Euler's formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

Chapter 1

1.2.2 Bounded signals

A continuous-time signal $x(t)$ is bounded if there exists an M such that $0 < M < \infty$ and $\forall t\ |x(t)| \le M$ (has an upper and lower range limit)

1.2.3 Absolutely integrable signals

A continuous-time signal $x(t)$ is absolutely integrable if

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

1.2.4 Periodc and aperiodic signals

Periodic: there is a non-zero positive value, T, satisfying

$$x(t) = x(t+T)\ \forall t \tag{1.1}$$

Aperiodic: not periodic

1.2.6 Energy and Power Signals

Energy signals

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \tag{1.3a}$$

$$x(t) \text{ is an energy signal } \iff 0 < E < \infty \tag{1.3b}$$

Power signals

$$P = \lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} |x(t)|^2 dt \tag{1.4a}$$

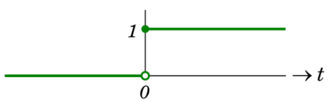
$$x(t) \text{ is a power signal } \iff 0 < P < \infty \tag{1.4b}$$

If $x(t)$ is a periodic signal, average power may be computed by

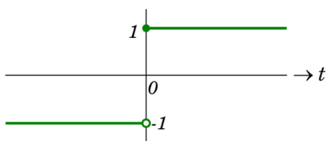
$$\frac{1}{T}\int_0^T |x(t)|^2 dt$$

- Energy signals have 0 average power, bc E = finite implies P = 0
- Power signals have infinite total energy, bc P = finite implies $E = \infty$
- All bounded periodic signals are power signals

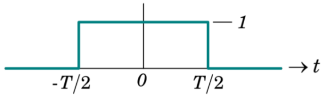
u(t):



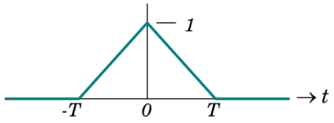
sgn(t):



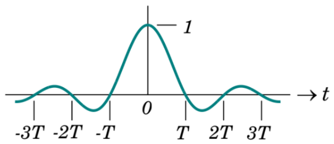
rect($\frac{t}{T}$):



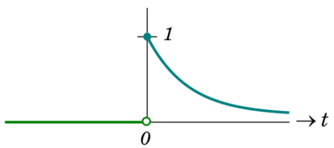
tri($\frac{t}{T}$):



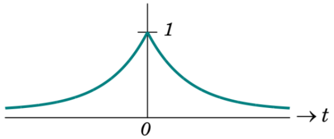
sinc($\frac{t}{T}$):



$e^{-\alpha t}u(t)$**:**



$e^{-\alpha|t|}$**:**



Sinusoidal signals

$$x(t) = \mu\cos(\omega_0 t + \phi)$$
$$= \mu\cos(2\pi f_0 t + \phi)$$
$$= \mu\cos(\frac{2\pi t}{T} + \phi)$$
$$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$$

Chapter 2

2.1 Time-domain Operations

2.1.1 Time-Scaling

$x(\alpha t)$: Scale x-axis by a factor of $\frac{1}{\alpha}$

$x(-t)$: Reflect about x-axis

2.1.2 Time-Shifting

$x(t-\beta)$:

$\beta > 0$: Delaying $x(t)$ by β units of time (translate right along x-axis)

$\beta > 0$: Advancing $x(t)$ by β units of time (translate left along x-axis)

2.1.5 Convolution of 2 Signals

$x(t) * y(t) = \int_{-\infty}^{\infty} x(\alpha)y(t-\alpha)\ d\alpha$

Properties of convolutions

1. Commutative: $f * g = g * f$
2. Associative: $f * (g * h) = (f * g) * h$
3. Distributive: $f * (g + h) = (f * g) + (f * h)$

2.2 Dirac-δ function

$$\delta(t) = \begin{cases}\infty; & t = 0\\ 0; & t \ne 0\end{cases}$$

Properties:

1. Symmetry:
$$\delta(t) = \delta(-t) \tag{2.3}$$
2. Sampling:
$$x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda) \tag{2.4}$$
3. Sifting
$$\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt$$
$$= x(\lambda)\int_{-\infty}^{\infty}\delta(t-\lambda)dt = x(\lambda) \tag{2.5}$$
4. Replication
$$x(t) * \delta(t-\lambda)$$
$$= \int_{-\infty}^{\infty} x(\zeta)\delta(t-\zeta-\lambda)d\zeta$$
$$= \int_{-\infty}^{\infty} x(\zeta)\delta(\zeta-(t-\lambda))d\zeta = x(t-\lambda) \tag{2.6}$$

