

Euler's formula

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$e^{-j\theta} = \cos(\theta) - j \sin(\theta)$$

Chapter 1

1.2.2 Bounded signals

A continuous-time signal $x(t)$ is bounded if there exists an M such that $0 < M < \infty$ and $\forall t |x(t)| \leq M$ (has an upper and lower range limit)

1.2.3 Absolutely integrable signals

A continuous-time signal $x(t)$ is absolutely integrable if

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

1.2.4 Periodic and aperiodic signals

Periodic: there is a non-zero positive value, T , satisfying

$$x(t) = x(t + T) \quad \forall t \quad (1.1) \quad e^{-\alpha t} u(t):$$

Aperiodic: not periodic

1.2.6 Energy and Power Signals

Energy signals

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (1.3a)$$

$$x(t) \text{ is an energy signal} \iff 0 < E < \infty \quad (1.3b)$$

Power signals

$$P = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt \quad (1.4a)$$

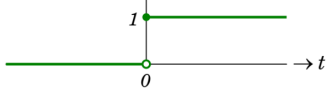
$$x(t) \text{ is a power signal} \iff 0 < P < \infty \quad (1.4b)$$

If $x(t)$ is a periodic signal, average power may be computed by

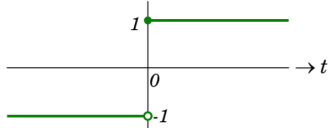
$$\frac{1}{T} \int_0^T |x(t)|^2 dt$$

- Energy signals have 0 average power, bc $E = \text{finite}$ implies $P = 0$
- Power signals have infinite total energy, bc $P = \text{finite}$ implies $E = \infty$
- All bounded periodic signals are power signals

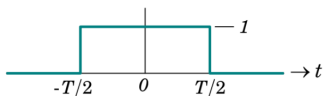
$u(t)$:



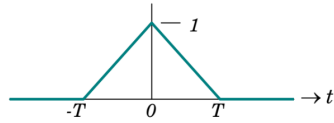
$\text{sgn}(t)$:



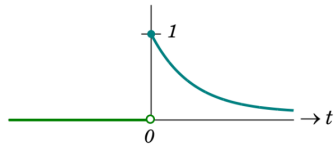
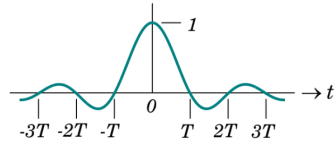
$\text{rect}(\frac{t}{T})$:



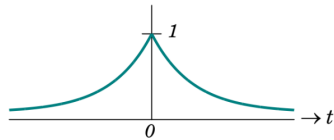
$\text{tri}(\frac{t}{T})$:



$\text{sinc}(\frac{t}{T})$:



$e^{-\alpha|t|}$:



Sinusoidal signals

$$x(t) = \mu \cos(\omega_0 t + \phi)$$

$$= \mu \cos(2\pi f_0 t + \phi)$$

$$= \mu \cos(\frac{2\pi t}{T} + \phi)$$

$$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$$

Chapter 2

2.1 Time-domain Operations

2.1.1 Time-Scaling

$x(\alpha t)$: Scale x-axis by a factor of $\frac{1}{\alpha}$
 $x(-t)$: Reflect about x-axis

2.1.2 Time-Shifting

$x(t - \beta)$:

$\beta > 0$: Delaying $x(t)$ by β units of time (translate right along x-axis)

$\beta > 0$: Advancing $x(t)$ by β units of time (translate left along x-axis)

2.1.5 Convolution of 2 Signals

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\alpha) y(t - \alpha) d\alpha$$

Properties of convolutions

1. Commutative: $f * g = g * f$
2. Associative: $f * (g * h) = (f * g) * h$
3. Distributive: $f * (g + h) = (f * g) + (f * h)$

2.2 Dirac- δ function

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \neq 0 \end{cases}$$

Properties:

