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Homework 11

数值计算方法，2021 春

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周一作业

1、设 $f(x)$ 在各节点处的数据为

i	0	1	2	3	4	5
x_i	0.30	0.40	0.55	0.65	0.80	1.05
y_i	0.30163	0.41075	0.57815	0.69675	0.87335	1.18885

求 $f(x)$ 在 $x = 0.36, 0.42, 0.75, 0.98, 1.1$ 处的近似值。

(用分段线性、二次插值)

创建数据如下：

```python
x = np.array([0.30,0.40,0.55,0.65,0.80,1.05])
y = np.array([0.30163,0.41075,0.57815,0.69675,0.87335,1.18885])
data = np.array([0.36,0.42,0.75,0.98,1.1])
```

分段线性插值

对 $\forall x \in [x_i, x_{i+1}]$,

$$f(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}} f(x_i) + \frac{x - x_i}{x_{i+1} - x_i} f(x_{i+1})$$

编写函数如下：

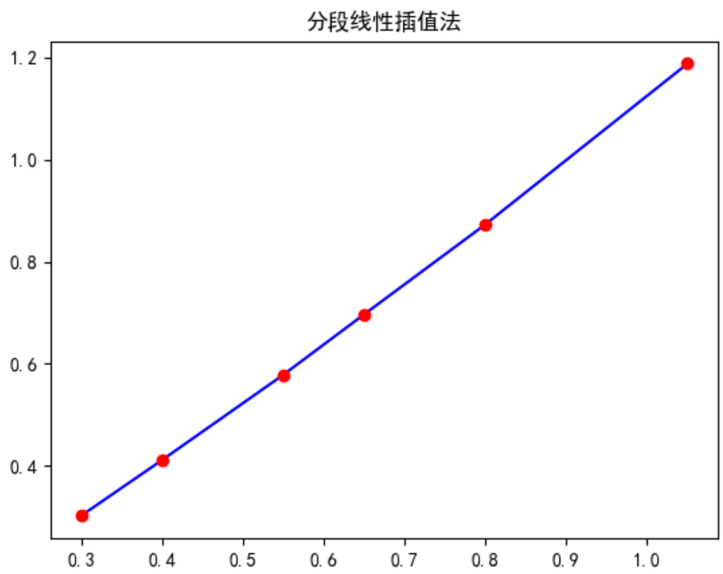
```python
def Linear(x:np.array, y:np.array, n, data):
    # 选取最近的段
    i = 0
    while i < n and data > x[i]:
        i += 1
    a = i - 1
    b = i
    # 处理边缘情况
    if i == 0:
        a += 1
        b += 1
    elif i == n:
        a -= 1
        b -= 1
```

```
    result = ((data - x[b]) * y[a] / (x[a] - x[b])) + ((data - x[a]) * y[b] / (x[b] - x[a]))
    return result

plt.plot(x, y, 'b-', x, y, 'ro') # 原始点为蓝色线条，插值点为红色圆点
plt.title("分段线性插值法")
plt.show()

for i in range(5):
    print("f(%f) = " %(data[i]), Linear(x, y, 6, data[i]))
```

可绘出如下函数图像：



结果如下：

```
------------分段线性插值------------
f(0.360000) =  0.367102
f(0.420000) =  0.43307
f(0.750000) =  0.8144833333333333
f(0.980000) =  1.1005099999999999
f(1.100000) =  1.2519500000000001
```

二次插值

任取三个相邻节点 x_{k-1}, x_k, x_{k+1}，以 $[x_{k-1}, x_{k+1}]$ 为插值区间构造二次 Lagrange 插值多项式：

$$L_2^{(k)}(x) = y_{k-1}\frac{(x-x_k)(x-x_{k+1})}{(x_{k-1}-x_k)(x_{k-1}-x_{k+1})} + y_k\frac{(x-x_{k-1})(x-x_{k+1})}{(x_k-x_{k-1})(x_k-x_{k+1})} + y_{k+1}\frac{(x-x_{k-1})(x-x_k)}{(x_{k+1}-x_{k-1})(x_{k+1}-x_k)}$$

编写函数如下：

```
def Lagrange2(x:np.array, y:np.array, n, data):
    # 选取最近的Xk
```

```
    k = 0
    dst = abs(data - x[k])
    for i in range(1, n):
        if abs(data - x[i]) < dst:
            dst = abs(data - x[i])
            k = i
    a = k - 1
    b = k + 1

    if k == 0:
        a += 1
        b += 1
    elif k == n-1:
        a -= 1
        b -= 1

    result = 0
    for i in range(a, b+1):
        af = 1
        for j in range(a, b+1):
            if j != i:
                af *= ( 1.0 * (data - x[j]) / (x[i] - x[j]) )
        result += y[i] * af
    return result
```

结果如下：

```
------------二次插值------------
f(0.360000) =  0.36686391999999995
f(0.420000) =  0.43281207999999993
f(0.750000) =  0.813425
f(0.980000) =  1.0978429999999997
f(1.100000) =  1.2551249999999996
```

