Euler's formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

Chapter 1

1.2.2 Bounded signals

$x(t)$ is bounded if

$\exists M\ [(0 < M < \infty) \wedge (\forall t\ |x(t)| \leq M)]$

(got upper n lower range limit)

1.2.3 Absolutely integrable signals

$x(t)$ is absolutely integrable if

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

1.2.6 Energy and Power Signals

Energy signals

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad (1.3a)$$

$$x(t) \text{ is an energy signal } \iff 0 < E < \infty \qquad (1.3b)$$

Power signals

$$P = \lim_{\tau \to \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt \qquad (1.4a)$$

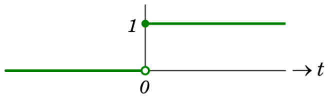
$$x(t) \text{ is a power signal } \iff 0 < P < \infty \qquad (1.4b)$$

If $x(t)$ is a periodic signal, average power may be computed by

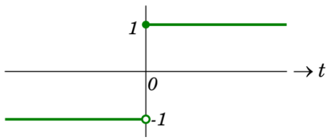
$$\frac{1}{T}\int_0^T |x(t)|^2 dt$$

- Energy signals have 0 average power, bc E = finite implies P = 0
- Power signals have infinite total energy, bc P = finite implies $E = \infty$
- All bounded periodic signals are power signals

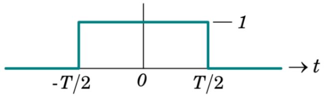
u(t):



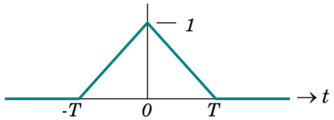
sgn(t):



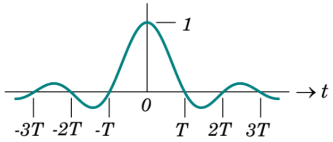
rect($\frac{t}{T}$):



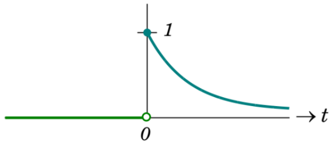
tri($\frac{t}{T}$):



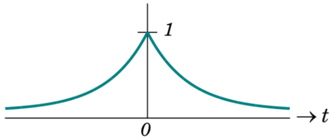
sinc($\frac{t}{T}$):



$e^{-\alpha t}u(t)$:



$e^{-\alpha|t|}$:



Sinusoidal signals

$$x(t) = \mu\cos(\omega_0 t + \phi)$$
$$= \mu\cos(2\pi f_0 t + \phi)$$
$$= \mu\cos(\frac{2\pi t}{T} + \phi)$$
$$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$$

Chapter 2

2.1 Time-domain Operations

2.1.5 Convolution of 2 Signals

$x(t) * y(t) = \int_{-\infty}^{\infty} x(\alpha)y(t-\alpha)\, d\alpha$

2.2 Dirac-δ function

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \neq 0 \end{cases}$$

Properties:

1. Symmetry: $$\delta(t) = \delta(-t) \qquad (2.3)$$
2. Sampling: $$x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda) \qquad (2.4)$$
3. Sifting $$\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt$$ $$= x(\lambda)\int_{-\infty}^{\infty} \delta(t-\lambda)dt = x(\lambda) \qquad (2.5)$$
4. Replication $$x(t) * \delta(t-\lambda)$$ $$= \int_{-\infty}^{\infty} x(\zeta)\delta(t-\zeta-\lambda)d\zeta$$ $$= \int_{-\infty}^{\infty} x(\zeta)\delta(\zeta-(t-\lambda))d\zeta = x(t-\lambda) \qquad (2.6)$$

2.2.1 Dirac-δ Comb function

$$\sum_{n=-\infty}^{\infty} \delta(t - nT)$$
$$= \ldots + \delta(t+T) + \delta(t) + \delta(t-T) + \ldots$$

Convolution with Dirac-δ Comb function

$$x_p(t) = x(t) * \sum_n \delta(t-nT)$$
$$= \sum_n x(t-nT)$$

$x(t)$ is known as the generating function.

Multiplication with the Dirac-δ Comb function

Used for sampling

$$x_s(t) = x(t) \times \sum_n \delta(t-nT)$$
$$= \sum_n x(t) \times \delta(t-nT)$$
$$= \sum_n x(nT)\delta(t-nT)$$

Chapter 3

3.2 Spectrum of a Sinusoid

Spectrum of a Complex Exponential Signal

$\tilde{x}(t) = \mu e^{j(2\pi f_0 t + \phi)} = \mu e^{j\phi} \times e^{j2\pi f_0 t}$,

where μ: magnitude spectrum, ϕ: phase spectrum, f_0: frequency

Spectrum of a Cosine Signal

$$\mu\cos(2\pi f_0 t + \phi)$$
$$= \frac{\mu}{2}e^{j\phi}e^{j2\pi f_0 t} + \frac{\mu}{2}e^{j(-\phi)}e^{j2\pi(-f_0)t}$$

Spectrum of a Sine Signal

$$\mu\sin(2\pi f_0 t + \phi) = \frac{\mu}{2}e^{j(\phi - 0.5\pi)}e^{j2\pi f_0 t}$$
$$+ \frac{\mu}{2}e^{j(-\phi+0.5\pi)}e^{j2\pi(-f_0)t}$$

