

## Euler's formula

$$e^{j\theta} = \cos(\theta) + j \sin(\theta)$$

$$e^{-j\theta} = \cos(\theta) - j \sin(\theta)$$

## Chapter 1

### 1.2.2 Bounded signals

A continuous-time signal  $x(t)$  is bounded if there exists an  $M$  such that  $0 < M < \infty$  and  $\forall t |x(t)| \leq M$  (has an upper and lower range limit)

### 1.2.3 Absolutely integrable signals

A continuous-time signal  $x(t)$  is absolutely integrable if

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

### 1.2.4 Periodic and aperiodic signals

Periodic: there is a non-zero positive value,  $T$ , satisfying

$$x(t) = x(t + T) \quad \forall t \quad (1.1) \quad e^{-\alpha t} u(t):$$

Aperiodic: not periodic

### 1.2.6 Energy and Power Signals

Energy signals

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (1.3a)$$

$$x(t) \text{ is an energy signal} \iff 0 < E < \infty \quad (1.3b)$$

Power signals

$$P = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt \quad (1.4a)$$

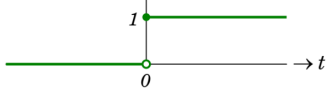
$$x(t) \text{ is a power signal} \iff 0 < P < \infty \quad (1.4b)$$

If  $x(t)$  is a periodic signal, average power may be computed by

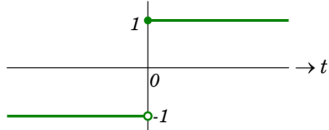
$$\frac{1}{T} \int_0^T |x(t)|^2 dt$$

- Energy signals have 0 average power, bc  $E = \text{finite}$  implies  $P = 0$
- Power signals have infinite total energy, bc  $P = \text{finite}$  implies  $E = \infty$
- All bounded periodic signals are power signals

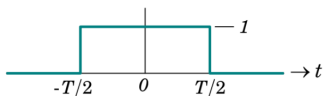
$u(t)$ :



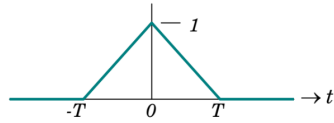
$\text{sgn}(t)$ :



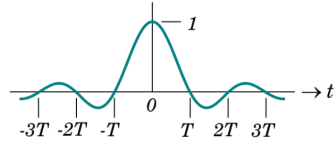
$\text{rect}(\frac{t}{T})$ :



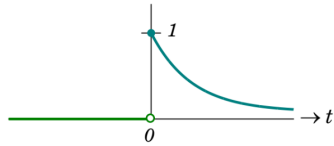
$\text{tri}(\frac{t}{T})$ :



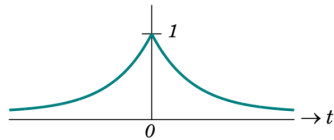
$\text{sinc}(\frac{t}{T})$ :



$e^{-\alpha t} u(t)$ :



$e^{-\alpha|t|}$ :



Sinusoidal signals

$$x(t) = \mu \cos(\omega_0 t + \phi)$$

$$= \mu \cos(2\pi f_0 t + \phi)$$

$$= \mu \cos(\frac{2\pi t}{T} + \phi)$$

$$T_0 = \frac{2\pi}{\omega_0} = \frac{1}{f_0}$$

## Chapter 2

### 2.1 Time-domain Operations

#### 2.1.1 Time-Scaling

$x(\alpha t)$ : Scale x-axis by a factor of  $\frac{1}{\alpha}$   
 $x(-t)$ : Reflect about x-axis

#### 2.1.2 Time-Shifting

$x(t - \beta)$ :

$\beta > 0$ : Delaying  $x(t)$  by  $\beta$  units of time (translate right along x-axis)

$\beta > 0$ : Advancing  $x(t)$  by  $\beta$  units of time (translate left along x-axis)

#### 2.1.5 Convolution of 2 Signals

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\alpha) y(t - \alpha) d\alpha$$

