

## Euler's formula

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$e^{-j\theta} = \cos(\theta) - j \sin(\theta)$$

## Chapter 1

### 1.2.2 Bounded signals

A continuous-time signal  $x(t)$  is bounded if there exists an  $M$  such that  $0 < M < \infty$  and  $\forall t |x(t)| \leq M$  (has an upper and lower range limit)

### 1.2.3 Absolutely integrable signals

A continuous-time signal  $x(t)$  is absolutely integrable if 
$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

### 1.2.4 Periodic and aperiodic signals

Periodic: there is a non-zero positive value,  $T$ , satisfying  $x(t) = x(t + T) \forall t$  (1.1)

Aperiodic: not periodic

### 1.2.6 Energy and Power Signals

#### Energy signals

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (1.3a)$$

$$x(t) \text{ is an energy signal} \iff 0 < E < \infty \quad (1.3b)$$

#### Power signals

$$P = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt \quad (1.4a)$$

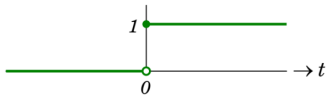
$$x(t) \text{ is a power signal} \iff 0 < P < \infty \quad (1.4b)$$

If  $x(t)$  is a periodic signal, average power may be computed by

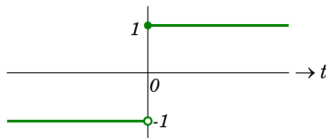
$$\frac{1}{T} \int_0^T |x(t)|^2 dt$$

- Energy signals have 0 average power, bc  $E = \text{finite}$  implies  $P = 0$
- Power signals have infinite total energy, bc  $P = \text{finite}$  implies  $E = \infty$
- All bounded periodic signals are power signals

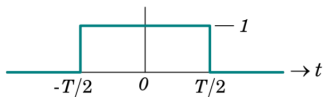
**u(t):**



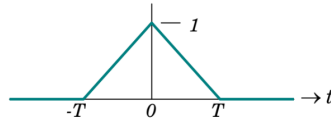
**sgn(t):**



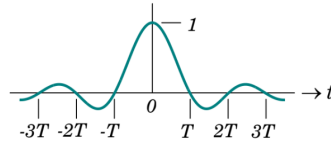
**rect(t/T):**



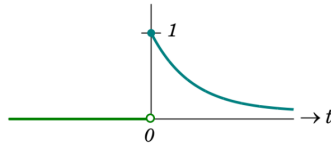
**tri(t/T):**



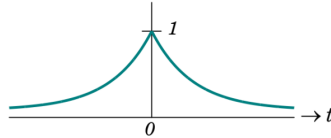
**sinc(t/T):**



**e^{-\alpha t} u(t):**



**e^{-\alpha|t|}:**



### Sinusoidal signals

$$x(t) = \mu \cos(\omega_0 t + \phi)$$

$$= \mu \cos(2\pi f_0 t + \phi)$$

$$= \mu \cos\left(\frac{2\pi t}{T} + \phi\right)$$

$$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$$

## Chapter 2

### 2.1 Time-domain Operations

#### 2.1.1 Time-Scaling

$x(\alpha t)$ : Scale x-axis by a factor of  $\frac{1}{\alpha}$

$x(-t)$ : Reflect about x-axis

#### 2.1.2 Time-Shifting

$x(t - \beta)$ :

$\beta > 0$ : Delaying  $x(t)$  by  $\beta$  units of time (translate right along x-axis)

$\beta > 0$ : Advancing  $x(t)$  by  $\beta$  units of time (translate left along x-axis)

#### 2.1.5 Convolution of 2 Signals

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\alpha) y(t - \alpha) d\alpha$$

#### Properties of convolutions

1. Commutative:  $f * g = g * f$

2. Associative:  $f * (g * h) = (f * g) * h$

3. Distributive:  $f * (g + h) = (f * g) + (f * h)$

### 2.2 Dirac- $\delta$ function

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \neq 0 \end{cases}$$

**Properties:**

$$1. \text{ Symmetry: } \delta(t) = \delta(-t) \quad (2.3)$$

$$2. \text{ Sampling: } x(t)\delta(t - \lambda) = x(\lambda)\delta(t - \lambda) \quad (2.4)$$

$$3. \text{ Sifting } \int_{-\infty}^{\infty} x(t)\delta(t - \lambda) dt = x(\lambda) \int_{-\infty}^{\infty} \delta(t - \lambda) dt = x(\lambda) \quad (2.5)$$

$$4. \text{ Replication } x(t) * \delta(t - \lambda) = \int_{-\infty}^{\infty} x(\zeta)\delta(t - \zeta - \lambda) d\zeta = \int_{-\infty}^{\infty} x(\zeta)\delta(\zeta - (t - \lambda)) d\zeta = x(t - \lambda) \quad (2.6)$$

### 2.2.1 Dirac- $\delta$ Comb function

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) = \dots + \delta(t + T) + \delta(t) + \delta(t - T) + \dots$$

#### Convolution with Dirac- $\delta$ Comb function

$$x_p(t) = x(t) * \sum_n \delta(t - nT) = \sum_n x(t - nT)$$

$x(t)$  is known as the generating function.

#### Multiplication with the Dirac- $\delta$ Comb function

Used for sampling

$$x_s(t) = x(t) \times \sum_n \delta(t - nT) = \sum_n x(t) \times \delta(t - nT) = \sum_n x(nT)\delta(t - nT)$$

## Chapter 3

### 3.2 Spectrum of a Sinusoid

#### Spectrum of a Complex Exponential Signal

$$\hat{x}(t) = \mu e^{j(2\pi f_0 t + \phi)} = \mu e^{j\phi} \times e^{j2\pi f_0 t}$$

where  $\mu$ : magnitude spectrum,  $\phi$ : phase spectrum,  $f_0$ : frequency

#### Spectrum of a Cosine Signal

$$\mu \cos(2\pi f_0 t + \phi) = \frac{\mu}{2} e^{j\phi} e^{j2\pi f_0 t} + \frac{\mu}{2} e^{j(\phi - 0.5\pi)} e^{j2\pi(-f_0)t}$$

#### Spectrum of a Sine Signal

$$\mu \sin(2\pi f_0 t + \phi) = \frac{\mu}{2} e^{j(\phi - 0.5\pi)} e^{j2\pi f_0 t} + \frac{\mu}{2} e^{j(-\phi + 0.5\pi)} e^{j2\pi(-f_0)t}$$