Complex exponential Fourier Series

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T_p}$$
$$= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} \qquad (3.1a)$$
$$c_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)e^{-j2\pi kt/T_p}dt, k \in \mathbb{Z} \qquad (3.1b)$$

Trigonometric Fourier Series

$$x_p(t) = a_0 + 2\sum_{k=1}^{\infty}[a_k\cos(2\pi kt/T_p)$$
$$+ b_k\sin(2\pi kt/T_p)]$$
$$a_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)\cos(2\pi kt/T_p)dt; k \geq 0$$
$$b_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)\sin(2\pi kt/T_p)dt; k > 0 \qquad (3.2)$$

Chapter 4

4.1 Fourier Transform

Forward Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \qquad (4.1a)$$

Inverse Fourier Transform

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \qquad (4.1b)$$

Spectrum of exponentially decaying pulse

$$x(t) = Ae^{-\alpha t}u(t)$$

Assume $\alpha > 0$

$$X(f) = \frac{A}{\alpha + j2\pi f}$$

4.3 Spectral properties of a REAL signal

- If $x(t)$ is **REAL** ($x^*(t) = x(t)$), then
  - $X(f)$ is conjugate symmetric ($X^*(f) = X(f)$)
  - $|X(f)|$ is even ($|X(f)| = |X(-f)|$)
  - $\angle X(f)$ is odd ($\angle X(f) = -\angle X(-f)$)
- If $x(t)$ is **REAL and EVEN** ($x^*(t) = x(t) \wedge x(-t) = x(t)$), then
  - $X(f)$ is real ($X^*f = X(f)$)
  - $X(f)$ is even ($X(-f) = X(f)$)
- If $x(t)$ is **REAL and ODD** ($x^*(t) = x(t) \wedge x(-t) = -x(t)$), then
  - $X(f)$ is imaginary ($X^*(f) = -X(f)$)
  - $X(f)$ is odd ($X(-f) = -X(f)$)

The above can apply to Fourier series coefficients of periodic signals too:

- $x_p(t)$ is **REAL**
  - c_k is conjugate symmetric ($c_k^* = c_{-k}$)
  - $|c_k|$ has even symmetry ($|c_k| = |c_{-k}|$)
  - $\angle c_k$ has odd symmetry ($\angle c_k = -\angle c_{-k}$)
- $x_p(t)$ is **REAL and EVEN**
  - c_k is real ($c_k^* = c_k$)
  - c_k is even ($c_k = c_{-k}$)
- $x_p(t)$ is **REAL and ODD**
  - c_k is imaginary ($c_k^* = -c_k$)
  - c_k is odd ($c_k = -c_{-k}$)

4.4 Spectrum of Signals that are not Absolutely Integrable

$$\Im\{K\delta(t)\} = \int_{-\infty}^{\infty} K\delta(t)e^{-j2\pi ft}dt = K \qquad (4.13)$$

By duality, $\Im\{K\} = K\delta(f)$

4.4.1 Spectrm of Unit Step and Signum function

$$\Im\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$
$$\Im\{\mathrm{Sgn}(t)\} = \frac{1}{j\pi f}$$

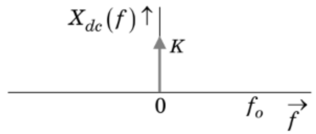
4.4.2 Continuous-Frequency Spectrum of Periodic Signals

The following make use of the fact that

$$\Im\{k\} = K\delta(f) \qquad (4.14)$$

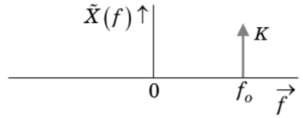
DC

$$x_{dc}(t) = K$$
$$X_{dc}(f) = \Im\{k\} = K\Im\{1\} = K\delta(f)$$



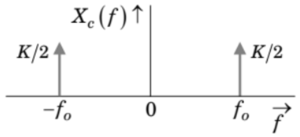
Complex Exponential

$$\tilde{x}(t) = Ke^{j2\pi f_0 t}$$
$$\tilde{X}(f) = \Im\{Ke^{j2\pi f_0 t}\} = K\delta(f - f_0)$$



Cosine

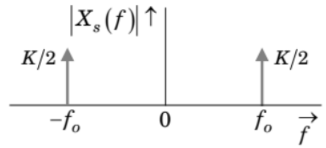
$$\Im\{K\cos(2\pi f_0 t)\}$$
$$= \frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0)$$

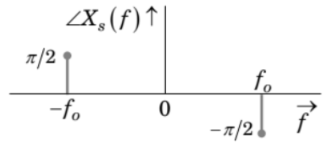


Sine

$$\Im\{K\sin(2\pi f_0 t)\}$$
$$= \frac{K}{j2}\delta(f - f_0) - \frac{K}{j2}\delta(f + f_0)$$

where
$$\begin{cases} |X_s(f)| = \frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0) \\ \angle X_s(f) = \begin{cases} -\pi/2, & f = f_0 \\ \pi/2, & f = -f_0 \end{cases} \end{cases}$$



