Numerical ODEs and stochastic models in Python

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About me

- Name: Wenzhong **Wang**
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- Junior. Math major.
- Research: Dynamical systems, climate science.
- SIAM member. UNC student chapter of SIAM member.
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About MCRN

- Website: mcrn.hubzero.org
- Mathematics and Climate Research Network (MCRN) is a virtual organization of ~200 active researchers in mathematics and the geosciences in the US and beyond, aims to develop the applied mathematics of climate that is tailored to the needs of climate science.
- MCRN will hold its academic year engagement program with a 1-week winter school in academic year 2020-2021 with stipend. Announcement coming soon.
- Stickers.

Contents

- 1. ODE with previous problems, plot in Python
- 2. simple stochastic model: Markov chain model in Python

1. ODE and plot in Python

- 2018 MCM Problem E statement:

…

Task 1: Develop a model that determines a country's fragility and simultaneously measures the impact of climate change … It should also identify how climate change increases fragility through direct means or indirectly as it influences other factors and indicators.

Task 2: …

- An example of simple ODE model:

Earth's temperature with energy, where no atmosphere

$$\boxed{\frac{dx}{dt} = \ldots}$$

Step 1. Identify x, t.

x: temperature (in K)

t: time

- Step 2: Convert to something we could quantify

$$\boxed{C\frac{dx}{dt} = \dots}$$

convert ‘temp. change’ to ‘energy change’

Step 3: Figure out what makes energy increase/decrease

$$\boxed{C\frac{dx}{dt} = Energy_{in} - Energy_{out}}$$

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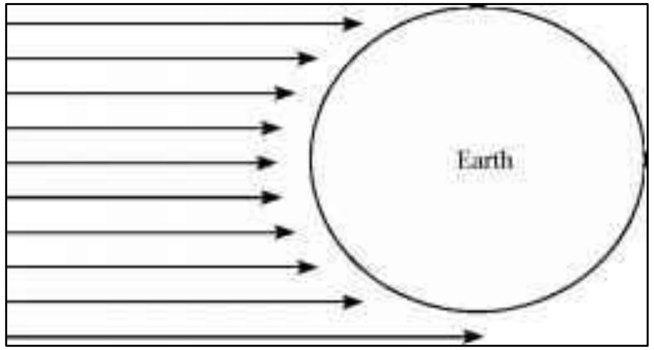
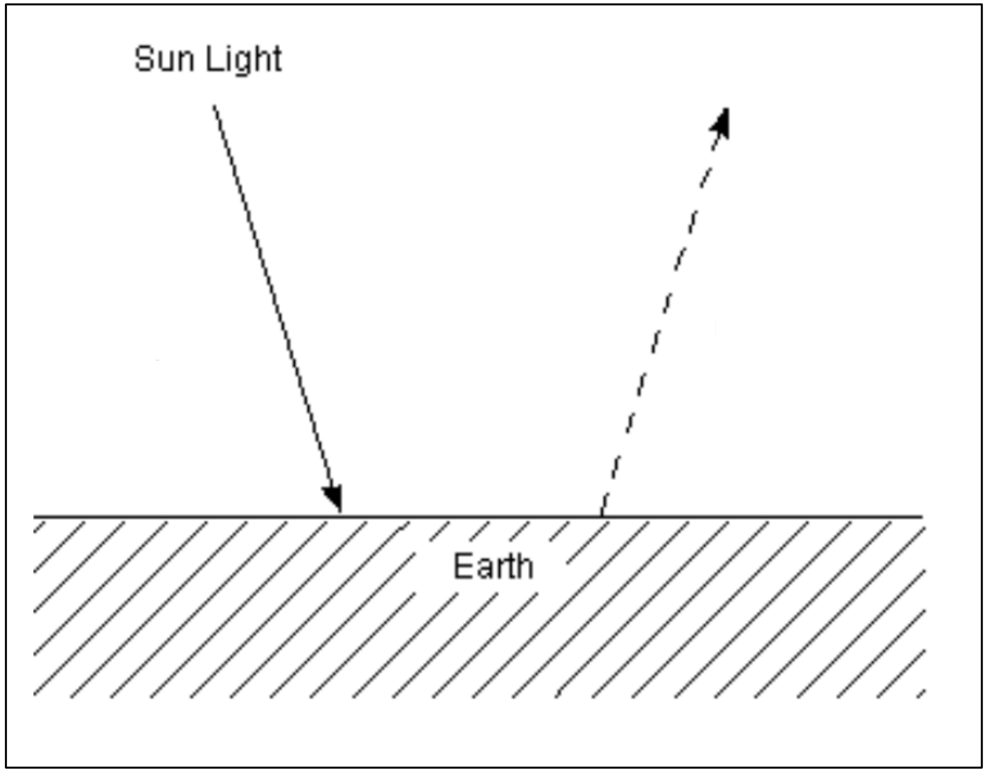
$$C\frac{dx}{dt} = Energy_{in} - Energy_{out}$$