#### Complex exponential Fourier Series

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t / T_p} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} \quad (3.1a)$$

$$c_k = \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) e^{-j2\pi k t / T_p} dt, k \in \mathbb{Z} \quad (3.1b)$$

### Trigonometric Fourier Series

$$x_p(t) = a_0 + 2 \sum_{k=1}^{\infty} [a_k \cos(2\pi k t / T_p) + b_k \sin(2\pi k t / T_p)]$$

$$a_k = \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) \cos(2\pi k t / T_p) dt; k \geq 0$$
$$b_k = \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) \sin(2\pi k t / T_p) dt; k > 0 \quad (3.2)$$

## Chapter 4

### Dirichlet Conditions

Conditions for existence of Fourier Transform:

1.  $x(t)$  has only a finite number of maxima and minima in any finite time interval
  2.  $x(t)$  has only a finite number of discontinuities in any finite time interval
  3.  $x(t)$  is absolutely integrable
- 3 is weak Dirichlet condition: satisfied by most energy signals, violated by all power signals.

### 4.1 Fourier Transform

#### Forward Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \quad (4.1a)$$

#### Inverse Fourier Transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \quad (4.1b)$$

#### Spectrum of exponentially decaying pulse

$$x(t) = A e^{-\alpha t} u(t) = \begin{cases} A e^{-\alpha t}; & t > 0 \\ 0; & t < 0 \end{cases}$$
$$\text{Assume } \alpha > 0$$
$$X(f) = \frac{A}{\alpha + j2\pi f}$$

### 4.2 Properties of Fourier Transform

- $X(f) = \mathfrak{F}\{x(t)\}$  denotes the Fourier transform of  $x(t)$
- $x(t) = \mathfrak{F}^{-1}\{X(f)\}$  denotes the inverse Fourier transform of  $X(f)$
- $x(t) \rightleftharpoons X(f)$  denotes a Fourier transform pair with the time-domain on the LHS and frequency-domain on the RHS.

#### Linearity

If  $x_1(t) \rightleftharpoons X_1(f)$  and  $x_2(t) \rightleftharpoons X_2(f)$ , then

$$\alpha x_1(t) + \beta x_2(t) \rightleftharpoons \alpha X_1(f) + \beta X_2(f) \quad (4.2)$$

#### Time Scaling

$$x(\beta t) \rightleftharpoons \frac{1}{|\beta|} X\left(\frac{f}{\beta}\right) \quad (4.3)$$

#### Duality

$$X(t) \rightleftharpoons x(-f) \quad (4.4)$$

or

$$X(-t) \rightleftharpoons x(f)$$

#### Time Shifting

$$x(t - t_0) \rightleftharpoons X(f) e^{-j2\pi f t_0} \quad (4.5)$$

$$x(t + t_0) \rightleftharpoons X(f) e^{j2\pi f t_0}$$

#### Frequency Shifting (Modulation)

$$x(t) e^{j2\pi f_0 t} \rightleftharpoons X(f - f_0) \quad (4.6)$$

$$x(t) e^{-j2\pi f_0 t} \rightleftharpoons X(f + f_0)$$

### Differentiation in the Time Domain

$$\frac{d}{dt} x(t) \rightleftharpoons j2\pi f \cdot X(f) \quad (4.7)$$

### Integration in the Time Domain

$$\int_{-\infty}^t x(\tau) d\tau \rightleftharpoons \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0) \delta(f) \quad (4.8)$$

### Convolution in the Time Domain / Multiplication in the Frequency Domain

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\alpha) x_2(t - \alpha) d\alpha \rightleftharpoons X_1(f) X_2(f) \quad (4.9a)$$

### Multiplication in the Time Domain / Convolution in the Frequency Domain

$$x_1(t) x_2(t) \rightleftharpoons \int_{-\infty}^{\infty} X_1(\alpha) X_2(f - \alpha) d\alpha = X_1(f) * X_2(f) \quad (4.9b)$$

### 4.3 Spectral properties of a REAL signal

- If  $x(t)$  is **REAL** ( $x^*(t) = x(t)$ ), then
  - $X(f)$  is conjugate symmetric ( $X^*(f) = X(-f)$ )
  - $|X(f)|$  is even ( $|X(f)| = |X(-f)|$ )
  - $\angle X(f)$  is odd ( $\angle X(f) = -\angle X(-f)$ )
- If  $x(t)$  is **REAL** and **EVEN** ( $x^*(t) = x(t) \wedge x(-t) = x(t)$ ), then
  - $X(f)$  is real ( $X^* f = X(f)$ )
  - $X(f)$  is even ( $X(-f) = X(f)$ )
- If  $x(t)$  is **REAL** and **ODD** ( $x^*(t) = x(t) \wedge x(-t) = -x(t)$ ), then
  - $X(f)$  is imaginary ( $X^*(f) = -X(f)$ )
  - $X(f)$  is odd ( $X(-f) = -X(f)$ )

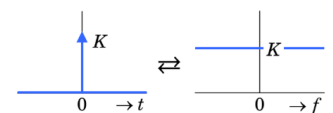
The above can apply to Fourier series coefficients of periodic signals too:

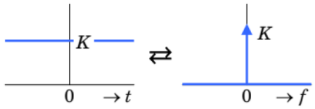
- $x_p(t)$  is **REAL**
  - $c_k$  is conjugate symmetric ( $c_k^* = c_{-k}$ )
  - $|c_k|$  has even symmetry ( $|c_k| = |c_{-k}|$ )
  - $\angle c_k$  has odd symmetry ( $\angle c_k = -\angle c_{-k}$ )
- $x_p(t)$  is **REAL** and **EVEN**
  - $c_k$  is real ( $c_k^* = c_k$ )
  - $c_k$  is even ( $c_k = c_{-k}$ )
- $x_p(t)$  is **REAL** and **ODD**
  - $c_k$  is imaginary ( $c_k^* = -c_k$ )
  - $c_k$  is odd ( $c_k = -c_{-k}$ )

### 4.4 Spectrum of Signals that are not Absolutely Integrable

$$\mathfrak{F}\{K\delta(t)\} = \int_{-\infty}^{\infty} K\delta(t) e^{-j2\pi f t} dt = K \quad (4.13)$$

