1) $E[a+bx] = \int_x (a+bx) f(x)\,dx =$

$= \int_x [a f(x) + b x f(x)]\,dx$

$= \int a f(x)\,dx + \int bx f(x)\,dx$

$= a \underbrace{\int f(x)\,dx}_{1} + b \underbrace{\int x f(x)\,dx}_{E[x]} = a + b E[x]$

$Var[x] = E[(x-\mu_x)^2]$

$Var[a+bx] = E[[(a+bx) - \underbrace{E[a+bx]}]^2] =$

$= E[[a+bx-(a+b\mu_x)]^2] = E[[b(x-\mu_x)]^2] =$

$= E[b^2(x-\mu_x)^2] = b^2 \underbrace{E[(x-\mu_x)^2]}_{var\,x} = b^2\,Var[x]$

$Var[x] = E[(x-\mu_x)^2] =$

$= E[x^2 - 2\mu_x x + \mu_x^2]$

$= E[x^2] - 2\mu_x \underbrace{E[x]}_{\mu_x} + \mu_x^2 = E[x^2] - 2\mu_x^2 + \mu_x^2 =$

$= E[x^2] - \mu_x^2$

$\equiv E[x^2] - E[x]^2$

$Cov[x,y] = E[(x-\mu_x)(y-\mu_y)]$

$= E[xy - x\mu_y - y\mu_x + \mu_x\mu_y]$

$= E[xy] - \mu_x\mu_y \cancel{-\mu_x\mu_y} + \cancel{\mu_x\mu_y} =$

$= E[xy] - \mu_x\mu_y$

$X \sim N(\mu_x, \sigma_x^2)$

$\boxed{z = \frac{x-\mu_x}{\sigma_x}}$ $E[z] = ?$ $Var[z] = ?$

$z = \frac{X}{\sigma_x} - \frac{\mu_x}{\sigma_x} = -\frac{\mu_x}{\sigma_x} + \frac{1}{\sigma_x} X$

$a + bx$

$$E[a+bx] = a + b\,E[x]$$
$$= -\frac{\mu_x}{\sigma_x} + \frac{1}{\sigma_x} E[x] = -\frac{\mu_x}{\sigma_x} + \frac{\mu_x}{\sigma_x} = 0$$

$$Var[z] = Var[a+bx] = b^2\,Var[x]$$
$$= \frac{1}{\sigma_x^2}\,\sigma_x^2 = \underline{1}$$

$b \equiv \hat{\beta} \quad e \equiv \hat{\varepsilon} \equiv u$

2) Population model

$$\underset{n\times1}{Y} = \underset{n\times k}{X}\,\underset{k\times1}{\beta} + \underset{n\times1}{\varepsilon}$$

Sample model

$$y = xb \boxed{+e}$$

RSS	VCM
scalar $\rightarrow e^2$	ee'
matrix $\rightarrow e'e$	$n\times1\times1\times n$
$1\times n \times n\times1$	$n\times n$
1×1	

OLS Problem:

$$\min_b \; RSS = e'e$$
$$= (y-xb)'(y-xb)$$
$$= y'y - (xb)'y - y'xb + (xb)'(xb)$$
$$= y'y \underbrace{- (xb)'y - (xb)'y}\, + b'x'xb =$$
$$= y'y - 2(xb)'y + b'x'xb =$$
$$= \underbrace{y'y} \underbrace{-2b'x'y} + \underbrace{b'x'xb} \longrightarrow$$

