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1. Introduction



2. Neuron Dynamics

Spiking neural network's neuron model describe how the membrane potential of a neuron change over the time. In this chapter, we dive into the biological neuron's dynamics and exhibit several examples neuron models in the spiking neural network.

2.1 Biological Neuron

2.2 Neuron Model Abstract and Taxonomy

In spiking neural network, Equation 2.2.1 to Equation 2.2.4 provide a highly abstract description for modeling a biological neuron. In these equation, \mathbf{x}_t is the state vector at time t , $\Delta\mathbf{x}_t$ is the state variation at time t , \mathbf{u}_t is the input vector at time t , V is the membrane potential of the neuron, \mathbf{y}_t is the neuron's output at time t , and V_{out} is the output voltage that will be sent to the synapses that departure from this neuron. $(\mathbf{x}_t)_i$ is the i -th element of the state vector.

$$\Delta\mathbf{x}_t = f(\mathbf{x}_t, \mathbf{u}_t) \quad (2.2.1)$$

$$\mathbf{y}_t = g(\mathbf{x}_t, \mathbf{u}_t) \quad (2.2.2)$$

$$s.t., (\exists i \in [0, |\mathbf{x}_t|])(\mathbf{x}_t)_i = V \quad (2.2.3)$$

$$(\exists i \in [0, |\mathbf{y}|])_i \mathbf{y} = V_{out} \quad (2.2.4)$$

These equations are in the form of a **state-space model (SMM)**.

Equation 2.2.3 indicates that, a neuron should maintain a membrane potential V , and Equation 2.2.4 indicates that, the neuron's output should contains a voltage V_{out} .

2.2.1 Current-based Neuron

In a current-based neuron model, the synaptic input is represented as an injected current directly added to the membrane potential equation. we have the input \mathbf{u}_t contains the current I_t , and \mathbf{x}_t contains the membrane potential V_t .

We will later to see that a current-based LIF neuron (Section 2.3.2) can hold a form of Equation 2.2.5.

$$\dot{V}_t = \frac{1}{\tau_m} (-(V_t - V_{rest}) + I_t) = f_{CurrentLIF_1}(V_t, \mathbf{u}_t = [I_t]) \quad (2.2.5)$$

2.2.2 Conductance-based Neuron

In a conductance-based neuron model, the synaptic input is modeled by changing the conductance of the membrane, which then affects the current flow.

In the state space representation, the input \mathbf{u}_t is the synapse conductance. let $\mathbf{u}_t = g_{syn}(t)$, $\mathbf{x}_t = V_t$, we will see that in conductance-based LIF neuron model, it hold Equation 2.2.6

$$\dot{V}_t = \frac{1}{\tau_m} (-(V_t - V_{rest}) + g_{syn}(t)(E_{syn} - V_t)) = f_{ConductanceLIF_1}(V_t, \mathbf{u}_t = g_{syn}(t)) \quad (2.2.6)$$

2.3 Neuron Model Examples

2.3.1 Hodgkin-Huxley (HH) Model

The Hodgkin-Huxley (HH) model is a conductance-based model , which can be utilize to accurately reproduce the bio-neuron's dynamics. Its form is shown in Equation 2.3.1 to Equation 2.3.5. Combine all these equations, we get Equation 2.3.6.

$$I = C_m \frac{dV_m}{dt} + I_i \quad (2.3.1)$$

$$I_i = I_{Na} + I_K + I_l \quad (2.3.2)$$

$$I_{Na} = g_{Na}(V_m - V_{Na}) \quad (2.3.3)$$

$$I_K = g_K(V_m - V_K) \quad (2.3.4)$$

$$I_l = \bar{g}_l(V_m - V_l) \quad (2.3.5)$$

$$I_l = C_m \frac{dV_m}{dt} + g_{Na}(V_m - V_{Na}) + g_K(V_m - V_K) + \bar{g}_l(V_m - V_l) \quad (2.3.6)$$

Ion channel function g . are function respect to time t and membrane potential V . Specifically, Equation 2.3.7 is held.