By duality,  $\mathfrak{F}\{K\} = K\delta(f)$





#### 4.4.1 Spectrum of Unit Step and Signum function

$$\Im\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

$$\Im\{\text{Sgn}(t)\} = \frac{1}{j\pi f}$$

#### 4.4.2 Continuous-Frequency Spectrum of Periodic Signals

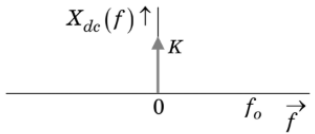
The following make use of the fact that

$$\Im\{k\} = K\delta(f) \quad (4.14)$$

#### DC

$$x_{dc}(t) = K$$

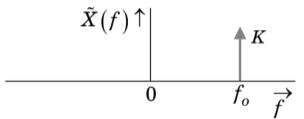
$$X_{dc}(f) = \Im\{k\} = K\Im\{1\} = K\delta(f)$$



#### Complex Exponential

$$\hat{x}(t) = Ke^{j2\pi f_0 t}$$

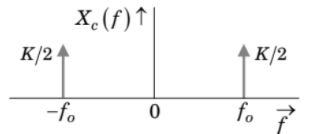
$$\hat{X}(f) = \Im\{Ke^{j2\pi f_0 t}\} = K\delta(f - f_0)$$



#### Cosine

$$\Im\{K \cos(2\pi f_0 t)\}$$

$$= \frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0)$$



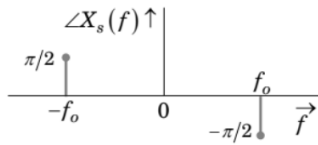
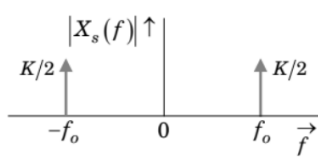
#### Sine

$$\Im\{K \sin(2\pi f_0 t)\}$$

$$= \frac{K}{j2}\delta(f - f_0) - \frac{K}{j2}\delta(f + f_0)$$

where

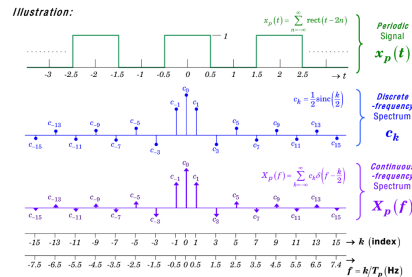
$$\begin{cases} |X_s(f)| &= \frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0) \\ \angle X_s(f) &= \begin{cases} -\pi/2, & f = f_0 \\ \pi/2, & f = -f_0 \end{cases} \end{cases}$$



#### Arbitrary periodic signals

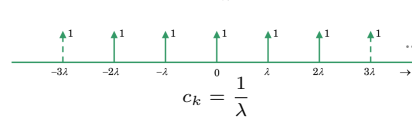
Let  $x_p(t)$  be a periodic signal with period  $T_p$  and fundamental frequency  $f_p$

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p) \quad (4.16)$$



#### 4.4.2.1 Spectrum of Dirac-δ Comb function

$$\text{comb}_\lambda(t) \triangleq \sum_n \delta(t - n\lambda)$$



$$\Im\{\text{comb}_\lambda(t)\} = \text{COMB}_\lambda(f)$$

$$= \frac{1}{\lambda} \sum_k \delta(f - k/\lambda)$$



### Chapter 5

#### 5.1 Energy Spectral Density (ESD)

Total energy of a signal  $x(t)$  is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (\text{Joules}) \quad (5.1)$$

#### Rayleigh Energy Theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df, \quad (5.2)$$

where  $X(f) = \Im\{x(t)\}$  is the spectrum of the signal.

#### Energy Spectral Density

$$E_x(f) = |X(f)|^2 \quad \text{Joules Hz}^{-1} \quad (5.3)$$

#### Properties of $E_x(f)$

1.  $E_x(f)$  is a real function of  $f$
2.  $E_x(f) \geq 0 \quad \forall f$
3.  $E_x(f)$  is an even function of  $f$  if  $x(t)$  is real.

#### 5.2 Power Spectral Density (PSD)

In the time-domain, the average power of a signal  $x(t)$  is defined as

$$P = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt \quad (5.4)$$

Windowed version of  $x(t)$ :

$$x_W(t) = x(t) \text{rect}\left(\frac{t}{2W}\right) \quad (5.5)$$

#### Parseval Power Theorem

$$P = \lim_{W \rightarrow \infty} \frac{1}{2W} \int_{-W}^W |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} \lim_{W \rightarrow \infty} \frac{1}{2W} |X_W(f)|^2 df \quad (5.9)$$

#### Power Spectral Density

$$P_x(f) = \lim_{W \rightarrow \infty} \frac{1}{2W} |X_W(f)|^2 \quad \text{Watts Hz}^{-1} \quad (5.10)$$

#### Properties of $P_x(f)$

1.  $P_x(f)$  is a real function of  $f$
2.  $P_x(f) \geq 0 \quad \forall f$
3.  $P_x(f)$  is an even function of  $f$  if  $x(t)$  is real.

#### 5.2.1 PSD of Periodic Signals

From chapter 4 equation 4.16:

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p)$$

#### PSD of $x_p(t)$

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p) \quad (5.12)$$

#### Average power of $x_p(t)$

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad (5.13)$$

#### 5.3 Bandwidth

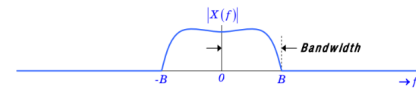
##### 5.3.1 Bandlimited Signals

##### Lowpass signal

A signal  $x(t)$  is said to be a bandlimited lowpass signal if its magnitude spectrum is concentrated around 0 Hz, and at the same time satisfies

$$|X(f)| = 0; \quad |f| > B \quad (5.14)$$

where B is defined as the bandwidth of the signal.



##### Bandpass signal

A signal  $x(t)$  is said to be a bandlimited bandpass signal if its magnitude spectrum is concentrated around a non-zero center frequency  $f_c$ , and at the same time satisfies

$$|X(f)| = 0; \quad ||f| - f_c| > B/2 \quad (5.15)$$

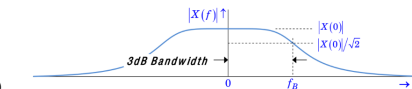
where B is defined as the bandwidth of the signal



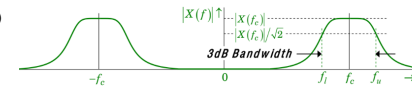
#### 5.3.2 Signals with Unrestricted Band

##### 5.3.2.1 3dB Bandwidth

**Lowpass signal:** The frequency where  $|X(f)| = |X(0)|/\sqrt{2}$  first occurs (or where  $|X(f)|^2 = |X(0)|^2/2$  first occurs) when  $f$  is increased from 0 Hz.



