Euler's formula

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

Chapter 1

1.2.2 Bounded signals

A continuous-time signal $x(t)$ is bounded if there exists an M such that $0 < M < \infty$ and $\forall t\ |x(t)| \leq M$ (has an upper and lower range limit)

1.2.3 Absolutely integrable signals

A continuous-time signal $x(t)$ is absolutely integrable if
$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

1.2.4 Periodc and aperiodic signals

Periodic: there is a non-zero positive value, T, satisfying
$$x(t) = x(t+T)\ \forall t \qquad (1.1)$$

Aperiodic: not periodic

1.2.6 Energy and Power Signals

Energy signals

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad (1.3a)$$

$$x(t) \text{ is an energy signal} \iff 0 < E < \infty \qquad (1.3b)$$

Power signals

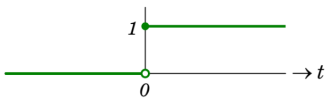
$$P = \lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} |x(t)|^2 dt \qquad (1.4a)$$

$$x(t) \text{ is a power signal} \iff 0 < P < \infty \qquad (1.4b)$$

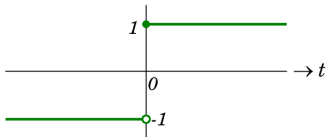
If $x(t)$ is a periodic signal, average power may be computed by
$$\frac{1}{T}\int_0^T |x(t)|^2 dt$$

- Energy signals have 0 average power, bc E = finite implies P = 0
- Power signals have infinite total energy, bc P = finite implies $E = \infty$
- All bounded periodic signals are power signals

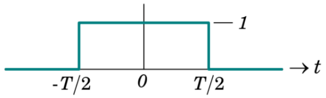
u(t):



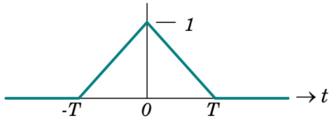
sgn(t):



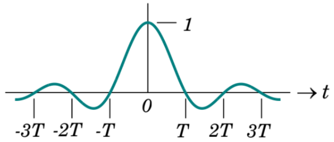
rect$(\frac{t}{T})$**:**



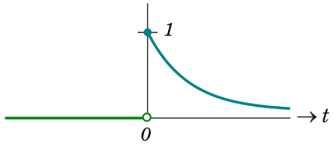
tri$(\frac{t}{T})$**:**



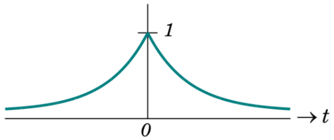
sinc$(\frac{t}{T})$**:**



$e^{-\alpha t}u(t)$**:**



$e^{-\alpha|t|}$**:**



Sinusoidal signals

$$x(t) = \mu\cos(\omega_0 t + \phi)$$
$$= \mu\cos(2\pi f_0 t + \phi)$$
$$= \mu\cos(\frac{2\pi t}{T} + \phi)$$
$$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$$

Chapter 2

2.1 Time-domain Operations

2.1.1 Time-Scaling

$x(\alpha t)$: Scale x-axis by a factor of $\frac{1}{\alpha}$

$x(-t)$: Reflect about x-axis

2.1.2 Time-Shifting

$x(t-\beta)$:

$\beta > 0$: Delaying $x(t)$ by β units of time (translate right along x-axis)

$\beta > 0$: Advancing $x(t)$ by β units of time (translate left along x-axis)

2.1.5 Convolution of 2 Signals

$x(t) * y(t) = \int_{-\infty}^{\infty} x(\alpha)y(t-\alpha)\, d\alpha$

Properties of convolutions

1. Commutative: $f * g = g * f$
2. Associative: $f * (g * h) = (f * g) * h$
3. Distributive: $f * (g + h) = (f * g) + (f * h)$

2.2 Dirac-δ function

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \neq 0 \end{cases}$$

Properties:

1. Symmetry: $\delta(t) = \delta(-t) \qquad (2.3)$
2. Sampling:
$$x(t)\delta(t-\lambda) = x(\lambda)\delta(t-\lambda) \qquad (2.4)$$
3. Sifting
$$\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt$$
$$= x(\lambda)\int_{-\infty}^{\infty}\delta(t-\lambda)dt = x(\lambda) \qquad (2.5)$$
4. Replication
$$x(t) * \delta(t-\lambda)$$
$$= \int_{-\infty}^{\infty} x(\zeta)\delta(t-\zeta-\lambda)d\zeta$$
$$= \int_{-\infty}^{\infty} x(\zeta)\delta(\zeta-(t-\lambda))d\zeta = x(t-\lambda) \qquad (2.6)$$

2.2.1 Dirac-δ Comb function

$$\sum_{n=-\infty}^{\infty}\delta(t-nT)$$
$$= \ldots + \delta(t+T) + \delta(t) + \delta(t-T) + \ldots$$

Convolution with Dirac-δ Comb function

$$x_p(t) = x(t) * \sum_n \delta(t-nT)$$
$$= \sum_n x(t-nT)$$

$x(t)$ is known as the generating function.

Multiplication with the Dirac-δ Comb function

Used for sampling
$$x_s(t) = x(t) \times \sum_n \delta(t-nT)$$
$$= \sum_n x(t) \times \delta(t-nT)$$
$$= \sum_n x(nT)\delta(t-nT)$$

Chapter 3

3.2 Spectrum of a Sinusoid

Spectrum of a Complex Exponential Signal

$\tilde{x}(t) = \mu e^{j(2\pi f_0 t + \phi)} = \mu e^{j\phi} \times e^{j2\pi f_0 t}$,

where μ: magnitude spectrum, ϕ: phase spectrum, f_0: frequency

Spectrum of a Cosine Signal

$$\mu\cos(2\pi f_0 t + \phi)$$
$$= \frac{\mu}{2}e^{j\phi}e^{j2\pi f_0 t} + \frac{\mu}{2}e^{j(-\phi)}e^{j2\pi(-f_0)t}$$