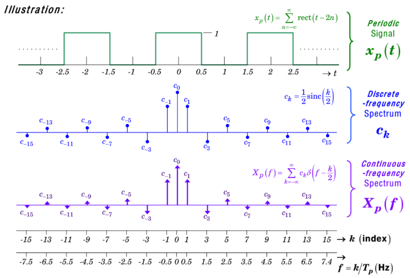


Arbitrary periodic signals

Let $x_p(t)$ be a periodic signal with period T_p and fundamental frequency f_p

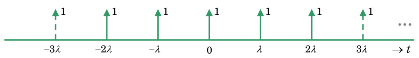
$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k\delta(f - kf_p) \qquad (4.16)$$

Illustration:

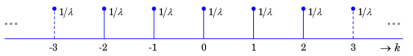


4.4.2.1 Spectrum of Dirac-δ Comb function

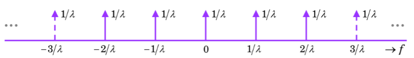
$$\mathrm{comb}_\lambda(t) \triangleq \sum_n \delta(t - n\lambda)$$



$$c_k = \frac{1}{\lambda}$$



$$\Im\{\mathrm{comb}_\lambda(t)\} = \mathrm{COMB}_\lambda(f)$$
$$= \frac{1}{\lambda}\sum_k \delta(f - k/\lambda)$$



Chapter 5

5.1 Energy Spectral Density (ESD)

Total energy of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \text{ (Joules)} \quad (5.1)$$

Rayleigh Energy Theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df, \quad (5.2)$$

where $X(f) = \Im\{x(t)\}$ is the spectrum of the signal.

Energy Spectral Density

$$E_x(f) = |X(f)|^2 \text{ Joules Hz}^{-1} \quad (5.3)$$

Properties of $E_x(f)$

1. $E_x(f)$ is a real function of f
2. $E_x(f) \geq 0 \quad \forall f$
3. $E_x(f)$ is an even function of f if $x(t)$ is real.

5.2 Power Spectral Density (PSD)

In the time-domain, the average power of a signal $x(t)$ is defined as

$$P = \lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} |x(t)|^2 dt \quad (5.4)$$

Windowed version of $x(t)$:

$$x_W(t) = x(t)\mathrm{rect}\left(\frac{t}{2W}\right) \quad (5.5)$$

Parseval Power Theorem

$$P = \lim_{W\to\infty} \frac{1}{2W}\int_{-W}^{W} |x(t)|^2 dt$$
$$= \int_{-\infty}^{\infty} \lim_{W\to\infty} \frac{1}{2W} |X_W(f)|^2 df \quad (5.9)$$

Power Spectral Density

$$P_x(f) = \lim_{W\to\infty} \frac{1}{2W} |X_W(f)|^2 \text{ Watts Hz}^{-1} \quad (5.10)$$

Properties of $P_x(f)$

1. $P_x(f)$ is a real function of f
2. $P_x(f) \geq 0 \quad \forall f$
3. $P_x(f)$ is an even function of f if $x(t)$ is real.

5.2.1 PSD of Periodic Signals

From chapter 4 equation 4.16:

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p)$$

PSD of $x_p(t)$

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p) \quad (5.12)$$

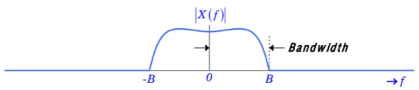
Average power of $x_p(t)$

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad (5.13)$$

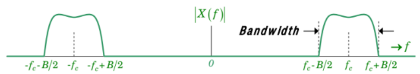
5.3 Bandwidth

5.3.1 Bandlimited Signals

Lowpass signal



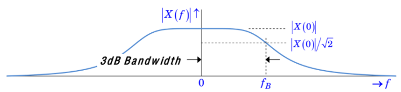
Bandpass signal



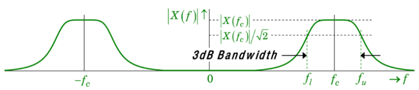
5.3.2 Signals with Unrestricted Band

5.3.2.1 3dB Bandwidth

Lowpass signal:

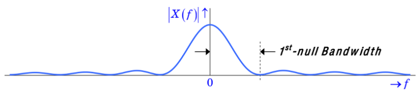


Bandpass signal:

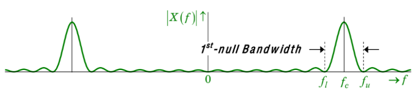


5.3.2.2 1st-null Bandwidth

Lowpass signal:



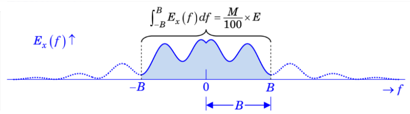
Bandpass signal:



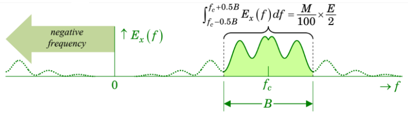
5.3.2.3 M% Energy Containment Bandwidth

Smallest bandwidth that contains at least M% of the total signal energy $E = \int_{-\infty}^{\infty} E_x(f)\, df$

Lowpass:



Bandpass:

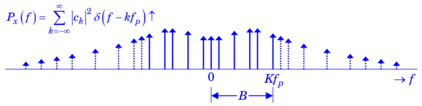


5.3.2.4 M% Power Containment Bandwidth

The smallest bandwidth that contains at least M% of the average signal power. For a periodic signal, the aerage power is given by

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2$$

where f_p(Hz) is the fundamental frequency and c_k's are the Fourier series coefficients.



$$B = Kf_p$$

where K is the smallest positive integer that satisfies

$$\sum_{k=-K}^{K} |c_k|^2 \geq \frac{M}{100} \times P$$

Chapter 6

6.1 Systems

6.2 Classification of Systems

6.2.1 Systems with Memory and Without Memory

Memoryless: output at a given time is dependent on only the input at that time.