$$g_{Na} = \bar{g}_{Na} m^3 h \quad g_K = \bar{g}_K n^4 \quad g_l = \bar{g}_l \quad (2.3.7)$$

Combine Equation 2.3.1 to Equation 2.3.7, we get Equation 2.3.8

$$I_l = C_m \frac{dV_m}{dt} + \bar{g}_{Na} m^3 h (V_m - V_{Na}) + \bar{g}_K n^4 (V_m - V_K) + \bar{g}_l (V_m - V_l) \quad (2.3.8)$$

$\frac{d}{dt} = \alpha \cdot (V_m)(1 - \cdot) - \beta \cdot (V_m) \cdot$ is held. Where \cdot is a placeholder for m, n and h . As such, Equation 2.3.9 to Equation 2.3.11 are held.

$$\frac{dn}{dt} = \alpha_n(V_m)(1 - n) - \beta_n(V_m)n \quad (2.3.9)$$

$$\frac{dm}{dt} = \alpha_m(V_m)(1 - m) - \beta_m(V_m)m \quad (2.3.10)$$

$$\frac{dh}{dt} = \alpha_h(V_m)(1 - h) - \beta_h(V_m)h \quad (2.3.11)$$

$$(2.3.12)$$

From experiment, we have Equation 2.3.13 to Equation 2.3.18.

$$\alpha_n(V_m) = \frac{0,01(10 - V)}{\exp(\frac{10-V}{10}) - 1} \quad (2.3.13)$$

$$\alpha_m(V_m) = \frac{0,1(25 - V)}{\exp(\frac{25-V}{10}) - 1} \quad (2.3.14)$$

$$\alpha_h(V_m) = 0,07\exp(-\frac{V}{20}) \quad (2.3.15)$$

$$\beta_n(V_m) = 0,125\exp(-\frac{V}{80}) \quad (2.3.16)$$

$$\beta_m(V_m) = 4\exp(-\frac{V}{18}) \quad (2.3.17)$$

$$\beta_h(V_m) = \frac{1}{\exp(\frac{30-V}{10}) + 1} \quad (2.3.18)$$

Hodgkin-Huxley could be seen as a current-based neuron model, which may represent by space state model, with $\mathbf{x}_t = [V_t, m_t, h_t, n_t]$, and $\mathbf{u}_t = I_t$.

2.3.2 Leaky Integrate-and-fire Model

Leaky Integrate-and-fire model is a computational effective model, in which a threshold is set, when membrane potential cross the threshold, the neuro emit a spike. In implementation, we may set the voltage output at time t , V_{out} be 1 mV. The scale problem has the potential to be solved by automatically adjusting the synapses' weights in training. A LIF neuron i 's output form a **spike train** $S_i(t) = \sum_m \delta(t - t_m)$.

$$\dot{V}_t = \frac{1}{\tau_m} (-(V_t - V_{rest}) + g_{syn}(t)(E_{syn} - V_t)) \quad (2.3.19)$$

2.3.3 Izhikevich Model

2.3.4 FitzHugh-Nagumo Model

2.3.5 Morris-Lecar Model

2.3.6 Hindmarsh-Rose Model

2.3.7 Cable theory

2.3.8 Perfect Integrate-and-fire

2.3.9 Adaptive Integrate-and-fire

2.3.10 Fring Rate Model

2.3.11 Discussion

Spike Representation You may note that, although different neuron models have different dynamics and spike representations, they can still communicate with each other through synapses. Some neuron models require current input, while others do not. Nonetheless, spikes can transmit over synapses and cause current variations across them by utilizing a common output element V_{out} .

One challenge that may arise is the normalization of spike representations. To address this issue, one approach is to standardize the spike events by converting all spike representations into binary form. Another approach can involve adjusting the synaptic weights. Although we may face scaling issues with different types of spike representations, these can be mitigated during training by appropriately adjusting the synapse weights.



3. Synapse Dynamics

3.1 Biological Synapse

3.2 Synapse Abstract and Taxonomy

Similar to neuron dynamics, the synapse dynamics could be modeled a state space model, and the current I is included in the state \mathbf{x}_t . We only need modified the constraint term, as shown in Equation ?? to Equation ??.

