



Numerical ODEs and stochastic models in Python

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About me

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- Junior. Math major.
- Research: Dynamical systems, climate science.
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Contents

- 1. ODE with previous problems, plot in Python
- 2. simple stochastic model: Markov chain model in Python

1. ODE and plot in Python

- 2018 MCM Problem E statement:

...

Task 1: Develop a model that determines a country's fragility and simultaneously measures the impact of climate change ... It should also identify how climate change increases fragility through direct means or indirectly as it influences other factors and indicators.

Task 2: ...

- An example of simple ODE model:

Earth's temperature with energy, where no atmosphere

$$\frac{dx}{dt} = \dots$$

Step 1. Identify x, t.

x: temperature (in K)

t: time

- Step 2: Convert to something we could quantify

$$C \frac{dx}{dt} = \dots$$

convert 'temp. change' to 'energy change'

Step 3: Figure out what makes energy increase/decrease

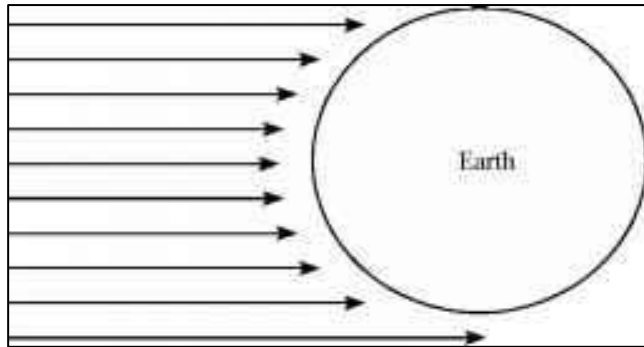
$$C \frac{dx}{dt} = Energy_{in} - Energy_{out}$$

Step 3: Figure out what makes energy increase/decrease

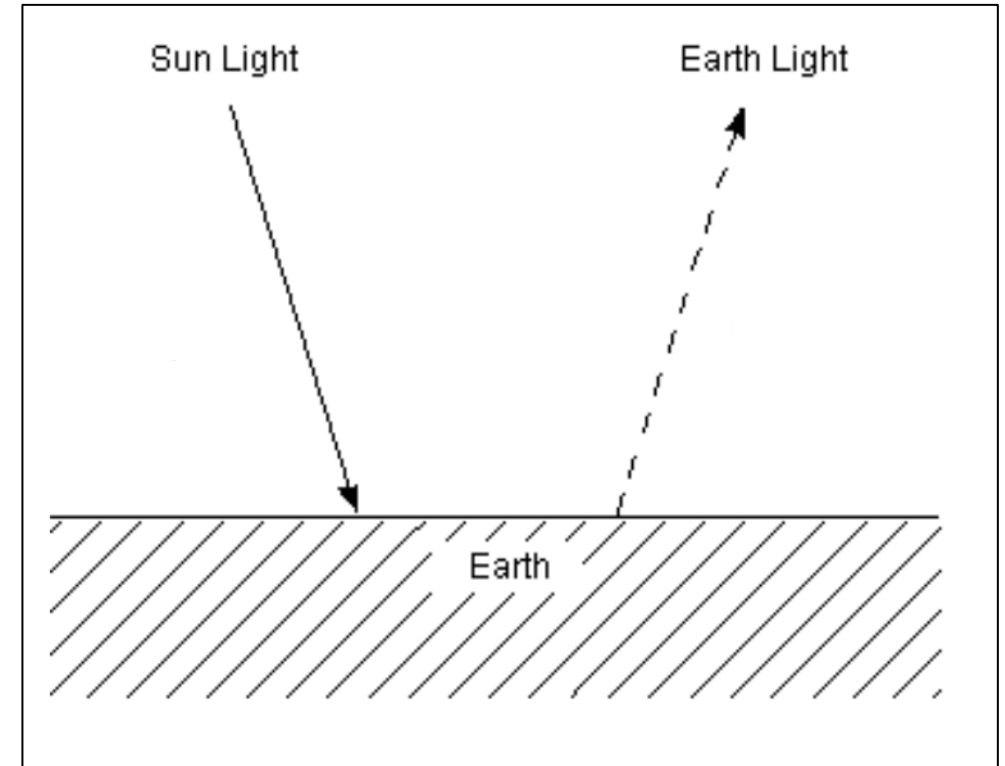
$$C \frac{dx}{dt} = Energy_{in} - Energy_{out}$$