1. Symmetry:

$$\delta(t) = \delta(-t) \quad (2.3)$$

2. Sampling:

$$x(t)\delta(t - \lambda) = x(\lambda)\delta(t - \lambda) \quad (2.4)$$

3. Sifting

$$\begin{aligned} & \int_{-\infty}^{\infty} x(t)\delta(t - \lambda) dt \\ &= x(\lambda) \int_{-\infty}^{\infty} \delta(t - \lambda) dt = x(\lambda) \end{aligned} \quad (2.5)$$

4. Replication

$$\begin{aligned} & x(t) * \delta(t - \lambda) \\ &= \int_{-\infty}^{\infty} x(\zeta)\delta(t - \zeta - \lambda) d\zeta \\ &= \int_{-\infty}^{\infty} x(\zeta)\delta(\zeta - (t - \lambda)) d\zeta = x(t - \lambda) \end{aligned} \quad (2.6)$$

2.2.1 Dirac- δ Comb function

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \dots + \delta(t + T) + \delta(t) + \delta(t - T) + \dots \end{aligned}$$

Convolution with Dirac- δ Comb function

$$\begin{aligned} x_p(t) &= x(t) * \sum_n \delta(t - nT) \\ &= \sum_n x(t - nT) \end{aligned}$$

$x(t)$ is known as the generating function.

Multiplication with the Dirac- δ Comb function

Used for sampling

$$\begin{aligned} x_s(t) &= x(t) * \sum_n \delta(t - nT) \\ &= \sum_n x(t) \times \delta(t - nT) \\ &= \sum_n x(nT)\delta(t - nT) \end{aligned}$$

Chapter 3

3.2 Spectrum of a Sinusoid

Spectrum of a Complex Exponential Signal

$$\tilde{x}(t) = \mu e^{j(2\pi f_0 t + \phi)} = \mu e^{j\phi} \times e^{j2\pi f_0 t},$$

where μ : magnitude spectrum, ϕ : phase spectrum, f_0 : frequency

Spectrum of a Cosine Signal

$$\begin{aligned} & \mu \cos(2\pi f_0 t + \phi) \\ &= \frac{\mu}{2} e^{j\phi} e^{j2\pi f_0 t} + \frac{\mu}{2} e^{j(-\phi)} e^{j2\pi(-f_0)t} \end{aligned}$$

Spectrum of a Sine Signal

$$\begin{aligned} \mu \sin(2\pi f_0 t + \phi) &= \frac{\mu}{2} e^{j(\phi - 0.5\pi)} e^{j2\pi f_0 t} \\ &+ \frac{\mu}{2} e^{j(-\phi + 0.5\pi)} e^{j2\pi(-f_0)t} \end{aligned}$$

Complex exponential Fourier Series

$$\begin{aligned} x_p(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t / T_p} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} \end{aligned} \quad (3.1a)$$

$$c_k = \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) e^{-j2\pi k t / T_p} dt, k \in \mathbb{Z} \quad (3.1b)$$

Trigonometric Fourier Series

$$\begin{aligned} x_p(t) &= a_0 + 2 \sum_{k=1}^{\infty} [a_k \cos(2\pi k t / T_p) \\ &+ b_k \sin(2\pi k t / T_p)] \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) \cos(2\pi k t / T_p) dt; k \geq 0 \\ b_k &= \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) \sin(2\pi k t / T_p) dt; k > 0 \end{aligned} \quad (3.2)$$

Chapter 4

Dirichlet Conditions

Conditions for existence of Fourier Transform:

1. $x(t)$ has only a finite number of maxima and minima in any finite time interval
 2. $x(t)$ has only a finite number of discontinuities in any finite time interval
 3. $x(t)$ is absolutely integrable
- 3 is weak Dirichlet condition: satisfied by most energy signals, violated by all power signals.

4.1 Fourier Transform

Forward Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \quad (4.1a)$$

Inverse Fourier Transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \quad (4.1b)$$

4.2 Properties of Fourier Transform

- $X(f) = \mathfrak{F}\{x(t)\}$ denotes the Fourier transform of $x(t)$
- $x(t) = \mathfrak{F}^{-1}\{X(f)\}$ denotes the inverse Fourier transform of $X(f)$
- $x(t) \rightleftharpoons X(f)$ denotes a Fourier transform pair with the time-domain on the LHS and frequency-domain on the RHS.