Spectrum of a Sine Signal

$$\mu\sin(2\pi f_0 t + \phi) = \frac{\mu}{2}e^{j(\phi-0.5\pi)}e^{j2\pi f_0 t}$$
$$+ \frac{\mu}{2}e^{j(-\phi+0.5\pi)}e^{j2\pi(-f_0)t}$$

Complex exponential Fourier Series

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kt/T_p}$$
$$= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} \qquad (3.1a)$$
$$c_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)e^{-j2\pi kt/T_p}dt, k \in \mathbb{Z} \qquad (3.1b)$$

Trigonometric Fourier Series

$$x_p(t) = a_0 + 2\sum_{k=1}^{\infty}[a_k\cos(2\pi kt/T_p) + b_k\sin(2\pi kt/T_p)]$$
$$a_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)\cos(2\pi kt/T_p)dt; k \geq 0$$
$$b_k = \frac{1}{T_p}\int_{t_0}^{t_0+T_p} x_p(t)\sin(2\pi kt/T_p)dt; k > 0 \qquad (3.2)$$

Chapter 4

Dirichlet Conditions

Conditions for existence of Fourier Transform:

1. $x(t)$ has only a finite number of maxima and minima in any finite time interval
2. $x(t)$ has only a finite number of discontinuities in any finite time interval
3. $x(t)$ is absolutely integrable

3 is weak Dirichlet condition: satisfied by most energy signals, violated by all power signals.

4.1 Fourier Transform

Forward Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \qquad (4.1a)$$

Inverse Fourier Transform

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \qquad (4.1b)$$

Spectrum of exponentially decaying pulse

$$x(t) = Ae^{-\alpha t}u(t)$$
$$= \begin{cases} Ae^{-\alpha t}; & t > 0 \\ 0; & t < 0 \end{cases}$$

Assume $\alpha > 0$
$$X(f) = \frac{A}{\alpha + j2\pi f}$$

4.2 Properties of Fourier Transform

- $X(f) = \Im\{x(t)\}$ denotes the Fourier transform of $x(t)$
- $x(t) = \Im^{-1}\{X(f)\}$ denotes the inverse Fourier transform of $X(f)$
- $x(t) \rightleftarrows X(f)$ denotes a Fourier transform pair with the time-domain on the LHS and frequency-domain on the RHS.

Linearity

If $x_1(t) \rightleftarrows X_1(f)$ and $x_2(t) \rightleftarrows X_2(f)$, then
$$\alpha x_1(t) + \beta x_2(t) \rightleftarrows \alpha X_1(f) + \beta X_2(f) \qquad (4.2)$$

Time Scaling

$$x(\beta t) \rightleftarrows \frac{1}{|\beta|}X\left(\frac{f}{\beta}\right) \qquad (4.3)$$

Duality

$$X(t) \rightleftarrows x(-f) \qquad (4.4)$$
or
$$X(-t) \rightleftarrows x(f)$$

Time Shifting

$$x(t-t_0) \rightleftarrows X(f)e^{-j2\pi ft_0} \qquad (4.5)$$
$$x(t+t_0) \rightleftarrows X(f)e^{j2\pi ft_0}$$

Frequency Shifting (Modulation)

$$x(t)e^{j2\pi f_0 t} \rightleftarrows X(f-f_0) \qquad (4.6)$$
$$x(t)e^{-j2\pi f_0 t} \rightleftarrows X(f+f_0)$$

Differentiation in the Time Domain

$$\frac{d}{dt}x(t) \rightleftarrows j2\pi f \cdot X(f) \qquad (4.7)$$

Integration in the Time Domain

$$\int_{-\infty}^{t} x(\tau)\, d\tau \rightleftarrows \frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f) \qquad (4.8)$$

Convolution in the Time Domain / Multiplication in the Frequency Domain

$$x_1(t) * x_2(t)$$
$$= \int_{-\infty}^{\infty} x_1(\alpha)x_2(t-\alpha)\, d\alpha \rightleftarrows X_1(f)X_2(f) \qquad (4.9a)$$

Multiplication in the Time Domain / Convolution in the Frequency Domain

$$x_1(t)x_2(t) \rightleftarrows \int_{-\infty}^{\infty} X_1(\alpha)X_2(f-\alpha)\, d\alpha$$
$$= X_1(f) * X_2(f) \qquad (4.9b)$$

4.3 Spectral properties of a REAL signal

- If $x(t)$ is **REAL** ($x^*(t) = x(t)$), then
  - $X(f)$ is conjugate symmetric ($X^*(f) = X(f)$)
  - $|X(f)|$ is even ($|X(f)| = |X(-f)|$)
  - $\angle X(f)$ is odd ($\angle X(f) = -\angle X(-f)$)
- If $x(t)$ is **REAL** and **EVEN** ($x^*(t) = x(t) \wedge x(-t) = x(t)$), then
  - $X(f)$ is real ($X^*f = X(f)$)
  - $X(f)$ is even ($X(-f) = X(f)$)
- If $x(t)$ is **REAL** and **ODD** ($x^*(t) = x(t) \wedge x(-t) = -x(t)$), then
  - $X(f)$ is imaginary ($X^*(f) = -X(f)$)
  - $X(f)$ is odd ($X(-f) = -X(f)$)

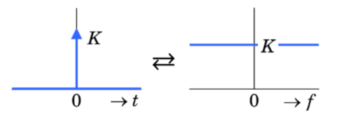
The above can apply to Fourier series coefficients of periodic signals too:

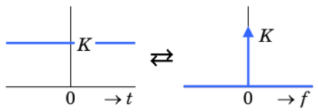
- $x_p(t)$ is **REAL**
  - c_k is conjugate symmetric ($c_k^* = c_{-k}$)
  - $|c_k|$ has even symmetry ($|c_k| = |c_{-k}|$)
  - $\angle c_k$ has odd symmetry ($\angle c_k = -\angle c_{-k}$)
- $x_p(t)$ is **REAL** and **EVEN**
  - c_k is real ($c_k^* = c_k$)
  - c_k is even ($c_k = c_{-k}$)
- $x_p(t)$ is **REAL** and **ODD**
  - c_k is imaginary ($c_k^* = -c_k$)
  - c_k is odd ($c_k = -c_{-k}$)

