

Euler's formula

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$e^{-j\theta} = \cos(\theta) - j \sin(\theta)$$

Chapter 1

1.2.2 Bounded signals

A continuous-time signal $x(t)$ is bounded if there exists an M such that $0 < M < \infty$ and $\forall t |x(t)| \leq M$ (has an upper and lower range limit)

1.2.3 Absolutely integrable signals

A continuous-time signal $x(t)$ is absolutely integrable if
$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

1.2.4 Periodic and aperiodic signals

Periodic: there is a non-zero positive value, T , satisfying $x(t) = x(t + T) \forall t$ (1.1)

Aperiodic: not periodic

1.2.6 Energy and Power Signals

Energy signals

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (1.3a)$$

$$x(t) \text{ is an energy signal} \iff 0 < E < \infty \quad (1.3b)$$

Power signals

$$P = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt \quad (1.4a)$$

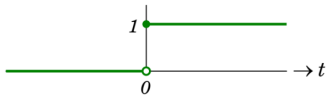
$$x(t) \text{ is a power signal} \iff 0 < P < \infty \quad (1.4b)$$

If $x(t)$ is a periodic signal, average power may be computed by

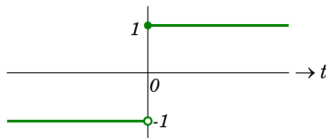
$$\frac{1}{T} \int_0^T |x(t)|^2 dt$$

- Energy signals have 0 average power, bc $E = \text{finite}$ implies $P = 0$
- Power signals have infinite total energy, bc $P = \text{finite}$ implies $E = \infty$
- All bounded periodic signals are power signals

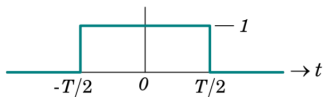
u(t):



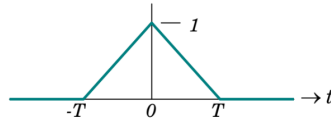
sgn(t):



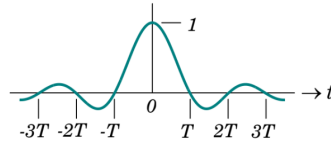
rect(t/T):



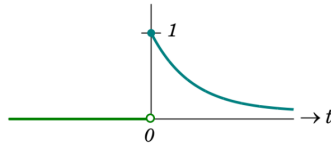
tri(t/T):



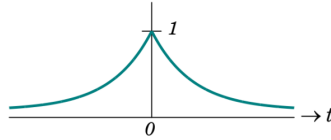
sinc(t/T):



e^{-\alpha t}u(t):



e^{-\alpha|t|}:



Sinusoidal signals

$$\begin{aligned} x(t) &= \mu \cos(\omega_0 t + \phi) \\ &= \mu \cos(2\pi f_0 t + \phi) \\ &= \mu \cos\left(\frac{2\pi t}{T} + \phi\right) \\ T_0 &= \frac{2\pi}{\omega_0} = \frac{1}{f_0} \end{aligned}$$

Chapter 2

2.1 Time-domain Operations

2.1.1 Time-Scaling

$x(\alpha t)$: Scale x-axis by a factor of $\frac{1}{\alpha}$

$x(-t)$: Reflect about x-axis

2.1.2 Time-Shifting

$x(t - \beta)$:

$\beta > 0$: Delaying $x(t)$ by β units of time (translate right along x-axis)

$\beta > 0$: Advancing $x(t)$ by β units of time (translate left along x-axis)

2.1.5 Convolution of 2 Signals

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\alpha) y(t - \alpha) d\alpha$$

Properties of convolutions

1. Commutative: $f * g = g * f$
2. Associative: $f * (g * h) = (f * g) * h$
3. Distributive: $f * (g + h) = (f * g) + (f * h)$

2.2 Dirac- δ function

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \neq 0 \end{cases}$$

Properties:

$$1. \text{ Symmetry: } \delta(t) = \delta(-t) \quad (2.3)$$

$$2. \text{ Sampling: } x(t)\delta(t - \lambda) = x(\lambda)\delta(t - \lambda) \quad (2.4)$$

$$\begin{aligned} 3. \text{ Sifting } \int_{-\infty}^{\infty} x(t)\delta(t - \lambda) dt \\ = x(\lambda) \int_{-\infty}^{\infty} \delta(t - \lambda) dt = x(\lambda) \end{aligned} \quad (2.5)$$

$$\begin{aligned} 4. \text{ Replication } x(t) * \delta(t - \lambda) \\ = \int_{-\infty}^{\infty} x(\zeta)\delta(t - \zeta - \lambda) d\zeta \\ = \int_{-\infty}^{\infty} x(\zeta)\delta(\zeta - (t - \lambda)) d\zeta = x(t - \lambda) \end{aligned} \quad (2.6)$$

2.2.1 Dirac- δ Comb function

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ = \dots + \delta(t + T) + \delta(t) + \delta(t - T) + \dots \end{aligned}$$

Convolution with Dirac- δ Comb function

$$\begin{aligned} x_p(t) = x(t) * \sum_n \delta(t - nT) \\ = \sum_n x(t - nT) \end{aligned}$$

$x(t)$ is known as the generating function.