Properties of convolutions

1. Commutative:  $f * g = g * f$
2. Associative:  $f * (g * h) = (f * g) * h$
3. Distributive:  $f * (g + h) = (f * g) + (f * h)$

## 2.2 Dirac- $\delta$ function

$$\delta(t) = \begin{cases} \infty; & t = 0 \\ 0; & t \neq 0 \end{cases}$$

Properties:

1. Symmetry:

$$\delta(t) = \delta(-t) \quad (2.3)$$

2. Sampling:

$$x(t)\delta(t - \lambda) = x(\lambda)\delta(t - \lambda) \quad (2.4)$$

3. Sifting

$$\begin{aligned} & \int_{-\infty}^{\infty} x(t)\delta(t - \lambda) dt \\ &= x(\lambda) \int_{-\infty}^{\infty} \delta(t - \lambda) dt = x(\lambda) \end{aligned} \quad (2.5)$$

4. Replication

$$\begin{aligned} & x(t) * \delta(t - \lambda) \\ &= \int_{-\infty}^{\infty} x(\zeta)\delta(t - \zeta - \lambda) d\zeta \\ &= \int_{-\infty}^{\infty} x(\zeta)\delta(\zeta - (t - \lambda)) d\zeta = x(t - \lambda) \end{aligned} \quad (2.6)$$

### 2.2.1 Dirac- $\delta$ Comb function

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \dots + \delta(t + T) + \delta(t) + \delta(t - T) + \dots \end{aligned}$$

Convolution with Dirac- $\delta$  Comb function

$$\begin{aligned} x_p(t) &= x(t) * \sum_n \delta(t - nT) \\ &= \sum_n x(t - nT) \end{aligned}$$

$x(t)$  is known as the generating function.

Multiplication with the Dirac- $\delta$  Comb function

Used for sampling

$$\begin{aligned} x_s(t) &= x(t) \times \sum_n \delta(t - nT) \\ &= \sum_n x(t) \times \delta(t - nT) \\ &= \sum_n x(nT)\delta(t - nT) \end{aligned}$$

## Chapter 3

### 3.2 Spectrum of a Sinusoid

Spectrum of a Complex Exponential Signal

$$\tilde{x}(t) = \mu e^{j(2\pi f_0 t + \phi)} = \mu e^{j\phi} \times e^{j2\pi f_0 t}$$

where  $\mu$ : magnitude spectrum,  $\phi$ : phase spectrum,  $f_0$ : frequency

Spectrum of a Cosine Signal

$$\begin{aligned} & \mu \cos(2\pi f_0 t + \phi) \\ &= \frac{\mu}{2} e^{j\phi} e^{j2\pi f_0 t} + \frac{\mu}{2} e^{j(-\phi)} e^{j2\pi(-f_0)t} \end{aligned}$$

Spectrum of a Sine Signal

$$\begin{aligned} \mu \sin(2\pi f_0 t + \phi) &= \frac{\mu}{2} e^{j(\phi - 0.5\pi)} e^{j2\pi f_0 t} \\ &+ \frac{\mu}{2} e^{j(-\phi + 0.5\pi)} e^{j2\pi(-f_0)t} \end{aligned}$$

## Complex exponential Fourier Series

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k t / T_p}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_p t} \quad (3.1a)$$

$$c_k = \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) e^{-j2\pi k t / T_p} dt, k \in \mathbb{Z} \quad (3.1b)$$

### Trigonometric Fourier Series

$$\begin{aligned} x_p(t) &= a_0 + 2 \sum_{k=1}^{\infty} [a_k \cos(2\pi k t / T_p) \\ &+ b_k \sin(2\pi k t / T_p)] \end{aligned}$$

$$a_k = \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) \cos(2\pi k t / T_p) dt; k \geq 0$$

$$b_k = \frac{1}{T_p} \int_{t_0}^{t_0 + T_p} x_p(t) \sin(2\pi k t / T_p) dt; k > 0 \quad (3.2)$$

## Chapter 4

### Dirichlet Conditions

Conditions for existence of Fourier Transform:

1.  $x(t)$  has only a finite number of maxima and minima in any finite time interval
  2.  $x(t)$  has only a finite number of discontinuities in any finite time interval
  3.  $x(t)$  is absolutely integrable
- 3 is weak Dirichlet condition: satisfied by most energy signals, violated by all power signals.