4.4 Spectrum of Signals that are not Absolutely Integrable

$$\Im\{K\delta(t)\} = \int_{-\infty}^{\infty} K\delta(t)e^{-j2\pi ft}dt = K \qquad (4.13)$$

By duality, $\Im\{K\} = K\delta(f)$





4.4.1 Spectrm of Unit Step and Signum function

$$\Im\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$
$$\Im\{\text{Sgn}(t)\} = \frac{1}{j\pi f}$$

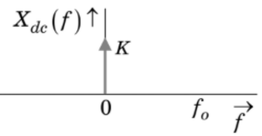
4.4.2 Continuous-Frequency Spectrum of Periodic Signals

The following make use of the fact that

$$\Im\{k\} = K\delta(f) \tag{4.14}$$

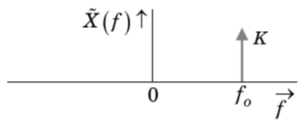
DC

$$x_{dc}(t) = K$$
$$X_{dc}(f) = \Im\{k\} = K\Im\{1\} = K\delta(f)$$



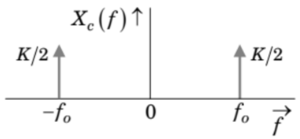
Complex Exponential

$$\tilde{x}(t) = Ke^{j2\pi f_0 t}$$
$$\tilde{X}(f) = \Im\{Ke^{j2\pi f_0 t}\} = K\delta(f - f_0)$$



Cosine

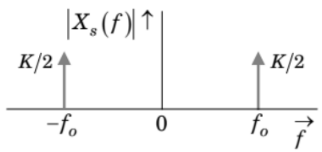
$$\Im\{K\cos(2\pi f_0 t)\}$$
$$= \frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0)$$

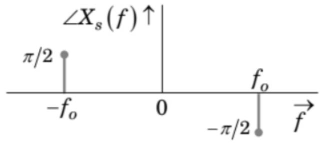


Sine

$$\Im\{K\sin(2\pi f_0 t)\}$$
$$= \frac{K}{j2}\delta(f - f_0) - \frac{K}{j2}\delta(f + f_0)$$

$$\text{where} \begin{cases} |X_s(f)| & = \frac{K}{2}\delta(f - f_0) \\ & \quad + \frac{K}{2}\delta(f + f_0) \\ \angle X_s(f) & = \begin{cases} -\pi/2, & f = f_0 \\ \pi/2, & f = -f_0 \end{cases} \end{cases}$$

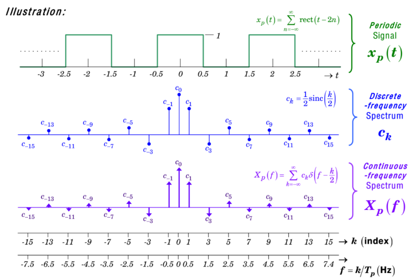




Arbitrary periodic signals

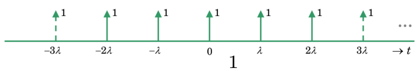
Let $x_p(t)$ be a periodic signal with period T_p and fundamental frequency f_p

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p) \tag{4.16}$$

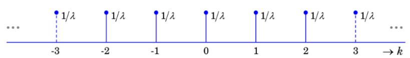


4.4.2.1 Spectrum of Dirac-δ Comb function

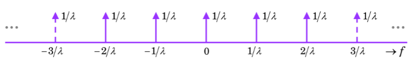
$$\text{comb}_\lambda(t) \triangleq \sum_n \delta(t - n\lambda)$$



$$c_k = \frac{1}{\lambda}$$



$$\Im\{\text{comb}_\lambda(t)\} = \text{COMB}_\lambda(f)$$
$$= \frac{1}{\lambda}\sum_k \delta(f - k/\lambda)$$



Chapter 5

5.1 Energy Spectral Density (ESD)

Total energy of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \text{ (Joules)} \tag{5.1}$$

Rayleigh Energy Theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df, \tag{5.2}$$

where $X(f) = \Im\{x(t)\}$ is the spectrum of the signal.

Energy Spectral Density

$$E_x(f) = |X(f)|^2 \text{ Joules Hz}^{-1} \tag{5.3}$$

Properties of $E_x(f)$

1. $E_x(f)$ is a real function of f
2. $E_x(f) \geq 0 \quad \forall f$
3. $E_x(f)$ is an even function of f if $x(t)$ is real.

5.2 Power Spectral Density (PSD)

In the time-domain, the average power of a signal $x(t)$ is defined as

$$P = \lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} |x(t)|^2 dt \tag{5.4}$$

Windowed version of $x(t)$:

$$x_W(t) = x(t)\text{rect}\left(\frac{t}{2W}\right) \tag{5.5}$$

Parseval Power Theorem

$$P = \lim_{W\to\infty} \frac{1}{2W}\int_{-W}^{W} |x(t)|^2 dt$$
$$= \int_{-\infty}^{\infty} \lim_{W\to\infty} \frac{1}{2W}|X_W(f)|^2 df \tag{5.9}$$

Power Spectral Density

$$P_x(f) = \lim_{W\to\infty} \frac{1}{2W}|X_W(f)|^2 \text{ Watts Hz}^{-1} \tag{5.10}$$

Properties of $P_x(f)$

1. $P_x(f)$ is a real function of f
2. $P_x(f) \geq 0 \quad \forall f$
3. $P_x(f)$ is an even function of f if $x(t)$ is real.

5.2.1 PSD of Periodic Signals

From chapter 4 equation 4.16:

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p)$$

PSD of $x_p(t)$

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p) \tag{5.12}$$

Average power of $x_p(t)$

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2 \tag{5.13}$$

5.3 Bandwidth

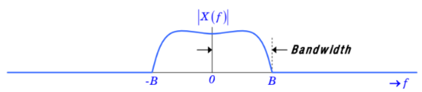
5.3.1 Bandlimited Signals

Lowpass signal

A signal $x(t)$ is said to be a bandlimited lowpass signal if its magnitude spectrum is concentrated around 0 Hz, and at the same time satisfies

$$|X(f)| = 0; \quad |f| > B \tag{5.14}$$

where B is defined as the bandwidth of the signal.