Multiplication with the Dirac- δ Comb function

Used for sampling

$$\begin{aligned} x_s(t) = x(t) \times \sum_n \delta(t - nT) \\ = \sum_n x(t) \times \delta(t - nT) \\ = \sum_n x(nT)\delta(t - nT) \end{aligned}$$

Chapter 3

3.2 Spectrum of a Sinusoid

Spectrum of a Complex Exponential Signal

$$\tilde{x}(t) = \mu e^{j(2\pi f_0 t + \phi)} = \mu e^{j\phi} \times e^{j2\pi f_0 t}$$

where μ : magnitude spectrum, ϕ : phase spectrum, f_0 : frequency

Spectrum of a Cosine Signal

$$\begin{aligned} \mu \cos(2\pi f_0 t + \phi) \\ = \frac{\mu}{2} e^{j\phi} e^{j2\pi f_0 t} + \frac{\mu}{2} e^{-j\phi} e^{j2\pi(-f_0)t} \end{aligned}$$

Spectrum of a Sine Signal

$$\begin{aligned} \mu \sin(2\pi f_0 t + \phi) = \frac{\mu}{2} e^{j(\phi - 0.5\pi)} e^{j2\pi f_0 t} \\ + \frac{\mu}{2} e^{j(-\phi + 0.5\pi)} e^{j2\pi(-f_0)t} \end{aligned}$$

Complex exponential Fourier Series

$$\begin{aligned} x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t / T_p} \\ = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} \end{aligned} \quad (3.1a)$$

$$c_k = \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) e^{-j2\pi k t / T_p} dt, k \in \mathbb{Z} \quad (3.1b)$$

Trigonometric Fourier Series

$$x_p(t) = a_0 + 2 \sum_{k=1}^{\infty} [a_k \cos(2\pi k t / T_p) + b_k \sin(2\pi k t / T_p)]$$

$$\begin{aligned} a_k &= \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) \cos(2\pi k t / T_p) dt; k \geq 0 \\ b_k &= \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) \sin(2\pi k t / T_p) dt; k > 0 \end{aligned} \quad (3.2)$$

Chapter 4

Dirichlet Conditions

Conditions for existence of Fourier Transform:

1. $x(t)$ has only a finite number of maxima and minima in any finite time interval
 2. $x(t)$ has only a finite number of discontinuities in any finite time interval
 3. $x(t)$ is absolutely integrable
- 3 is weak Dirichlet condition: satisfied by most energy signals, violated by all power signals.

4.1 Fourier Transform

Forward Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \quad (4.1a)$$

Inverse Fourier Transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \quad (4.1b)$$

Spectrum of exponentially decaying pulse

$$\begin{aligned} x(t) &= A e^{-\alpha t} u(t) \\ &= \begin{cases} A e^{-\alpha t}; & t > 0 \\ 0; & t < 0 \end{cases} \\ \text{Assume } \alpha > 0 \\ X(f) &= \frac{A}{\alpha + j2\pi f} \end{aligned}$$

4.2 Properties of Fourier Transform

- $X(f) = \mathfrak{F}\{x(t)\}$ denotes the Fourier transform of $x(t)$
- $x(t) = \mathfrak{F}^{-1}\{X(f)\}$ denotes the inverse Fourier transform of $X(f)$
- $x(t) \rightleftharpoons X(f)$ denotes a Fourier transform pair with the time-domain on the LHS and frequency-domain on the RHS.

Linearity

If $x_1(t) \rightleftharpoons X_1(f)$ and $x_2(t) \rightleftharpoons X_2(f)$, then $\alpha x_1(t) + \beta x_2(t) \rightleftharpoons \alpha X_1(f) + \beta X_2(f)$ (4.2)

Time Scaling

$$x(\beta t) \rightleftharpoons \frac{1}{|\beta|} X\left(\frac{f}{\beta}\right) \quad (4.3)$$

Duality

$$\begin{aligned} X(t) &\rightleftharpoons x(-f) \\ \text{or} \\ X(-t) &\rightleftharpoons x(f) \end{aligned} \quad (4.4)$$

Time Shifting

$$\begin{aligned} x(t - t_0) &\rightleftharpoons X(f) e^{-j2\pi f t_0} \\ x(t + t_0) &\rightleftharpoons X(f) e^{j2\pi f t_0} \end{aligned} \quad (4.5)$$

Frequency Shifting (Modulation)

$$\begin{aligned} x(t) e^{j2\pi f_0 t} &\rightleftharpoons X(f - f_0) \\ x(t) e^{-j2\pi f_0 t} &\rightleftharpoons X(f + f_0) \end{aligned} \quad (4.6)$$

Differentiation in the Time Domain

$$\frac{d}{dt} x(t) \rightleftharpoons j2\pi f \cdot X(f) \quad (4.7)$$

Integration in the Time Domain

$$\int_{-\infty}^t x(\tau) d\tau \rightleftharpoons \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0) \delta(f) \quad (4.8)$$

Convolution in the Time Domain / Multiplication in the Frequency Domain

$$\begin{aligned} x_1(t) * x_2(t) \\ = \int_{-\infty}^{\infty} x_1(\alpha) x_2(t - \alpha) d\alpha \rightleftharpoons X_1(f) X_2(f) \end{aligned} \quad (4.9a)$$

Multiplication in the Time Domain / Convolution in the Frequency Domain

$$\begin{aligned} x_1(t) x_2(t) \rightleftharpoons \int_{-\infty}^{\infty} X_1(\alpha) X_2(f - \alpha) d\alpha \\ = X_1(f) * X_2(f) \end{aligned} \quad (4.9b)$$

4.3 Spectral properties of a REAL signal

- If $x(t)$ is **REAL** ($x^*(t) = x(t)$), then
 - $X(f)$ is conjugate symmetric ($X^*(f) = X(-f)$)
 - $|X(f)|$ is even ($|X(f)| = |X(-f)|$)
 - $\angle X(f)$ is odd ($\angle X(f) = -\angle X(-f)$)
- If $x(t)$ is **REAL** and **EVEN** ($x^*(t) = x(t) \wedge x(-t) = x(t)$), then
 - $X(f)$ is real ($X^* f = X(f)$)
 - $X(f)$ is even ($X(-f) = X(f)$)
- If $x(t)$ is **REAL** and **ODD** ($x^*(t) = x(t) \wedge x(-t) = -x(t)$), then
 - $X(f)$ is imaginary ($X^*(f) = -X(f)$)
 - $X(f)$ is odd ($X(-f) = -X(f)$)

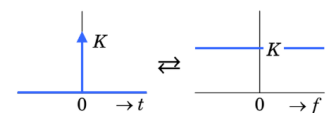
The above can apply to Fourier series coefficients of periodic signals too:

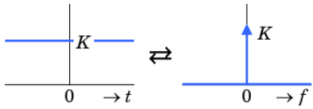
- $x_p(t)$ is **REAL**
 - c_k is conjugate symmetric ($c_k^* = c_{-k}$)
 - $|c_k|$ has even symmetry ($|c_k| = |c_{-k}|$)
 - $\angle c_k$ has odd symmetry ($\angle c_k = -\angle c_{-k}$)
- $x_p(t)$ is **REAL** and **EVEN**
 - c_k is real ($c_k^* = c_k$)
 - c_k is even ($c_k = c_{-k}$)
- $x_p(t)$ is **REAL** and **ODD**
 - c_k is imaginary ($c_k^* = -c_k$)
 - c_k is odd ($c_k = -c_{-k}$)

4.4 Spectrum of Signals that are not Absolutely Integrable

$$\mathfrak{F}\{K\delta(t)\} = \int_{-\infty}^{\infty} K\delta(t) e^{-j2\pi f t} dt = K \quad (4.13)$$

By duality, $\mathfrak{F}\{K\} = K\delta(f)$





4.4.1 Spectrum of Unit Step and Signum function

$$\Im\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

$$\Im\{\text{Sgn}(t)\} = \frac{1}{j\pi f}$$

4.4.2 Continuous-Frequency Spectrum of Periodic Signals

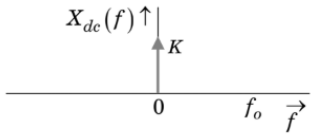
The following make use of the fact that

$$\Im\{k\} = K\delta(f) \quad (4.14)$$

DC

$$x_{dc}(t) = K$$

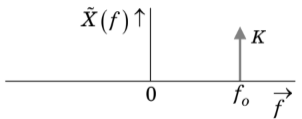
$$X_{dc}(f) = \Im\{k\} = K\Im\{1\} = K\delta(f)$$



Complex Exponential

$$\hat{x}(t) = Ke^{j2\pi f_0 t}$$

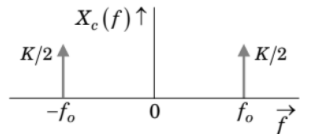
$$\hat{X}(f) = \Im\{Ke^{j2\pi f_0 t}\} = K\delta(f - f_0)$$



Cosine

$$\Im\{K \cos(2\pi f_0 t)\}$$

$$= \frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0)$$



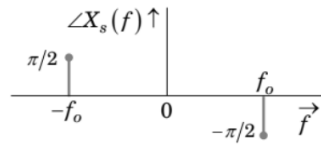
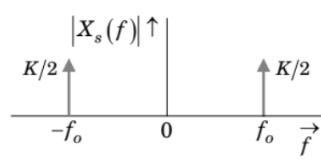
Sine

$$\Im\{K \sin(2\pi f_0 t)\}$$

$$= \frac{K}{j2}\delta(f - f_0) - \frac{K}{j2}\delta(f + f_0)$$

where

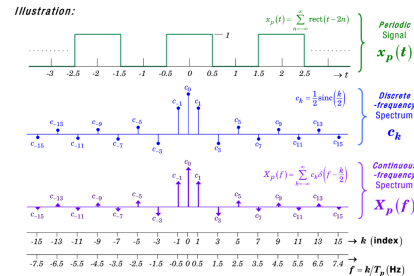
$$\begin{cases} |X_s(f)| &= \frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0) \\ \angle X_s(f) &= \begin{cases} -\pi/2, & f = f_0 \\ \pi/2, & f = -f_0 \end{cases} \end{cases}$$



Arbitrary periodic signals

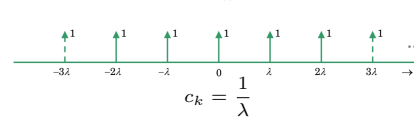
Let $x_p(t)$ be a periodic signal with period T_p and fundamental frequency f_p

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p) \quad (4.16)$$



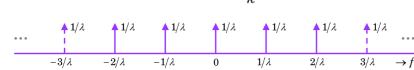
4.4.2.1 Spectrum of Dirac-δ Comb function

$$\text{comb}_\lambda(t) \triangleq \sum_n \delta(t - n\lambda)$$



$$\Im\{\text{comb}_\lambda(t)\} = \text{COMB}_\lambda(f)$$

$$= \frac{1}{\lambda} \sum_k \delta(f - k/\lambda)$$



Chapter 5

5.1 Energy Spectral Density (ESD)

Total energy of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \text{ (Joules)} \quad (5.1)$$

Rayleigh Energy Theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df, \quad (5.2)$$

where $X(f) = \Im\{x(t)\}$ is the spectrum of the signal.

Energy Spectral Density

$$E_x(f) = |X(f)|^2 \text{ Joules Hz}^{-1} \quad (5.3)$$

Properties of $E_x(f)$

1. $E_x(f)$ is a real function of f
2. $E_x(f) \geq 0 \quad \forall f$
3. $E_x(f)$ is an even function of f if $x(t)$ is real.

5.2 Power Spectral Density (PSD)

In the time-domain, the average power of a signal $x(t)$ is defined as

$$P = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt \quad (5.4)$$

Windowed version of $x(t)$:

$$x_W(t) = x(t) \text{rect}\left(\frac{t}{2W}\right) \quad (5.5)$$

Parseval Power Theorem

$$P = \lim_{W \rightarrow \infty} \frac{1}{2W} \int_{-W}^W |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} \lim_{W \rightarrow \infty} \frac{1}{2W} |X_W(f)|^2 df \quad (5.9)$$

Power Spectral Density

$$P_x(f) = \lim_{W \rightarrow \infty} \frac{1}{2W} |X_W(f)|^2 \text{ Watts Hz}^{-1} \quad (5.10)$$

Properties of $P_x(f)$

1. $P_x(f)$ is a real function of f
2. $P_x(f) \geq 0 \quad \forall f$
3. $P_x(f)$ is an even function of f if $x(t)$ is real.