2.2.1 Dirac-δ Comb function

$$\sum_{n=-\infty}^{\infty}\delta(t-nT)$$
$$= \ldots + \delta(t+T) + \delta(t) + \delta(t-T) + \ldots$$

Convolution with Dirac-δ Comb function

$$x_p(t) = x(t) * \sum_n \delta(t-nT)$$
$$= \sum_n x(t-nT)$$

$x(t)$ is known as the generating function.

Multiplication with the Dirac-δ Comb function

Used for sampling

$$x_s(t) = x(t) \times \sum_n \delta(t-nT)$$
$$= \sum_n x(t) \times \delta(t-nT)$$
$$= \sum_n x(nT)\delta(t-nT)$$

Chapter 3

3.2 Spectrum of a Sinusoid

Spectrum of a Complex Exponential Signal

$$\tilde{x}(t) = \mu e^{j(2\pi f_0 t + \phi)} = \mu e^{j\phi} \times e^{j2\pi f_0 t},$$

where μ: magnitude spectrum, ϕ: phase spectrum, f_0: frequency

Spectrum of a Cosine Signal

$$\mu\cos(2\pi f_0 t + \phi)$$
$$= \frac{\mu}{2}e^{j\phi}e^{j2\pi f_0 t} + \frac{\mu}{2}e^{j(-\phi)}e^{j2\pi(-f_0)t}$$

Spectrum of a Sine Signal

$$\mu\sin(2\pi f_0 t + \phi) = \frac{\mu}{2}e^{j(\phi-0.5\pi)}e^{j2\pi f_0 t} + \frac{\mu}{2}e^{j(-\phi+0.5\pi)}e^{j2\pi(-f_0)t}$$

Complex exponential Fourier Series

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T_p}$$
$$= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} \tag{3.1a}$$
$$c_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)e^{-j2\pi kt/T_p}dt, k \in \mathbb{Z} \tag{3.1b}$$

Trigonometric Fourier Series

$$x_p(t) = a_0 + 2\sum_{k=1}^{\infty}[a_k\cos(2\pi kt/T_p) + b_k\sin(2\pi kt/T_p)]$$
$$a_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)\cos(2\pi kt/T_p)dt; k \ge 0$$
$$b_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)\sin(2\pi kt/T_p)dt; k > 0 \tag{3.2}$$

Chapter 4

Dirichlet Conditions

Conditions for existence of Fourier Transform:

1. $x(t)$ has only a finite number of maxima and minima in any finite time interval
2. $x(t)$ has only a finite number of discontinuities in any finite time interval
3. $x(t)$ is absolutely integrable

3 is weak Dirichlet condition: satisfied by most energy signals, violated by all power signals.

4.1 Fourier Transform

Forward Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \tag{4.1a}$$

Inverse Fourier Transform

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \tag{4.1b}$$

Spectrum of exponentially decaying pulse

$$x(t) = Ae^{-\alpha t}u(t)$$
$$= \begin{cases}Ae^{-\alpha t}; & t > 0\\ 0; & t < 0\end{cases}$$

Assume $\alpha > 0$

$$X(f) = \frac{A}{\alpha + j2\pi f}$$

4.2 Properties of Fourier Transform

- $X(f) = \Im\{x(t)\}$ denotes the Fourier transform of $x(t)$
- $x(t) = \Im^{-1}\{X(f)\}$ denotes the inverse Fourier transform of $X(f)$
- $x(t) \rightleftarrows X(f)$ denotes a Fourier transform pair with the time-domain on the LHS and frequency-domain on the RHS.