##### Bandpass signal:

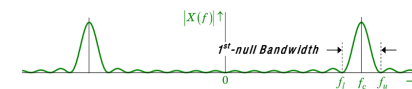


##### 5.3.2.2 1st-null Bandwidth

**Lowpass signal:** The frequency at which  $|X(f)| = 0$  first occurs when  $f$  is increased from 0 Hz:



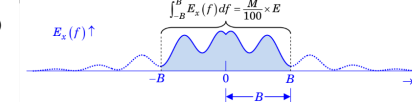
##### Bandpass signal:



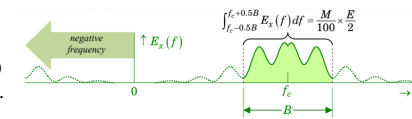
##### 5.3.2.3 M% Energy Containment Bandwidth

Smallest bandwidth that contains at least M% of the total signal energy  $E = \int_{-\infty}^{\infty} E_x(f) df$

##### Lowpass:



##### Bandpass:



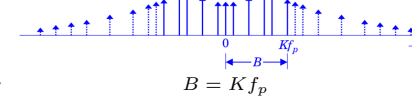
##### 5.3.2.4 M% Power Containment Bandwidth

The smallest bandwidth that contains at least M% of the average signal power. For a periodic signal, the average power is given by

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2$$

where  $f_p$  (Hz) is the fundamental frequency and  $c_k$ 's are the Fourier series coefficients.

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p)$$



where  $K$  is the smallest positive integer that satisfies

$$\sum_{k=-K}^K |c_k|^2 \geq \frac{M}{100} \times P$$

### Chapter 6

#### 6.1 Systems

- A system is a mathematical model of a physical process that relates the input (or excitation) signal to the output (or response) signal.
- With an input  $x(t)$  and an output  $y(t)$ , the system may be viewed as a transformation (or mapping) of  $x(t)$  into  $y(t)$ , mathematically expressed as  $y(t) = \mathbf{T}[x(t)]$  (6.1)

#### 6.2 Classification of Systems

##### 6.2.1 Systems with Memory and Without Memory

A system is said to be memoryless (or static) if its output at a given time is dependent on only the input at that time.

Otherwise, the system is said to have memory (or to be dynamic).

##### 6.2.2 Causal and Noncausal Systems

A system is said to be causal (or non-anticipative) if its output,  $y(t)$ , at the present time depends on only the present and/or past values of its input,  $x(t)$ .

$\therefore$  not possible for a causal system to produce an output before an input is applied.  $\therefore \forall t < 0 \quad y(t) = 0$ .

##### 6.2.3 Stable and Unstable Systems

A system is BIBO stable (bounded-input/bounded-output) if for every bounded input  $x(t)$  where

$$\forall t \quad |x(t)| \leq K \quad (6.2)$$

the system produces a bounded output  $y(t)$  where

$$\forall t \quad |y(t)| \leq L \quad (6.3)$$

in which  $K$  and  $L$  are positive constants.

##### 6.2.4 Linear and Nonlinear Systems

A linear system is one that satisfies the following two conditions:

$$\mathbf{T}[x_1(t) + x_2(t)] = \mathbf{T}[x_1(t)] + \mathbf{T}[x_2(t)]$$

$$= y_1(t) + y_2(t) \quad (6.4)$$

$$\mathbf{T}[\alpha x(t)] = \alpha \mathbf{T}[x(t)] = \alpha y(t) \quad (6.5)$$

(6.4) and (6.5) can be combined into:

$$\mathbf{T}[\alpha_1 x_1(t) + \alpha_2 x_2(t)]$$

$$= \alpha_1 \mathbf{T}[x_1(t)] + \alpha_2 \mathbf{T}[x_2(t)] \quad (6.6)$$

$$= \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

(6.6) is known as the superposition property.

Important property of linear systems:

$$x(t) = 0 \implies y(t) = 0$$

##### 6.2.5 Time-Invariant and Time-Varying Systems

A system is time-invariant if a time shift (delay or advance) in the input signal,  $x(t)$ , causes the same time shift in the output signal,  $y(t)$ .

$$\mathbf{T}[x(t - \tau)] = y(t - \tau) \quad (6.7)$$

A time-varying system is one which does not satisfy (6.7).

#### Laplace Transform

$$\tilde{F}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st} dt \quad (6.8)$$

where  $s$  is a complex variable.

### Inverse Laplace Transform

$$f(t) = \mathcal{L}^{-1}\{\tilde{F}(s)\} = \frac{1}{2\pi j} \int_{\gamma-j\infty}^{\gamma+j\infty} \tilde{F}(s) ds \quad (6.9)$$

## Chapter 7

### 7.1 Impulse Response

Impulse response,  $h(t)$ , of a continuous-time LTI system is defined as the response/output of the system when the input is a unit impulse,  $\delta(t)$ .

$$\delta(t) \rightarrow \text{LTI system} \rightarrow h(t)$$

where

$$h(t) = \mathbf{T}[\delta(t)] \quad (7.1)$$

From replication property,

$$x(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \quad (7.2)$$

Substituting (7.2) into (6.1),

$$y(t) = \mathbf{T}[x(t)] = \mathbf{T}\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right] \quad (7.3)$$

$$= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}[\delta(t-\tau)]d\tau$$

As the system is time-invariant, by applying (6.7) to (7.1),

$$h(t-\tau) = \mathbf{T}[\delta(t-\tau)] \quad (7.4)$$

Sub (7.4) into (4.3):

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t) \quad (7.5)$$

Therefore

$$\mathbf{T}[x(t)] = y(t) = x(t) * h(t)$$

### 7.1.1 Step Response

Step response: the output of the system when input is unit step function

$$u(t) \rightarrow h(t) \rightarrow o(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^t h(\tau)d\tau$$

Step response equals integration of impulse response:

$$o(t) = \int_{-\infty}^t h(\tau)d\tau$$

Impulse response equals differentiation of step response:

$$h(t) = \frac{d}{dt} o(t)$$

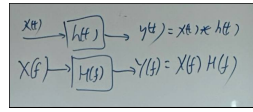
### 7.2 Frequency Response

The frequency response ( $H(f)$ ) of an LTI system is defined as the Fourier transform of the system impulse response  $h(t)$

$$H(f) = \mathfrak{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft} dt \quad (7.6)$$

$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f) \cdot H(f) \quad (7.7)$$



$$H(f) = |H(f)|e^{j\angle H(f)} \quad (7.8)$$

where  $|H(f)|$  is called the magnitude response and  $\angle H(f)$  is called the phase response of the system.