5.2.1 PSD of Periodic Signals

From chapter 4 equation 4.16:

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p)$$

PSD of $x_p(t)$

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p) \quad (5.12)$$

Average power of $x_p(t)$

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad (5.13)$$

5.3 Bandwidth

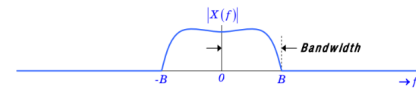
5.3.1 Bandlimited Signals

Lowpass signal

A signal $x(t)$ is said to be a bandlimited lowpass signal if its magnitude spectrum is concentrated around 0 Hz, and at the same time satisfies

$$|X(f)| = 0; \quad |f| > B \quad (5.14)$$

where B is defined as the bandwidth of the signal.



Bandpass signal

A signal $x(t)$ is said to be a bandlimited bandpass signal if its magnitude spectrum is concentrated around a non-zero center frequency f_c , and at the same time satisfies

$$|X(f)| = 0; \quad ||f| - f_c| > B/2 \quad (5.15)$$

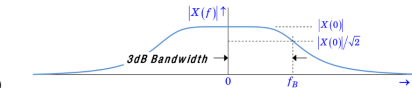
where B is defined as the bandwidth of the signal



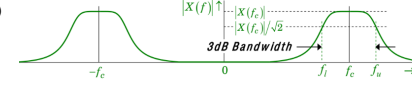
5.3.2 Signals with Unrestricted Band

5.3.2.1 3dB Bandwidth

Lowpass signal: The frequency where $|X(f)| = |X(0)|/\sqrt{2}$ first occurs (or where $|X(f)|^2 = |X(0)|^2/2$ first occurs) when f is increased from 0 Hz.



Bandpass signal:

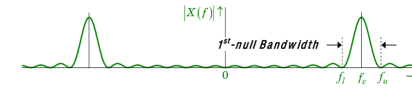


5.3.2.2 1st-null Bandwidth

Lowpass signal: The frequency at which $|X(f)| = 0$ first occurs when f is increased from 0 Hz:



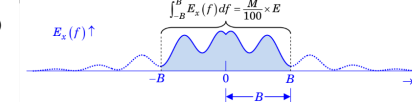
Bandpass signal:



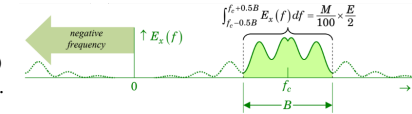
5.3.2.3 M% Energy Containment Bandwidth

Smallest bandwidth that contains at least M% of the total signal energy $E = \int_{-\infty}^{\infty} E_x(f) df$

Lowpass:



Bandpass:

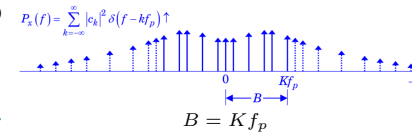


5.3.2.4 M% Power Containment Bandwidth

The smallest bandwidth that contains at least M% of the average signal power. For a periodic signal, the average power is given by

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2$$

where f_p (Hz) is the fundamental frequency and c_k 's are the Fourier series coefficients.



where K is the smallest positive integer that satisfies

$$\sum_{k=-K}^K |c_k|^2 \geq \frac{M}{100} \times P$$

Chapter 6

6.1 Systems

- A system is a mathematical model of a physical process that relates the input (or excitation) signal to the output (or response) signal.
- With an input $x(t)$ and an output $y(t)$, the system may be viewed as a transformation (or mapping) of $x(t)$ into $y(t)$, mathematically expressed as $y(t) = \mathbf{T}[x(t)]$ (6.1)

6.2 Classification of Systems

6.2.1 Systems with Memory and Without Memory

A system is said to be memoryless (or static) if its output at a given time is dependent on only the input at that time.

Otherwise, the system is said to have memory (or to be dynamic).

6.2.2 Causal and Noncausal Systems

A system is said to be causal (or non-anticipative) if its output, $y(t)$, at the present time depends on only the present and/or past values of its input, $x(t)$.

∴ not possible for a causal system to produce an output before an input is applied. ∴ $\forall t < 0$ $y(t) = 0$.

6.2.3 Stable and Unstable Systems

A system is BIBO stable (bounded-input/bounded-output) if for every bounded input $x(t)$ where

$$\forall t \quad |x(t)| \leq K \quad (6.2)$$

the system produces a bounded output $y(t)$ where

$$\forall t \quad |y(t)| \leq L \quad (6.3)$$

in which K and L are positive constants.

6.2.4 Linear and Nonlinear Systems

A linear system is one that satisfies the following two conditions:

$$\mathbf{T}[x_1(t) + x_2(t)] = \mathbf{T}[x_1(t)] + \mathbf{T}[x_2(t)] \quad (6.4)$$

$$= y_1(t) + y_2(t)$$

$$\mathbf{T}[\alpha x(t)] = \alpha \mathbf{T}[x(t)] = \alpha y(t) \quad (6.5)$$

(6.4) and (6.5) can be combined into:

$$\mathbf{T}[\alpha_1 x_1(t) + \alpha_2 x_2(t)]$$

$$= \alpha_1 \mathbf{T}[x_1(t)] + \alpha_2 \mathbf{T}[x_2(t)] \quad (6.6)$$

$$= \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

(6.6) is known as the superposition property.

Important property of linear systems:

$$x(t) = 0 \implies y(t) = 0$$

6.2.5 Time-Invariant and Time-Varying Systems

A system is time-invariant if a time shift (delay or advance) in the input signal, $x(t)$, causes the same time shift in the output signal, $y(t)$.

$$\mathbf{T}[x(t - \tau)] = y(t - \tau) \quad (6.7)$$

A time-varying system is one which does not satisfy (6.7).

Laplace Transform

$$\tilde{F}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st} dt \quad (6.8)$$

where s is a complex variable.

Inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}\{\tilde{F}(s)\} = \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} \tilde{F}(s) ds \quad (6.9)$$

Chapter 7

7.1 Impulse Response

Impulse response, $h(t)$, of a continuous-time LTI system is defined as the response/output of the system when the input is a unit impulse, $\delta(t)$.

$$\delta(t) \rightarrow \text{LTI system} \rightarrow h(t)$$

where

$$h(t) = \mathbf{T}[\delta(t)] \quad (7.1)$$

From replication property,

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^\infty x(\tau)\delta(t-\tau) d\tau \quad (7.2)$$

Substituting (7.2) into (6.1),

$$y(t) = \mathbf{T}[x(t)] = \mathbf{T}\left[\int_{-\infty}^\infty x(\tau)\delta(t-\tau) d\tau\right] \quad (7.3)$$

$$= \int_{-\infty}^\infty x(\tau)\mathbf{T}[\delta(t-\tau)] d\tau$$

As the system is time-invariant, by applying (6.7) to (7.1),

$$h(t-\tau) = \mathbf{T}[\delta(t-\tau)] \quad (7.4)$$

Sub (7.4) into (4.3):

$$y(t) = \int_{-\infty}^\infty x(\tau)h(t-\tau) d\tau = x(t) * h(t) \quad (7.5)$$

Therefore

$$\mathbf{T}[x(t)] = y(t) = x(t) * h(t)$$

7.1.1 Step Response

Step response: the output of the system when input is unit step function

$$u(t) \rightarrow h(t) \rightarrow o(t) = \int_{-\infty}^\infty h(\tau)u(t-\tau) d\tau = \int_{-\infty}^t h(\tau) d\tau$$

Step response equals integration of impulse response:

$$o(t) = \int_{-\infty}^t h(\tau) d\tau$$

Impulse response equals differentiation of step response:

$$h(t) = \frac{d}{dt} o(t)$$

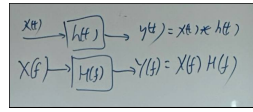
7.2 Frequency Response

The frequency response ($H(f)$) of an LTI system is defined as the Fourier transform of the system impulse response $h(t)$

$$H(f) = \mathfrak{F}\{h(t)\} = \int_{-\infty}^\infty h(t)e^{-j2\pi ft} dt \quad (7.6)$$

$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f) \cdot H(f) \quad (7.7)$$



$$H(f) = |H(f)|e^{j\angle H(f)} \quad (7.8)$$

where $|H(f)|$ is called the magnitude response and $\angle H(f)$ is called the phase response of the system.

7.3 Transfer Function

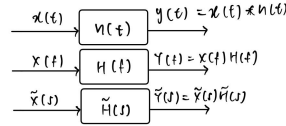
The transfer function $\tilde{H}(s)$ of an LTI system is defined as the Laplace transform of $h(t)$

$$\tilde{H}(s) = \mathcal{L}\{h(t)\} = \int_0^\infty h(t)e^{-st} dt \quad (7.9)$$

where $s = \sigma + j\omega$ is a complex variable.

$$y(t) = x(t) * h(t)$$

$$\tilde{Y}(s) = \tilde{X}(s) \cdot \tilde{H}(s) \quad (7.10)$$



7.4 Relationship between Transfer Function and Frequency Response

Substituting $s = j\omega$ into (7.9), we get

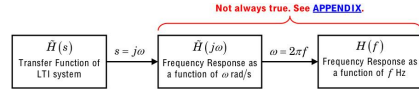
$$\tilde{H}(s)\Big|_{s=j\omega} = \tilde{H}(j\omega) = \int_0^\infty h(t)e^{-j\omega t} dt \quad (7.11)$$

Sub $\omega = 2\pi f$ into (7.11):

$$\tilde{H}(j\omega)\Big|_{\omega=2\pi f} = \int_0^\infty h(t)e^{-j2\pi ft} dt \quad (7.12)$$

For causal LTI systems, $\forall t < 0$ $h(t) = 0$. Hence (7.6) and (7.12) are equivalent.

$$H(f) = \tilde{H}(j\omega)\Big|_{\omega=2\pi f} \quad (7.13)$$



$$\tilde{H}(j\omega) = |\tilde{H}(j\omega)|e^{j\angle \tilde{H}(j\omega)} \quad (7.14)$$

where $|\tilde{H}(j\omega)|$ is called the magnitude response and $\angle \tilde{H}(j\omega)$ is called the phase response of the system.

7.4 Sinusoidal Response at Steady-State

Let system input at steady-state be

$$x(t) = Ae^{j(2\pi f_0 t + \psi)} \quad (7.15)$$

Then

$$X(f) = Ae^{j\psi} \delta(f - f_0) \quad (7.16)$$

$$Y(f) = A |H(f_0)| e^{j(\psi + \angle H(f_0))} \delta(f - f_0) \quad (7.17)$$

$$y(t) = \mathfrak{F}^{-1}\{Y(f)\} = A |H(f_0)| e^{j(2\pi f_0 t + \psi + \angle H(f_0))} \quad (7.18)$$

System Input : $Ae^{j(2\pi f_0 t + \psi)}$
System Output : $A|H(f_0)|e^{j(2\pi f_0 t + \psi + \angle H(f_0))}$

$$x(t) = Ae^{j(2\pi f_0 t + \psi)} \rightarrow H(f) \rightarrow y(t) = A|H(f_0)|e^{j(2\pi f_0 t + \psi + \angle H(f_0))}$$

Steady-state Sinusoidal Response of a LTI System in f -domain

$$x(t) = Ae^{j(\omega_0 t + \psi)} \rightarrow \tilde{H}(j\omega) \rightarrow y(t) = A|\tilde{H}(j\omega_0)|e^{j(\omega_0 t + \psi + \angle \tilde{H}(j\omega_0))}$$

$$x(t) = A\cos(\omega_0 t + \psi) \rightarrow \tilde{H}(j\omega) \rightarrow y(t) = A|\tilde{H}(j\omega_0)|\cos(\omega_0 t + \psi + \angle \tilde{H}(j\omega_0))$$

$$x(t) = A\sin(\omega_0 t + \psi) \rightarrow \tilde{H}(j\omega) \rightarrow y(t) = A|\tilde{H}(j\omega_0)|\sin(\omega_0 t + \psi + \angle \tilde{H}(j\omega_0))$$

Steady-state Sinusoidal Response of a LTI System in ω -domain

7.6 LTI Systems Described by Differential Equations

LTI systems represented by linear constant-coefficient differential equations have the general form

$$\sum_{n=0}^N a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^M b_m \frac{d^m x(t)}{dt^m} \quad (7.21)$$

where $x(t)$ is input, $y(t)$ is output, and a_n, b_m are real constants.

