

Chapter 1

1.2.2 Bounded signals

$x(t)$ is bounded if

$$\exists M [(0 < M < \infty) \wedge (\forall t |x(t)| \leq M)]$$

(got upper n lower range limit)

1.2.3 Absolutely integrable signals

$x(t)$ is absolutely integrable if

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

1.2.6 Energy and Power Signals

Energy signals

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (1.3a)$$

$$x(t) \text{ is an energy signal} \iff 0 < E < \infty \quad (1.3b)$$

Power signals

$$P = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt \quad (1.4a)$$

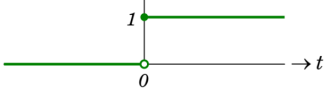
$$x(t) \text{ is a power signal} \iff 0 < P < \infty \quad (1.4b)$$

If $x(t)$ is a periodic signal, average power may be computed by

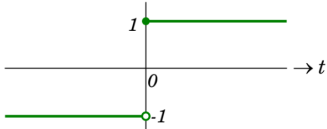
$$\frac{1}{T} \int_0^T |x(t)|^2 dt$$

- Energy signals have 0 average power, bc E = finite implies P = 0
- Power signals have infinite total energy, bc P = finite implies E = ∞
- All bounded periodic signals are power signals

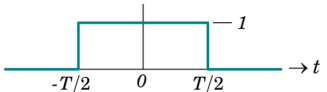
u(t):



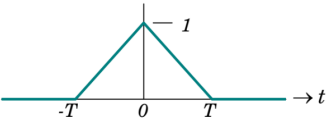
sgn(t):



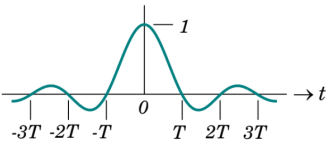
rect(t/T):



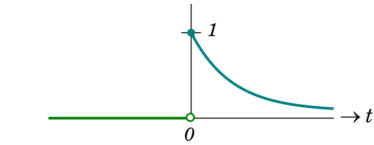
tri(t/T):



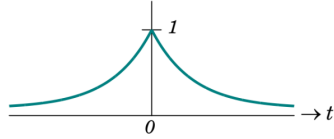
sinc(t/T):



$$e^{-\alpha t} u(t):$$



$$e^{-\alpha|t|};$$



Chapter 2

2.1 Time-domain Operations

2.1.5 Convolution of 2 Signals

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\alpha) y(t - \alpha) d\alpha$$

Properties of Dirac- δ :

$$1. \text{ Symmetry: } \delta(t) = \delta(-t) \quad (2.3)$$

$$2. \text{ Sampling: } x(t)\delta(t - \lambda) = x(\lambda)\delta(t - \lambda) \quad (2.4)$$

$$3. \text{ Sifting } \int_{-\infty}^{\infty} x(t)\delta(t - \lambda) dt = x(\lambda) \quad (2.5)$$

$$4. \text{ Replication } x(t) * \delta(t - \lambda) = x(t - \lambda) \quad (2.6)$$

Convolution with Dirac- δ Comb function

$$\begin{aligned} x_p(t) &= x(t) * \sum_n \delta(t - nT) \\ &= \sum_n x(t - nT) \end{aligned}$$

Multiplication with the Dirac- δ Comb function

Used for sampling

$$\begin{aligned} x_s(t) &= x(t) \times \sum_n \delta(t - nT) \\ &= \sum_n x(t) \times \delta(t - nT) \\ &= \sum_n x(nT) \delta(t - nT) \end{aligned}$$

Chapter 3

3.2 Spectrum of a Sinusoid

Spectrum of a Complex Exponential Signal

$$\begin{aligned} \tilde{x}(t) &= \mu e^{j(2\pi f_0 t + \phi)} = \mu e^{j\phi} \times e^{j2\pi f_0 t} \\ \text{Spectrum of a Cosine Signal} \\ \mu \cos(2\pi f_0 t + \phi) &= \frac{\mu}{2} e^{j\phi} e^{j2\pi f_0 t} + \frac{\mu}{2} e^{j(-\phi)} e^{j2\pi(-f_0)t} \\ &= \frac{\mu}{2} e^{j\phi} e^{j2\pi f_0 t} + \frac{\mu}{2} e^{j(-\phi + 0.5\pi)} e^{j2\pi(-f_0)t} \end{aligned}$$