Otherwise, the system has memory / is dynamic.

6.2.2 Causal and Noncausal Systems

Causal (or non-anticipative): Its output, $y(t)$, at the present time depends on only the present and/or past values of its input, $x(t)$.

$\therefore$ not possible for a causal system to produce an output before an input is applied. $\therefore\ \forall t < 0\ y(t) = 0$.

6.2.3 Stable and Unstable Systems

BIBO stable (bounded-input/bounded-output): For every bounded input $x(t)$ where

$$\forall t\ |x(t)| \leq k \quad (6.2)$$

the system produces a bounded output $y(t)$ where

$$\forall t\ |y(t)| \leq L \quad (6.3)$$

in which K and L are positive constants.

6.2.4 Linear and Nonlinear Systems

Linear system satisfies the following:

$$\mathbf{T}[\alpha_1 x_1(t) + \alpha_2 x_2(t)]$$
$$= \alpha_1 \mathbf{T}[x_1(t)] + \alpha_2 \mathbf{T}[x_2(t)] \quad (6.6)$$
$$= \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

(6.6) is known as the superposition property.

Important property of linear systems:
$x(t) = 0 \implies y(t) = 0$

6.2.5 Time-Invariant and Time-Varying Systems

Time-invariant: a time shift (delay or advance) in the input signal, $x(t)$, causes the same time shift in the output signal, $y(t)$.

$$\mathbf{T}[x(t-\tau)] = y(t-\tau) \quad (6.7)$$

A time-varying system is one which does not satisfy (6.7).

Laplace Transform

$$\tilde{F}(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt \quad (6.8)$$

where s is a complex variable.

Inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}\left\{\tilde{F}(s)\right\} = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} \tilde{F}(s)ds \quad (6.9)$$

Chapter 7

7.1 Impulse Response

Impulse response, $h(t)$: The response/output when the input is a unit impulse, $\delta(t)$.

$$\delta(t) \to \text{LTI system} \to h(t)$$

where

$$h(t) = \mathbf{T}[\delta(t)] \quad (7.1)$$

$$\mathbf{T}[x(t)] = y(t) = x(t) * h(t) \quad (7.5)$$

7.1.1 Step Response

Step response: the output of the system when input is unit step function

$$u(t) \to h(t) \to o(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$
$$= \int_{-\infty}^{t} h(\tau)d\tau$$

Step response equals integration of impulse response: $\quad o(t) = \int_{-\infty}^{t} h(\tau)d\tau$

Impulse response equals differentiation of step response: $\quad h(t) = \frac{d}{dt}o(t)$

7.2 Frequency Response

Frequency response ($H(f)$): The Fourier transform of the system impulse response $h(t)$

$$H(f) = \Im\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt \quad (7.6)$$

$$Y(f) = X(f) \cdot H(f) \quad (7.7)$$
$$H(f) = |H(f)|e^{j\angle H(f)} \quad (7.8)$$

where $|H(f)|$ is called the magnitude response and $\angle H(f)$ is called the phase response of the system.

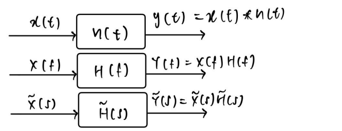
7.3 Transfer Function

Transfer function $\tilde{H}(s)$: Laplace transform of $h(t)$

$$\tilde{H}(s) = \mathcal{L}\{h(t)\} = \int_0^{\infty} h(t)e^{-st}dt \quad (7.9)$$

where $s = \sigma + j\omega$ is a complex variable.

$$y(t) = x(t) * h(t)$$
$$\tilde{Y}(s) = \tilde{X}(s) \cdot \tilde{H}(s) \quad (7.10)$$



7.4 Relationship between Transfer Function and Frequency Response

Substituting $s = j\omega$ into (7.9), we get

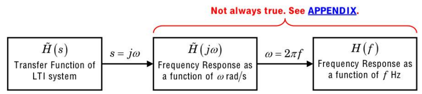
$$\tilde{H}(s)\Big|_{s=j\omega} = \tilde{H}(j\omega) = \int_0^{\infty} h(t)e^{-j\omega t}dt \quad (7.11)$$

Sub $\omega = 2\pi f$ into (7.11):

$$\tilde{H}(j\omega)\Big|_{\omega=2\pi f} = \int_0^{\infty} h(t)e^{-j2\pi ft}dt \quad (7.12)$$

For causal LTI systems, $\forall t < 0\ h(t) = 0$. Hence (7.6) and (7.12) are equivalent.

$$H(f) = \tilde{H}(j\omega)\Big|_{\omega=2\pi f} \quad (7.13)$$



$$\tilde{H}(j\omega) = |\tilde{H}(j\omega)|e^{j\angle\tilde{H}(j\omega)} \quad (7.14)$$

where $|\tilde{H}(j\omega)|$ is called the magnitude response and $\angle\tilde{H}(j\omega)$ is called the phase response of the system.