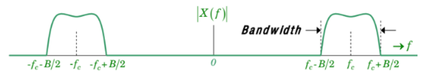


Bandpass signal

A signal $x(t)$ is said to be a bandlimited bandpass signal if its magnitude spectrum is concentrated around a non-zero center frequency f_c, and at the same time satisfies

$$|X(f)| = 0; \quad ||f| - f_c| > B/2 \tag{5.15}$$

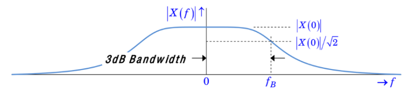
where B is defined as the bandwidth of the signal



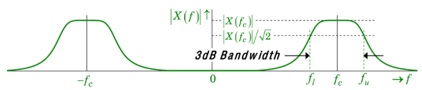
5.3.2 Signals with Unrestricted Band

5.3.2.1 3dB Bandwidth

Lowpass signal: The frequency where $|X(f)| = |X(0)|/\sqrt{2}$ first occurs (or where $|X(f)|^2 = |X(0)|^2/2$ first occurs) when f is increased from 0 Hz.

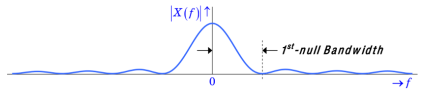


Bandpass signal:

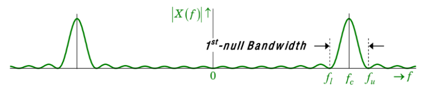


5.3.2.2 1st-null Bandwidth

Lowpass signal: The frequency at which $|X(f)| = 0$ first occurs when f is increased from 0 Hz:



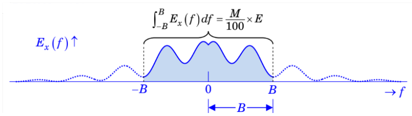
Bandpass signal:



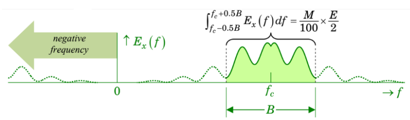
5.3.2.3 M% Energy Containment Bandwidth

Smallest bandwidth that contains at least M% of the total signal energy $E = \int_{-\infty}^{\infty} E_x(f)\, df$

Lowpass:



Bandpass:

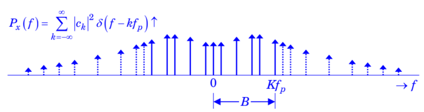


5.3.2.4 M% Power Containment Bandwidth

The smallest bandwidth that contains at least M% of the average signal power. For a periodic signal, the aerage power is given by

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2$$

where f_p(Hz) is the fundamental frequency and c_k's are the Fourier series coefficients.



$$B = Kf_p$$

where K is the smallest positive integer that satisfies

$$\sum_{k=-K}^{K} |c_k|^2 \geq \frac{M}{100} \times P$$

Chapter 6

6.1 Systems

- A system is a mathematical model of a physical process that relates the input (or excitation) signal to the output (or response) signal.
- With an input $x(t)$ and an output $y(t)$, the system may be viewed as a transformation (or mapping) of $x(t)$ into $y(t)$, mathematically expressed as $\quad y(t) = \mathbf{T}[x(t)] \quad (6.1)$

6.2 Classification of Systems

6.2.1 Systems with Memory and Without Memory

A system is said to be memoryless (or static) if its output at a given time is dependent on only the input at that time.

Otherwise, the system is said to have memory (or to be dynamic).

6.2.2 Causal and Noncausal Systems

A system is said to be causal (or non-anticipative) if its output, $y(t)$, at the present time depends on only the present and/or past values of its input, $x(t)$.

$\therefore$ not possible for a causal system to produce an output before an input is applied. $\therefore \forall t < 0\ y(t) = 0$.

6.2.3 Stable and Unstable Systems

A system is BIBO stable (bounded-input/bounded-output) if for every bounded input $x(t)$ where

$$\forall t\ |x(t)| \leq k \tag{6.2}$$

the system produces a bounded output $y(t)$ where

$$\forall t\ |y(t)| \leq L \tag{6.3}$$

in which K and L are positive constants.

6.2.4 Linear and Nonlinear Systems

A linear system is one that satisfies the following two conditions:

$$\mathbf{T}[x_1(t) + x_2(t)] = \mathbf{T}[x_1(t)] + \mathbf{T}[x_2(t)]$$
$$= y_1(t) + y_2(t) \tag{6.4}$$

$$\mathbf{T}[\alpha x(t)] = \alpha\mathbf{T}[x(t)] = \alpha y(t) \tag{6.5}$$

(6.4) and (6.5) can be combined into:

$$\mathbf{T}[\alpha_1 x_1(t) + \alpha_2 x_2(t)]$$
$$= \alpha_1\mathbf{T}[x_1(t)] + \alpha_2\mathbf{T}[x_2(t)] \tag{6.6}$$
$$= \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

(6.6) is known as the superposition property.

Important property of linear systems:

$x(t) = 0 \implies y(t) = 0$

6.2.5 Time-Invariant and Time-Varying Systems

A system is time-invariant if a time shift (delay or advance) in the input signal, $x(t)$, causes the same time shift in the output signal, $y(t)$.

$$\mathbf{T}[x(t - \tau)] = y(t - \tau) \tag{6.7}$$

A time-varying system is one which does not satisfy (6.7).

Laplace Transform

$$\tilde{F}(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^{\infty} f(t)e^{-st}dt \quad (6.8)$$

where s is a complex variable.

Inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}\left\{\tilde{F}(s)\right\} = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} \tilde{F}(s)ds \quad (6.9)$$

Chapter 7

7.1 Impulse Response

Impulse response, $h(t)$, of a continuous-time LTI system is defined as the response/output of the system when the input is a unit impulse, $\delta(t)$.

$$\delta(t) \rightarrow \text{LTI system} \rightarrow h(t)$$

where

$$h(t) = \mathbf{T}[\delta(t)] \quad (7.1)$$

From replication property,

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \quad (7.2)$$

Substituting (7.2) into (6.1),

$$\begin{aligned} y(t) &= \mathbf{T}[x(t)] \\ &= \mathbf{T}\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right] \\ &= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}[\delta(t-\tau)]d\tau \end{aligned} \quad (7.3)$$

As the system is time-invariant, by applying (6.7) to (7.1),

$$h(t-\tau) = \mathbf{T}\left[\delta(t-\tau)\right] \quad (7.4)$$

Sub (7.4) into (4.3):

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= x(t) * h(t) \end{aligned} \quad (7.5)$$

Therefore

$$\mathbf{T}[x(t)] = y(t) = x(t) * h(t)$$

7.1.1 Step Response

Step response: the output of the system when input is unit step function

$$\begin{aligned} u(t) \rightarrow h(t) \rightarrow o(t) &= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau \\ &= \int_{-\infty}^{t} h(\tau)d\tau \end{aligned}$$

Step response equals integration of impulse response: $\quad o(t) = \int_{-\infty}^{t} h(\tau)d\tau$

Impulse response equals differentiation of step response: $\quad h(t) = \frac{d}{dt}o(t)$

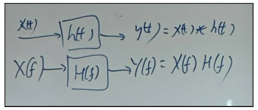
7.2 Frequency Response

The frequency response ($H(f)$) of an LTI system is defined as the Fourier transform of the system impulse response $h(t)$

$$H(f) = \Im\left\{h(t)\right\} = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt \quad (7.6)$$

$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f) \cdot H(f) \quad (7.7)$$



$$H(f) = |H(f)|e^{j\angle H(f)} \quad (7.8)$$

where $|H(f)|$ is called the magnitude response and $\angle H(f)$ is called the phase response of the system.

7.3 Transfer Function

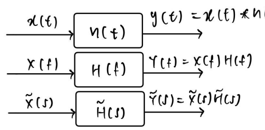
The transfer function $\tilde{H}(s)$ of an LTI system is defined as the Laplace transform of $h(t)$

$$\tilde{H}(s) = \mathcal{L}\left\{h(t)\right\} = \int_0^{\infty} h(t)e^{-st}dt \quad (7.9)$$

where $s = \sigma + j\omega$ is a complex variable.

$$y(t) = x(t) * h(t)$$

$$\tilde{Y}(s) = \tilde{X}(s) \cdot \tilde{H}(s) \quad (7.10)$$



7.4 Relationship between Transfer Function and Frequency Response

Substituting $s = j\omega$ into (7.9), we get

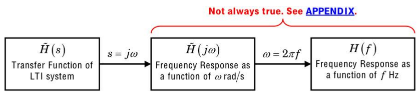
$$\tilde{H}(s)\Big|_{s=j\omega} = \tilde{H}(j\omega) = \int_0^{\infty} h(t)e^{-j\omega t}dt \quad (7.11)$$

Sub $\omega = 2\pi f$ into (7.11):

$$\tilde{H}(j\omega)\Big|_{\omega=2\pi f} = \int_0^{\infty} h(t)e^{-j2\pi ft}dt \quad (7.12)$$

For causal LTI systems, $\forall t < 0$ $h(t) = 0$. Hence (7.6) and (7.12) are equivalent.

$$H(f) = \tilde{H}(j\omega)\Big|_{\omega=2\pi f} \quad (7.13)$$



$$\tilde{H}(j\omega) = |\tilde{H}(j\omega)|e^{j\angle\tilde{H}(j\omega)} \quad (7.14)$$

where $|\tilde{H}(j\omega)|$ is called the magnitude response and $\angle\tilde{H}(j\omega)$ is called the phase response of the system.

7.4 Sinusoidal Response at Steady-State

Let system input at steady-state be

$$x(t) = Ae^{j(2\pi f_0 t+\psi)} \quad (7.15)$$

Then

$$X(f) = Ae^{j\psi}\delta(f-f_0) \quad (7.16)$$

$$Y(f) = A\left|H(f_0)\right|e^{j(\psi+\angle H(f_0))}\delta(f-f_0) \quad (7.17)$$

$$\begin{aligned} y(t) &= \Im^{-1}\left\{Y(f)\right\} \\ &= A\left|H(f_0)\right|e^{j(2\pi f_0 t+\psi+\angle H(f_0))} \end{aligned} \quad (7.18)$$

System Input : $\boxed{A}e^{j(2\pi f_0 t+\boxed{\psi})}$

System Output : $\boxed{A\left|H(f_0)\right|}e^{j(2\pi f_0 t+\boxed{\psi+\angle H(f_0)})}$

$$x(t) = Ae^{j(2\pi f_0 t+\psi)} \rightarrow \boxed{H(f)} \rightarrow y(t) = A\left|H(f_0)\right|e^{j(2\pi f_0 t+\psi+\angle H(f_0))}$$

$$x(t) = A\cos(2\pi f_0 t+\psi) \rightarrow \boxed{H(f)} \rightarrow y(t) = A\left|H(f_0)\right|\cos(2\pi f_0 t+\psi+\angle H(f_0))$$

$$x(t) = A\sin(2\pi f_0 t+\psi) \rightarrow \boxed{H(f)} \rightarrow y(t) = A\left|H(f_0)\right|\sin(2\pi f_0 t+\psi+\angle H(f_0))$$

Steady-state Sinusoidal Response of a LTI System in f-domain

$$x(t) = Ae^{j(\omega_0 t+\psi)} \rightarrow \boxed{\tilde{H}(j\omega)} \rightarrow y(t) = A\left|\tilde{H}(j\omega_0)\right|e^{j(\omega_0 t+\psi+\angle\tilde{H}(j\omega_0))}$$

$$x(t) = A\cos(\omega_0 t+\psi) \rightarrow \boxed{\tilde{H}(j\omega)} \rightarrow y(t) = A\left|\tilde{H}(j\omega_0)\right|\cos(\omega_0 t+\psi+\angle\tilde{H}(j\omega_0))$$

$$x(t) = A\sin(\omega_0 t+\psi) \rightarrow \boxed{\tilde{H}(j\omega)} \rightarrow y(t) = A\left|\tilde{H}(j\omega_0)\right|\sin(\omega_0 t+\psi+\angle\tilde{H}(j\omega_0))$$

Steady-state Sinusoidal Response of a LTI System in ω-domain

7.6 LTI Systems Described by Differential Equations

LTI systems represented by linear constant-coefficient differential equations have the general form

$$\sum_{n=0}^{N} a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m} \quad (7.21)$$

where $x(t)$ is input, $y(t)$ is output, and a_n, b_m are real constants.