7.6.1 Transfer Function

Applying Laplace to both sides of (7.21) with initial conditions set to 0,

$$\sum_{n=0}^N a_n \tilde{Y}(s)s^n = \sum_{m=0}^M b_m \tilde{X}(s)s^m \quad (7.22)$$

$$\tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \left(\sum_{m=0}^M b_m s^m \right) / \left(\sum_{n=0}^N a_n s^n \right) = \frac{b_M s^M + b_{M-1} s^{M-1} + \dots + b_0}{a_N s^N + a_{N-1} s^{N-1} + \dots + a_0} \quad (7.23a)$$

$$\tilde{H}(s) = K \frac{\left(\frac{s}{z_1} + 1\right) \left(\frac{s}{z_2} + 1\right) \dots \left(\frac{s}{z_M} + 1\right)}{\left(\frac{s}{p_1} + 1\right) \left(\frac{s}{p_2} + 1\right) \dots \left(\frac{s}{p_N} + 1\right)}$$

$$K = \frac{a_0}{b_0} \quad (7.23b)$$

$$\tilde{H}(s) = K' \frac{(s + z_1)(s + z_2) \dots (s + z_M)}{(s + p_1)(s + p_2) \dots (s + p_N)}$$

$$K = \frac{b_M}{a_N} \quad (7.23c)$$

$\forall n \in \{1, 2, \dots, N\}$

- $-p_n$ are roots of the denominator polynomial of $\tilde{H}(s)$
- $\tilde{H}(-p_n) = \infty$
- $-p_n$ are called **poles** of $\tilde{H}(s)$
- $\forall m \in \{1, 2, \dots, M\}$
- $-z_m$ are roots of the numerator polynomial of $\tilde{H}(s)$
- $\tilde{H}(-z_m) = 0$
- $-z_m$ are called **zeros** of $\tilde{H}(s)$

The system is said to have N poles and M zeros, and the difference $N - M$ is called pole-zero excess.

7.6.2 System Stability

BIBO Stable

- All system poles lying on the left-half s-plane
- $h(t)$ will converge to 0 as t tends to infinity

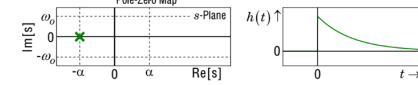
$$\lim_{t \rightarrow \infty} h(t) = 0$$

E.g.

$$\tilde{H}(s) = \frac{1}{s + \alpha}$$

$$\text{Pole: } s = -\alpha$$

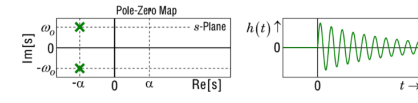
$$h(t) = e^{-\alpha t} u(t)$$



$$\tilde{H}(s) = \frac{\omega_0}{(s + \alpha)^2 + \omega_0^2}$$

$$\text{Poles: } s_{1,2} = -\alpha \pm j\omega_0$$

$$h(t) = e^{-\alpha t} \sin(\omega_0 t) u(t)$$



Marginally Stable

- One or more non-repeated system poles lying on the imaginary axis of the s-plane and no system pole lying on the right half s-plane.
- $h(t)$ will not “blow up” and become unbounded, but neither will it converge to zero as t tends to infinity.

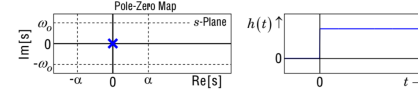
$$\lim_{t \rightarrow \infty} |h(t)| \neq \infty \text{ and } \lim_{t \rightarrow \infty} h(t) \neq 0$$

E.g.

$$\tilde{H}(s) = \frac{1}{s}$$

$$\text{Pole: } s = 0$$

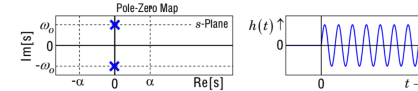
$$h(t) = u(t)$$



$$\tilde{H}(s) = \frac{\omega_0}{s^2 + \omega_0^2}$$

$$\text{Poles: } s_{1,2} = \pm j\omega_0$$

$$h(t) = \sin(\omega_0 t) u(t)$$



Unstable (Case 1)

- One or more system poles lying on the right-half s-plane
- $h(t)$ will “blow up” and become unbounded as t tends to infinity

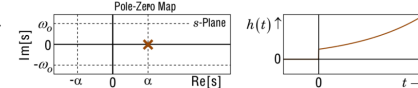
$$\lim_{t \rightarrow \infty} |h(t)| = \infty$$

E.g.

$$\tilde{H}(s) = \frac{1}{s - \alpha}$$

$$\text{Pole: } s = \alpha$$

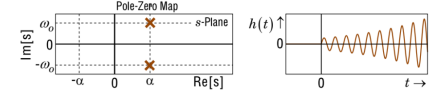
$$h(t) = e^{\alpha t} u(t)$$



$$\tilde{H}(s) = \frac{\omega_0}{(s - \alpha)^2 + \omega_0^2}$$

$$\text{Poles: } s_{1,2} = \alpha \pm j\omega_0$$

$$h(t) = e^{\alpha t} \sin(\omega_0 t) u(t)$$



Unstable (Case 2)

- One or more repeated system poles lying on the imaginary axis
- $h(t)$ will “blow up” and become unbounded as t tends to infinity

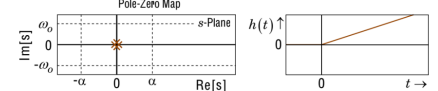
$$\lim_{t \rightarrow \infty} |h(t)| = \infty$$

