

论题 1-8 作业

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1 [UD] Problem 6.7

- $B \setminus A$;
- $(A \cup B) \setminus (A \cap B)$;
- $A \cap B \cap C$;
- $(B \cap C) \setminus A$;
- $((A \cap B) \cup (A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C)$.

2 [UD] Problem 6.16

- (a) For every n in A , $n = x^2$ where x is an integer, therefore n is an integer, i.e. $n \in B$, so $A \subseteq B$. \square
- (b) For every t in A , t is a real number, there exists a real number $x = t/2$, such that $t = 2x$, so $t \in B$.
Therefore $A \subseteq B$. \square
- (c) For every point (x, y) in A , we have $y = (5 - 3x)/2$, therefore $2y + 3x = 5$, which means that (x, y) is also an element of B . So $A \subseteq B$. \square

3 [UD] Problem 6.17

- (a) A is a proper subset of B . For every (x, y) in A , we have $xy > 0$, so both x and y are nonzero, thus $x^2 + y^2 > 0$, therefore A is a subset of B . However, $(1, -1)$ is an element of B , but not an element of A , so A is a proper subset of B . \square
- (b) A is a proper subset of B . By theorem 6.10, we have $A \subseteq B$. However, $(0, 0)$ is an element of B , but not an element of A , so A is a proper subset of B . \square

4 [UD] Problem 7.1

- (a) For every x in universe, by definition of complement, if $x \in A$, then $x \notin A^c$ and if $x \notin A^c$ then $x \in (A^c)^c$, therefore we have if $x \in A$, then $x \in (A^c)^c$, i.e. A is a subset of $(A^c)^c$. $(A^c)^c$ is a subset of A likewise. So $(A^c)^c = A$. \square
- (b) For every x in $A \cap (B \cup C)$, we have $x \in A$, and $x \in B$ or $x \in C$, so $x \in A$ and B or $x \in A$ and C , thus $A \cap (B \cup C)$ is a subset of $(A \cap B) \cup (A \cap C)$. For every x in $(A \cap B) \cup (A \cap C)$, we have $x \in A$ and B or $x \in A$ and C , so $x \in A$, and $x \in B$ or $x \in C$, thus $(A \cap B) \cup (A \cap C)$ is a subset of $A \cap (B \cup C)$. So $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. \square
- (c) For every x in $X \setminus (A \cap B)$, we have $x \in X$ and, $x \notin A$ or $x \notin B$, thus $x \in X$ and $x \notin A$, or $x \in X$ and $x \notin B$, therefore $X \setminus (A \cap B)$ is a subset of $(X \setminus A) \cup (X \setminus B)$. For every x in $(X \setminus A) \cup (X \setminus B)$, we have $x \in X$ and $x \notin A$, or $x \in X$ and $x \notin B$, thus $x \in X$ and, $x \notin A$ or $x \notin B$, so $(X \setminus A) \cup (X \setminus B)$ is a subset of $X \setminus (A \cap B)$. Therefore $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. \square
- (d) Since A, B are subsets of X , for every $x \in X$, “if $x \in A$ then $x \in B$ ” and “if $x \notin B$ then $x \notin A$ ” are equivalent, so $A \subseteq B$ if and only if $(X \setminus B) \subseteq (X \setminus A)$. \square
- (e) If $A \cap B = B$, then for every x , “ $x \in B$ ” and “ $x \in A$ and B ” are equivalent, so $x \in B$ implies $x \in A$, i.e. A is a subset of B . If $B \subseteq A$, for every x , $x \in B$ implies $x \in A$, thus $x \in B$ and $x \in A$ and B are equivalent, so $A \cap B = B$. Therefore, $A \cap B = B$ if and only if $B \subseteq A$. \square

5 [UD] Problem 7.8

- (a) (ii);
- (b) (i), (ii), (iii), (iv), (v);
- (c) For every x in $(A \cap B) \setminus C$, we have $x \in A, B$ and $x \notin C$, so $x \in A$ and $x \notin C$, and $x \in B$ and $x \notin C$, thus $(A \cap B) \setminus C$ is a subset of $(A \setminus C) \cap (B \setminus C)$. Likewise $(A \setminus C) \cap (B \setminus C)$ is a subset of $(A \cap B) \setminus C$. Therefore $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$. \square