We need to reference physics principles,

$$Energy_{in} = S(1 - \alpha)\pi R^2$$



$$S=1350 \text{ W/m}^2, \\ \alpha=0.3$$



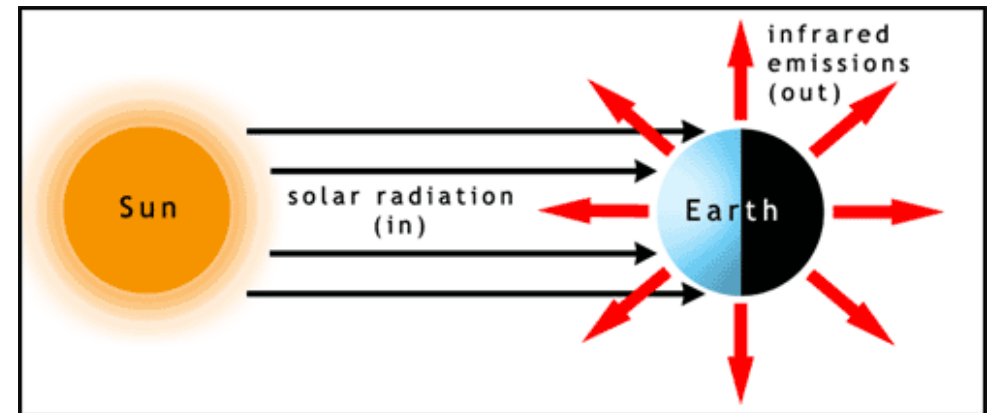
Step 3: Figure out what makes energy increase/decrease

$$C \frac{dx}{dt} = Energy_{in} - Energy_{out}$$

We need to reference physics principles: blackbody radiation

$$Energy_{out} = \sigma x^4 \cdot 4\pi R^2$$

$$\sigma = 5.67 \times 10^{-8}$$



- Step 4: Summary and simplify, plug in parameters

$$C \frac{dx}{dt} = S(1 - \alpha)\pi R^2 - \sigma x^4 \cdot 4\pi R^2$$

$$\frac{dx}{dt} = \frac{S(1-\alpha)}{4} - \sigma x^4$$

$$\frac{dx}{dt} = \frac{1350 \cdot (1-0.7)}{4} - 5.67 \times 10^{-8} \cdot x^4$$

Step 5: Now good enough to solve and plot in Python, given initial condition.

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- Any thoughts of the plot?

Logic: Based on this model, if there is no atmosphere, the Earth will tend to be extremely cold. $\sim 220\text{K}$

Fact: Currently the average temperature of Earth is $\sim 288\text{K}$.

Conclusion: Based on this model, atmosphere helps us stay warm!

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- Reference from PSU

2. Markov chain model

- Problem: Suppose that whether or not it rains today depends on previous weather conditions through the last two days.
- state 0 (S_0): if it rained both today and yesterday,
state 1 (S_1): if it rained today but not yesterday,
state 2 (S_2): if it rained yesterday but not today,
state 3 (S_3): if it did not rain either yesterday or today.

- state 0 (S_0): if it rained both today and yesterday,
state 1 (S_1): if it rained today but not yesterday,
state 2 (S_2): if it rained yesterday but not today,
state 3 (S_3): if it did not rain either yesterday or today.