### 7.3 Transfer Function

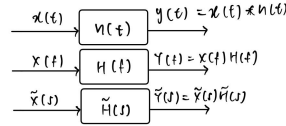
The transfer function  $\tilde{H}(s)$  of an LTI system is defined as the Laplace transform of  $h(t)$

$$\tilde{H}(s) = \mathcal{L}\{h(t)\} = \int_0^\infty h(t)e^{-st} dt \quad (7.9)$$

where  $s = \sigma + j\omega$  is a complex variable.

$$y(t) = x(t) * h(t)$$

$$\tilde{Y}(s) = \tilde{X}(s) \cdot \tilde{H}(s) \quad (7.10)$$



### 7.4 Relationship between Transfer Function and Frequency Response

Substituting  $s = j\omega$  into (7.9), we get

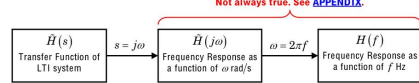
$$\tilde{H}(s)\Big|_{s=j\omega} = \tilde{H}(j\omega) = \int_0^\infty h(t)e^{-j\omega t} dt \quad (7.11)$$

Sub  $\omega = 2\pi f$  into (7.11):

$$\tilde{H}(j\omega)\Big|_{\omega=2\pi f} = \int_0^\infty h(t)e^{-j2\pi ft} dt \quad (7.12)$$

For causal LTI systems,  $\forall t < 0$   $h(t) = 0$ . Hence (7.6) and (7.12) are equivalent.

$$H(f) = \tilde{H}(j\omega)\Big|_{\omega=2\pi f} \quad (7.13)$$



$$\tilde{H}(j\omega) = |\tilde{H}(j\omega)|e^{j\angle \tilde{H}(j\omega)} \quad (7.14)$$

where  $|\tilde{H}(j\omega)|$  is called the magnitude response and  $\angle \tilde{H}(j\omega)$  is called the phase response of the system.

### 7.4 Sinusoidal Response at Steady-State

Let system input at steady-state be

$$x(t) = Ae^{j(2\pi f_0 t + \psi)} \quad (7.15)$$

Then

$$X(f) = Ae^{j\psi}\delta(f-f_0) \quad (7.16)$$

$$Y(f) = A|H(f_0)|e^{j(\psi + \angle H(f_0))}\delta(f-f_0) \quad (7.17)$$

$$y(t) = \mathfrak{F}^{-1}\{Y(f)\} = A|H(f_0)|e^{j(2\pi f_0 t + \psi + \angle H(f_0))} \quad (7.18)$$

System Input :  $Ae^{j(2\pi f_0 t + \psi)}$   
System Output :  $A|H(f_0)|e^{j(2\pi f_0 t + \psi + \angle H(f_0))}$

$$x(t) = Ae^{j(2\pi f_0 t + \psi)} \rightarrow H(f) \rightarrow y(t) = A|H(f_0)|e^{j(2\pi f_0 t + \psi + \angle H(f_0))}$$

### Steady-state Sinusoidal Response of a LTI System in f-domain

$$x(t) = Ae^{j(\omega_0 t + \psi)} \rightarrow \tilde{H}(j\omega) \rightarrow y(t) = A|\tilde{H}(j\omega_0)|e^{j(\omega_0 t + \psi + \angle \tilde{H}(j\omega_0))}$$

$$x(t) = A\cos(\omega_0 t + \psi) \rightarrow \tilde{H}(j\omega) \rightarrow y(t) = A|\tilde{H}(j\omega_0)|\cos(\omega_0 t + \psi + \angle \tilde{H}(j\omega_0))$$

$$x(t) = A\sin(\omega_0 t + \psi) \rightarrow \tilde{H}(j\omega) \rightarrow y(t) = A|\tilde{H}(j\omega_0)|\sin(\omega_0 t + \psi + \angle \tilde{H}(j\omega_0))$$

### Steady-state Sinusoidal Response of a LTI System in \omega-domain

### 7.6 LTI Systems Described by Differential Equations

LTI systems represented by linear constant-coefficient differential equations have the general form

$$\sum_{n=0}^N a_n \frac{d^n y(t)}{dt^n} = \sum_{m=0}^M b_m \frac{d^m x(t)}{dt^m} \quad (7.21)$$

where  $x(t)$  is input,  $y(t)$  is output, and  $a_n, b_m$  are real constants.

### 7.6.1 Transfer Function

Applying Laplace to both sides of (7.21) with initial conditions set to 0,

$$\sum_{n=0}^N a_n \tilde{Y}(s)s^n = \sum_{m=0}^M b_m \tilde{X}(s)s^m \quad (7.22)$$

$$\tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \left( \sum_{m=0}^M b_m s^m \right) / \left( \sum_{n=0}^N a_n s^n \right) = \frac{b_M s^M + b_{M-1} s^{M-1} + \dots + b_0}{a_N s^N + a_{N-1} s^{N-1} + \dots + a_0} \quad (7.23a)$$

$$\tilde{H}(s) = K \frac{\left(\frac{s}{z_1} + 1\right)\left(\frac{s}{z_2} + 1\right)\dots\left(\frac{s}{z_M} + 1\right)}{\left(\frac{s}{p_1} + 1\right)\left(\frac{s}{p_2} + 1\right)\dots\left(\frac{s}{p_N} + 1\right)} \quad (7.23b)$$

$$K = \frac{a_0}{b_0} \quad (7.23c)$$

$$\tilde{H}(s) = K' \frac{(s+z_1)(s+z_2)\dots(s+z_M)}{(s+p_1)(s+p_2)\dots(s+p_N)}$$

$\forall n \in \{1, 2, \dots, N\}$

- $-p_n$  are roots of the denominator polynomial of  $\tilde{H}(s)$
- $\tilde{H}(-p_n) = \infty$
- $-p_n$  are called **poles** of  $\tilde{H}(s)$
- $\forall m \in \{1, 2, \dots, M\}$
- $-z_m$  are roots of the numerator polynomial of  $\tilde{H}(s)$
- $\tilde{H}(-z_m) = 0$
- $-z_m$  are called **zeros** of  $\tilde{H}(s)$

The system is said to have  $N$  poles and  $M$  zeros, and the difference  $N - M$  is called pole-zero excess.

