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Homework Assignment 7

数值计算方法，2021 春

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求下列矩阵的 QR 分解

$$\begin{bmatrix} 4 & 2 & 3 & 0 \\ -2 & 3 & -1 & 1 \\ 1 & 3 & -4 & 2 \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 3 & -2 \end{bmatrix}$$

解：记 $x_1 = \begin{bmatrix} 4 \\ -2 \\ 1 \\ 1 \\ 3 \end{bmatrix}$，$x_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 0 \\ 1 \end{bmatrix}$，$x_3 = \begin{bmatrix} 3 \\ -1 \\ -4 \\ 1 \\ 3 \end{bmatrix}$，$x_4 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \\ -2 \\ = \end{bmatrix}$，补充向量 $x_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$，$Q = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \end{bmatrix}$，$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}$。

将 x_1、x_2、x_3、x_4、x_5 正交化

$$\begin{cases} y_1 = x_1 \\ y_2 = x_2 - q_1 q_1^T x_2 \\ y_3 = x_3 - q_1 q_1^T x_3 - q_2 q_2^T x_3 \\ y_4 = x_4 - q_1 q_1^T x_4 - q_2 q_2^T x_4 - q_3 q_3^T x_4 \\ y_5 = x_5 - q_1 q_1^T x_5 - q_2 q_2^T x_5 - q_3 q_3^T x_5 - -q_4 q_4^T x_5 \end{cases}$$

因此 $y_1 = x_1 = \begin{bmatrix} 4 \\ -2 \\ 1 \\ 1 \\ 3 \end{bmatrix}$, 故 $r_{11} = ||y_1||_2 = \sqrt{4^2 + (-2)^2 + 1^2 + 1^2 + 3^2} = \sqrt{31}$,

$$q_1 = \frac{y_1}{||y_1||_2} = \frac{1}{\sqrt{31}} \begin{bmatrix} 4 \\ -2 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

.

$y_2 = x_2 - q_1 q_1^T x_2 = x_2 - \frac{8}{\sqrt{31}} q_1 = \frac{1}{31} \begin{bmatrix} 30 \\ 109 \\ 85 \\ -8 \\ 7 \end{bmatrix}$, 故 $r_{12} = \frac{8}{\sqrt{31}}$, $r_{22} = ||y_2||_2 = \frac{\sqrt{20119}}{31}$,

$$q_2 = \frac{y_2}{||y_2||_2} = \frac{1}{\sqrt{20119}} \begin{bmatrix} 30 \\ 109 \\ 85 \\ -8 \\ 7 \end{bmatrix}$$

$y_3 = x_3 - q_1 q_1^T x_3 - q_2 q_2^T x_3 = x_3 - \frac{20}{\sqrt{31}} q_1 + \frac{346}{\sqrt{20119}} q_2 = \frac{1}{20119} \begin{bmatrix} 18817 \\ 43555 \\ -64046 \\ 4371 \\ 23839 \end{bmatrix}$, 故 $r_{13} = \frac{20}{\sqrt{31}}$, $r_{23} = -\frac{346}{\sqrt{20119}}$,$r_{33} = ||y_3||_2 = \frac{\sqrt{6940411192}}{20119}$,

$$q_3 = \frac{y_3}{||y_3||_2} = \frac{1}{\sqrt{6940411192}} \begin{bmatrix} 18817 \\ 43555 \\ -64046 \\ 4371 \\ 23839 \end{bmatrix}$$

$y_4 == x_4 - q_1 q_1^T x_4 - q_2 q_2^T x_4 - q_3 q_3^T x_4 = x_4 + \frac{7}{\sqrt{31}} q_1 - \frac{273}{\sqrt{20119}} q_2 + \frac{136586}{\sqrt{6940411192}} q_3 = \frac{1}{6940411192} \begin{bmatrix} 6013609338 \\ -510177602 \\ -1304757388 \\ -4022794050 \\ -6582413706 \end{bmatrix}$,

故 $r_{14}=-\frac{7}{\sqrt{31}}$, $r_{24}=\frac{273}{\sqrt{20119}}$,$r_{34}=-\frac{136586}{\sqrt{6940411192}}$,$r_{44}=||y_4||_2=\frac{\sqrt{97637212462855908128}}{6940411192}$,

$$q_4=\frac{y_4}{||y_4||_2}=\frac{1}{\sqrt{97637212462855908128}}\begin{bmatrix}6013609338\\-510177602\\-1304757388\\-4022794050\\-6582413706\end{bmatrix}$$

$y_5 = x_5 - q_1q_1^Tx_5 - q_2q_2^Tx_5 - q_3q_3^Tx_5 - -q_4q_4^Tx_5 = x_5 - \frac{3}{\sqrt{31}}q_1 - \frac{30}{\sqrt{20119}}q_2 - \frac{18817}{\sqrt{6940411192}}q_3 - \frac{6013609338}{\sqrt{97637212462855908128}} = \frac{1}{97637212462855908128}\begin{bmatrix}1731463246370915200\\865731623185457600\\-173146324637091520\\11600803750685131840\\-5540683118919241008\end{bmatrix}$，故

$$||y_5||_2=\frac{\sqrt{169055252963920205791897348744815352064}}{97637212462855908128}$$