Linearity

If $x_1(t) \rightleftarrows X_1(f)$ and $x_2(t) \rightleftarrows X_2(f)$, then

$$\alpha x_1(t) + \beta x_2(t) \rightleftarrows \alpha X_1(f) + \beta X_2(f) \tag{4.2}$$

Time Scaling

$$x(\beta t) \rightleftarrows \frac{1}{|\beta|}X\left(\frac{f}{\beta}\right) \tag{4.3}$$

Duality

$$X(t) \rightleftarrows x(-f) \tag{4.4}$$

or

$$X(-t) \rightleftarrows x(f)$$

Time Shifting

$$x(t-t_0) \rightleftarrows X(f)e^{-j2\pi ft_0} \tag{4.5}$$
$$x(t+t_0) \rightleftarrows X(f)e^{j2\pi ft_0}$$

Frequency Shifting (Modulation)

$$x(t)e^{j2\pi f_0 t} \rightleftarrows X(f-f_0) \tag{4.6}$$
$$x(t)e^{-j2\pi f_0 t} \rightleftarrows X(f+f_0)$$

Differentiation in the Time Domain

$$\frac{d}{dt}x(t) \rightleftarrows j2\pi f \cdot X(f) \tag{4.7}$$

Integration in the Time Domain

$$\int_{-\infty}^{t} x(\tau)\ d\tau \rightleftarrows \frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f) \tag{4.8}$$

Convolution in the Time Domain / Multiplication in the Frequency Domain

$$x_1(t) * x_2(t)$$
$$= \int_{-\infty}^{\infty} x_1(\alpha)x_2(t-\alpha)\ d\alpha \rightleftarrows X_1(f)X_2(f) \tag{4.9a}$$

Multiplication in the Time Domain / Convolution in the Frequency Domain

$$x_1(t)x_2(t) \rightleftarrows \int_{-\infty}^{\infty} X_1(\alpha)X_2(f-\alpha)\ d\alpha$$
$$= X_1(f) * X_2(f) \tag{4.9b}$$

4.3 Spectral properties of a REAL signal

- If $x(t)$ is **REAL** ($x^*(t) = x(t)$), then
  - $X(f)$ is conjugate symmetric ($X^*(f) = X(f)$)
  - $|X(f)|$ is even ($|X(f)| = |X(-f)|$)
  - $\angle X(f)$ is odd ($\angle X(f) = -\angle X(-f)$)
- If $x(t)$ is **REAL** and **EVEN** ($x^*(t) = x(t) \wedge x(-t) = x(t)$), then
  - $X(f)$ is real ($X^*f = X(f)$)
  - $X(f)$ is even ($X(-f) = X(f)$)
- If $x(t)$ is **REAL** and **ODD** ($x^*(t) = x(t) \wedge x(-t) = -x(t)$), then
  - $X(f)$ is imaginary ($X^*(f) = -X(f)$)
  - $X(f)$ is odd ($X(-f) = -X(f)$)

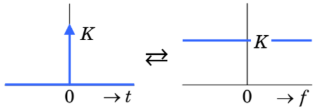
The above can apply to Fourier series coefficients of periodic signals too:

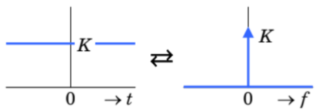
- $x_p(t)$ is **REAL**
  - c_k is conjugate symmetric ($c_k^* = c_{-k}$)
  - $|c_k|$ has even symmetry ($|c_k| = |c_{-k}|$)
  - $\angle c_k$ has odd symmetry ($\angle c_k = -\angle c_{-k}$)
- $x_p(t)$ is **REAL** and **EVEN**
  - c_k is real ($c_k^* = c_k$)
  - c_k is even ($c_k = c_{-k}$)
- $x_p(t)$ is **REAL** and **ODD**
  - c_k is imaginary ($c_k^* = -c_k$)
  - c_k is odd ($c_k = -c_{-k}$)

4.4 Spectrum of Signals that are not Absolutely Integrable

$$\Im\{K\delta(t)\} = \int_{-\infty}^{\infty} K\delta(t)e^{-j2\pi ft}dt = K \tag{4.13}$$

By duality, $\Im\{K\} = K\delta(f)$





4.4.1 Spectrm of Unit Step and Signum function

$$\Im\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

$$\Im\{\mathrm{Sgn}(t)\} = \frac{1}{j\pi f}$$

4.4.2 Continuous-Frequency Spectrum of Periodic Signals

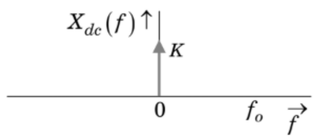
The following make use of the fact that

$$\Im\{k\} = K\delta(f) \tag{4.14}$$

DC

$$x_{dc}(t) = K$$

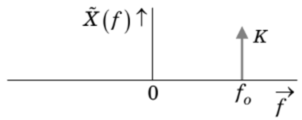
$$X_{dc}(f) = \Im\{k\} = K\Im\{1\} = K\delta(f)$$