We need to reference physics principles,

$$Energy_{in} = S(1-\alpha)\pi R^2$$



S=1350 W/m^2,
α=0.3



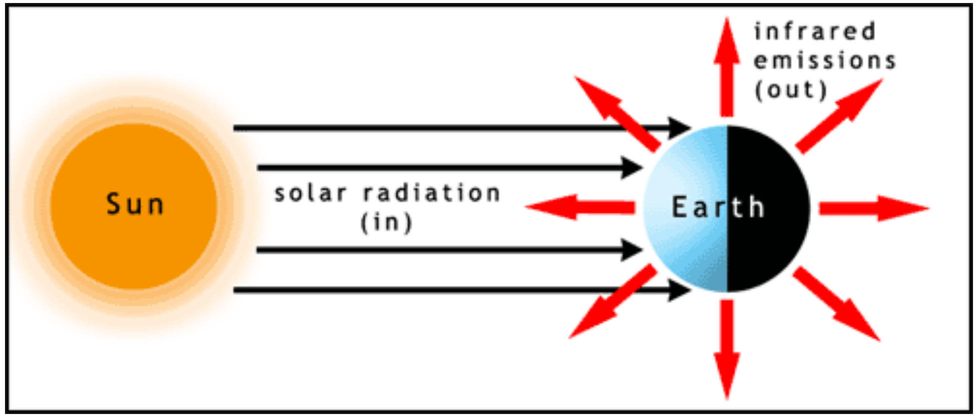
Step 3: Figure out what makes energy increase/decrease

$$C\frac{dx}{dt} = Energy_{in} - Energy_{out}$$

We need to reference physics principles: blackbody radiation

$$Energy_{out} = \sigma x^4 \cdot 4\pi R^2$$

$\sigma = 5.67 \times 10^{-8}$



- Step 4: Summary and simplify, plug in parameters

$$\boxed{C\frac{dx}{dt} = S(1-\alpha)\pi R^2 - \sigma x^4 \cdot 4\pi R^2}$$

$$\boxed{\frac{dx}{dt} = \frac{S(1-\alpha)}{4} - \sigma x^4}$$

$$\boxed{\frac{dx}{dt} = \frac{1350 \cdot (1-0.3)}{4} - 5.67 \times 10^{-8} \cdot x^4}$$

Step 5: Now good enough to solve and plot in Python, given initial condition.

- Any thoughts of the plot?

Logic: Based on this model, if there is no atmosphere, the Earth will tend to be extremely cold. ~254K = ~-19℃ = ~-4°F

Fact: Currently the average temperature of Earth is ~288K = ~15℃.

Conclusion: Based on this model, atmosphere helps us stay warm!

- Reference from PSU

2. Markov chain model

- Problem: Suppose that whether or not it rains today depends on previous weather conditions through the last two days.
- state 0 (S_0): if it rained both today and yesterday,

  state 1 (S_1): if it rained today but not yesterday,

  state 2 (S_2): if it rained yesterday but not today,

  state 3 (S_3): if it did not rain either yesterday or today.

- state 0 (S_0): if it rained both today and yesterday,

  state 1 (S_1): if it rained today but not yesterday,

  state 2 (S_2): if it rained yesterday but not today,

  state 3 (S_3): if it did not rain either yesterday or today.