,

$$q_5=\frac{y_5}{||y_5||_2}=\frac{1}{\sqrt{169055252963920205791897348744815352064}}\begin{bmatrix}1731463246370915200\\865731623185457600\\-173146324637091520\\11600803750685131840\\-5540683118919241008\end{bmatrix}.$$

综上可得

$$Q=\begin{bmatrix}\frac{4}{\sqrt{31}} & \frac{30}{\sqrt{20119}} & \frac{18817}{\sqrt{6940411192}} & \frac{6013609338}{\sqrt{97637212462855908128}} & \frac{1731463246370915200}{\sqrt{169055252963920205791897348744815352064}}\\ \frac{-2}{\sqrt{31}} & \frac{109}{\sqrt{20119}} & \frac{43555}{\sqrt{6940411192}} & \frac{-510177602}{\sqrt{97637212462855908128}} & \frac{865731623185457600}{\sqrt{169055252963920205791897348744815352064}}\\ \frac{1}{\sqrt{31}} & \frac{85}{\sqrt{20119}} & \frac{-64046}{\sqrt{6940411192}} & \frac{-1304757388}{\sqrt{97637212462855908128}} & \frac{-173146324637091520}{\sqrt{169055252963920205791897348744815352064}}\\ \frac{1}{\sqrt{31}} & \frac{-8}{\sqrt{20119}} & \frac{4371}{\sqrt{6940411192}} & \frac{-4022794050}{\sqrt{97637212462855908128}} & \frac{11600803750685131840}{\sqrt{169055252963920205791897348744815352064}}\\ \frac{3}{\sqrt{31}} & \frac{7}{\sqrt{20119}} & \frac{23839}{\sqrt{6940411192}} & \frac{-6582413706}{\sqrt{97637212462855908128}} & \frac{-5540683118919241008}{\sqrt{169055252963920205791897348744815352064}}\end{bmatrix}$$

,

$$R=\begin{bmatrix}\sqrt{31} & \frac{8}{\sqrt{31}} & \frac{20}{\sqrt{31}} & -\frac{7}{\sqrt{31}}\\ 0 & \frac{\sqrt{20119}}{31} & -\frac{346}{\sqrt{20119}} & \frac{273}{\sqrt{20119}}\\ 0 & 0 & \frac{\sqrt{6940411192}}{20119} & -\frac{136586}{\sqrt{6940411192}}\\ 0 & 0 & 0 & \frac{\sqrt{97637212462855908128}}{6940411192}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

验证：

$$QR = \begin{bmatrix} 4 & 2 & 3 & 0 \\ -2 & 3 & -1 & 1 \\ 1 & 3 & -4 & 2 \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 3 & -2 \end{bmatrix} = A$$

将上述运算结果的 Q、R 矩阵和 Matlab 自带函数分解的 Q、R 矩阵作比较，手算的 Q、R 矩阵转化为小数形式如下：

```
Q =

   0.718421208107100   0.211503743913783   0.225869600743011   0.608593802791436   0.133167713302033
  -0.359210604053550   0.768463602886746   0.522811843564961  -0.051631376341360   0.066583856651016
   0.179605302026775   0.599260607755719  -0.768775280288405  -0.132045035826559  -0.013316771330203
   0.179605302026775  -0.056400998377009   0.052467238393352  -0.407117820784564   0.892223679123619
   0.538815906080325   0.049350873579883   0.286151108684309  -0.666158369079114  -0.426136738752114
```

```
R =

   5.567764362830022   1.436842416214199   3.592106040535498  -1.257237114187424
                   0   4.575530993334844  -2.439343179805633   1.924684069615427
                   0                   0   4.140818644263973   1.639508172773820
                   0                   0                   0   1.423713110948314
                   0                   0                   0                   0
```

Matlab 自带函数分解的 Q、R 矩阵如下：

```
A =

     4     2     3     0
    -2     3    -1     1
     1     3    -4     2
     1     0     1    -1
     3     1     3    -2

>> [Q,R]=qr(A)

Q =

  -0.718421208107100  -0.211503743913783   0.225869600743012   0.608593802791436  -0.133167716494799
   0.359210604053550  -0.768463602886746   0.522811843564961  -0.051631376341360  -0.066583858247399
  -0.179605302026775  -0.599260607755719  -0.768775280288405  -0.132045035826559   0.013316771649480
  -0.179605302026775   0.056400998377009   0.052467238393352  -0.407117820784564  -0.892223700515151
  -0.538815906080325  -0.049350873579883   0.286151108684309  -0.666158369079114   0.426136692783356


R =

  -5.567764362830022  -1.436842416214200  -3.592106040535499   1.257237114187425
                   0  -4.575530993334843   2.439343179805634  -1.924684069615428
                   0                   0   4.140818644263972  -1.639508172773820
                   0                   0                   0   1.423713110948314
                   0                   0                   0                   0
```

可以看到，Matlab 自带函数分解的 Q 矩阵在数值上等于 $-q_1$、$-q_2$、q_3、q_4、$-q_5$，相应的，R 矩阵的第 1、2、5 行是手算 R 矩阵对应行的相反数，这也证明了上述计算结果的正确性。