### 7.6.2 System Stability

#### BIBO Stable

- All system poles lying on the left-half s-plane
- $h(t)$  will converge to 0 as  $t$  tends to infinity

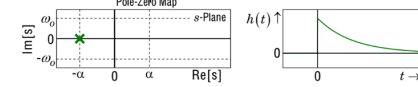
$$\lim_{t \rightarrow \infty} h(t) = 0$$

E.g.

$$\tilde{H}(s) = \frac{1}{s+\alpha}$$

$$\text{Pole: } s = -\alpha$$

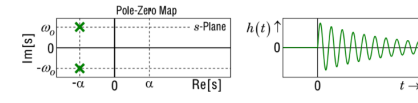
$$h(t) = e^{-\alpha t} u(t)$$



$$\tilde{H}(s) = \frac{\omega_0}{(s+\alpha)^2 + \omega_0^2}$$

$$\text{Poles: } s_{1,2} = -\alpha \pm j\omega_0$$

$$h(t) = e^{-\alpha t} \sin(\omega_0 t) u(t)$$



#### Marginally Stable

- One or more non-repeated system poles lying on the imaginary axis of the s-plane and no system pole lying on the right half s-plane.
- $h(t)$  will not “blow up” and become unbounded, but neither will it converge to zero as  $t$  tends to infinity.

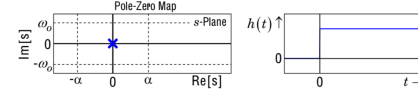
$$\lim_{t \rightarrow \infty} |h(t)| \neq \infty \text{ and } \lim_{t \rightarrow \infty} h(t) \neq 0$$

E.g.

$$\tilde{H}(s) = \frac{1}{s}$$

$$\text{Pole: } s = 0$$

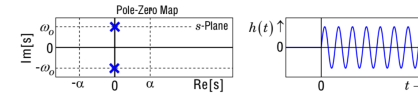
$$h(t) = u(t)$$



$$\tilde{H}(s) = \frac{\omega_0}{s^2 + \omega_0^2}$$

$$\text{Poles: } s_{1,2} = \pm j\omega_0$$

$$h(t) = \sin(\omega_0 t) u(t)$$



#### Unstable (Case 1)

- One or more system poles lying on the right-half s-plane
- $h(t)$  will “blow up” and become unbounded as  $t$  tends to infinity

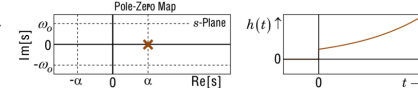
$$\lim_{t \rightarrow \infty} |h(t)| = \infty$$

E.g.

$$\tilde{H}(s) = \frac{1}{s-\alpha}$$

$$\text{Pole: } s = \alpha$$

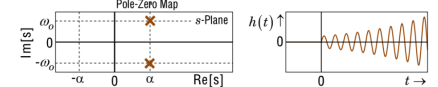
$$h(t) = e^{\alpha t} u(t)$$



$$\tilde{H}(s) = \frac{\omega_0}{(s-\alpha)^2 + \omega_0^2}$$

$$\text{Poles: } s_{1,2} = \alpha \pm j\omega_0$$

$$h(t) = e^{\alpha t} \sin(\omega_0 t) u(t)$$



#### Unstable (Case 2)

- One or more repeated system poles lying on the imaginary axis
- $h(t)$  will “blow up” and become unbounded as  $t$  tends to infinity

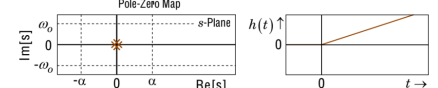
$$\lim_{t \rightarrow \infty} |h(t)| = \infty$$

E.g.

$$\tilde{H}(s) = \frac{1}{s^2}$$

$$\text{Pole: } s_{1,2} = 0$$

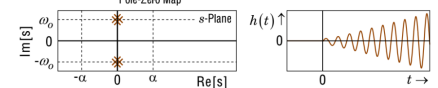
$$h(t) = tu(t)$$



$$\tilde{H}(s) = \frac{\omega_0}{(s^2 + \omega_0^2)^2}$$

$$\text{Poles: } s_{1,2,3,4} = \pm j\omega_0, \pm j\omega_0$$

$$h(t) = \frac{1}{2} [\omega_0^{-1} \sin(\omega_0 t) - t \cos(\omega_0 t)] u(t)$$



### 7.7 First Order System (Standard Form)

#### 7.7.1 Differential Eqn, Transfer Func, Impulse Response and Step Response

Differential equation:

$$T \frac{dy(t)}{dt} + y(t) = Kx(t) \quad (7.26)$$

where

- $x(t)$ : system input
- $y(t)$ : system output
- $K$ : DC gain
- $T$ : time-constant

Transfer Function  $\tilde{H}(s)$ :

$$Ts\tilde{Y}(s) + \tilde{Y}(s) = K\tilde{X}(s)$$

$$\rightarrow \tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{K}{Ts + 1} \quad (7.27)$$

$$\text{Pole: } s_1 = -\frac{1}{T}$$

- Impulse Response  $h(t)$
- $h(t) = \mathcal{L}^{-1}\{\tilde{H}(s)\} = \frac{K}{T} e^{-t/T} u(t)$
- Step Response  $o(t)$

$$o(t) = \int_{-\infty}^t h(\tau)d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}\tilde{H}(s)\right\} = K \left[1 - e^{-t/T}\right] u(t)$$

### 7.8 Second Order System (Standard Form)

#### 7.8.1 Differential Eqn and Transfer Func

Differential equation:

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 x(t) \quad (7.28)$$



where

- $x(t)$ : system input
- $y(t)$ : system output
- $\zeta$ : damping ratio
- $\omega_n$ : undamped natural frequency (when  $\zeta < 1$ )
- $K$ : DC gain