Linearity

$$\begin{aligned} \text{If } x_1(t) &\rightleftharpoons X_1(f) \text{ and } x_2(t) \rightleftharpoons X_2(f), \text{ then} \\ \alpha x_1(t) + \beta x_2(t) &\rightleftharpoons \alpha X_1(f) + \beta X_2(f) \end{aligned} \quad (4.2)$$

Time Scaling

$$x(\beta t) \rightleftharpoons \frac{1}{|\beta|} X\left(\frac{f}{\beta}\right) \quad (4.3)$$

Duality

$$\begin{aligned} X(t) &\rightleftharpoons x(-f) \\ \text{or} \\ X(-t) &\rightleftharpoons x(f) \end{aligned} \quad (4.4)$$

Time Shifting

$$x(t - t_0) \rightleftharpoons X(f) e^{-j2\pi f t_0} \quad (4.5)$$

Frequency Shifting (Modulation)

$$x(t) e^{j2\pi f_0 t} \rightleftharpoons X(f - f_0) \quad (4.6)$$

Differentiation in the Time Domain

$$\frac{d}{dt} x(t) \rightleftharpoons j2\pi f \cdot X(f) \quad (4.7)$$

Integration in the Time Domain

$$\int_{-\infty}^t x(\tau) d\tau \rightleftharpoons \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0) \delta(f) \quad (4.8)$$

Convolution in the Time Domain / Multiplication in the Frequency Domain

$$\begin{aligned} x_1(t) * x_2(t) \\ &= \int_{-\infty}^{\infty} x_1(\alpha) x_2(t - \alpha) d\alpha \rightleftharpoons X_1(f) X_2(f) \end{aligned} \quad (4.9a)$$

Multiplication in the Time Domain / Convolution in the Frequency Domain

$$\begin{aligned} x_1(t) x_2(t) &\rightleftharpoons \int_{-\infty}^{\infty} X_1(\alpha) X_2(f - \alpha) d\alpha \\ &= X_1(f) * X_2(f) \end{aligned} \quad (4.9b)$$

4.3 Spectral properties of a REAL signal

- If $x(t)$ is **REAL** ($x^*(t) = x(t)$), then
 - $X(f)$ is conjugate symmetric ($X^*(f) = X(-f)$)
 - $|X(f)|$ is even ($|X(f)| = |X(-f)|$)
 - $\angle X(f)$ is odd ($\angle X(f) = -\angle X(-f)$)
- If $x(t)$ is **REAL** and **EVEN** ($x^*(t) = x(t) \wedge x(-t) = x(t)$), then
 - $X(f)$ is real ($X^* f = X(f)$)
 - $X(f)$ is even ($X(-f) = X(f)$)
- If $x(t)$ is **REAL** and **ODD** ($x^*(t) = x(t) \wedge x(-t) = -x(t)$), then
 - $X(f)$ is imaginary ($X^*(f) = -X(f)$)
 - $X(f)$ is odd ($X(-f) = -X(f)$)

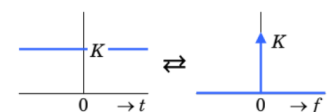
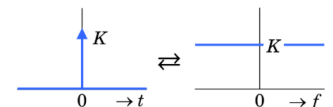
The above can apply to Fourier series coefficients of periodic signals too:

- $x_p(t)$ is **REAL**
 - c_k is conjugate symmetric ($c_k^* = c_{-k}$)
 - $|c_k|$ has even symmetry ($|c_k| = |c_{-k}|$)
 - $\angle c_k$ has odd symmetry ($\angle c_k = -\angle c_{-k}$)
- $x_p(t)$ is **REAL** and **EVEN**
 - c_k is real ($c_k^* = c_k$)
 - c_k is even ($c_k = c_{-k}$)
- $x_p(t)$ is **REAL** and **ODD**
 - c_k is imaginary ($c_k^* = -c_k$)
 - c_k is odd ($c_k = -c_{-k}$)