### 4.1 Fourier Transform

Forward Fourier Transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \quad (4.1a)$$

Inverse Fourier Transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \quad (4.1b)$$

Spectrum of exponentially decaying pulse

$$\begin{aligned} x(t) &= A e^{-\alpha t} u(t) \\ &= \begin{cases} A e^{-\alpha t}; & t > 0 \\ 0; & t < 0 \end{cases} \end{aligned}$$

Assume  $\alpha > 0$

$$X(f) = \frac{A}{\alpha + j2\pi f}$$

### 4.2 Properties of Fourier Transform

- $X(f) = \mathfrak{F}\{x(t)\}$  denotes the Fourier transform of  $x(t)$
- $x(t) = \mathfrak{F}^{-1}\{X(f)\}$  denotes the inverse Fourier transform of  $X(f)$
- $x(t) \rightleftharpoons X(f)$  denotes a Fourier transform pair with the time-domain on the LHS and frequency-domain on the RHS.

Linearity

If  $x_1(t) \rightleftharpoons X_1(f)$  and  $x_2(t) \rightleftharpoons X_2(f)$ , then

$$\alpha x_1(t) + \beta x_2(t) \rightleftharpoons \alpha X_1(f) + \beta X_2(f) \quad (4.2)$$

Time Scaling

$$x(\beta t) \rightleftharpoons \frac{1}{|\beta|} X\left(\frac{f}{\beta}\right) \quad (4.3)$$

Duality

$$X(t) \rightleftharpoons x(-f) \quad (4.4)$$

or

$$X(-t) \rightleftharpoons x(f)$$

### Time Shifting

$$x(t - t_0) \rightleftharpoons X(f) e^{-j2\pi f t_0} \quad (4.5)$$

$$x(t + t_0) \rightleftharpoons X(f) e^{j2\pi f t_0}$$

### Frequency Shifting (Modulation)

$$x(t) e^{j2\pi f_0 t} \rightleftharpoons X(f - f_0) \quad (4.6)$$

$$x(t) e^{-j2\pi f_0 t} \rightleftharpoons X(f + f_0)$$

### Differentiation in the Time Domain

$$\frac{d}{dt} x(t) \rightleftharpoons j2\pi f \cdot X(f) \quad (4.7)$$

### Integration in the Time Domain

$$\int_{-\infty}^t x(\tau) d\tau \rightleftharpoons \frac{1}{j2\pi f} X(f) + \frac{1}{2} X(0) \delta(f) \quad (4.8)$$

### Convolution in the Time Domain / Multiplication in the Frequency Domain

$$x_1(t) * x_2(t)$$

$$\begin{aligned} &= \int_{-\infty}^{\infty} x_1(\alpha) x_2(t - \alpha) d\alpha \rightleftharpoons X_1(f) X_2(f) \\ &= X_1(f) * X_2(f) \end{aligned} \quad (4.9a)$$

### Multiplication in the Time Domain / Convolution in the Frequency Domain

$$\begin{aligned} & x_1(t) x_2(t) \rightleftharpoons \int_{-\infty}^{\infty} X_1(\alpha) X_2(f - \alpha) d\alpha \\ &= X_1(f) * X_2(f) \end{aligned} \quad (4.9b)$$