$(\text{scalar})' = \text{scalar}$

$$(AB)' = B'A'$$

Quadratic form $\frac{\partial x^2}{\partial x} = 2x$

$$\frac{\partial x'M}{\partial x} = M$$

$$\frac{\partial RSS}{\partial b} = 0 - 2x'y + 2x'xb = 0$$

$$x'xb = x'y \qquad \Big|\,(x'x)^{-1} \text{ premultiply}$$

$$(x'x)^{-1}(x'x)b = (x'x)^{-1}x'y$$

$$\boxed{b = (x'x)^{-1}x'y}$$

$$\boxed{\hat{\beta} = \frac{\sum xy}{\sum x^2}}$$

Univariate

Unbiased? $E[b] \overset{?}{=} \beta \Rightarrow$ need to link b and β

$$b = (X'X)^{-1}X'Y = (X'X)^{-1}X'(X\beta + \varepsilon) =$$
$$= \underbrace{(X'X)^{-1}X'X}_{I}\beta + (X'X)^{-1}X'\varepsilon$$
$$= \beta + (X'X)^{-1}X'\varepsilon$$

$$E[b|X] = \beta + E[\underbrace{(X'X)^{-1}X'}\varepsilon|X] =$$
$$= \beta + \underbrace{(X'X)^{-1}X'E[\varepsilon|X]}_{\emptyset} = \beta$$

assumption A3 (Greene T2.1 or T4.1)

$$E[b] = E_X\{E[b|X]\} = E_X[\beta] = \beta \Rightarrow \text{unbiased} \checkmark$$

$$\mathrm{Var}[b|X] = E[(b - \underbrace{E[b]}_{\beta})(b - E[b])'|X]$$
$$= E[(b-\beta)(b-\beta)'|X]$$
$$= E[(X'X)^{-1}X'\varepsilon\varepsilon'X(X'X)^{-1}|X]$$
$$= (X'X)^{-1}X'\underbrace{E[\varepsilon\varepsilon'|X]}X(X'X)^{-1}$$
$$= (X'X)^{-1}X'\sigma_\varepsilon^2 I X(X'X)^{-1} =$$
$$= \sigma_\varepsilon^2 \underbrace{(X'X)^{-1}X'X}(X'X)^{-1} =$$
$$\boxed{= \sigma_\varepsilon^2 (X'X)^{-1}}$$

$ee' \equiv$ VCM

$$\boxed{(AB)' = B'A'}$$

$\sigma^2 \to s^2 = RSS = e'e$

Unbiased?

$$E[e'e|X] \overset{?}{=} \sigma_\varepsilon^2$$

Need to link e and $\varepsilon \Rightarrow \begin{cases} \text{projection mtrx} \\ \text{res. maker mtrx} \end{cases}$

$$\hat{Y} = Xb = \underbrace{X(X'X)^{-1}X'}_{P}Y$$

$$M = I - P \qquad\qquad X - PX = X - X = 0$$

$$e = MY = M(\underbrace{X\beta}_{0} + \varepsilon) = M\varepsilon$$

$P' = P = P^2$

$M' = M = M^2$

symetric & idempotent

$$E[e'e|X] = E[\varepsilon' M' M \varepsilon | X] =$$

$$= E[\underbrace{\varepsilon' M \varepsilon}_{\text{trace}(\varepsilon' M \varepsilon)} | X]$$

$$= E[tr(\varepsilon' M \varepsilon) | X] =$$

$$= tr(E[\varepsilon' M \varepsilon | X]) = tr(M \underbrace{E[\varepsilon\varepsilon' | X]}_{\sigma^2})$$

$$= tr\, M \sigma_\varepsilon^2 I$$

$$= \sigma^2 tr(M)$$

$$= \sigma^2 tr(I_n - X(X'X)^{-1}X') =$$

$$= \sigma^2 tr(I_n) - tr(\underbrace{(X'X)^{-1}X'X}_{I_k})$$

$$= \sigma^2 (n - k)$$

$tr(ABC) = tr(BCA)$

$$s^2 = \frac{e'e}{n-k}$$

$$E[s^2] = E\left[\frac{e'e}{n-k}\right] = \frac{(n-k)\sigma^2}{n-k} = \sigma^2 \Rightarrow \text{unbiased now}$$

$$\Rightarrow \text{Est. } Var[b|X] = s^2 (X'X)^{-1} \Rightarrow$$ standard error of b_k is square root of kth diagonal element

$$\Rightarrow S.E[b_k] = \left\{ [s^2 (X'X)^{-1}]_{kk} \right\}^{1/2}$$