4.4 Spectrum of Signals that are not Absolutely Integrable

$$\mathfrak{F}\{K\delta(t)\} = \int_{-\infty}^{\infty} K\delta(t) e^{-j2\pi f t} dt = K \quad (4.13)$$

By duality, $\mathfrak{F}\{K\} = K\delta(f)$



4.4.1 Spectrm of Unit Step and Signum function

$$\Im\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

$$\Im\{\text{Sgn}(t)\} = \frac{1}{j\pi f}$$

4.4.2 Continuous-Frequency Spectrum of Periodic Signals

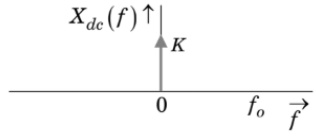
The following make use of the fact that

$$\Im\{k\} = K\delta(f) \quad (4.14)$$

DC

$$x_{dc}(t) = K$$

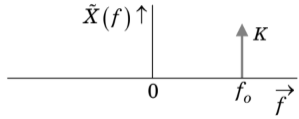
$$X_{dc}(f) = \Im\{k\} = K\Im\{1\} = K\delta(f)$$



Complex Exponential

$$\tilde{x}(t) = Ke^{j2\pi f_0 t}$$

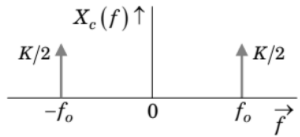
$$\tilde{X}(f) = \Im\{Ke^{j2\pi f_0 t}\} = K\delta(f - f_0)$$



Cosine

$$\Im\{K \cos(2\pi f_0 t)\}$$

$$= \frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0)$$



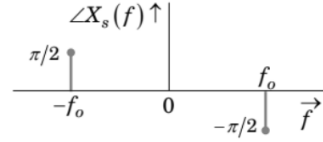
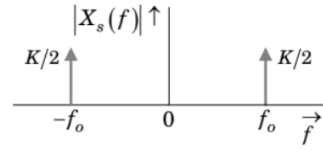
Sine

$$\Im\{K \sin(2\pi f_0 t)\}$$

$$= \frac{K}{j2}\delta(f - f_0) - \frac{K}{j2}\delta(f + f_0)$$

where

$$\begin{cases} |X_s(f)| &= \frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0) \\ \angle X_s(f) &= \begin{cases} -\pi/2, & f = f_0 \\ \pi/2, & f = -f_0 \end{cases} \end{cases}$$

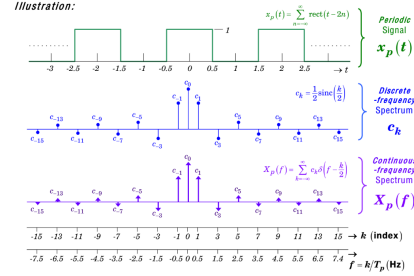


Arbitrary periodic signals

Let $x_p(t)$ be a periodic signal with period T_p and fundamental frequency f_p

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p) \quad (4.16)$$

Illustration:



4.4.2.1 Spectrum of Dirac-δ Comb function

$$\text{comb}_\lambda(t) \triangleq \sum_n \delta(t - n\lambda)$$

$$c_k = \frac{1}{\lambda}$$



$$\Im\{\text{comb}_\lambda(t)\} = \text{COMB}_\lambda(f)$$

$$= \frac{1}{\lambda} \sum_k \delta(f - k/\lambda)$$



Chapter 5

5.1 Energy Spectral Density (ESD)

Total energy of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (\text{Joules}) \quad (5.1)$$

Rayleigh Energy Theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df, \quad (5.2)$$

where $X(f) = \Im\{x(t)\}$ is the spectrum of the signal.

Energy Spectral Density

$$E_x(f) = |X(f)|^2 \text{ Joules Hz}^{-1} \quad (5.3)$$

Properties of $E_x(f)$

1. $E_x(f)$ is a real function of f
2. $E_x(f) \geq 0 \quad \forall f$
3. $E_x(f)$ is an even function of f if $x(t)$ is real.