6 [UD] Problem 7.9

- (a) For every x in $A \setminus B$, we have $x \in A$ and $x \notin B$, so $A \setminus B$ and B are disjoint. \square
- (b) For every x in $A \cup B$, we have $x \in A$ or $x \in B$, so $x \in A$, or $x \in B$ and $x \notin A$, therefore $A \cup B$ is a subset of $(A \setminus B) \cup B$. For every x in $(A \setminus B) \cup B$, we have $x \in A$, or $x \in B$ and $x \notin A$, so $x \in A$ or $x \in B$, therefore $(A \setminus B) \cup B$ is a subset of $A \cup B$. So $A \cup B = (A \setminus B) \cup B$. \square

7 [UD] Problem 7.10

This statement is false. Here is a counterexample. Let $A = \{1, 2\}$, $B = \{1\}$ and $C = \{2\}$, then $A \cup B = A \cup C$, but $B \neq C$. \square

8 [UD] Problem 7.11

This statement is true. We know that for every x , $x \in S$ if and only if $\{x\} \cap S = \{x\}$. For every $x \in A$, let $Y = \{x\}$, then $B \cap Y = A \cap Y = \{x\}$, so $x \in B$, thus A is a subset of B . B is a subset of A likewise. So the statement is true. \square

9 [UD] Problem 8.1

- (a) $\bigcup_{n=1}^{\infty} A_n = [0, 1) \cup [0, 1/2) \cup [0, 1/3) \cdots = [0, 1) ;$
 $\bigcup_{n=1}^{\infty} B_n = [0, 1] \cup [0, 1/2] \cup [0, 1/3] \cdots = [0, 1] ;$
 $\bigcup_{n=1}^{\infty} C_n = (0, 1) \cup (0, 1/2) \cup (0, 1/3) \cdots = (0, 1)$
- (b) $\bigcap_{n=1}^{\infty} A_n = [0, 1) \cap [0, 1/2) \cap [0, 1/3) \cdots = \{0\} ;$
 $\bigcap_{n=1}^{\infty} B_n = [0, 1] \cap [0, 1/2] \cap [0, 1/3] \cdots = \{0\} ;$
 $\bigcap_{n=1}^{\infty} C_n = (0, 1) \cap (0, 1/2) \cap (0, 1/3) \cdots = \emptyset$
- (c) No. Because A_0 is undefined.

10 [UD] Problem 8.4

This statement is false. Here is a counterexample: let $A_n = (n, n + \frac{1}{2})$, $B_n = [n, n + \frac{1}{2}]$, for all positive integer n , $A_n \subset B_n$ holds, however,

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} B_n = \emptyset$$

which does not satisfy the definition of strict inclusion.

11 [UD] Problem 8.7

- (a) Suppose, to the contrary, that $\bigcap_{\alpha \in I} A_\alpha \neq \emptyset$, then there exists $x \in \bigcap_{\alpha \in I} A_\alpha$, however, there exists $\alpha_0 \in I$ such that $A_{\alpha_0} = \emptyset$, so $x \in \emptyset$, which leads to a contradiction. Therefore $\bigcap_{\alpha \in I} A_\alpha = \emptyset$. \square
- (b) Let X be the universe. For all x in X , there exists $\alpha_0 \in I$, such that $A_{\alpha_0} = X$, thus $x \in X$, so $x \in \bigcup_{\alpha \in I} A_\alpha = X$. \square
- (c) For all $x \in B$, for all $\alpha \in I$, we have $x \in A_\alpha$, so $x \in \bigcap_{\alpha \in I} A_\alpha$, therefore $B \subseteq \bigcap_{\alpha \in I} A_\alpha$. \square