7.4 Sinusoidal Response at Steady-State

Let system input at steady-state be

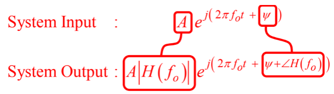
$$x(t) = Ae^{j(2\pi f_0 t + \psi)} \quad (7.15)$$

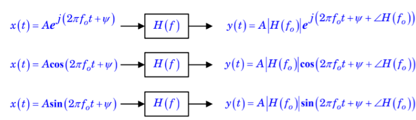
Then

$$X(f) = Ae^{j\psi}\delta(f - f_0) \quad (7.16)$$

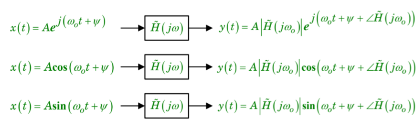
$$Y(f) = A|H(f_0)|e^{j(\psi + \angle H(f_0))}\delta(f - f_0) \quad (7.17)$$

$$y(t) = \Im^{-1}\{Y(f)\}$$
$$= A|H(f_0)|e^{j(2\pi f_0 t + \psi + \angle H(f_0))} \quad (7.18)$$





Steady-state Sinusoidal Response of a LTI System in f-domain



Steady-state Sinusoidal Response of a LTI System in ω-domain

7.6 LTI Systems Described by Differential Equations

LTI systems represented by linear constant-coefficient differential equations have the general form

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m} \qquad (7.21)$$

where $x(t)$ is input, $y(t)$ is output, and a_n, b_m are real constants.

7.6.1 Transfer Function

Applying Laplace to both sides of (7.21) with initial conditions set to 0,

$$\sum_{n=0}^{N} a_n \tilde{Y}(s) s^n = \sum_{m=0}^{M} b_m \tilde{X}(s) s^m \qquad (7.22)$$

$$\tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \ldots + b_0}{a_N s^N + a_{N-1} s^{N-1} + \ldots + a_0} \qquad (7.23a)$$

$$\tilde{H}(s) = K \frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\ldots\left(\frac{s}{z_M}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\ldots\left(\frac{s}{p_N}+1\right)}$$

$$K = \frac{a_0}{b_0} \qquad (7.23b)$$

$$\tilde{H}(s) = K' \frac{(s+z_1)(s+z_2)\ldots(s+z_M)}{(s+p_1)(s+p_2)\ldots(s+p_N)}$$

$$K = \frac{b_M}{a_N} \qquad (7.23c)$$

$\forall n \in \{1, 2, \ldots, N\}$
- $\tilde{H}(-p_n) = \infty$
- $-p_n$ are called **poles** of $\tilde{H}(s)$

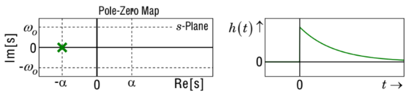
$\forall m \in \{1, 2, \ldots, M\}$
- $\tilde{H}(-z_m) = 0$
- $-z_m$ are called **zeros** of $\tilde{H}(s)$

The system is said to have N poles and M zeros, and the difference $N - M$ is called pole-zero excess.

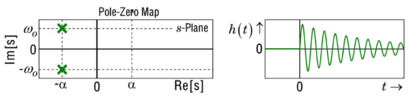
7.6.2 System Stability

BIBO Stable
- All system poles lying on the left-half s-plane
- $h(t)$ will converge to 0 as t tends to infinity
  $\lim_{t\to\infty} h(t) = 0$

E.g.
$\tilde{H}(s) = \frac{1}{s+\alpha}$
Pole: $s = -\alpha$
$h(t) = e^{-\alpha t} u(t)$

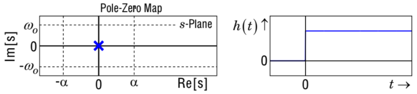


$\tilde{H}(s) = \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$
Poles: $s_{1,2} = -\alpha \pm j\omega_0$
$h(t) = e^{-\alpha t} \sin(\omega_0 t) u(t)$

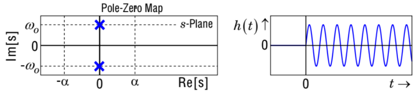


Marginally Stable
- One or more **non-repeated** system poles lying on the imaginary axis of the s-plane and no system pole lying on the right half s-plane.
- $h(t)$ will not "blow up" and become unbounded, but neither will it converge to zero as t tends to infinity.
  $\lim_{t\to\infty} |h(t)| \neq \infty$ and $\lim_{t\to\infty} h(t) \neq 0$

E.g.
$\tilde{H}(s) = \frac{1}{s}$
Pole: $s = 0$
$h(t) = u(t)$

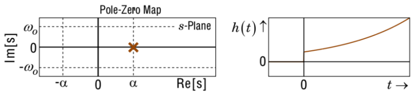


$\tilde{H}(s) = \frac{\omega_0}{s^2+\omega_0^2}$
Poles: $s_{1,2} = \pm j\omega_0$
$h(t) = \sin(\omega_0 t) u(t)$