- And given probabilities for tomorrow's weather conditions,
state 00 (S_{00}): tomorrow will rain if it rained today and yesterday $\rightarrow P_{00} = 0.7$,
state 10 (S_{10}): tomorrow will rain if it rained today but not yesterday $\rightarrow P_{10} = 0.5$,
state 21 (S_{21}): tomorrow will rain if it rained yesterday but not today $\rightarrow P_{21} = 0.4$,
state 31 (S_{31}): tomorrow will rain if it did not rain yesterday or today $\rightarrow P_{31} = 0.2$.

- Can we calculate more probabilities?

- It follows immediately that,

state 02 (S_{02}): tomorrow won't rain if it rained today and yesterday →

$$P_{02} = 1 - 0.7 = 0.3,$$

state 12 (S_{12}): tomorrow won't rain if it rained today but not yesterday →

$$P_{12} = 1 - 0.5 = 0.5,$$

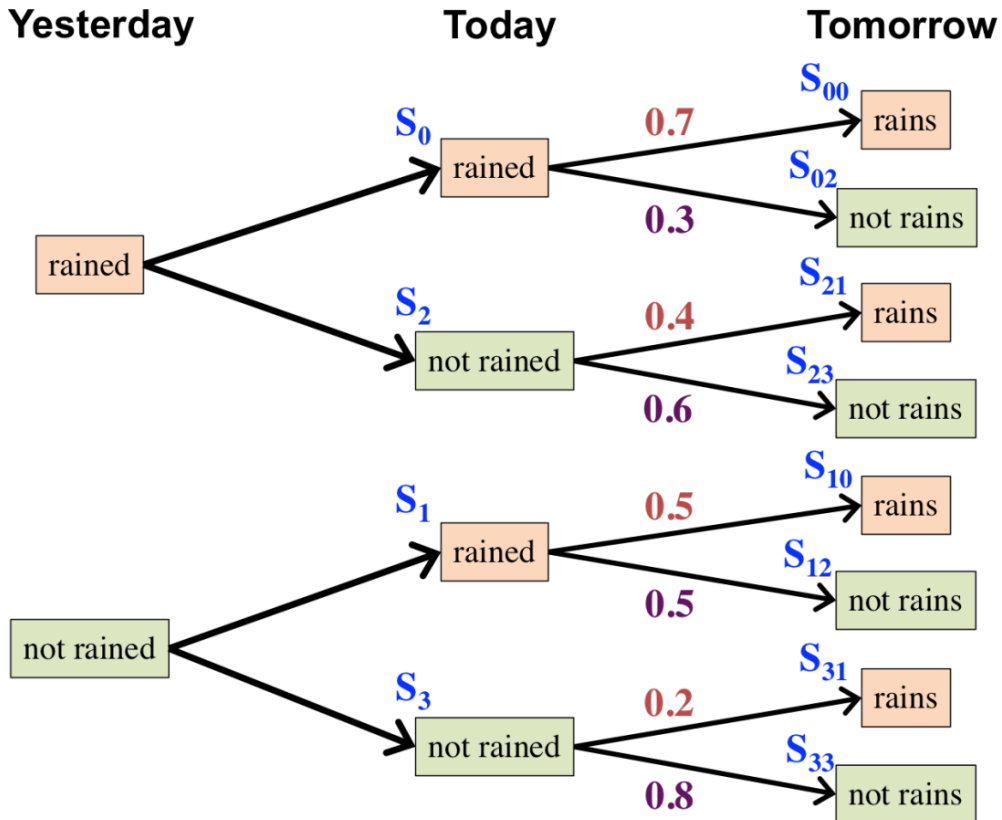
state 23 (S_{23}): tomorrow won't rain if it rained yesterday but not today →

$$P_{23} = 1 - 0.4 = 0.6,$$

state 33 (S_{33}): tomorrow won't rain if it did not rain either yesterday or today →