Spectrum of a Sine Signal

$$\begin{aligned} \mu \sin(2\pi f_0 t + \phi) &= \frac{\mu}{2} e^{j(\phi - 0.5\pi)} e^{j2\pi f_0 t} \\ &\quad + \frac{\mu}{2} e^{j(-\phi + 0.5\pi)} e^{j2\pi(-f_0)t} \end{aligned}$$

Complex exponential Fourier Series

$$\begin{aligned} x_p(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t / T_p} \\ &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} \end{aligned} \quad (3.1a)$$

$$c_k = \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) e^{-j2\pi k t / T_p} dt, k \in \mathbb{Z} \quad (3.1b)$$

Trigonometric Fourier Series

$$x_p(t) = a_0 + 2 \sum_{k=1}^{\infty} [a_k \cos(2\pi k t / T_p) + b_k \sin(2\pi k t / T_p)]$$

$$a_k = \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) \cos(2\pi k t / T_p) dt; k \geq 0$$

$$b_k = \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) \sin(2\pi k t / T_p) dt; k > 0 \quad (3.2)$$

Chapter 4

4.1 Fourier Transform

Forward Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \quad (4.1a)$$

Inverse Fourier Transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \quad (4.1b)$$

Spectrum of exponentially decaying pulse

$$x(t) = A e^{-\alpha t} u(t)$$

Assume $\alpha > 0$

$$X(f) = \frac{A}{\alpha + j2\pi f}$$

4.3 Spectral properties of a REAL signal

- If $x(t)$ is **REAL** ($x^*(t) = x(t)$), then
 - $X(f)$ is conjugate symmetric ($X^*(f) = X(-f)$)
 - $|X(f)|$ is even ($|X(f)| = |X(-f)|$)
 - $\angle X(f)$ is odd ($\angle X(f) = -\angle X(-f)$)
- If $x(t)$ is **REAL** and **EVEN** ($x^*(t) = x(t) \wedge x(-t) = x(t)$), then
 - $X(f)$ is real ($X^*(f) = X(f)$)
 - $X(f)$ is even ($X(-f) = X(f)$)
- If $x(t)$ is **REAL** and **ODD** ($x^*(t) = x(t) \wedge x(-t) = -x(t)$), then
 - $X(f)$ is imaginary ($X^*(f) = -X(f)$)
 - $X(f)$ is odd ($X(-f) = -X(f)$)

The above can apply to Fourier series coefficients of periodic signals too:

- $x_p(t)$ is **REAL**
 - c_k is conjugate symmetric ($c_k^* = c_{-k}$)
 - $|c_k|$ has even symmetry ($|c_k| = |c_{-k}|$)
 - $\angle c_k$ has odd symmetry ($\angle c_k = -\angle c_{-k}$)
- $x_p(t)$ is **REAL** and **EVEN**
 - c_k is real ($c_k^* = c_k$)
 - c_k is even ($c_k = c_{-k}$)
- $x_p(t)$ is **REAL** and **ODD**
 - c_k is imaginary ($c_k^* = -c_k$)
 - c_k is odd ($c_k = -c_{-k}$)

4.4 Spectrum of Signals that are not Absolutely Integrable

$$\Im\{K\delta(t)\} = \int_{-\infty}^{\infty} K\delta(t) e^{-j2\pi f t} dt = K \quad (4.13)$$

By duality, $\Im\{K\} = K\delta(f)$

4.4.1 Spectrm of Unit Step and Signum function

$$\begin{aligned} \Im\{u(t)\} &= \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) \\ \Im\{\text{sgn}(t)\} &= \frac{1}{j\pi f} \end{aligned}$$

4.4.2 Continuous-Frequency Spectrum of Periodic Signals

The following make use of the fact that

$$\Im\{k\} = K\delta(f) \quad (4.14)$$

DC

$$\begin{aligned} x_{dc}(t) &= K \\ x_{dc}(f) &= \Im\{k\} = K\Im\{1\} = K\delta(f) \end{aligned}$$

Complex Exponential

$$\tilde{x}(t) = K e^{j2\pi f_0 t}$$

$$\tilde{X}(f) = \Im\{K e^{j2\pi f_0 t}\} = K\delta(f - f_0)$$

Cosine

$$\begin{aligned} \Im\{K \cos(2\pi f_0 t)\} \\ = \frac{K}{2} \delta(f - f_0) + \frac{K}{2} \delta(f + f_0) \end{aligned}$$