Complex Exponential

$$\tilde{x}(t) = Ke^{j2\pi f_0 t}$$

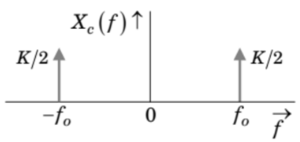
$$\tilde{X}(f) = \Im\{Ke^{j2\pi f_0 t}\} = K\delta(f - f_0)$$



Cosine

$$\Im\{K\cos(2\pi f_0 t)\}$$

$$= \frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0)$$

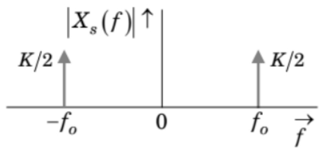


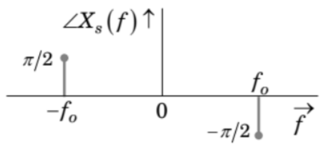
Sine

$$\Im\{K\sin(2\pi f_0 t)\}$$

$$= \frac{K}{j2}\delta(f - f_0) - \frac{K}{j2}\delta(f + f_0)$$

where
$$\begin{cases} |X_s(f)| &= \frac{K}{2}\delta(f - f_0) \\ & \quad + \frac{K}{2}\delta(f + f_0) \\ \angle X_s(f) &= \begin{cases} -\pi/2, & f = f_0 \\ \pi/2, & f = -f_0 \end{cases} \end{cases}$$

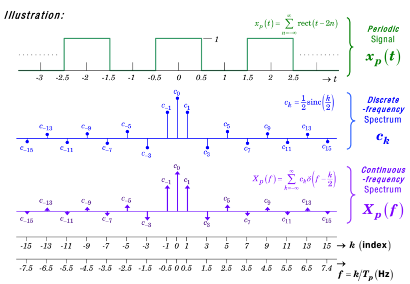




Arbitrary periodic signals

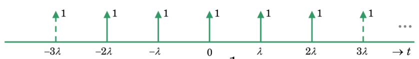
Let $x_p(t)$ be a periodic signal with period T_p and fundamental frequency f_p

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k\delta(f - kf_p) \tag{4.16}$$

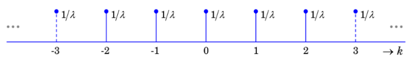


4.4.2.1 Spectrum of Dirac-δ Comb function

$$\mathrm{comb}_\lambda(t) \triangleq \sum_n \delta(t - n\lambda)$$

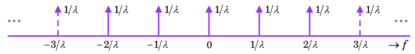


$$c_k = \frac{1}{\lambda}$$



$$\Im\{\mathrm{comb}_\lambda(t)\} = \mathrm{COMB}_\lambda(f)$$

$$= \frac{1}{\lambda}\sum_k \delta(f - k/\lambda)$$



Chapter 5

5.1 Energy Spectral Density (ESD)

Total energy of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \text{ (Joules)} \tag{5.1}$$

Rayleigh Energy Theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df, \tag{5.2}$$

where $X(f) = \Im\{x(t)\}$ is the spectrum of the signal.

Energy Spectral Density

$$E_x(f) = |X(f)|^2 \text{ Joules Hz}^{-1} \tag{5.3}$$

Properties of $E_x(f)$

1. $E_x(f)$ is a real function of f
2. $E_x(f) \geq 0 \quad \forall f$
3. $E_x(f)$ is an even function of f if $x(t)$ is real.

5.2 Power Spectral Density (PSD)

In the time-domain, the average power of a signal $x(t)$ is defined as

$$P = \lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} |x(t)|^2 dt \tag{5.4}$$

Windowed version of $x(t)$:

$$x_W(t) = x(t)\mathrm{rect}\left(\frac{t}{2W}\right) \tag{5.5}$$

Parseval Power Theorem

$$P = \lim_{W\to\infty} \frac{1}{2W}\int_{-W}^{W} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} \lim_{W\to\infty} \frac{1}{2W}|X_W(f)|^2 df \tag{5.9}$$

Power Spectral Density

$$P_x(f) = \lim_{W\to\infty} \frac{1}{2W}|X_W(f)|^2 \text{ Watts Hz}^{-1} \tag{5.10}$$

Properties of $P_x(f)$

1. $P_x(f)$ is a real function of f
2. $P_x(f) \geq 0 \quad \forall f$
3. $P_x(f)$ is an even function of f if $x(t)$ is real.