7.6.1 Transfer Function

Applying Laplace to both sides of (7.21) with initial conditions set to 0,

$$\sum_{n=0}^{N} a_n\tilde{Y}(s)s^n = \sum_{m=0}^{M} b_m\tilde{X}(s)s^m \quad (7.22)$$

$$\begin{aligned} \tilde{H}(s) &= \frac{\tilde{Y}(s)}{\tilde{X}(s)} \\ &= \left(\sum_{m=0}^{M} b_m s^m \Big/ \sum_{n=0}^{N} a_n s^n\right) \\ &= \frac{b_M s^M + b_{M-1}s^{M-1} + \ldots + b_0}{a_N s^N + a_{N-1}s^{N-1} + \ldots + a_0} \end{aligned} \quad (7.23a)$$

$$\begin{aligned} \tilde{H}(s) &= K\frac{\left(\frac{s}{z_1}+1\right)\left(\frac{s}{z_2}+1\right)\ldots\left(\frac{s}{z_M}+1\right)}{\left(\frac{s}{p_1}+1\right)\left(\frac{s}{p_2}+1\right)\ldots\left(\frac{s}{p_N}+1\right)} \\ K &= \frac{a_0}{b_0} \end{aligned} \quad (7.23b)$$

$$\begin{aligned} \tilde{H}(s) &= K'\frac{(s+z_1)(s+z_2)\ldots(s+z_M)}{(s+p_1)(s+p_2)\ldots(s+p_N)} \\ K &= \frac{b_M}{a_N} \end{aligned} \quad (7.23c)$$

$\forall n \in \{1, 2, \ldots, N\}$

- $-p_n$ are roots of the denominator polynomial of $\tilde{H}(s)$
- $\tilde{H}(-p_n) = \infty$
- $-p_n$ are called **poles** of $\tilde{H}(s)$

$\forall m \in \{1, 2, \ldots, M\}$

- $-z_m$ are roots of the numerator polynomial of $\tilde{H}(s)$
- $\tilde{H}(-z_m) = 0$
- $-z_m$ are called **zeros** of $\tilde{H}(s)$

The system is said to have N poles and M zeros, and the difference $N - M$ is called pole-zero excess.

7.6.2 System Stability

BIBO Stable

- All system poles lying on the left-half s-plane
- $h(t)$ will converge to 0 as t tends to infinity

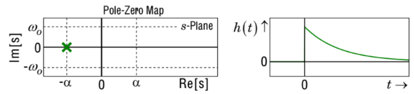
  $\lim_{t\to\infty} h(t) = 0$

E.g.

$\tilde{H}(s) = \frac{1}{s+\alpha}$

Pole: $s = -\alpha$

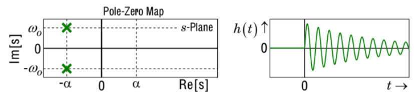
$h(t) = e^{-\alpha t}u(t)$



$\tilde{H}(s) = \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$

Poles: $s_{1,2} = -\alpha \pm j\omega_0$

$h(t) = e^{-\alpha t}\sin(\omega_0 t)u(t)$



Marginally Stable

- One or more **non-repeated** system poles lying on the imaginary axis of the s-plane and no system pole lying on the right half s-plane.
- $h(t)$ will not "blow up" and become unbounded, but neither will it converge to zero as t tends to infinity.

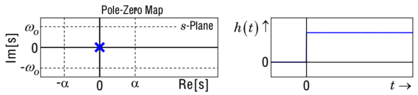
  $\lim_{t\to\infty} |h(t)| \neq \infty$ and $\lim_{t\to\infty} h(t) \neq 0$

E.g.

$\tilde{H}(s) = \frac{1}{s}$

Pole: $s = 0$

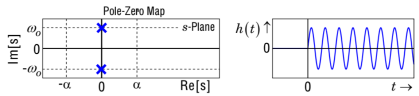
$h(t) = u(t)$



$\tilde{H}(s) = \frac{\omega_0}{s^2+\omega_0^2}$

Poles: $s_{1,2} = \pm j\omega_0$

$h(t) = \sin(\omega_0 t)u(t)$



Unstable (Case 1)

- One or more system poles lying on the right-half s-plane
- $h(t)$ will "blow up" and become unbounded as t tends to infinity

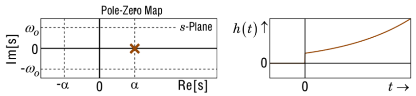
  $\lim_{t\to\infty} |h(t)| = \infty$

E.g.

$\tilde{H}(s) = \frac{1}{s-\alpha}$

Pole: $s = \alpha$

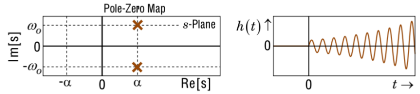
$h(t) = e^{\alpha t}u(t)$



$\tilde{H}(s) = \frac{\omega_0}{(s-\alpha)^2+\omega_0^2}$

Poles: $s_{1,2} = \alpha \pm j\omega_0$

$h(t) = e^{\alpha t}\sin(\omega_0 t)u(t)$



Unstable (Case 2)

- One or more repeated system poles lying on the imaginary axis
- $h(t)$ will "blow up" and become unbounded as t tends to infinity

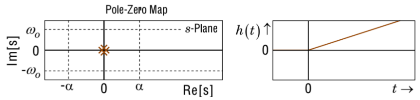
  $\lim_{t\to\infty} |h(t)| = \infty$

E.g.