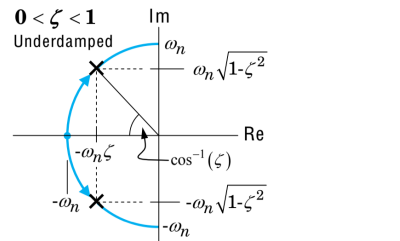
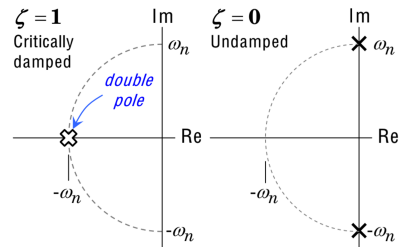
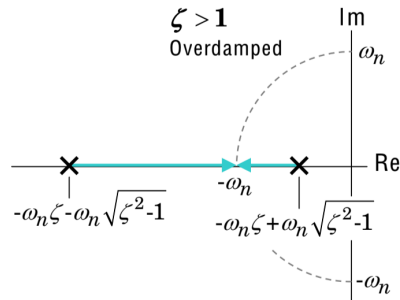
- Transfer function  $\tilde{H}(s)$   
 $s^2 \tilde{Y}(s) + 2\zeta\omega_n s \tilde{Y}(s) + \omega_n^2 \tilde{Y}(s)$

$$= K\omega_n^2 \tilde{X}(s)$$

$$\Rightarrow \tilde{H}(s) = \frac{\tilde{Y}(s)}{\tilde{X}(s)} = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (7.29)$$

Poles:  $s_{1,2} = -\omega_n \zeta \pm \omega_n (\zeta^2 - 1)^{1/2}$

- Damping



## 7.8.2 Impulse Response and Step Response

### 7.8.2.1 Overdamped System ( $\zeta > 1$ )

#### Chapter 8

Bode plot: approximate visualization of frequency response,  $\tilde{H}(j\omega)$  of a system

- Magnitude plot: plot of  $|\tilde{H}(j\omega)|_{dB} = 20 \log_{10} (|\tilde{H}(j\omega)|)$  dB

- Phase plot: plot of  $\angle \tilde{H}(j\omega)$  in degrees
- x-axis is logarithmically scaled (semilog-x: scale is log, but labels are still linear)
- Only positive frequency side visualized (which suffices for real systems as  $|\tilde{H}(j\omega)|$  and  $\angle \tilde{H}(j\omega)$  are even and odd functions of  $\omega$  respectively)
- 0 is not in the axis cuz it goes from 1 to 0.1 to 0.01 to 0.001...

### 8.1 Construction of Bode Plots

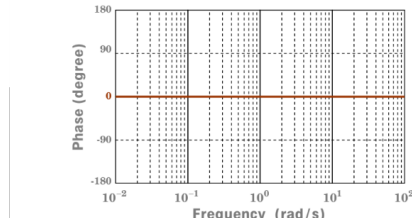
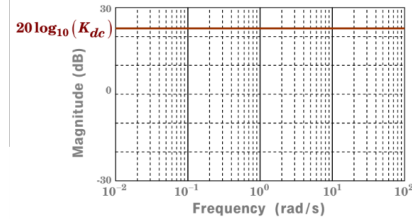
Need to express (7.23b) in a suitable form for each of the following cases:

- Systems without integrator and differentiator
- Systems with differentiators
- Systems with integrators

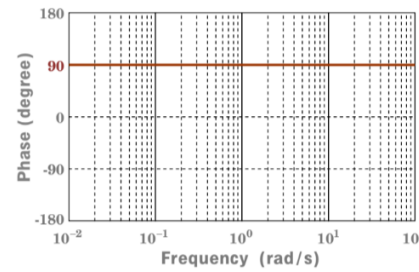
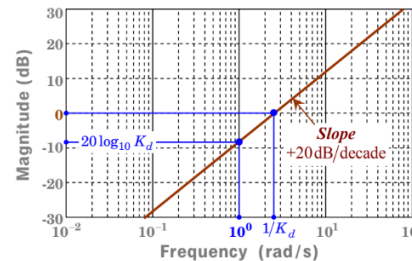
Basic systems:

1.  $\tilde{H}(s) = K_{dc}$ : DC gain (constant)
2.  $\tilde{H}(s) = K_d s$ : differentiator with gain  $K_d$
3.  $\tilde{H}(s) = K_i / s$ : integrator with gain  $K_i$
4.  $\tilde{H}(s) = s / z_m + 1$ : zero factor with unity DC gain ( $\tilde{H}(0) = 1$ )
5.  $\tilde{H}(s) = \frac{1}{s / p_n + 1}$ : pole factor with unity DC gain
6.  $\tilde{H}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ : 2nd-order factor with unity DC gain

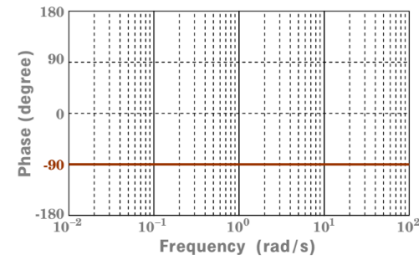
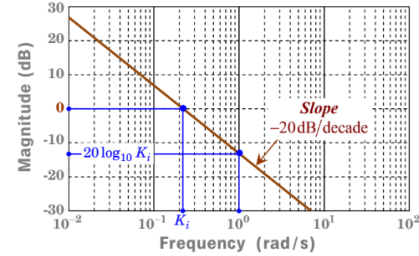
#### 1. DC gain ( $\tilde{H}(s) = K_{dc}$ )



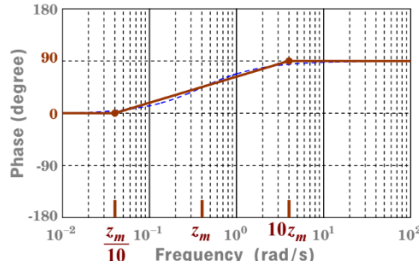
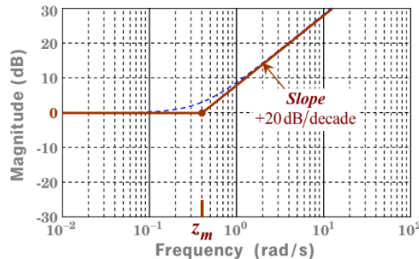
#### 2. Differentiator ( $\tilde{H}(s) = K_d s$ )



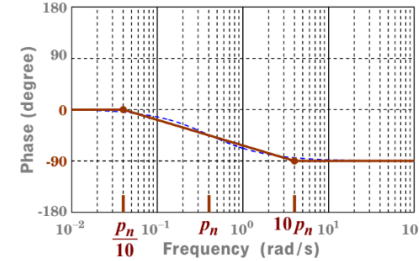
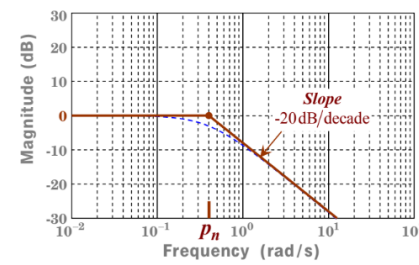
#### 3. Integrator ( $\tilde{H}(s) = K_i / s$ )