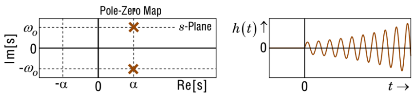


Unstable (Case 1)
- One or more system poles lying on the right-half s-plane
- $h(t)$ will "blow up" and become unbounded as t tends to infinity
  $\lim_{t\to\infty} |h(t)| = \infty$

E.g.
$\tilde{H}(s) = \frac{1}{s-\alpha}$
Pole: $s = \alpha$
$h(t) = e^{\alpha t} u(t)$

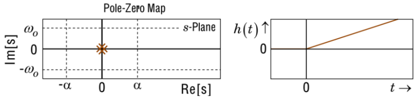


$\tilde{H}(s) = \frac{\omega_0}{(s-\alpha)^2+\omega_0^2}$
Poles: $s_{1,2} = \alpha \pm j\omega_0$
$h(t) = e^{\alpha t} \sin(\omega_0 t) u(t)$

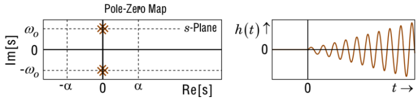


Unstable (Case 2)
- One or more repeated system poles lying on the imaginary axis
- $h(t)$ will "blow up" and become unbounded as t tends to infinity
  $\lim_{t\to\infty} |h(t)| = \infty$

E.g.
$\tilde{H}(s) = \frac{1}{s^2}$
Pole: $s_{1,2} = 0$
$h(t) = t u(t)$



$\tilde{H}(s) = \frac{\omega_0}{(s^2+\omega_0^2)^2}$
Poles: $s_{1,2,3,4} = \pm j\omega_0, \pm j\omega_0$
$h(t) = \frac{1}{2}\left[\omega_0^{-1}\sin(\omega_0 t) - t\cos(\omega_0 t)\right] u(t)$



7.7 First Order System (Standard Form)

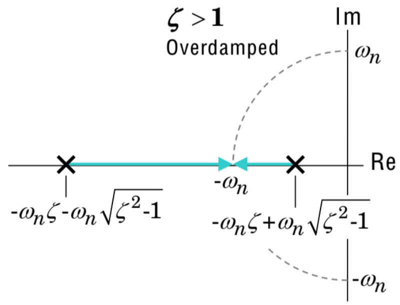
7.7.1 Differential Eqn, Transfer Func, Impulse Response and Step Response

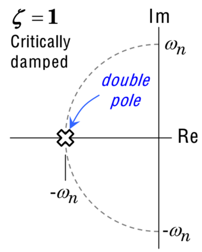
- Differential equation:
  $$T\frac{dy(t)}{dt} + y(t) = Kx(t) \qquad (7.26)$$
  where
  - $x(t)$: system input
  - $y(t)$: system output
  - K: DC gain
  - T: time-constant
- Transfer Function $\tilde{H}(s)$:
  $$Ts\tilde{Y}(s) + \tilde{Y}(s) = K\tilde{X}(s)$$
  $$\to \tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{K}{Ts+1} \qquad (7.27)$$
  Pole: $s_1 = -\frac{1}{T}$
- Impulse Response $h(t)$
  $$h(t) = \mathcal{L}^{-1}\left\{\tilde{H}(s)\right\} = \frac{K}{T}e^{-t/T}u(t)$$
- Step Response $o(t)$
  $$o(t) = \int_{-\infty}^{t} h(\tau)d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}\tilde{H}(s)\right\} = K\left[1-e^{-t/T}\right]u(t)$$

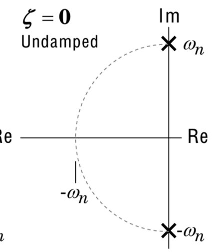
7.8 Second Order System (Standard Form)

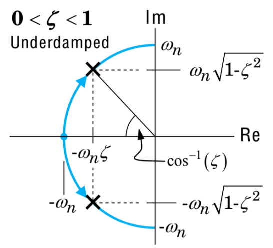
7.8.1 Differential Eqn and Transfer Func

- Differential equation:
  $$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 x(t) \qquad (7.28)$$
  where
  - $x(t)$: system input
  - $y(t)$: system output
  - ζ: damping ratio
  - ω_n: undamped natural frequency (when $\zeta < 1$)
  - K: DC gain
- Transfer function $\tilde{H}(s)$
  $$s^2\tilde{Y}(s) + 2\zeta\omega_n s\tilde{Y}(s) + \omega_n^2\tilde{Y}(s) = K\omega_n^2\tilde{X}(s)$$
  $$\implies \tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad (7.29)$$
  Poles: $s_{1,2} = -\omega_n\zeta \pm \omega_n\left(\zeta^2-1\right)^{1/2}$
- Damping









- Overdamped system: distinct real poles
- Critically damped system: repeated real poles
- Underdamped system: conjugate complex poles