E.g.

$$\tilde{H}(s) = \frac{1}{s^2}$$

$$\text{Pole: } s_{1,2} = 0$$

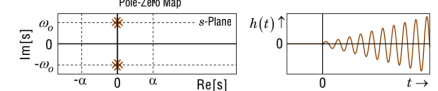
$$h(t) = tu(t)$$



$$\tilde{H}(s) = \frac{\omega_0}{(s^2 + \omega_0^2)^2}$$

$$\text{Poles: } s_{1,2,3,4} = \pm j\omega_0, \pm j\omega_0$$

$$h(t) = \frac{1}{2} [\omega_0^{-1} \sin(\omega_0 t) - t \cos(\omega_0 t)] u(t)$$



7.7 First Order System (Standard Form)

7.7.1 Differential Eqn, Transfer Func, Impulse Response and Step Response

- Differential equation:

$$T \frac{dy(t)}{dt} + y(t) = Kx(t) \quad (7.26)$$

where

- $x(t)$: system input
- $y(t)$: system output
- K : DC gain
- T : time-constant

- Transfer Function $\tilde{H}(s)$:

$$Ts\tilde{Y}(s) + \tilde{Y}(s) = K\tilde{X}(s)$$

$$\rightarrow \tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{K}{Ts + 1} \quad (7.27)$$

$$\text{Pole: } s_1 = -\frac{1}{T}$$

- Impulse Response $h(t)$
- $h(t) = \mathcal{L}^{-1}\{\tilde{H}(s)\} = \frac{K}{T} e^{-t/T} u(t)$
- Step Response $o(t)$

$$o(t) = \int_{-\infty}^t h(\tau) d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s} \tilde{H}(s)\right\} = K \left[1 - e^{-t/T}\right] u(t)$$

7.8 Second Order System (Standard Form)

7.8.1 Differential Eqn and Transfer Func

- Differential equation:

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 x(t) \quad (7.28)$$

where

- $x(t)$: system input
- $y(t)$: system output
- ζ : damping ratio
- ω_n : undamped natural frequency (when $\zeta < 1$)
- K : DC gain

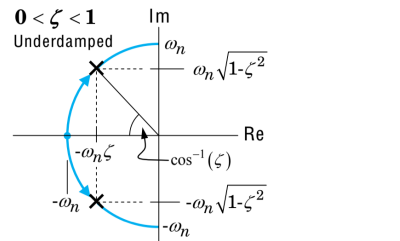
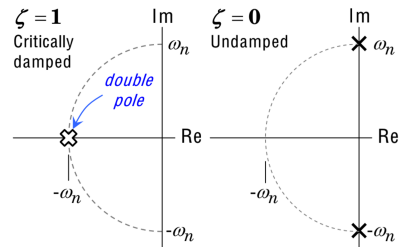
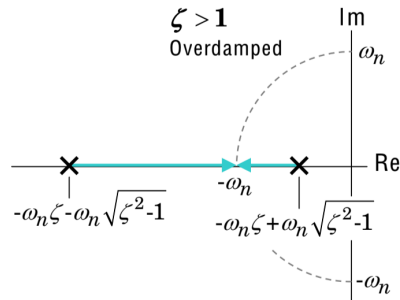
- Transfer function $\tilde{H}(s)$
 $s^2 \tilde{Y}(s) + 2\zeta\omega_n s \tilde{Y}(s) + \omega_n^2 \tilde{Y}(s)$

$$= K\omega_n^2 \tilde{X}(s)$$

$$\Rightarrow \tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (7.29)$$

Poles: $s_{1,2} = -\omega_n\zeta \pm \omega_n(\zeta^2 - 1)^{1/2}$

- Damping



7.8.2 Impulse Response and Step Response

7.8.2.1 Overdamped System ($\zeta > 1$)

Chapter 8

Bode plot: approximate visualization of frequency response, $\tilde{H}(j\omega)$ of a system

- Magnitude plot: plot of $|\tilde{H}(j\omega)|_{dB} = 20\log_{10}(|\tilde{H}(j\omega)|)$ dB

- Phase plot: plot of $\angle\tilde{H}(j\omega)$ in degrees
- x-axis is logarithmically scaled (semilog-x: scale is log, but labels are still linear)
- Only positive frequency side visualized (which suffices for real systems as $|\tilde{H}(j\omega)|$ and $\angle\tilde{H}(j\omega)$ are even and odd functions of ω respectively)
- 0 is not in the axis cuz it goes from 1 to 0.1 to 0.01 to 0.001...

8.1 Construction of Bode Plots

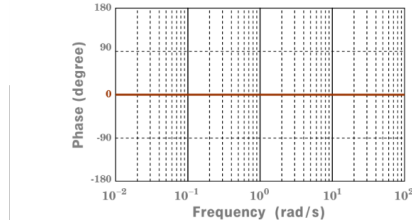
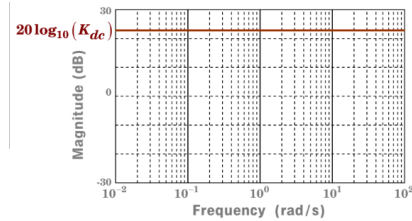
Need to express (7.23b) in a suitable form for each of the following cases:

- Systems without integrator and differentiator
- Systems with differentiators
- Systems with integrators

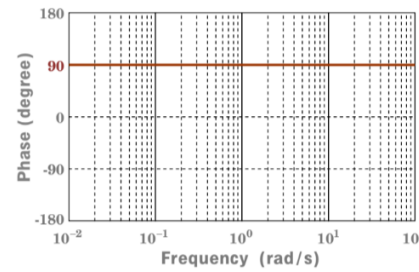
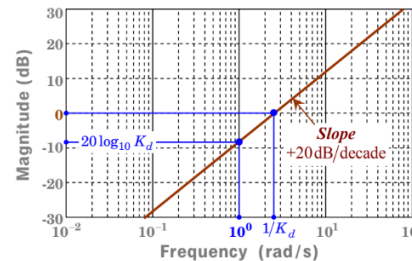
Basic systems:

1. $\tilde{H}(s) = K_{dc}$: DC gain (constant)
2. $\tilde{H}(s) = K_d s$: differentiator with gain K_d
3. $\tilde{H}(s) = K_i/s$: integrator with gain K_i
4. $\tilde{H}(s) = s/z_m + 1$: zero factor with unity DC gain ($\tilde{H}(0) = 1$)
5. $\tilde{H}(s) = \frac{1}{s/p_n + 1}$: pole factor with unity DC gain
6. $\tilde{H}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$: 2nd-order factor with unity DC gain

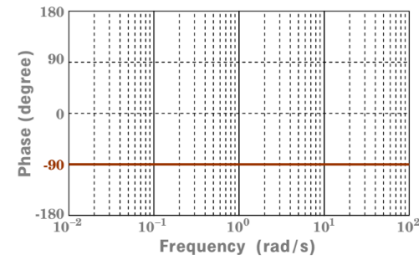
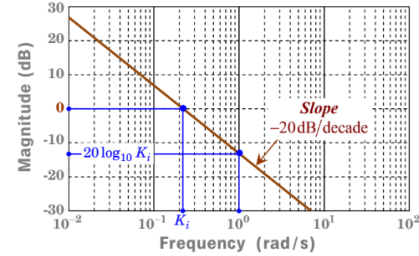
DC gain ($\tilde{H}(s) = K_{dc}$)



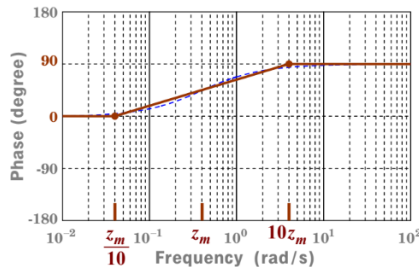
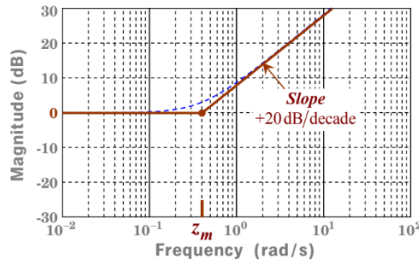
Differentiator ($\tilde{H}(s) = K_d s$)



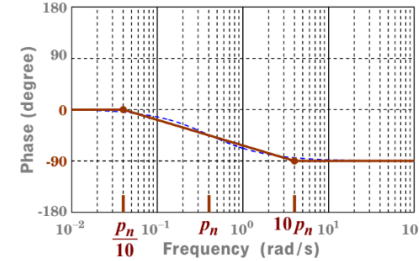
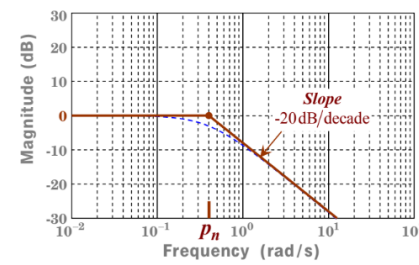
Integrator ($\tilde{H}(s) = K_i/s$)



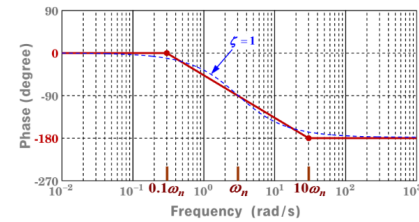
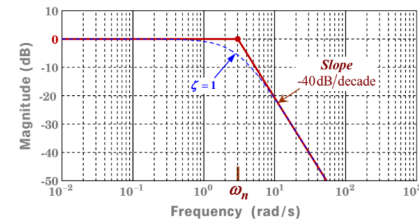
Zero factor ($\tilde{H}(s) = s/z_m + 1$)



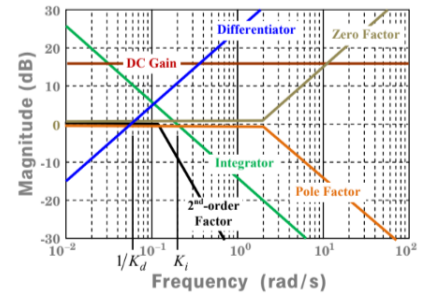
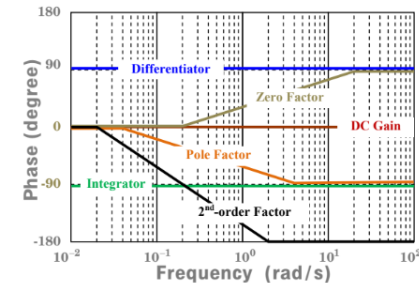
Pole factor ($\tilde{H}(s) = \frac{1}{s/p_n + 1}$)



2nd-order factor ($\tilde{H}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$)



8.2 Asymptotic Behavior of Bode Plots



Asymptotic phase of phase plot

High frequency:

$$\lim_{\omega \rightarrow \infty} \angle\tilde{H}(j\omega) = \text{Pole-zero excess} \times (-90^\circ) \quad (8.4a)$$

Low frequency:

$$\lim_{\omega \rightarrow 0} \angle\tilde{H}(j\omega) = \left[\text{No. of } \int dt - \text{No. of } \frac{d}{dt} \right] \times (-90^\circ) \quad (8.4b)$$

Asymptotic slope of magnitude plot

High frequency:

$$\lim_{\omega \rightarrow \infty} \left[\text{Slope of } |\tilde{H}(j\omega)| \right] = [\text{Pole-zero excess}] \times (-20 \text{ dB/decade}) \quad (8.5a)$$

Low frequency:

$$\lim_{\omega \rightarrow 0} \left[\text{Slope of } |\tilde{H}(j\omega)| \right] = \left[\text{No. of } \int dt - \text{No. of } \frac{d}{dt} \right] \times (-20 \text{ dB/decade}) \quad (8.5a)$$