$\tilde{H}(s) = \frac{1}{s^2}$

Pole: $s_{1,2} = 0$

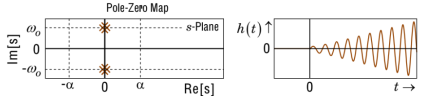
$h(t) = tu(t)$



$\tilde{H}(s) = \frac{\omega_0}{(s^2+\omega_0^2)^2}$

Poles: $s_{1,2,3,4} = \pm j\omega_0, \pm j\omega_0$

$h(t) = \frac{1}{2}\left[\omega_0^{-1}\sin(\omega_0 t) - t\cos(\omega_0 t)\right]u(t)$



7.7 First Order System (Standard Form)

7.7.1 Differential Eqn, Transfer Func, Impulse Response and Step Response

- Differential equation:

  $$T\frac{dy(t)}{dt} + y(t) = Kx(t) \quad (7.26)$$

  where
  - $x(t)$: system input
  - $y(t)$: system output
  - K: DC gain
  - T: time-constant
- Transfer Function $\tilde{H}(s)$:

  $$T s\tilde{Y}(s) + \tilde{Y}(s) = K\tilde{X}(s)$$

  $$\rightarrow \tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{K}{Ts+1} \quad (7.27)$$

  Pole: $s_1 = -\frac{1}{T}$
- Impulse Response $h(t)$

  $h(t) = \mathcal{L}^{-1}\left\{\tilde{H}(s)\right\} = \frac{K}{T}e^{-t/T}u(t)$
- Step Response $o(t)$

  $$\begin{aligned} o(t) &= \int_{-\infty}^{t} h(\tau)d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}\tilde{H}(s)\right\} \\ &= K\left[1 - e^{-t/T}\right]u(t) \end{aligned}$$

7.8 Second Order System (Standard Form)

7.8.1 Differential Eqn and Transfer Func

- Differential equation:

  $$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 x(t) \quad (7.28)$$

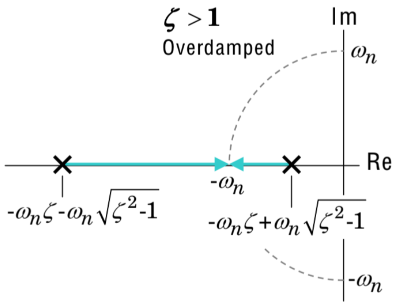
where
- $x(t)$: system input
- $y(t)$: system output
- ζ: damping ratio
- ω_n: undamped natural frequency (when $\zeta < 1$)
- K: DC gain

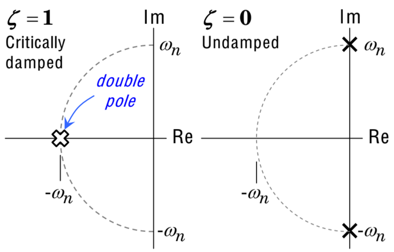
- Transfer function $\tilde{H}(s)$

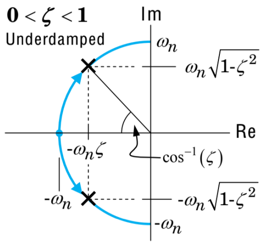
$$s^2\tilde{Y}(s) + 2\zeta\omega_n s\tilde{Y}(s) + \omega_n^2\tilde{Y}(s)$$
$$= K\omega_n^2\tilde{X}(s)$$
$$\implies \tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (7.29)$$

Poles: $s_{1,2} = -\omega_n\zeta \pm \omega_n\left(\zeta^2 - 1\right)^{1/2}$

- Damping







7.8.2 Impulse Response and Step Response

7.8.2.1 Overdamped System ($\zeta > 1$)

Chapter 8

Bode plot: approximate visualization of frequency response, $\tilde{H}(j\omega)$ of a system

- Magnitude plot: plot of $\left|\tilde{H}(j\omega)\right|_{dB} = 20\log_{10}\left(\left|\tilde{H}(j\omega)\right|\right) dB$
- Phase plot: plot of $\angle\tilde{H}(j\omega)$ in degrees
- x-axis is logarithmically scaled (semilog-x: scale is log, but labels are still linear)
- Only positive frequency side visualized (which suffices for real systems as $|\tilde{H}(j\omega)|$ and $\angle\tilde{H}(j\omega)$ are even and odd functions of ω respectively)
- 0 is not in the axis cuz it goes from 1 to 0.1 to 0.01 to 0.001...

8.1 Construction of Bode Plots

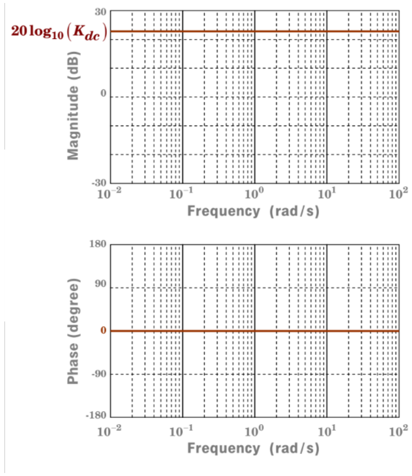
Need to express (7.23b) in a suitable form for each of the following cases:

- Systems without integrator and differentiator
- Systems with differentiators
- Systems with integrators

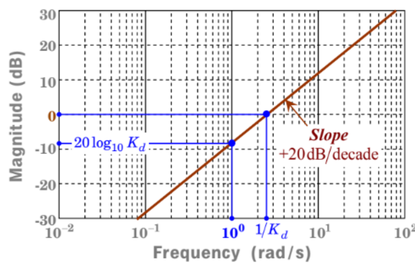
Basic systems:

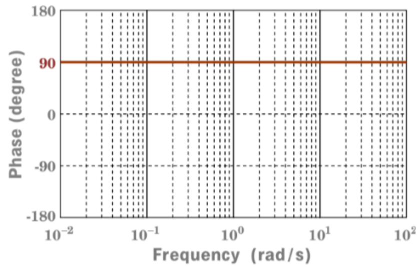
1. $\tilde{H}(s) = K_{dc}$: DC gain (constant)
2. $\tilde{H}(s) = K_d s$: differentiator with gain K_d
3. $\tilde{H}(s) = K_i/s$: integrator with gain K_i
4. $\tilde{H}(s) = s/z_m + 1$: zero factor with unity DC gain ($\tilde{H}(0) = 1$)
5. $\tilde{H}(s) = \frac{1}{s/p_n+1}$: pole factor with unity DC gain
6. $\tilde{H}(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$: 2nd-order factor with unity DC gain

DC gain ($\tilde{H}(s) = K_{dc}$)

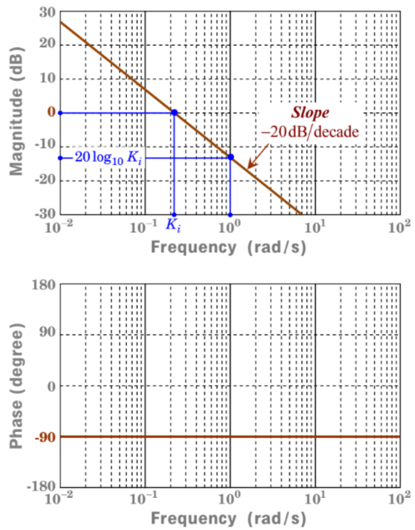


Differentiator ($\tilde{H}(s) = K_d s$)

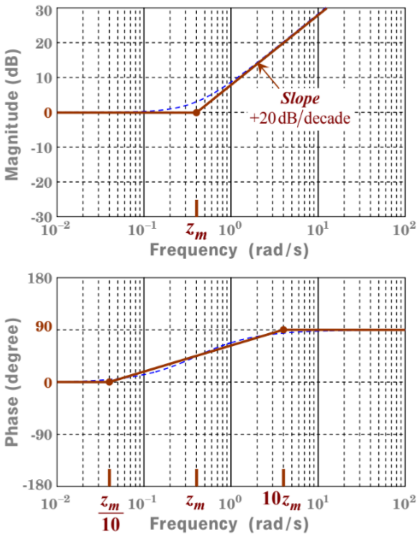




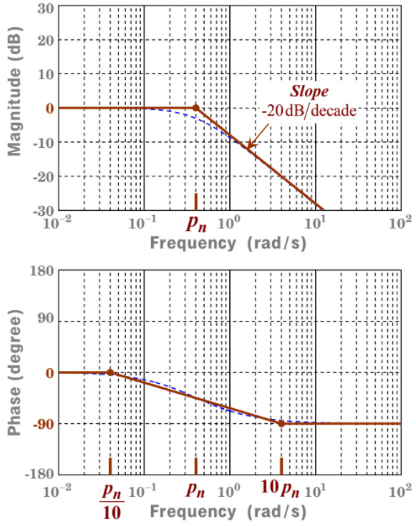
Integrator ($\tilde{H}(s) = K_i/s$)



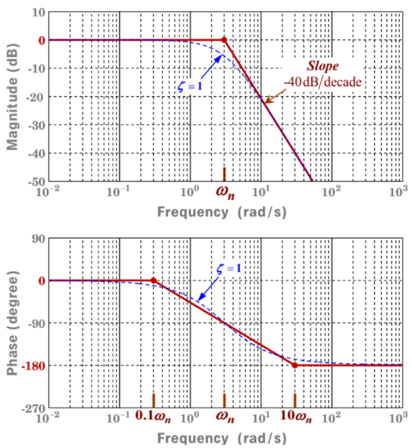
Zero factor ($\tilde{H}(s) = s/z_m + 1$)



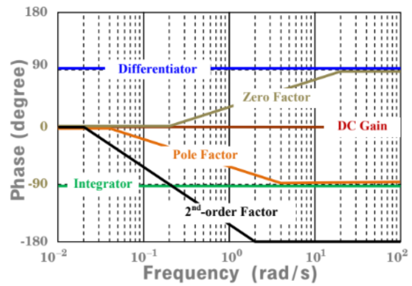
Pole factor ($\tilde{H}(s) = \frac{1}{s/p_n+1}$)

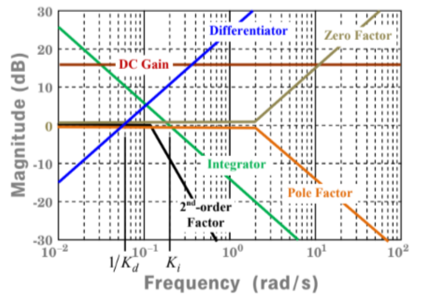


2nd-order factor ($\tilde{H}(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$)



8.2 Asymptotic Behavior of Bode Plots





Asymptotic phase of phase plot

High frequency:

$$\lim_{\omega\to\infty}\angle\tilde{H}(j\omega) = \text{Pole-zero excess} \times (-90^\circ) \quad (8.4a)$$

Low frequency:

$$\lim_{\omega\to 0}\angle\tilde{H}(j\omega)$$
$$= \left[\text{No. of} \int dt - \text{No. of} \frac{d}{dt}\right] \times (-90^\circ) \quad (8.4b)$$

Asymptotic slope of magnitude plot

High frequency:

$$\lim_{\omega\to\infty}\left[\text{Slope of }\left|\tilde{H}(j\omega)\right|\right]$$
$$= [\text{Pole-zero excess}] \times (-20 \text{ dB/decade}) \quad (8.5a)$$

Low frequency:

$$\lim_{\omega\to 0}\left[\text{Slope of }\left|\tilde{H}(j\omega)\right|\right]$$
$$= \left[\text{No. of} \int dt - \text{No. of} \frac{d}{dt}\right] \times (-20 \text{ dB/decade}) \quad (8.5a)$$