The synapse model can be categorized to current-based synapses and conductance-based synapses. Both of them can be seems as a convolution operations on the generalized weighted preneuron spike train. Specifically, we may write $I(t) = (\theta \cdot S(t)) * h(t)$, where $\theta \triangleq \{w_i, g_{max}\}$, $h(t)$ is a convolution kernel, and $S(t)$ is the pre-synapse neuron's spike train.

3.2.1 Current-based Synapse

3.2.2 Conductance-based Synapse

3.2.3 Chemical Synapse

Current-based Synapse

Conductance-based Synapse

3.3 Discussion



4. Training Algorithms

4.1 Unsupervised Learning

- 4.1.1 Spike-timing-dependent plasticity (STDP)
- 4.1.2 Growing Spiking Neural Networks
- 4.1.3 Artola, Bröcher, Singer (ABS) rule
- 4.1.4 Bienenstock, Cooper, Munro (BCM) rule
- 4.1.5 Relationship between BCM and STDP rules

4.2 Supervised Learning

4.2.1 STDP-based Methods

Supervised STDP (SSTDP)

Spike-Timing-Dependent Plasticity (STDP) with Supervision

4.2.2 Spike-Timing Dependent Backpropagation (STDBP)

4.2.3 Liquid State Machine (LSM) and Readout Training

4.2.4 SpikeProp

Extension (McKennoch et al., 2006; Booiij and tat Nguyen, 2005; Shrestha and Song, 2015; de Montigny and Mâsse, 2016; Banerjee, 2016; Shrestha and Song, 2017).

spike timing based methods is that they cannot learn starting from a quiescent state of no spiking. Bohte (2011)

Huh and Sejnowski (2017)

4.2.5 ReSuMe

Related Work (Sporea and Grüning, 2013) Pfister et al. (2006) Gardner et al. (2015) Fremaux et al. (2010)

4.2.6 SuperSpike

SuperSpike [**super-spike**] is a supervised learning algorithm dedicated to deterministic Leaky Integrate-and-Fire neuron model. While the backpropagation algorithm used in traditional neural

network cannot directly be used in the training of spiking neural network, the author provide a surrogate gradient-based method to tackling with the problems facing in solving the $S_i/\partial w_{ij}$, where S_i is the i -th neuron's spike train, and w_{ij} is the connection weight from neuron j to neuron i .

Specifically, $S_i(t) = \sum_k \delta(t - t_k)$, where t_k is the k -th spike emission time, and $\delta(\cdot)$ is the dirac delta function.

Approaches approximate the partial derivative of the hidden unit output by $f(S_{pre}, g(V_{post}))$.

Let \hat{S}_i be the target spike train of neuron i . The cost model for optimization that make \hat{S}_i approach the real S_i hold the form: $L = \frac{1}{2} \int_{-\infty}^t ds [(\alpha * \hat{S}_i - \alpha * S_i)(s)]^2$.

α is a normalized smooth temporal convolution kernel. The original SuperSPike use *double exponential causal kernel*.

$$\partial L / \partial w_{ij} = - \int_{-\infty}^t ds [(\alpha * \hat{S}_i - \alpha * S_i)(s)] (\alpha * \frac{\partial S_i}{\partial w_{ij}})(s)$$

Some existing methods for tackling the term $\frac{\partial S_i}{\partial w_{ij}}$: (1) making derivation directly to the membrane voltage, (2) introducing noisy which render the likelihood of $\langle S_i \rangle$ a smooth function of the membrane potential.

The superspike convert calculation of $\frac{\partial S_i}{\partial w_{ij}} \rightarrow \sigma'(U_i) \frac{\partial U_i}{\partial w_{ij}}$. In which U_i is the membrane voltage. Original superspike choose $\sigma(U)$ be the negative side of a fast sigmoid. This function is objective to increase steeply and peak at the spiking threshold. Other monotonic functions may also work.

For current-based LIF models the membrane potential $U_i(t)$ can be written in integral form as a spike response model (SRM0 (Gerstner et al., 2014)): $U_i(t) = \sum_j w_{ij} (\varepsilon * S_j(t)) + (\eta * S_i(t))$

In which, ε corresponds to the postsynaptic potential (PSP) shape, η captures spike dynamics and reset.