Sine

$$\begin{aligned} \Im\{K \sin(2\pi f_0 t)\} \\ = \frac{K}{j2} \delta(f - f_0) - \frac{K}{j2} \delta(f + f_0) \end{aligned}$$

$$\text{where } \begin{cases} |X_s(f)| = \frac{K}{2} \delta(f - f_0) + \frac{K}{2} \delta(f + f_0) \\ \angle X_s(f) = \begin{cases} -\pi/2, & f = f_0 \\ \pi/2, & f = -f_0 \end{cases} \end{cases}$$

Arbitrary periodic signals

Let $x_p(t)$ be a periodic signal with period T_p and fundamental frequency f_p

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p) \quad (4.16)$$

4.4.2.1 Spectrum of Dirac- δ Comb function

$$\text{comb}_\lambda(t) \triangleq \sum_n \delta(t - n\lambda)$$

$$c_k = \frac{1}{\lambda}$$

$$\begin{aligned} \Im\{\text{comb}_\lambda(t)\} &= \text{COMB}_\lambda(f) \\ &= \frac{1}{\lambda} \sum_k \delta(f - k/\lambda) \end{aligned}$$

Chapter 5

5.1 Energy Spectral Density (ESD)

Total energy of a signal $x(t)$ is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \text{ (Joules)} \quad (5.1)$$

Rayleigh Energy Theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df, \quad (5.2)$$

Energy Spectral Density

$$E_x(f) = |X(f)|^2 \text{ Joules Hz}^{-1} \quad (5.3)$$

Properties of $E_x(f)$

- $E_x(f)$ is a real function of f
- $E_x(f) \geq 0 \quad \forall f$
- $E_x(f)$ is an even function of f if $x(t)$ is real.

5.2 Power Spectral Density (PSD)

In the time-domain, the average power of a signal $x(t)$ is defined as

$$P = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt \quad (5.4)$$

Windowed version of $x(t)$:

$$x_W(t) = x(t) \text{rect}\left(\frac{t}{2W}\right) \quad (5.5)$$

Parseval Power Theorem

$$\begin{aligned} P &= \lim_{W \rightarrow \infty} \frac{1}{2W} \int_{-W}^W |x(t)|^2 dt \\ &= \int_{-\infty}^{\infty} \lim_{W \rightarrow \infty} \frac{1}{2W} |X_W(f)|^2 df \end{aligned} \quad (5.9)$$

Power Spectral Density

$$P_x(f) = \lim_{W \rightarrow \infty} \frac{1}{2W} |X_W(f)|^2 \text{ Watts Hz}^{-1} \quad (5.10)$$

Properties of $P_x(f)$

- $P_x(f)$ is a real function of f
- $P_x(f) \geq 0 \quad \forall f$
- $P_x(f)$ is an even function of f if $x(t)$ is real.

5.2.1 PSD of Periodic Signals

From chapter 4 equation 4.16:

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p)$$

PSD of $x_p(t)$

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p) \quad (5.12)$$

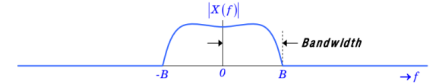
Average power of $x_p(t)$

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad (5.13)$$

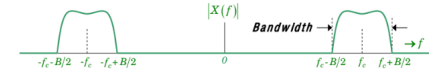
5.3 Bandwidth

5.3.1 Bandlimited Signals

Lowpass signal



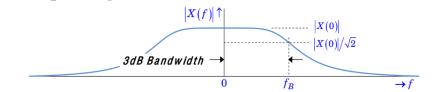
Bandpass signal



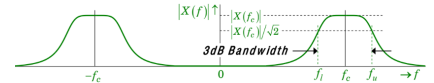
5.3.2 Signals with Unrestricted Band

5.3.2.1 3dB Bandwidth

Lowpass signal:

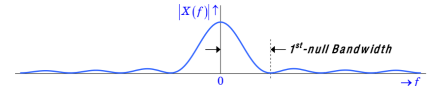


Bandpass signal:

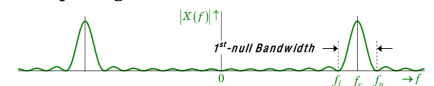


5.3.2.2 1st-null Bandwidth

Lowpass signal:



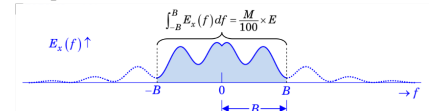
Bandpass signal:



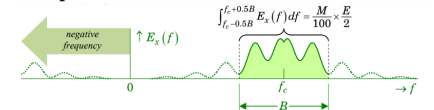
5.3.2.3 M% Energy Containment Bandwidth

Smallest bandwidth that contains at least M% of the total signal energy $E = \int_{-\infty}^{\infty} E_x(f) df$

Lowpass:



Bandpass:



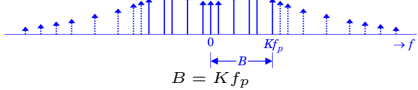
5.3.2.4 M% Power Containment Bandwidth

The smallest bandwidth that contains at least M% of the average signal power. For a periodic signal, the average power is given by

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2$$

where f_p (Hz) is the fundamental frequency and c_k 's are the Fourier series coefficients.

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p) \uparrow$$



where K is the smallest positive integer that satisfies

$$\sum_{k=-K}^K |c_k|^2 \geq \frac{M}{100} \times P$$

Chapter 6

6.1 Systems

6.2 Classification of Systems

6.2.1 Systems with Memory and Without Memory

Memoryless: output at a given time is dependent on only the input at that time.

Otherwise, the system has memory / is dynamic.

6.2.2 Causal and Noncausal Systems

Causal (or non-anticipative): Its output, $y(t)$, at the present time depends on only the present and/or past values of its input, $x(t)$.

∴ not possible for a causal system to produce an output before an input is applied. ∴ $\forall t < 0$ $y(t) = 0$.

6.2.3 Stable and Unstable Systems

BIBO stable (bounded-input/bounded-output): For every bounded input $x(t)$ where

$$\forall t |x(t)| \leq k \quad (6.2)$$

the system produces a bounded output $y(t)$ where

$$\forall t |y(t)| \leq L \quad (6.3)$$

in which K and L are positive constants.

6.2.4 Linear and Nonlinear Systems

Linear system satisfies the following:

$$\begin{aligned} & \mathbf{T}[\alpha_1 x_1(t) + \alpha_2 x_2(t)] \\ &= \alpha_1 \mathbf{T}[x_1(t)] + \alpha_2 \mathbf{T}[x_2(t)] \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned} \quad (6.6)$$

(6.6) is known as the superposition property.

Important property of linear systems:

$$x(t) = 0 \implies y(t) = 0$$

6.2.5 Time-Invariant and Time-Varying Systems

Time-invariant: a time shift (delay or advance) in the input signal, $x(t)$, causes the same time shift in the output signal, $y(t)$.

$$\mathbf{T}[x(t - \tau)] = y(t - \tau) \quad (6.7)$$

A time-varying system is one which does not satisfy (6.7).

Laplace Transform

$$\tilde{F}(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt \quad (6.8)$$

where s is a complex variable.

Inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}\{\tilde{F}(s)\} = \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} \tilde{F}(s) ds \quad (6.9)$$

$$\mathcal{L}\{y''\} = s^2 \mathcal{L}\{y\} - sy(0) - y'(0)$$

$$\begin{aligned} \mathcal{L}\{y'''\} &= s^3 \mathcal{L}\{y\} - s^2 y(0) - sy'(0) \\ &\quad - y''(0) \end{aligned}$$

$$\begin{aligned} \mathcal{L}\{y''''\} &= s^4 \mathcal{L}\{y\} - s^3 y(0) - s^2 y'(0) \\ &\quad - sy''(0) - y'''(0) \end{aligned}$$

Chapter 7

7.1 Impulse Response

Impulse response, $h(t)$: The response/output when the input is a unit impulse, $\delta(t)$.

$$\delta(t) \rightarrow \text{LTI system} \rightarrow h(t)$$

where

$$h(t) = \mathbf{T}[\delta(t)] \quad (7.1)$$

$$\mathbf{T}[x(t)] = y(t) = x(t) * h(t) \quad (7.5)$$

7.1.1 Step Response

Step response: the output of the system when input is unit step function

$$\begin{aligned} u(t) \rightarrow h(t) \rightarrow o(t) &= \int_{-\infty}^{\infty} h(\tau) u(t - \tau) d\tau \\ &= \int_{-\infty}^t h(\tau) d\tau \end{aligned}$$

Step response equals integration of impulse response:

$$o(t) = \int_{-\infty}^t h(\tau) d\tau$$

Impulse response equals differentiation of step response:

$$h(t) = \frac{d}{dt} o(t)$$

7.2 Frequency Response

Frequency response ($H(f)$): The Fourier transform of the system impulse response $h(t)$

$$H(f) = \mathfrak{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft} dt \quad (7.6)$$

$$Y(f) = X(f) \cdot H(f) \quad (7.7)$$

$$H(f) = |H(f)|e^{j\angle H(f)} \quad (7.8)$$

where $|H(f)|$ is called the magnitude response and $\angle H(f)$ is called the phase response of the system.

