as n → ∞, we see that the lower sums converge as the number of subintervals increases and the subinterval widths approach zero:

Without having to calculate limits of Riemann sums. Instead we find and evaluate an antiderivative at the upper and lower limits of integration(i.e. The Fundamental Theorem).