Numerical ODEs and stochastic models in Python

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About me

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Contents

- 1. ODEs from previous problems, plot in Python
- 2. simple stochastic model: Markov chain model in Python

2. Markov chain model

- Problem: Suppose that whether or not it rains today depends on previous weather conditions through the last two days.
- state 0 (S_0): if it rained both today and yesterday,

  state 1 (S_1): if it rained today but not yesterday,

  state 2 (S_2): if it rained yesterday but not today,

  state 3 (S_3): if it did not rain either yesterday or today.

- state 0 (S_0): if it rained both today and yesterday,

  state 1 (S_1): if it rained today but not yesterday,

  state 2 (S_2): if it rained yesterday but not today,

  state 3 (S_3): if it did not rain either yesterday or today.

- And given probabilities for tomorrow's weather conditions,

state 00 (S_{00}): tomorrow will rain if it rained today and yesterday → $P_{00} = 0.7$,

state 10 (S_{10}): tomorrow will rain if it rained today but not yesterday → $P_{10} = 0.5$,

state 21 (S_{21}): tomorrow will rain if it rained yesterday but not today → $P_{21} = 0.4$,

state 31 (S_{31}): tomorrow will rain if it did not rain yesterday or today → $P_{31} = 0.2$.

- Can we calculate more probabilities?
- It follows immediately that,

state 02 (S_{02}): tomorrow won't rain if it rained today and yesterday →

$P_{02} = 1 - 0.7 = 0.3$,

state 12 (S_{12}): tomorrow won't rain if it rained today but not yesterday →

$P_{12} = 1 - 0.5 = 0.5$,

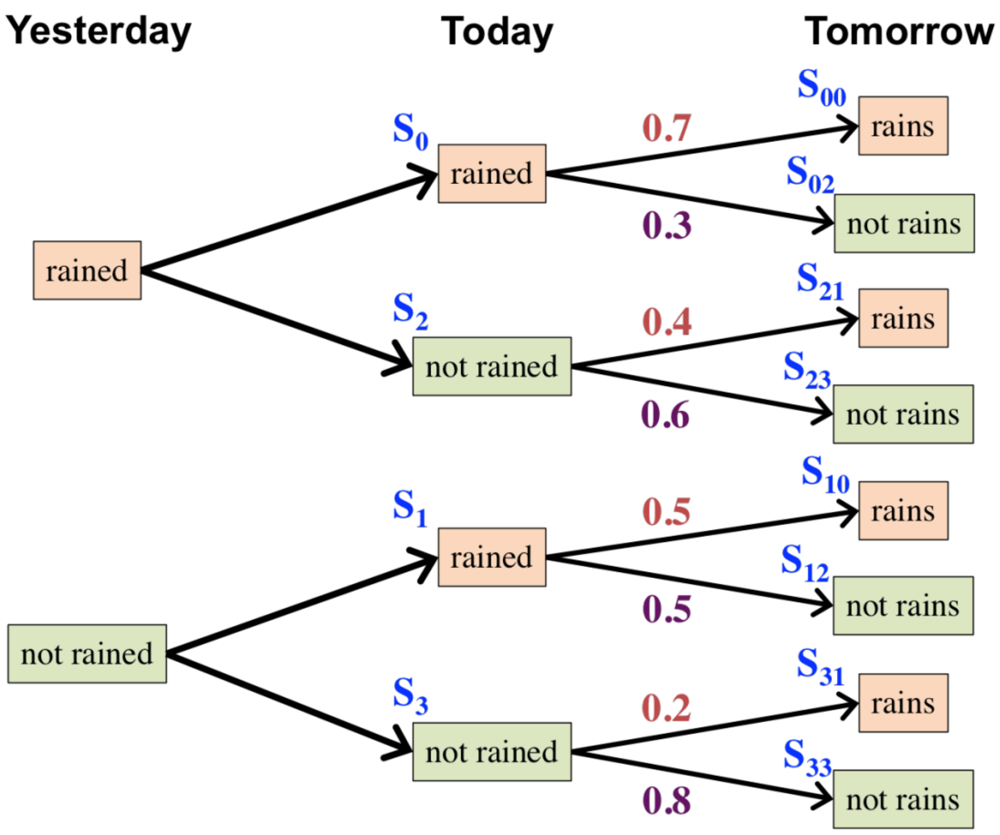
state 23 (S_{23}): tomorrow won't rain if it rained yesterday but not today →

$P_{23} = 1 - 0.4 = 0.6$,

state 33 (S_{33}): tomorrow won't rain if it did not rain either yesterday or today →