7.3 Transfer Function

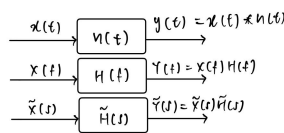
Transfer function $\tilde{H}(s)$: Laplace transform of $h(t)$

$$\tilde{H}(s) = \mathcal{L}\{h(t)\} = \int_0^{\infty} h(t)e^{-st} dt \quad (7.9)$$

where $s = \sigma + j\omega$ is a complex variable.

$$y(t) = x(t) * h(t)$$

$$\tilde{Y}(s) = \tilde{X}(s) \cdot \tilde{H}(s) \quad (7.10)$$



7.4 Relationship between Transfer Function and Frequency Response

Substituting $s = j\omega$ into (7.9), we get

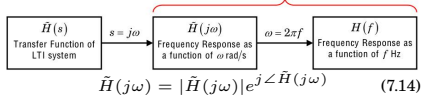
$$\tilde{H}(s) \Big|_{s=j\omega} = \tilde{H}(j\omega) = \int_0^{\infty} h(t)e^{-j\omega t} dt \quad (7.11)$$

Sub $\omega = 2\pi f$ into (7.11):

$$\tilde{H}(j\omega) \Big|_{\omega=2\pi f} = \int_0^{\infty} h(t)e^{-j2\pi ft} dt \quad (7.12)$$

For causal LTI systems, $\forall t < 0$ $h(t) = 0$. Hence (7.6) and (7.12) are equivalent.

$$H(f) = \tilde{H}(j\omega) \Big|_{\omega=2\pi f} \quad (7.13)$$



$$\tilde{H}(j\omega) = |\tilde{H}(j\omega)|e^{j\angle \tilde{H}(j\omega)} \quad (7.14)$$

where $|\tilde{H}(j\omega)|$ is called the magnitude response and $\angle \tilde{H}(j\omega)$ is called the phase response of the system.

7.4 Sinusoidal Response at Steady-State

Let system input at steady-state be

$$x(t) = Ae^{j(2\pi f_0 t + \psi)} \quad (7.15)$$

Then

$$X(f) = Ae^{j\psi} \delta(f - f_0) \quad (7.16)$$

$$Y(f) = A |H(f_0)| e^{j(\psi + \angle H(f_0))} \delta(f - f_0) \quad (7.17)$$

$$\begin{aligned} y(t) &= \mathfrak{F}^{-1}\{Y(f)\} \\ &= A |H(f_0)| e^{j(2\pi f_0 t + \psi + \angle H(f_0))} \end{aligned} \quad (7.18)$$

System Input : $Ae^{j(2\pi f_0 t + \psi)}$

System Output : $A|H(f_0)|e^{j(2\pi f_0 t + \psi + \angle H(f_0))}$

$$x(t) = Ae^{j(2\pi f_0 t + \psi)} \rightarrow [H(f)] \rightarrow y(t) = A|H(f_0)|e^{j(2\pi f_0 t + \psi + \angle H(f_0))}$$

$$x(t) = A\cos(2\pi f_0 t + \psi) \rightarrow [H(f)] \rightarrow y(t) = A|H(f_0)|\cos(2\pi f_0 t + \psi + \angle H(f_0))$$

$$x(t) = A\sin(2\pi f_0 t + \psi) \rightarrow [H(f)] \rightarrow y(t) = A|H(f_0)|\sin(2\pi f_0 t + \psi + \angle H(f_0))$$

Steady-state Sinusoidal Response of a LTI System in f -domain

$$x(t) = Ae^{j(\omega_0 t + \psi)} \rightarrow [\tilde{H}(j\omega)] \rightarrow y(t) = A|\tilde{H}(j\omega_0)|e^{j(\omega_0 t + \psi + \angle \tilde{H}(j\omega_0))}$$

$$x(t) = A\cos(\omega_0 t + \psi) \rightarrow [\tilde{H}(j\omega)] \rightarrow y(t) = A|\tilde{H}(j\omega_0)|\cos(\omega_0 t + \psi + \angle \tilde{H}(j\omega_0))$$

$$x(t) = A\sin(\omega_0 t + \psi) \rightarrow [\tilde{H}(j\omega)] \rightarrow y(t) = A|\tilde{H}(j\omega_0)|\sin(\omega_0 t + \psi + \angle \tilde{H}(j\omega_0))$$