### 4.3 Spectral properties of a REAL signal

- If  $x(t)$  is **REAL** ( $x^*(t) = x(t)$ ), then
  - $X(f)$  is conjugate symmetric ( $X^*(f) = X(f)$ )
  - $|X(f)|$  is even ( $|X(f)| = |X(-f)|$ )
  - $\angle X(f)$  is odd ( $\angle X(f) = -\angle X(-f)$ )
- If  $x(t)$  is **REAL** and **EVEN** ( $x^*(t) = x(t) \wedge x(-t) = x(t)$ ), then
  - $X(f)$  is real ( $X^* f = X(f)$ )
  - $X(f)$  is even ( $X(-f) = X(f)$ )
- If  $x(t)$  is **REAL** and **ODD** ( $x^*(t) = x(t) \wedge x(-t) = -x(t)$ ), then
  - $X(f)$  is imaginary ( $X^*(f) = -X(f)$ )
  - $X(f)$  is odd ( $X(-f) = -X(f)$ )

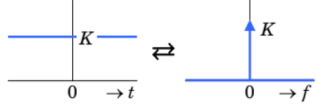
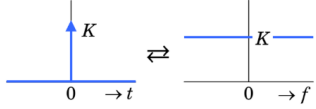
The above can apply to Fourier series coefficients of periodic signals too:

- $x_p(t)$  is **REAL**
  - $c_k$  is conjugate symmetric ( $c_k^* = c_{-k}$ )
  - $|c_k|$  has even symmetry ( $|c_k| = |c_{-k}|$ )
  - $\angle c_k$  has odd symmetry ( $\angle c_k = -\angle c_{-k}$ )
- $x_p(t)$  is **REAL** and **EVEN**
  - $c_k$  is real ( $c_k^* = c_k$ )
  - $c_k$  is even ( $c_k = c_{-k}$ )
- $x_p(t)$  is **REAL** and **ODD**
  - $c_k$  is imaginary ( $c_k^* = -c_k$ )
  - $c_k$  is odd ( $c_k = -c_{-k}$ )

### 4.4 Spectrum of Signals that are not Absolutely Integrable

$$\Im\{K\delta(t)\} = \int_{-\infty}^{\infty} K\delta(t)e^{-j2\pi ft} dt = K \quad (4.13)$$

By duality,  $\Im\{K\} = K\delta(f)$



#### 4.4.1 Spectrum of Unit Step and Signum function

$$\Im\{u(t)\} = \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

$$\Im\{\text{Sgn}(t)\} = \frac{1}{j\pi f}$$

#### 4.4.2 Continuous-Frequency Spectrum of Periodic Signals

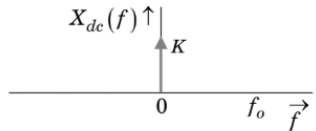
The following make use of the fact that

$$\Im\{k\} = K\delta(f) \quad (4.14)$$

DC

$$x_{dc}(t) = K$$

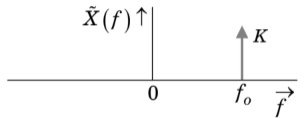
$$X_{dc}(f) = \Im\{k\} = K\Im\{1\} = K\delta(f)$$



#### Complex Exponential

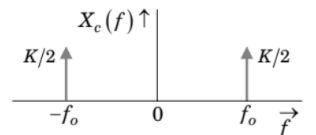
$$\tilde{x}(t) = Ke^{j2\pi f_0 t}$$

$$\tilde{X}(f) = \Im\{Ke^{j2\pi f_0 t}\} = K\delta(f - f_0)$$



Cosine

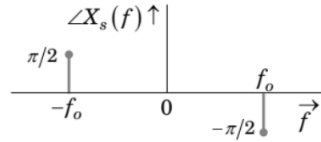
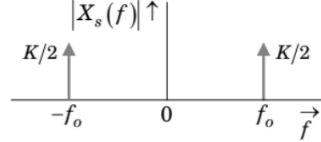
$$\Im\{K \cos(2\pi f_0 t)\} = \frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0)$$



Sine

$$\Im\{K \sin(2\pi f_0 t)\} = \frac{K}{j2}\delta(f - f_0) - \frac{K}{j2}\delta(f + f_0)$$