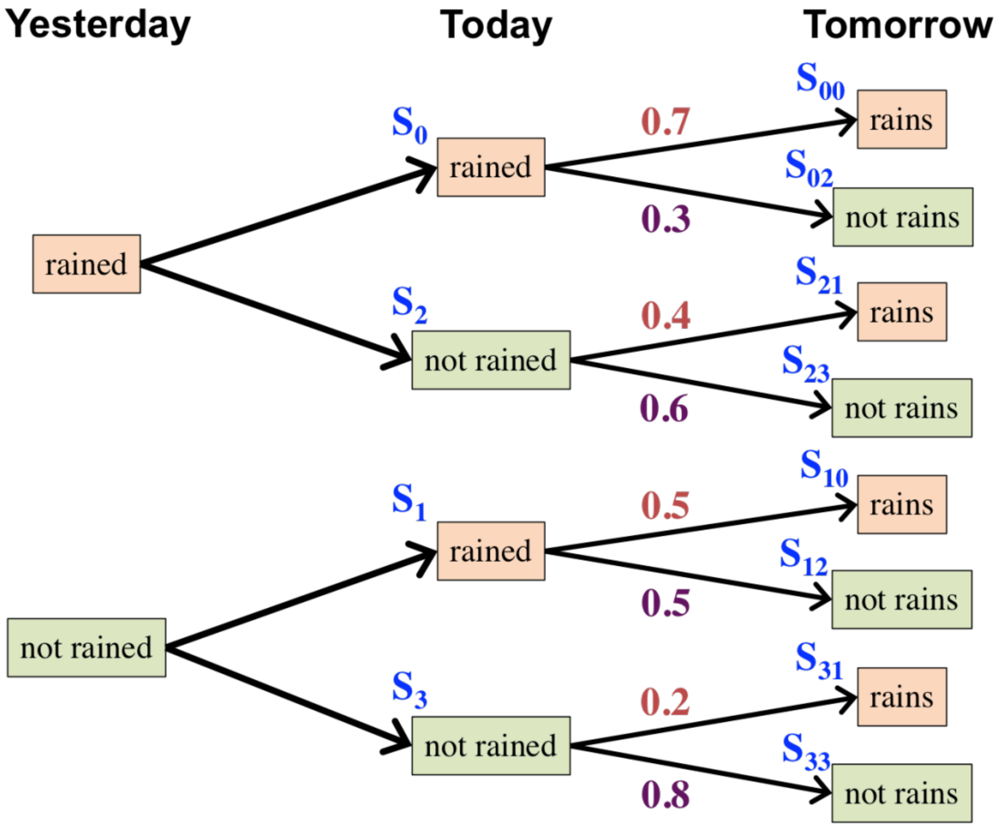
$P_{33} = 1 - 0.2 = 0.8$.

Summary



- Whenever the process in any state, it will be next in another state, but with a fixed probability in transition.
- Such a stochastic process is an example of **Markov chain**. (not formal definition)

- It's natural to express transition probabilities in a matrix:



$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

- state 0 (S_0): if it rained both today and yesterday,

  state 1 (S_1): if it rained today but not yesterday,

  state 2 (S_2): if it rained yesterday but not today,

  state 3 (S_3): if it did not rain either yesterday or today.

Transition probability matrix

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

- state 0 (S_0): if it rained both today and yesterday,

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Properties:

- 1. Entries of the matrix are non-negative
- 2. Sum of any row is 1

Transition probability matrix

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

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Chapman-Kolmogorov equations

$$\boxed{P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m}\,, \qquad \text{for } n, m \geq 0, \text{ all } i, j}$$

In this example, $\mathbf{P}^{(n)} = \mathbf{P}^{(n-1+1)} = \mathbf{P}^{n-1} \cdot \mathbf{P} = \mathbf{P}^{n}\,.$

It states that the n-step transition matrix $\mathbf{P}^{(n)}$ is just the nth power of $\mathbf{P}$.

Transition probability matrix

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

Chapman-Kolmogorov equations

In this example $\boxed{\mathbf{P}^{(n)} = \mathbf{P}^{(n-1+1)} = \mathbf{P}^{n-1} \cdot \mathbf{P} = \mathbf{P}^{n}\,.}$

For instance, transition from (today) to (the day after tomorrow),

$$\begin{aligned} \mathbf{P}^{(2)} = \mathbf{P}^{2} \quad &= \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix} \\ &= \begin{bmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{bmatrix} \end{aligned}$$