- And given probabilities for tomorrow's weather conditions,

state 00 (S_{00}): tomorrow will rain if it rained today and yesterday → $P_{00} = 0.7$,

state 10 (S_{10}): tomorrow will rain if it rained today but not yesterday → $P_{10} = 0.5$,

state 21 (S_{21}): tomorrow will rain if it rained yesterday but not today → $P_{21} = 0.4$,

state 31 (S_{31}): tomorrow will rain if it did not rain yesterday or today → $P_{31} = 0.2$.

- Can we calculate more probabilities?
- It follows immediately that,

state 02 (S_{02}): tomorrow won't rain if it rained today and yesterday →

$P_{02} = 1 - 0.7 = 0.3$,

state 12 (S_{12}): tomorrow won't rain if it rained today but not yesterday →

$P_{12} = 1 - 0.5 = 0.5$,

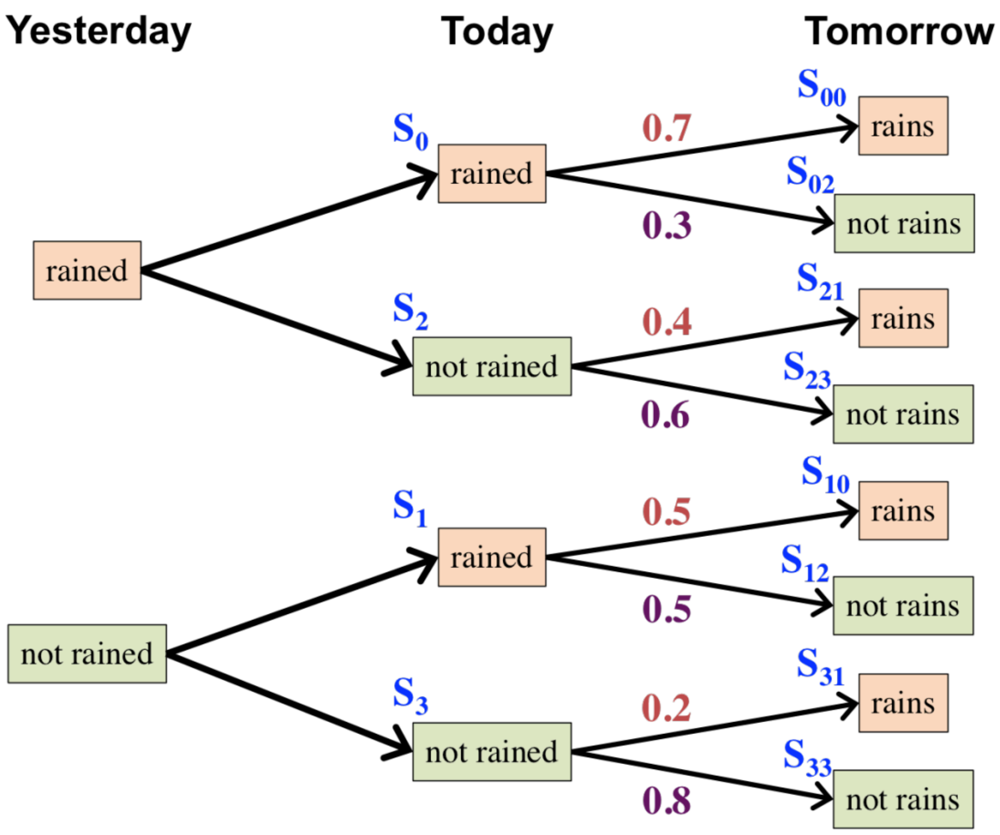
state 23 (S_{23}): tomorrow won't rain if it rained yesterday but not today →

$P_{23} = 1 - 0.4 = 0.6$,

state 33 (S_{33}): tomorrow won't rain if it did not rain either yesterday or today →