12 [UD] Problem 8.8

$$A = \mathbb{Z}.$$

Proof: for every integer m , there exists $n = |m|$ such that $m \notin \mathbb{R} \setminus \{-n, -n+1, \dots, 0, \dots, n-1, n\}$, thus $m \notin \bigcap_{n \in \mathbb{Z}^+} \mathbb{R} \setminus \{-n, -n+1, \dots, 0, \dots, n-1, n\}$, i.e. $m \in A$, so \mathbb{Z} is a subset of A . For every x in A , $x \notin \bigcap_{n \in \mathbb{Z}^+} \mathbb{R} \setminus \{-n, -n+1, \dots, 0, \dots, n-1, n\}$, that is, there exists a positive integer n , such that $x \notin \mathbb{R} \setminus \{-n, -n+1, \dots, 0, \dots, n-1, n\}$, i.e. $x \in \{-n, -n+1, \dots, 0, \dots, n-1, n\}$, so $x \in \mathbb{Z}$, therefore \mathbb{Z} is a subset of A . By the definition of equality of two sets, we get $\mathbb{Z} = A$. \square

13 [UD] Problem 8.9

$$A = \{n : n = 2m, m \in \mathbb{Z}\}.$$

Proof: let \mathbb{R} be the universe,

$$\begin{aligned}
 A &= \mathbb{Q} \setminus \bigcap_{n \in \mathbb{Z}} (\mathbb{R} \setminus \{2n\}) \\
 &= \mathbb{Q} \setminus (\mathbb{R} \setminus \bigcup_{n \in \mathbb{Z}} \{2n\}) && \text{(By De Morgan's laws)} \\
 &= \mathbb{Q} \setminus (\bigcup_{n \in \mathbb{Z}} \{2n\})^c && \text{(By the definition of complement)} \\
 &= \mathbb{Q} \setminus (\{n : n = 2m, m \in \mathbb{Z}\})^c \\
 &= \mathbb{Q} \cap (\{n : n = 2m, m \in \mathbb{Z}\})^c && \text{(By Theorem 7.4.17)} \\
 &= \mathbb{Q} \cap \{n : n = 2m, m \in \mathbb{Z}\} && \text{(By Theorem 7.4.2)} \\
 &= \{n : n = 2m, m \in \mathbb{Z}\} && \square
 \end{aligned}$$

14 [UD] Problem 8.11

- (a) $A_\alpha = \{\alpha\}, (\alpha \in \mathbb{Z});$

- (b) if $A_\alpha \neq A_\beta$, then $A_\alpha \cap A_\beta = \emptyset$;
- (c) if $A_\alpha = A_\beta$, then $A_\alpha \cap A_\beta \neq \emptyset$;
- (d) Yes.
- (e) Yes.
- (f) This assertion holds if and only if there is more than one element in I .
- (g) No. Here is a counterexample: $\{A_\alpha : A_\alpha = \{1, 2, 3\} \setminus \{\alpha\}, \alpha \in I\}$ ($I = \{1, 2, 3\}$).

15 [UD] Problem 9.2

- (a) By the definition of the power set, for every X in $\mathcal{P}(A) \cup \mathcal{P}(B)$, X is a subset of A or B , so X is a subset of $A \cup B$, therefore $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.
- (b) Let $A = \{1\}$, $B = \{2\}$, then $A \cup B = \{1, 2\}$, $\mathcal{P}(A) = \{\{1\}, \emptyset\}$, $\mathcal{P}(B) = \{\{2\}, \emptyset\}$, $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\{1\}, \{2\}, \emptyset\}$, $\mathcal{P}(A \cup B) = \{\{1\}, \{2\}, \{1, 2\}, \emptyset\}$, $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

16 [UD] Problem 9.4

If $A \subseteq B$, then for all set X such that X is a subset of A , X is also a subset of B , so $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
 If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \in \mathcal{P}(A) \subseteq \mathcal{P}(B)$, thus A is a subset of B .
 Therefore, $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. □

17 [UD] Problem 9.12

- (a) The sufficiency is obvious, and we only have to prove the necessity. Assume, to the contrary, that $A \neq C$ or $B \neq D$. Without loss of generality, suppose $A \neq C$. Therefore, there exists x such that $x \in A$ and $x \notin C$, or $x \notin A$ and $x \in C$. Suppose $x \in A$ and $x \notin C$ without loss of generality. Since B is a nonempty set, there exists y in B . Consider (x, y) , by the definition of Cartesian product, it is an element of $A \times B$, however it isn't an element of $C \times D$, because x is not an element of C . Therefore, $A \times B = C \times D$ if and only if $A = C$ and $B = D$. □
- (b) When constructing the pair (x, y) (which leads to contradiction), we need to take an element y in B , and this requires the sets are nonempty.