The existence of term $(\eta * S_i(t))$ make us difficult to perform derivation. Now, $U_i(t) \approx (\varepsilon * S_j(t))$.

The gradient calculation for a weight become:

$$\frac{\partial w_{ij}}{\partial t} = r \int_{-\infty}^t ds e_i(s) \alpha * (\sigma'(U_i(s)) (\varepsilon * S_j)(s))$$

r is the learning rate, $e_i(s) \equiv \alpha * (\hat{S}_i - S_i)$, $\lambda_{ij} = \alpha * (\sigma'(U_i(s)) (\varepsilon * S_j)(s))$ is the eligibility trace. The form above is also known as *non-vanishing surrogate gradient*.

The neuron model utilized by the SuperSPike is

$$\tau^{mem} \frac{dU_i}{dt} = (U^{rest} - U_i) + I_i^{syn}(t)$$

. And synapse evolution model is

$$\frac{d}{dt} I_i^{syn}(t) = -\frac{I_i^{syn}(t)}{\tau^{syn}} + \sum_{j \in pre} w_{ij} S_j(t)$$

- 4.2.7 SPAN (Mohammed et al., 2012)**
- 4.2.8 Remote Supervised Method (ReSuMe)**
- 4.2.9 FreqProp**
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- 4.4 Convert Traditional ANN to SNN**

5. Network Architecture Investigation

5.1 Liquid Neural Network

State Space Models (SSM)

Linear State Space Models

Structured State Space Model (S4) [Inn-s4]

Liquid Time Constant (LTC) Model [ltc]

Linear Liquid Time Constant Model

High-order Polynomial Projection Operator (HiPPO)

Normal Plus Low-Rank (NPLR)

Scaled Legendre Measure (LegS)

HiPPO Matrix

black-box Cauchy Kernel

Coupled Bilinear

Dynamics Causal Model

Liquid structure state space model [Inn-4s] introduce the liquid time constant into the structure state model. Specifically it hold the form in Equation 5.1.1.

$$\frac{d\mathbf{x}(t)}{dt} = -[\mathbf{A} + \mathbf{B} \otimes f(\mathbf{x}(t), \mathbf{u}(t), t, \theta)] \otimes \mathbf{x}(t) + \mathbf{B} \otimes f(\mathbf{x}(t), \mathbf{u}(t), t, \theta) \quad (5.1.1)$$

A linear version could be expressed by the form of Equation 5.1.2.

$$\dot{\mathbf{x}}(t) = [\mathbf{A} + \mathbb{I}_N \mathbf{B} \mathbf{u}(t)] \mathbf{x}(t) + \mathbf{B} \mathbf{u}(t), y(t) = \mathbf{C} \mathbf{x}(t) \quad (5.1.2)$$

The output y_k for input u_0 to u_k can be represented by $y = \bar{K} * u + \bar{K}_{liquid} * u_{correlation}$. In which the $*$ is the convolution operator. This equation divide the calculation of y_k to the calculation of the kernel \bar{K} , and the convolution between the kernel and input, and the convolution between the kernel and input correlation.

- 5.2** Feedforward Neural Network
- 5.3** Recurrent Neural Network
- 5.4** Synfire Chain
- 5.5** Reservoir Computing
- 5.5.1** Liquid State Machine



6. Quantum Neuromorphic Computing

- 6.1 System Dynamics-Based Methods
- 6.2 Neuron Simulation Methods



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The background of the slide features a complex, abstract composition. It includes a 3D bar chart with several grey bars of varying heights. Overlaid on this are various geometric elements: a green grid pattern, a blue and white grid, and a yellow square with a black border. A black arc is drawn across the scene, and a thin black line extends from the right side. The overall aesthetic is technical and mathematical.

7. Dynamical System Basis

7.1 Dynamical System Structures

7.1.1 Topology and Geometry

7.1.2 Flow and Manifold



8. Classic Dynamical System Analysis



9. Quantum Dynamical System Analysis



10. Symbolic Dynamics and Spiking Neural Network



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