5.2.1 PSD of Periodic Signals

From chapter 4 equation 4.16:

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k\delta(f - kf_p)$$

PSD of $x_p(t)$

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2\delta(f - kf_p) \tag{5.12}$$

Average power of $x_p(t)$

$$P = \int_{-\infty}^{\infty} P_x(f)df = \sum_{k=-\infty}^{\infty} |c_k|^2 \tag{5.13}$$

5.3 Bandwidth

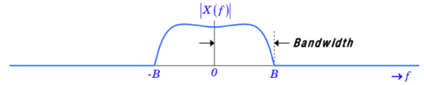
5.3.1 Bandlimited Signals

Lowpass signal

A signal $x(t)$ is said to be a bandlimited low-pass signal if its magnitude spectrum is concentrated around 0 Hz, and at the same time satisfies

$$|X(f)| = 0; \quad |f| > B \tag{5.14}$$

where B is defined as the bandwidth of the signal.

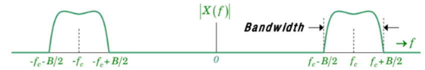


Bandpass signal

A signal $x(t)$ is said to be a bandlimited band-pass signal if its magnitude spectrum is concentrated around a non-zero center frequency f_c, and at the same time satisfies

$$|X(f)| = 0; \quad ||f| - f_c| > B/2 \tag{5.15}$$

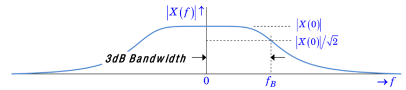
where B is defined as the bandwidth of the signal



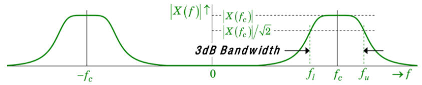
5.3.2 Signals with Unrestricted Band

5.3.2.1 3dB Bandwidth

Lowpass signal: The frequency where $|X(f)| = |X(0)|/\sqrt{2}$ first occurs (or where $|X(f)|^2 = |X(0)|^2/2$ first occurs) when f is increased from 0 Hz.

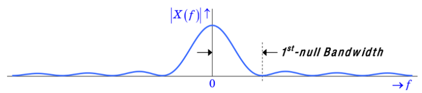


Bandpass signal:

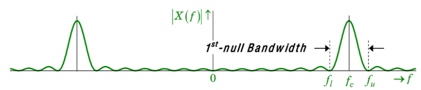


5.3.2.2 1st-null Bandwidth

Lowpass signal: The frequency at which $|X(f)| = 0$ first occurs when f is increased from 0 Hz:



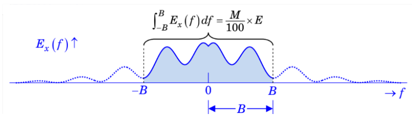
Bandpass signal:



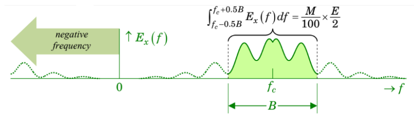
5.3.2.3 M% Energy Containment Bandwidth

Smallest bandwidth that contains at least M% of the total signal energy $E = \int_{-\infty}^{\infty} E_x(f)\, df$

Lowpass:



Bandpass:

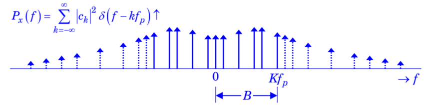


5.3.2.4 M% Power Containment Bandwidth

The smallest bandwidth that contains at least M% of the average signal power. For a periodic signal, the aerage power is given by

$$P = \int_{-\infty}^{\infty} P_x(f)df = \sum_{k=-\infty}^{\infty} |c_k|^2$$

where f_p(Hz) is the fundamental frequency and c_k's are the Fourier series coefficients.



$$B = Kf_p$$

where K is the smallest positive integer that satisfies

$$\sum_{k=-k}^{K} |c_k|^2 \geq \frac{M}{100} \times P$$