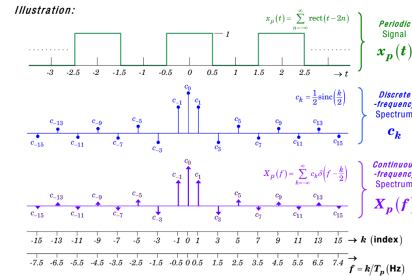
$$\text{where } \begin{cases} |X_s(f)| &= \frac{K}{2}\delta(f - f_0) + \frac{K}{2}\delta(f + f_0) \\ \angle X_s(f) &= \begin{cases} -\pi/2, & f = f_0 \\ \pi/2, & f = -f_0 \end{cases} \end{cases}$$



#### Arbitrary periodic signals

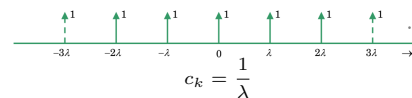
Let  $x_p(t)$  be a periodic signal with period  $T_p$  and fundamental frequency  $f_p$

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p) \quad (4.16)$$



#### 4.4.2.1 Spectrum of Dirac-δ Comb function

$$\text{comb}_\lambda(t) \triangleq \sum_n \delta(t - n\lambda)$$



$$\Im\{\text{comb}_\lambda(t)\} = \text{COMB}_\lambda(f)$$

$$= \frac{1}{\lambda} \sum_k \delta(f - k/\lambda)$$



### Chapter 5

#### 5.1 Energy Spectral Density (ESD)

Total energy of a signal  $x(t)$  is defined as

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (\text{Joules}) \quad (5.1)$$

#### Rayleigh Energy Theorem

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \quad (5.2)$$

where  $X(f) = \Im\{x(t)\}$  is the spectrum of the signal.

#### Energy Spectral Density

$$E_x(f) = |X(f)|^2 \quad \text{Joules Hz}^{-1} \quad (5.3)$$

#### Properties of $E_x(f)$

1.  $E_x(f)$  is a real function of  $f$
2.  $E_x(f) \geq 0 \quad \forall f$
3.  $E_x(f)$  is an even function of  $f$  if  $x(t)$  is real.

#### 5.2 Power Spectral Density (PSD)

In the time-domain, the average power of a signal  $x(t)$  is defined as

$$P = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} |x(t)|^2 dt \quad (5.4)$$

Windowed version of  $x(t)$ :

$$x_W(t) = x(t) \text{rect}\left(\frac{t}{2W}\right) \quad (5.5)$$

#### Parseval Power Theorem

$$P = \lim_{W \rightarrow \infty} \frac{1}{2W} \int_{-W}^W |x(t)|^2 dt = \lim_{W \rightarrow \infty} \frac{1}{2W} \int_{-W}^W |X_W(f)|^2 df \quad (5.9)$$

$$P_x(f) = \lim_{W \rightarrow \infty} \frac{1}{2W} |X_W(f)|^2 \quad \text{Watts Hz}^{-1} \quad (5.10)$$

#### Properties of $P_x(f)$

1.  $P_x(f)$  is a real function of  $f$
2.  $P_x(f) \geq 0 \quad \forall f$
3.  $P_x(f)$  is an even function of  $f$  if  $x(t)$  is real.

#### 5.2.1 PSD of Periodic Signals

From chapter 4 equation 4.16:

$$X_p(f) = \sum_{k=-\infty}^{\infty} c_k \delta(f - kf_p)$$

#### PSD of $x_p(t)$

$$P_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_p) \quad (5.12)$$

#### Average power of $x_p(t)$

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad (5.13)$$

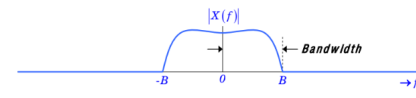
### 5.3 Bandwidth

#### 5.3.1 Bandlimited Signals

A signal  $x(t)$  is said to be a bandlimited low-pass signal if its magnitude spectrum is concentrated around 0 Hz, and at the same time satisfies

$$|X(f)| = 0; \quad |f| > B \quad (5.14)$$

where  $B$  is defined as the bandwidth of the signal.