18 [UD] Problem 9.13

No. Let $A = \{1\}$, $C = \{2\}$, $B = D = \emptyset$, then $A \times B = C \times D = \emptyset$, so $A \times B \subseteq C \times D$, however $A \not\subseteq C$.

19 [UD] Problem 9.14

- (a) True. For all (x, y) in $A \times (B \cup C)$, we have $x \in A$ and $y \in B \cup C$, so $x \in A$ and $y \in B$, or $x \in A$ and $y \in C$, hence $x \in (A \times B) \cup (A \times C)$, therefore, $A \times (B \cup C)$ is a subset of $x \in (A \times B) \cup (A \times C)$. For all (x, y) in $x \in (A \times B) \cup (A \times C)$, we have $x \in A$ and $y \in B$, or $x \in A$ and $y \in C$, so $x \in A$ and $y \in B \cup C$, therefore $x \in (A \times B) \cup (A \times C)$ is a subset of $A \times (B \cup C)$. By the definition of the equality of two sets, we get $A \times (B \cup C) = x \in (A \times B) \cup (A \times C)$. \square
- (b) True. For all (x, y) in $A \times (B \cap C)$, we have $x \in A$ and $y \in B \cap C$, so $x \in A$ and $y \in B$, and $x \in A$ and $y \in C$, hence $x \in (A \times B) \cap (A \times C)$, therefore, $A \times (B \cap C)$ is a subset of $x \in (A \times B) \cap (A \times C)$. For all (x, y) in $x \in (A \times B) \cap (A \times C)$, we have $x \in A$ and $y \in B$, and $x \in A$ and $y \in C$, so $x \in A$ and $y \in B \cap C$, therefore $x \in (A \times B) \cap (A \times C)$ is a subset of $A \times (B \cap C)$. By the definition of the equality of two sets, we get $A \times (B \cap C) = x \in (A \times B) \cap (A \times C)$. \square

20 [UD] Problem 9.16

- (a) If $a = b$, then $(a, b) = \{\{a\}, \{a, b\}\} = \{\{a\}, \{a\}\} = \{\{a\}\}$, thus $\{\{x\}, \{x, y\}\} = \{\{a\}\}$, so $\{x\} = \{x, y\} = \{a\}$, therefore $x = y = a$. Hence $a = x$ and $b = y$.
If $a \neq b$, then $\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\}$. Suppose $\{a\} = \{x, y\}$, then $a = x = y$, so $\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\} = \{\{x\}\}$, thus $\{a\} = \{a, b\}$, and therefore $a = b$, which contradicts $a \neq b$. So $\{a\} = \{x\}$, therefore $a = x$ and $\{a, b\} = \{x, y\}$. If $a = y$, then $a = x = y$, thus $\{a, b\} = \{x, y\} = \{x\}$, therefore $a = b$, which contradicts $a \neq b$. So $a \neq y$, therefore $a = x$ and $b = y$. \square
- (b) Since $\{a\}$ and $\{a, b\}$ are subsets of $A \cup B$, they are both the elements of $\mathcal{P}(A \cup B)$, therefore, $(a, b) = \{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$. \square
- (c) For all $x \in \mathcal{P}(\mathcal{P}(A \cup B))$ in $A \times B$, there exists $a \in A$ and $b \in B$, such that $x = (a, b)$. Therefore, $x \in \mathcal{P}(\mathcal{P}(C \cup D))$ (apply the conclusion in Problem 9.4 twice), and $a \in C$ and $b \in D$, so $x \in C \times D$. Therefore, $A \times B \subseteq C \times D$. \square