5.2 Power Spectral Density (PSD)

In the time-domain, the average power of a signal $x(t)$ is defined as

$$P = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt \quad (5.4)$$

Windowed version of $x(t)$:

$$x_W(t) = x(t) \text{rect}\left(\frac{t}{2W}\right) \quad (5.5)$$

Parseval Power Theorem

$$P = \lim_{W \rightarrow \infty} \frac{1}{2W} \int_{-W}^W |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} \lim_{W \rightarrow \infty} \frac{1}{2W} |X_W(f)|^2 df \quad (5.9)$$

Power Spectral Density

$$P_x(f) = \lim_{W \rightarrow \infty} \frac{1}{2W} |X_W(f)|^2 \text{ Watts Hz}^{-1} \quad (5.10)$$

Properties of $P_x(f)$

1. $P_x(f)$ is a real function of f
2. $P_x(f) \geq 0 \quad \forall f$
3. $P_x(f)$ is an even function of f if $x(t)$ is real.

5.2.1 PSD of Periodic Signals

From chapter 4 equation 4.16:

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p)$$

PSD of $x_p(t)$

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p) \quad (5.12)$$

Average power of $x_p(t)$

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad (5.13)$$

5.3 Bandwidth

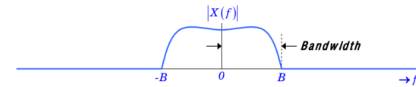
5.3.1 Bandlimited Signals

Lowpass signal

A signal $x(t)$ is said to be a bandlimited lowpass signal if its magnitude spectrum is concentrated around 0 Hz, and at the same time satisfies

$$|X(f)| = 0; \quad |f| > B \quad (5.14)$$

where B is defined as the bandwidth of the signal.

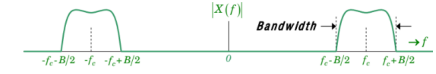


Bandpass signal

A signal $x(t)$ is said to be a bandlimited bandpass signal if its magnitude spectrum is concentrated around a non-zero center frequency f_c , and at the same time satisfies

$$|X(f)| = 0; \quad ||f| - f_c| > B/2 \quad (5.15)$$

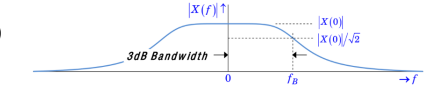
where B is defined as the bandwidth of the signal



5.3.2 Signals with Unrestricted Band

5.3.2.1 3dB Bandwidth

Lowpass signal: The frequency where $|X(f)| = |X(0)|/\sqrt{2}$ first occurs (or where $|X(f)|^2 = |X(0)|^2/2$ first occurs) when f is increased from 0 Hz.

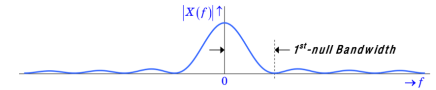


Bandpass signal:

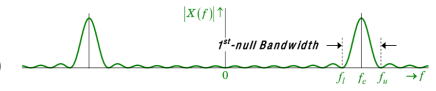


5.3.2.2 1st-null Bandwidth

Lowpass signal: The frequency at which $|X(f)| = 0$ first occurs when f is increased from 0 Hz:



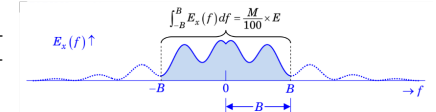
Bandpass signal:



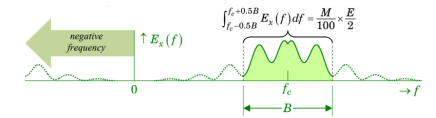
5.3.2.3 M% Energy Containment Bandwidth

Smallest bandwidth that contains at least M% of the total signal energy $E = \int_{-\infty}^{\infty} E_x(f) df$

Lowpass:



Bandpass:



5.3.2.4 M% Power Containment Bandwidth

The smallest bandwidth that contains at least M% of the average signal power. For a periodic signal, the average power is given by

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2$$

where f_p (Hz) is the fundamental frequency and c_k 's are the Fourier series coefficients.

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p)$$

$$B = K f_p$$

where K is the smallest positive integer that satisfies

$$\sum_{k=-K}^K |c_k|^2 \geq \frac{M}{100} \times P$$