#### Bandpass signal

A signal  $x(t)$  is said to be a bandlimited bandpass signal if its magnitude spectrum is concentrated around a non-zero center frequency  $f_c$ , and at the same time satisfies

$$|X(f)| = 0; \quad ||f| - f_c| > B/2 \quad (5.15)$$

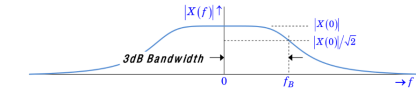
where  $B$  is defined as the bandwidth of the signal



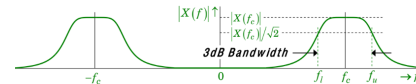
#### 5.3.2 Signals with Unrestricted Band

##### 5.3.2.1 3dB Bandwidth

**Lowpass signal:** The frequency where  $|X(f)| = |X(0)|/\sqrt{2}$  first occurs (or where  $|X(f)|^2 = |X(0)|^2/2$  first occurs) when  $f$  is increased from 0 Hz.

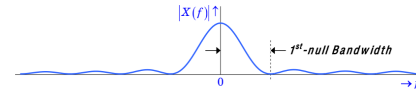


#### Bandpass signal:

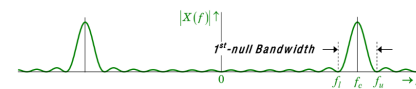


##### 5.3.2.2 1st-null Bandwidth

**Lowpass signal:** The frequency at which  $|X(f)| = 0$  first occurs when  $f$  is increased from 0 Hz:



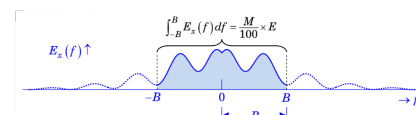
#### Bandpass signal:



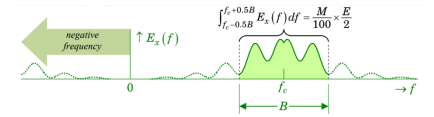
##### 5.3.2.3 M% Energy Containment Bandwidth

Smallest bandwidth that contains at least M% of the total signal energy  $E = \int_{-\infty}^{\infty} E_x(f) df$

#### Lowpass:



#### Bandpass:

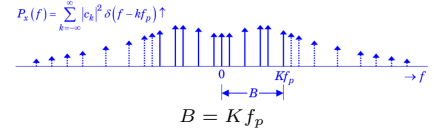


#### 5.3.2.4 M% Power Containment Bandwidth

The smallest bandwidth that contains at least M% of the average signal power. For a periodic signal, the average power is given by

$$P = \int_{-\infty}^{\infty} P_x(f) df = \sum_{k=-\infty}^{\infty} |c_k|^2$$

where  $f_p$  (Hz) is the fundamental frequency and  $c_k$ 's are the Fourier series coefficients.



where  $K$  is the smallest positive integer that satisfies

$$\sum_{k=-K}^K |c_k|^2 \geq \frac{M}{100} \times P$$

#### Random stuff

$$\angle j2\pi f = \tan^{-1}\left(\frac{2\pi f}{0}\right)$$

$$= \begin{cases} \tan^{-1}(\infty) & \text{if } f > 0 \\ \tan^{-1}(-\infty) & \text{if } f < 0 \end{cases}$$

$$= \begin{cases} 0.5\pi & \text{if } f > 0 \\ -0.5\pi & \text{if } f < 0 \end{cases}$$

$$= 0.5\pi \text{sgn}(f)$$

- Only 2 factors affect total energy: time scaling and amplitude
- A signal is real if each complex component got its conjugate then can add tgt
- DC value refers to the constant or average component of a signal over time, which is  $c_0$  in periodic signals
- ESD PSD disjoint
- Index in time domain use  $n$ , in freq domain use  $k$