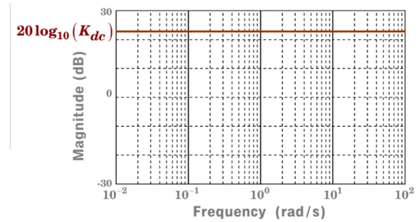
Chapter 8

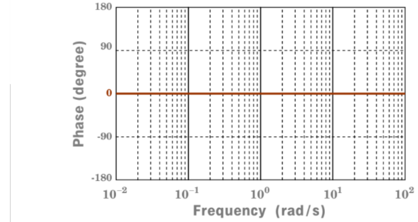
8.1 Construction of Bode Plots

Need to express (7.23b) in a suitable form for each of the following cases:
- Systems without integrator and differentiator
- Systems with differentiators
- Systems with integrators

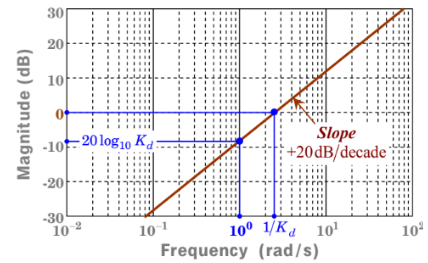
Basic systems:
1. $\tilde{H}(s) = K_{dc}$: DC gain (constant)
2. $\tilde{H}(s) = K_d s$: differentiator with gain K_d
3. $\tilde{H}(s) = K_i/s$: integrator with gain K_i
4. $\tilde{H}(s) = s/z_m + 1$: zero factor with unity DC gain ($\tilde{H}(0) = 1$)
5. $\tilde{H}(s) = \frac{1}{s/p_n+1}$: pole factor with unity DC gain
6. $\tilde{H}(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$: 2nd-order factor with unity DC gain

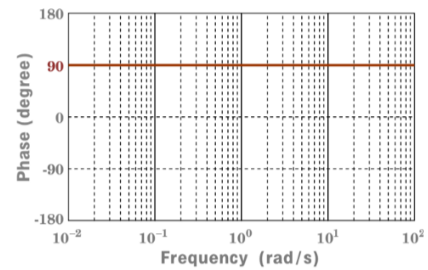
1. DC gain ($\tilde{H}(s) = K_{dc}$)



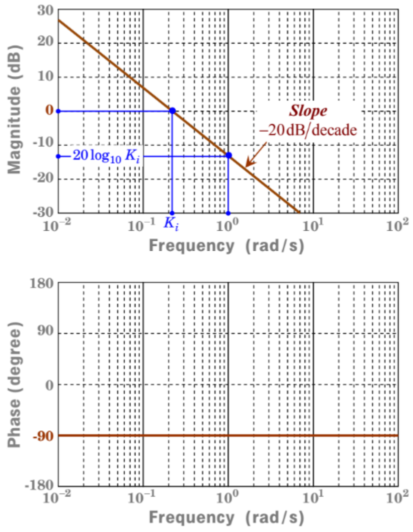


2. Differentiator ($\tilde{H}(s) = K_d s$)

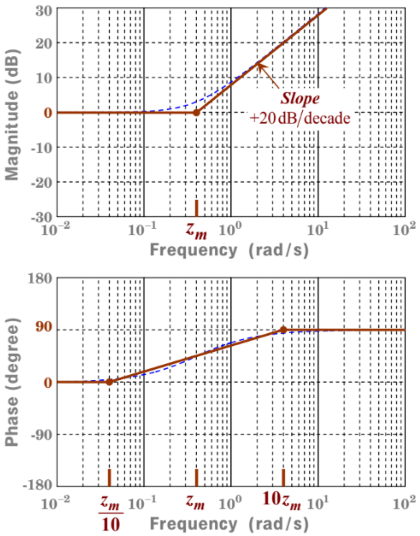




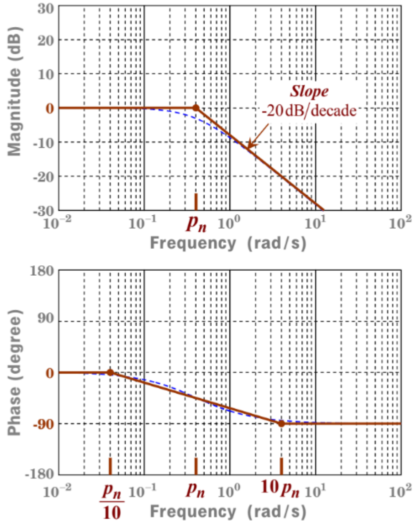
3. Integrator ($\tilde{H}(s) = K_i/s$)



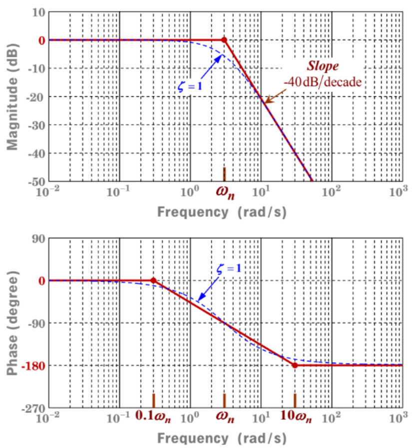
4. Zero factor ($\tilde{H}(s) = s/z_m + 1$)