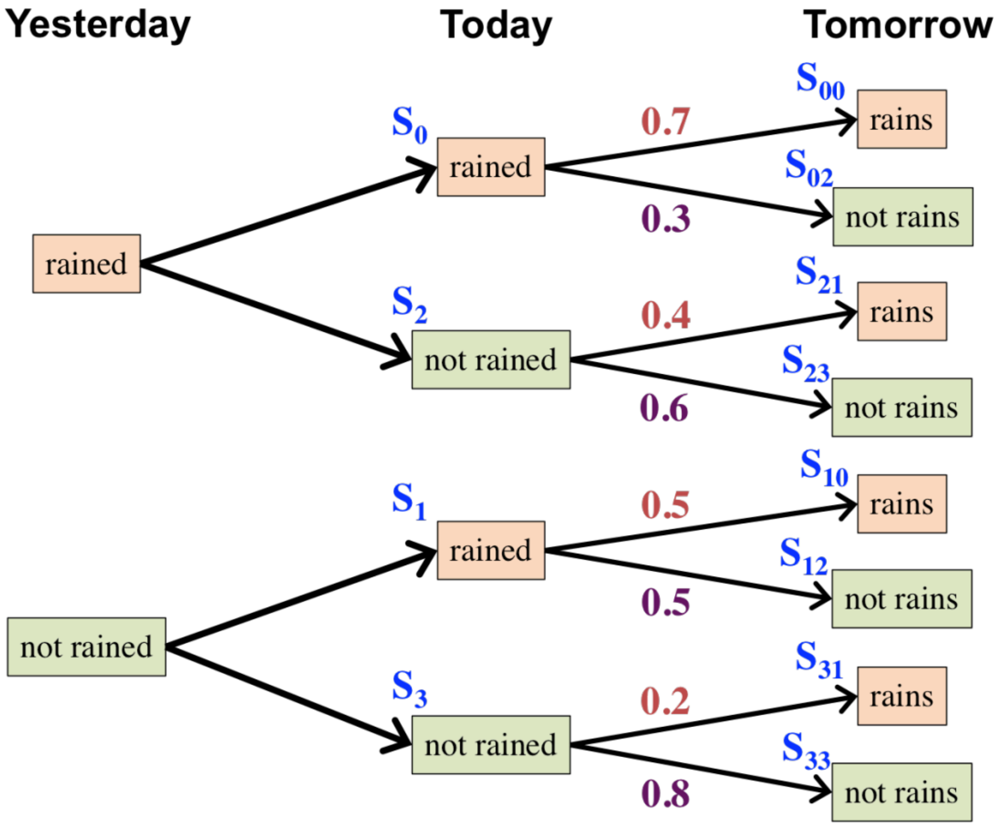
$P_{33} = 1 - 0.2 = 0.8$.

- Summary



- Whenever the process in any state, it will be next in another state, but with a fixed probability in transition.
- Such a stochastic process is an example of **Markov chain**. (not formal definition)

- It's natural to express transition probabilities in a matrix:



$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

- state 0 (S_0): if it rained both today and yesterday,

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  state 3 (S_3): if it did not rain either yesterday or today.

Transition probability matrix

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

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Properties:

- 1. Entries of the matrix are non-negative
- 2. Sum of any row is 1

Transition probability matrix

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

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Chapman-Kolmogorov equations

$$\boxed{P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m}, \qquad \text{for } n, m \geq 0, \text{ all } i, j}$$

In this example, $\mathbf{P}^{(n)} = \mathbf{P}^{(n-1+1)} = \mathbf{P}^{n-1} \cdot \mathbf{P} = \mathbf{P}^{n}$.

It states that the n-step transition matrix $\mathbf{P}^{(n)}$ is just the nth power of $\mathbf{P}$.

Transition probability matrix

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix}$$

Chapman-Kolmogorov equations

In this example $\boxed{\mathbf{P}^{(n)} = \mathbf{P}^{(n-1+1)} = \mathbf{P}^{n-1} \cdot \mathbf{P} = \mathbf{P}^{n}\,.}$

For instance, transition from (today) to (the day after tomorrow),

$$\begin{aligned} \mathbf{P}^{(2)} = \mathbf{P}^{2} &= \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix} \cdot \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix} \\ &= \begin{bmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{bmatrix} \end{aligned}$$

- Reference from Vivi Andasari