2、证明拉格朗日插值基函数是线性无关的。

证明：

用反证法，若不然，拉格朗日插值基函数是线性相关的，根据线性相关定义可知，对向量组 $\{l_1(x), l_2(x), ..., l_n(x)\}$，存在不全为零的数 $k_1, k_2, \cdots, k_n$，使

$$k_1 l_1(x) + k_2 l_2(x) + ... + k_n l_n(x) = 0$$

设 $k_1, k_2, \cdots, k_n$ 中不为 0 的数有 m 个，分别为为 $k_{i_1}, k_{i_2}, \cdots, k_{i_m}$，$1 \leq i_1 < i_2 < ... < i_m \leq n$，则任取 $x = x_{i_1}, x_{i_2}, \cdots, x_{i_m}$，不妨取 $x = x_{i_1}$，有

$$k_{i_1} = 0$$

与假设相矛盾，因此该假设不成立，拉格朗日插值基函数是线性无关的。

周四作业

例 4. 记 $\{x_i\}_{i=0}^n$ 上的 Lagrange 插值多项式为 $L_n(x)$，若有

$$L_n(x_n) - L_{n-1}(x_n) = A(x_n - x_0)...(x_n - x_{n-1})$$

试证明

$$A = f[x_0, x_1, ..., x_n]$$

证明:

Lagrange 插值多项式为

$$L_n(x) = f(x_0)l_0(x) + f(x_1)l_1(x) + ... + f(x_n)l_n(x)$$

其中

$$l_i(x) = \frac{(x-x_0)...(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_0)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}$$

且有

$$L_{n-1}(x) = f(x_0)l_0(x)\frac{x_0-x_n}{x-x_n} + f(x_1)l_1(x)\frac{x_1-x_n}{x-x_n} + ... + f(x_{n-1})l_{n-1}(x)\frac{x_{n-1}-x_n}{x-x_n}$$

因此

$$\begin{aligned} L_n(x) - L_{n-1}(x) =& f(x_0)l_0(x)(1-\frac{x_0-x_n}{x-x_n}) + f(x_1)l_1(x)(1-\frac{x_1-x_n}{x-x_n}) + ... \\ &+ f(x_{n-1})l_{n-1}(x)(1-\frac{x_{n-1}-x_n}{x-x_n}) + f(x_n)l_n(x) \\ =& f(x_0)l_0(x)\frac{x-x_0}{x-x_n} + f(x_1)l_1(x)\frac{x-x_1}{x-x_n} + ... + f(x_{n-1})l_{n-1}(x)\frac{x-x_{n-1}}{x-x_n} + f(x_n)l_n(x) \end{aligned}$$

故

$$\begin{aligned} \text{左边} =& (x-x_0)(x-x_1)...(x-x_{n-1})[\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)} + ... \\ &+ \frac{f(x_{n-1})}{(x_{n-1}-x_0)(x_0-x_1)...(x_{n-1}-x_n)} + \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)...(x_0-x_{n-1})}]|_{x=x_n} \\ =& (x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})\sum_{i=0}^{n}\frac{f(x_i)}{(x_i-x_0)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)} \end{aligned}$$

由差商的展开式

$$f[x_0, x_1, ..., x_n] = \sum_{i=0}^{n}\frac{f(x_i)}{(x_i-x_0)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}$$

可得

$$\begin{aligned}
\text{左边} =& f[x_0, x_1, ..., x_n](x_n - x_0)(x_n - x_1)...(x_n - x_{n-1}) \\
=& A(x_n - x_0)...(x_n - x_{n-1}) \\
=& \text{右边}
\end{aligned}$$

因此

$$A = f[x_0, x_1, ..., x_n]$$

得证。

在做 Newton 插值时，已知节点组 $\{x_0, x_1, ..., x_n\}$ 和各阶差商

$\{f(x_0), f[x_0, x_1], f[x_0, x_1, x_2], ..., f[x_0, x_1, x_2, ..., x_n]\}$；现增加了节点 $(x_{n+1}, f(x_{n+1}))$，试给出求差商 $f[x_0, x_1, ..., x_n, x_{n+1}]$ 的算法。

解：计算过程如下

$$\begin{aligned}
f[x_0, x_{n+1}] &= \frac{f(x_0) - f(x_{n+1})}{x_0 - x_{n+1}} \\
f[x_0, x_1, x_{n+1}] &= \frac{f[x_0, x_1] - f[x_0, x_{n+1}]}{x_1 - x_{n+1}} \\
f[x_0, x_1, x_2, x_{n+1}] &= \frac{f[x_0, x_1, x_2] - f[x_0, x_1, x_{n+1}]}{x_2 - x_{n+1}} \\
&... \\
f[x_0, x_1, .., x_{n-1}, x_{n+1}] &= \frac{f[x_0, x_1, x_2, ..., x_{n-2}, x_{n-1}] - f[x_0, x_1, ..., x_{n-2}, x_{n+1}]}{x_{n-1} - x_{n+1}} \\
f[x_0, x_1, ..., x_n, x_{n+1}] &= \frac{f[x_0, x_1, x_2, ..., x_{n-1}, x_n] - f[x_0, x_1, .., x_{n-1}, x_{n+1}]}{x_n - x_{n+1}}
\end{aligned}$$