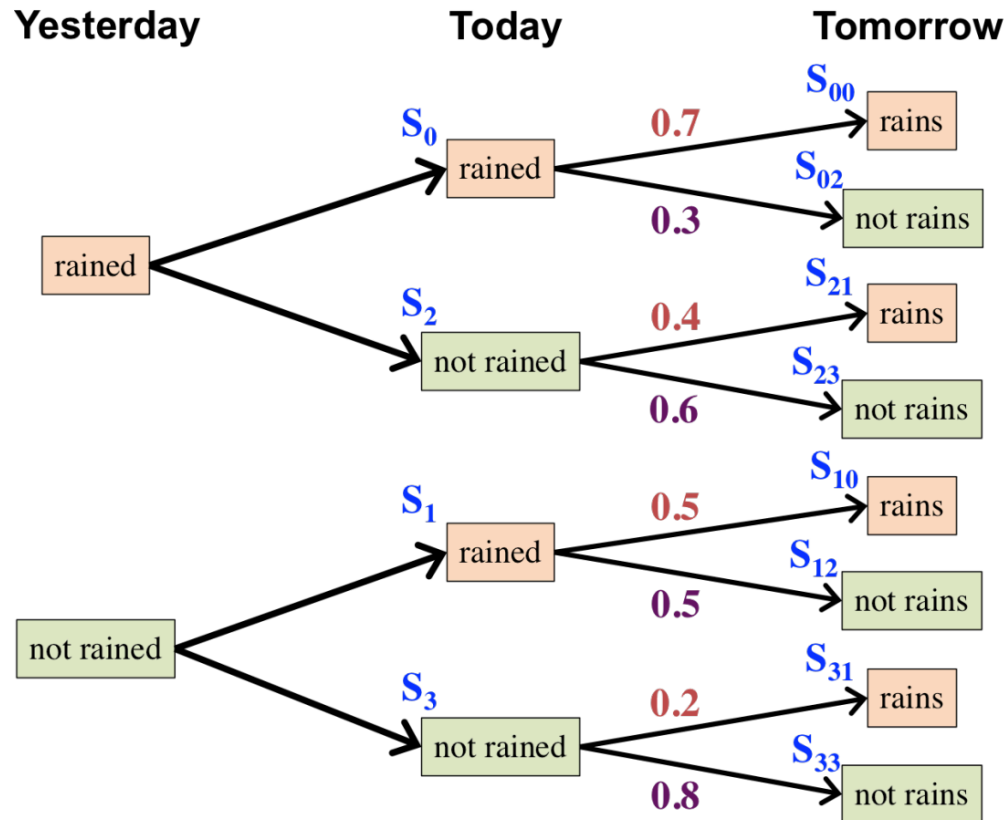
$$P_{33} = 1 - 0.2 = 0.8.$$

● Summary



- Whenever the process is in any state, it will be next in another state, but with a fixed probability in transition.
- Such a stochastic process is an example of **Markov chain**. (not formal definition)

- It's natural to express transition probabilities in a matrix:



$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

- state 0 (S_0): if it rained both today and yesterday,
- state 1 (S_1): if it rained today but not yesterday,
- state 2 (S_2): if it rained yesterday but not today,
- state 3 (S_3): if it did not rain either yesterday or today.

● Transition probability matrix

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

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Properties:

- 1. Entries of the matrix are non-negative
- 2. Sum of any row is 1

● Transition probability matrix

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

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Chapman-Kolmogorov equations

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m, \quad \text{for } n, m \geq 0, \text{ all } i, j$$

In this example, $\mathbf{P}^{(n)} = \mathbf{P}^{(n-1+1)} = \mathbf{P}^{n-1} \cdot \mathbf{P} = \mathbf{P}^n$.

It states that the n -step transition matrix $\mathbf{P}^{(n)}$ is just the n th power of \mathbf{P} .

● Transition probability matrix

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

Chapman-Kolmogorov equations

In this example $\mathbf{P}^{(n)} = \mathbf{P}^{(n-1+1)} = \mathbf{P}^{n-1} \cdot \mathbf{P} = \mathbf{P}^n$.

For instance, transition from (today) to (the day after tomorrow),

$$\begin{aligned} \mathbf{P}^{(2)} = \mathbf{P}^2 &= \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix} \\ &= \begin{bmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{bmatrix} \end{aligned}$$

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- Reference from Vivi Andasari