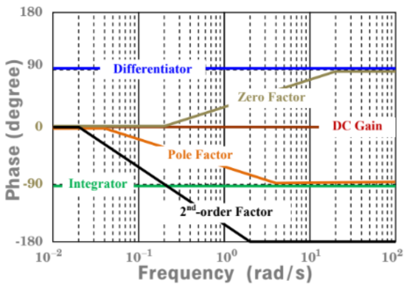
5. Pole factor ($\tilde{H}(s) = \frac{1}{s/p_n+1}$)

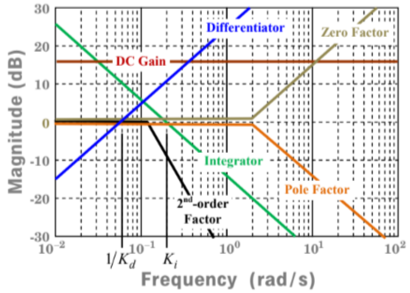


6. 2nd-order factor ($\tilde{H}(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$)



8.2 Asymptotic Behavior of Bode Plots





Asymptotic phase of phase plot

High frequency:

$$\lim_{\omega\to\infty} \angle\tilde{H}(j\omega) = \text{Pole-zero excess} \times (-90^\circ) \tag{8.4a}$$

Low frequency:

$$\lim_{\omega\to 0} \angle\tilde{H}(j\omega) = \left[\text{No. of} \int dt - \text{No. of} \frac{d}{dt}\right] \times (-90^\circ) \tag{8.4b}$$

Asymptotic slope of magnitude plot

High frequency:

$$\lim_{\omega\to\infty} \left[\text{Slope of } \left|\tilde{H}(j\omega)\right|\right] = [\text{Pole-zero excess}] \times (-20 \text{ dB/decade}) \tag{8.5a}$$

Low frequency:

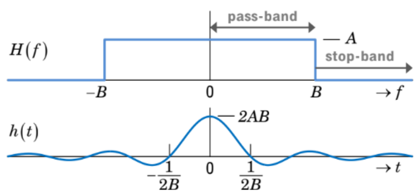
$$\lim_{\omega\to 0} \left[\text{Slope of } \left|\tilde{H}(j\omega)\right|\right] = \left[\text{No. of} \int dt - \text{No. of} \frac{d}{dt}\right] \times (-20 \text{ dB/decade}) \tag{8.5a}$$

Chapter 9

9.1 Idealized LTI filters

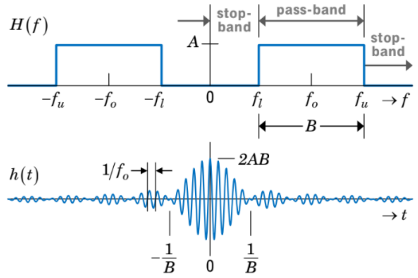
Ideal Low-Pass Filter (LPF)

- Frequency response: $H(f) = A\,\mathrm{rect}\left(\frac{f}{2B}\right)$
- Impulse response: $h(t) = 2AB\,\mathrm{sinc}\,(2Bt)$



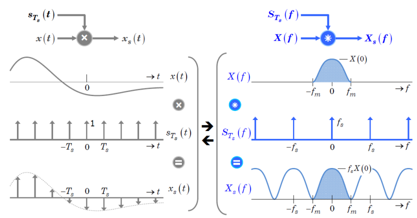
Ideal Band-Pass Filter (BPF)

- Frequency response: $H(f) = A\left[\mathrm{rect}\left(\frac{f+f_0}{B}\right) + \mathrm{rect}\left(\frac{f-f_0}{B}\right)\right]$
- Impulse response: $h(t) = 2AB\,\mathrm{sinc}\,(Bt)\cos(2\pi f_0 t)$

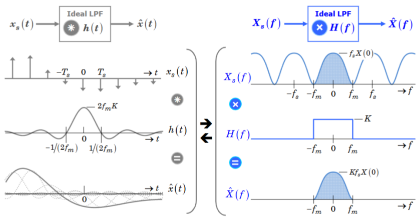


9.2 Continuous-time Sampling and Reconstruction of Signals

Sampling



Reconstruction



Nyquist Sampling Theorem:

- A band-limited signal, which has no frequency components higher than f_m Hz (f_m = bandwidth = highest freq component), may be completely described by specifying the values of the signal at insants of time separated by no more than $\frac{1}{2f_m}$ seconds.
- A band-limited signal, which has no frequency components higher than f_m Hz, may be completely recovered from a knowledge of its samples taken at a rate of no less than $2f_m$ samples/second.

Nyquist sampling frequency / Nyquist rate $f_s = 2f_m$

9.3 Sampling Band-limited Bandpass Signal below Nyquist Rate

(a) Overlapping spectral images ($f_c > 0.5B$)

$$f_s = 2f_c/k; \quad k = 1, 2, \ldots, \lfloor 2f_c/B \rfloor \tag{9.2a}$$

(b) Un-aliased spectral images ($f_c > 1.5B$)

$$\frac{2f_c + B}{k+1} \le f_s \le \frac{2f_c - B}{k}; \quad k = 1, 2, \ldots, \left\lfloor \frac{2f_c - B}{2B} \right\rfloor \tag{9.2b}$$