Steady-state Sinusoidal Response of a LTI System in ω -domain

7.6 LTI Systems Described by Differential Equations

LTI systems represented by linear constant-coefficient differential equations have the general form

$$\sum_{n=0}^N a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^M b_m \frac{d^m x(t)}{dt^m} \quad (7.21)$$

where $x(t)$ is input, $y(t)$ is output, and a_n, b_m are real constants.

7.6.1 Transfer Function

$$\begin{aligned} \tilde{H}(s) &= K \left(\frac{s}{z_1} + 1 \right) \left(\frac{s}{z_2} + 1 \right) \dots \left(\frac{s}{z_M} + 1 \right) \\ &\quad \left(\frac{s}{p_1} + 1 \right) \left(\frac{s}{p_2} + 1 \right) \dots \left(\frac{s}{p_N} + 1 \right) \\ K &= \frac{a_0}{b_0} \end{aligned} \quad (7.23b)$$

$$\tilde{H}(s) = K' \frac{(s + z_1)(s + z_2) \dots (s + z_M)}{(s + p_1)(s + p_2) \dots (s + p_N)} \quad (7.23c)$$

$$K = \frac{b_M}{a_N}$$

$\forall n \in \{1, 2, \dots, N\}$

- $\tilde{H}(-p_n) = \infty$
- $-p_n$ are called **poles** of $\tilde{H}(s)$
- $\forall m \in \{1, 2, \dots, M\}$
- $\tilde{H}(-z_m) = 0$
- $-z_m$ are called **zeros** of $\tilde{H}(s)$

The system is said to have N poles and M zeros, and the difference $N - M$ is called pole-zero excess.

7.6.2 System Stability

BIBO Stable

- All system poles lying on the left-half s -plane
- $h(t)$ will converge to 0 as t tends to infinity
- $\lim_{t \rightarrow \infty} h(t) = 0$

Marginally Stable

- One or more **non-repeated** system poles lying on the imaginary axis of the s -plane and no system pole lying on the right half s -plane.
- $h(t)$ will not “blow up” and become unbounded, but neither will it converge to zero as t tends to infinity.
- $\lim_{t \rightarrow \infty} |h(t)| \neq \infty$ and $\lim_{t \rightarrow \infty} h(t) \neq 0$

Unstable (Case 1)

- One or more system poles lying on the right-half s -plane

- $h(t)$ will “blow up” and become unbounded as t tends to infinity
 - $\lim_{t \rightarrow \infty} |h(t)| = \infty$
- Unstable (Case 2)**
- One or more repeated system poles lying on the imaginary axis
 - $h(t)$ will “blow up” and become unbounded as t tends to infinity
 - $\lim_{t \rightarrow \infty} |h(t)| = \infty$

7.7 First Order System (Standard Form)

7.7.1 Differential Eqn, Transfer Func, Impulse Response and Step Response

- Differential equation:

$$T \frac{dy(t)}{dt} + y(t) = Kx(t) \quad (7.26)$$

where

- $x(t)$: system input
- $y(t)$: system output
- K : DC gain
- T : time-constant

- Transfer Function $\tilde{H}(s)$:

$$\begin{aligned} Ts\tilde{Y}(s) + \tilde{Y}(s) &= K\tilde{X}(s) \\ \rightarrow \tilde{H}(s) &= \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{K}{Ts + 1} \end{aligned} \quad (7.27)$$

Pole: $s_1 = -\frac{1}{T}$

- Impulse Response $h(t)$
- $h(t) = \mathcal{L}^{-1}\{\tilde{H}(s)\} = \frac{K}{T}e^{-t/T}u(t)$
- Step Response $o(t)$
- $o(t) = \int_{-\infty}^t h(\tau) d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}\tilde{H}(s)\right\}$
- $= K[1 - e^{-t/T}]u(t)$

7.8 Second Order System (Standard Form) 7.8.1 Differential Eqn and Transfer Func

- Differential equation:

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 x(t) \quad (7.28)$$

where

- $x(t)$: system input
- $y(t)$: system output
- ζ : damping ratio
- ω_n : undamped natural frequency (when $\zeta < 1$)
- K : DC gain
- Transfer function $\tilde{H}(s)$
- $s^2 \tilde{Y}(s) + 2\zeta\omega_n s \tilde{Y}(s) + \omega_n^2 \tilde{Y}(s) = K\omega_n^2 \tilde{X}(s)$

$$\implies \tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (7.29)$$

Poles: $s_{1,2} = -\omega_n \zeta \pm j\omega_n (\zeta^2 - 1)^{1/2}$

- Overdamped system: distinct real poles, $\zeta > 1$
- Critically damped system: repeated real poles, $\zeta = 1$
- Underdamped system: conjugate complex poles, $0 < \zeta < 1$
- Undamped system: conjugate imaginary poles, $\zeta = 0$

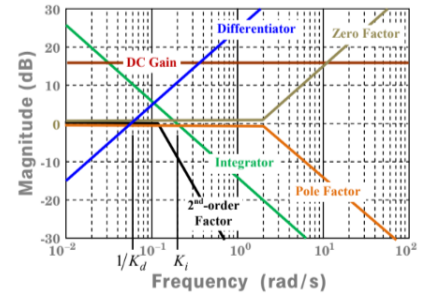
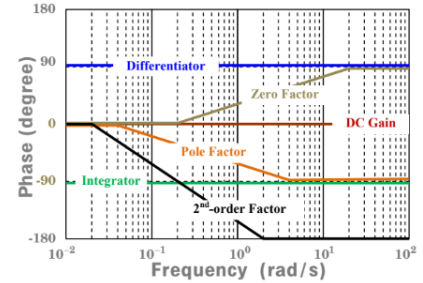
Chapter 8

8.1 Construction of Bode Plots

Basic systems:

- $\tilde{H}(s) = K_{dc}$: DC gain (constant)
- $\tilde{H}(s) = K_d s$: differentiator with gain K_d
- $\tilde{H}(s) = K_i / s$: integrator with gain K_i
- $\tilde{H}(s) = s / z_m + 1$: zero factor with unity DC gain ($\tilde{H}(0) = 1$)
- $\tilde{H}(s) = \frac{1}{s/p_n + 1}$: pole factor with unity DC gain
- $\tilde{H}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$: 2nd-order factor with unity DC gain

8.2 Asymptotic Behavior of Bode Plots



Asymptotic phase of phase plot

$$\text{High frequency: Pole-zero excess} \times (-90^\circ) \quad (8.4a)$$

$$\text{Low frequency: } \left[\text{No. of } \int dt - \text{No. of } \frac{d}{dt} \right] \times (-90^\circ) \quad (8.4b)$$

Asymptotic slope of magnitude plot

$$\text{High frequency: } [\text{Pole-zero excess}] \times (-20 \text{ dB/decade}) \quad (8.5a)$$

$$\text{Low frequency: } \left[\text{No. of } \int dt - \text{No. of } \frac{d}{dt} \right] \times (-20 \text{ dB/decade}) \quad (8.5a)$$

Chapter 9

9.1 Idealized LTI filters

Ideal Low-Pass Filter (LFP)

- Frequency response: $H(f) = A \text{ rect}\left(\frac{f}{2B}\right)$
- Impulse response: $h(t) = 2AB \text{ sinc}(2Bt)$

Ideal Band-Pass Filter (BPF)

- Frequency response:
- $H(f) = A \left[\text{rect}\left(\frac{f+f_0}{2B}\right) + \text{rect}\left(\frac{f-f_0}{2B}\right) \right]$

- Impulse response:
- $h(t) = 2AB \text{ sinc}(Bt) \cos(2\pi f_0 t)$

9.2 Continuous-time Sampling and Reconstruction of Signals

Nyquist Sampling Theorem:

Nyquist sampling frequency / Nyquist rate $f_s = 2f_m$

9.3 Sampling Bandpass Signal below Nyquist

- Overlapping spectral images ($f_c > 0.5B$)
- Un-aliased spectral images ($f_c > 1.5B$)

$$\begin{aligned} \frac{2f_c + B}{k + 1} &\leq f_s \leq \frac{2f_c - B}{k} \\ k &= 1, 2, \dots, \left\lfloor \frac{2f_c - B}{2B} \right\rfloor \end{aligned} \quad (9.2b)$$