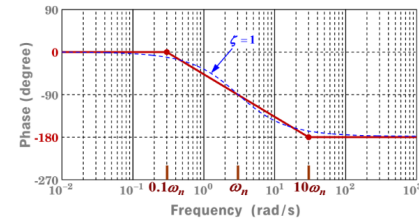
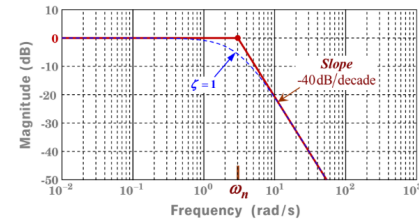
#### 4. Zero factor ( $\tilde{H}(s) = s / z_m + 1$ )



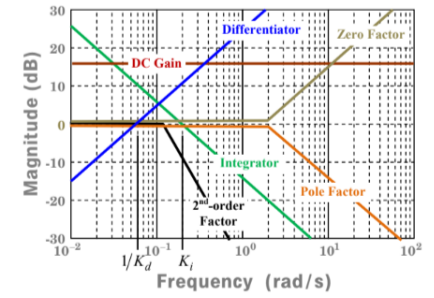
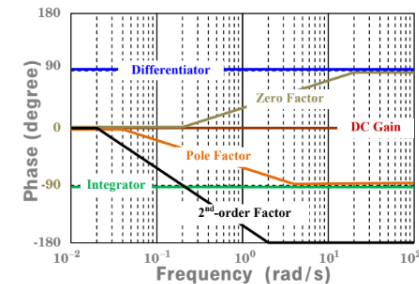
#### 5. Pole factor ( $\tilde{H}(s) = \frac{1}{s / p_n + 1}$ )



#### 6. 2nd-order factor ( $\tilde{H}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ )



### 8.2 Asymptotic Behavior of Bode Plots



### Asymptotic phase of phase plot

High frequency:

$$\lim_{\omega \rightarrow \infty} \angle \tilde{H}(j\omega) = \text{Pole-zero excess} \times (-90^\circ) \quad (8.4a)$$

Low frequency:

$$\lim_{\omega \rightarrow 0} \angle \tilde{H}(j\omega) = \left[ \text{No. of } \int dt - \text{No. of } \frac{d}{dt} \right] \times (-90^\circ) \quad (8.4b)$$

### Asymptotic slope of magnitude plot

High frequency:

$$\lim_{\omega \rightarrow \infty} \left[ \text{Slope of } |\tilde{H}(j\omega)| \right] = [\text{Pole-zero excess}] \times (-20 \text{ dB/decade}) \quad (8.5a)$$

Low frequency:

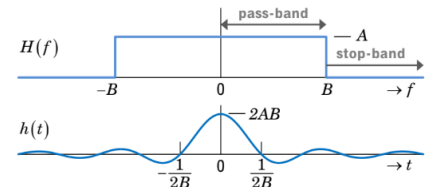
$$\lim_{\omega \rightarrow 0} \left[ \text{Slope of } |\tilde{H}(j\omega)| \right] = \left[ \text{No. of } \int dt - \text{No. of } \frac{d}{dt} \right] \times (-20 \text{ dB/decade}) \quad (8.5a)$$

## Chapter 9

### 9.1 Idealized LTI filters

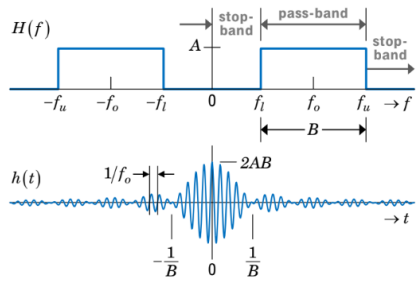
#### Ideal Low-Pass Filter (LPF)

- Frequency response:  $H(f) = A \text{ rect} \left( \frac{f}{2B} \right)$
- Impulse response:  $h(t) = 2AB \text{ sinc}(2Bt)$



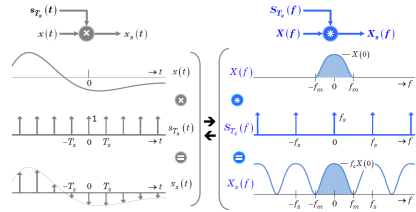
#### Ideal Band-Pass Filter (BPF)

- Frequency response:  $H(f) = A \left[ \text{rect} \left( \frac{f+f_0}{B} \right) + \text{rect} \left( \frac{f-f_0}{B} \right) \right]$
- Impulse response:  $h(t) = 2AB \text{ sinc}(Bt) \cos(2\pi f_0 t)$

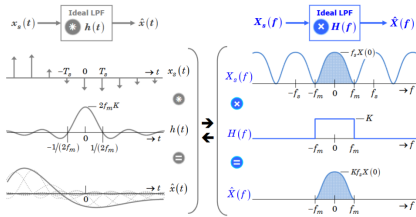


## 9.2 Continuous-time Sampling and Reconstruction of Signals

### Sampling



### Reconstruction



### Nyquist Sampling Theorem:

- A band-limited signal, which has no frequency components higher than  $f_m$  Hz ( $f_m$  = bandwidth = highest freq component), may be completely described by specifying the values of the signal at instants of time separated by no more than  $\frac{1}{2f_m}$  seconds.
- A band-limited signal, which has no frequency components higher than  $f_m$  Hz, may be completely recovered from a knowledge of its samples taken at a rate of no less than  $2f_m$  samples/second.

Nyquist sampling frequency / Nyquist rate  $f_s = 2f_m$

### 9.3 Sampling Band-limited Bandpass

#### Signal below Nyquist Rate

- (a) Overlapping spectral images ( $f_c > 0.5B$ )
- $$f_s = 2f_c/k; \quad k = 1, 2, \dots, \lfloor 2f_c/B \rfloor \quad (9.2a)$$
- (b) Un-aliased spectral images ( $f_c > 1.5B$ )

$$\frac{2f_c + B}{k + 1} \leq f_s \leq \frac{2f_c - B}{k};$$

$$k = 1, 2, \dots, \left\lfloor \frac{2f_c - B}{2B} \right\rfloor \quad (9.2b)$$