论题 1-8 作业

姓名：陈劭源　　　　学号：161240004

1 [UD] Problem 6.7

- $B \setminus A$;
- $(A \cup B) \setminus (A \cap B)$;
- $A \cap B \cap C$;
- $(B \cap C) \setminus A$;
- $((A \cap B) \cup (A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C)$.

2 [UD] Problem 6.16

(a) For every n in A, $n = x^2$ where x is an integer, therefore n is an integer, i.e. $n \in B$, so $A \subseteq B$. □

(b) For every t in A, t is a real number, there exists a real number $x = t/2$, such that $t = 2x$, so $t \in B$. Therefore $A \subseteq B$. □

(c) For every point (x, y) in A, we have $y = (5 - 3x)/2$, therefore $2y + 3x = 5$, which means that (x, y) is also an element of B. So $A \subseteq B$. □

3 [UD] Problem 6.17

(a) A is a proper subset of B. For every (x, y) in A, we have $xy > 0$, so both x and y are nonzero, thus $x^2 + y^2 > 0$, therefore A is a subset of B. However, $(1, -1)$ is an element of B, but not an element of A, so A is a proper subset of B. □

(b) A is a proper subset of B. By theorem 6.10, we have $A \subseteq B$. However, $(0, 0)$ is an element of B, but not an element of A, so A is a proper subset of B. □

4 [UD] Problem 7.1

(a) For every x in universe, by definition of complement, if $x \in A$, then $x \notin A^c$ and if $x \notin A^c$ then $x \in (A^c)^c$, therefore we have if $x \in A$, then $x \in (A^c)^c$, i.e. A is a subset of $(A^c)^c$. $(A^c)^c$ is a subset of A likewise. So $(A^c)^c = A$. □

(b) For every x in $A \cap (B \cup C)$, we have $x \in A$, and $x \in B$ or C, so $x \in A$ and B or $x \in A$ and C, thus $A \cap (B \cup C)$ is a subset of $(A \cap B) \cup (A \cap C)$. For every x in $(A \cap B) \cup (A \cap C)$, we have $x \in A$ and B or $x \in A$ and C, so $x \in A$, and $x \in B$ or C, thus $(A \cap B) \cup (A \cap C)$ is a subset of $A \cap (B \cup C)$. So $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. $\square$

(c) For every x in $X \setminus (A \cap B)$, we have $x \in X$ and, $x \notin A$ or $x \notin B$, thus $x \in X$ and $x \notin A$, or $x \in X$ and $x \notin B$, therefore $X \setminus (A \cap B)$ is a subset of $(X \setminus A) \cup (X \setminus B)$. For every x in $(X \setminus A) \cup (X \setminus B)$, we have $x \in X$ and $x \notin A$, or $x \in X$ and $x \notin B$, thus $x \in X$ and, $x \notin A$ or $x \notin B$, so $(X \setminus A) \cup (X \setminus B)$ is a subset of $X \setminus (A \cap B)$. Therefore $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. $\square$

(d) Since A, B are subsets of X, for every $x \in X$, "if $x \in A$ then $x \in B$" and "if $x \notin B$ then $x \notin A$" are equivalent, so $A \subseteq B$ if and only if $(X \setminus B) \subseteq (X \setminus A)$. $\square$

(e) If $A \cap B = B$, then for every x, "$x \in B$" and "$x \in A$ and B" are equivalent, so $x \in B$ implies $x \in A$, i.e. A is a subset of B. If $B \subseteq A$, for every x, $x \in B$ implies $x \in A$, thus $x \in B$ and $x \in A$ and B are equivalent, so $A \cap B = B$. Therefore, $A \cap B = B$ if and only if $B \subseteq A$. $\square$

5 [UD] Problem 7.8

(a) (ii);

(b) (i), (ii), (iii), (iv), (v);

(c) For every x in $(A \cap B) \setminus C$, we have $x \in A, B$ and $x \notin C$, so $x \in A$ and $x \notin C$, and $x \in B$ and $x \notin C$, thus $(A \cap B) \setminus C$ is a subset of $(A \setminus C) \cap (B \setminus C)$. Likewise $(A \setminus C) \cap (B \setminus C)$ is a subset of $(A \cap B) \setminus C$. Therefore $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$. $\square$

6 [UD] Problem 7.9

(a) For every x in $A \setminus B$, we have $x \in A$ and $x \notin B$, so $A \setminus B$ and B are disjoint. $\square$

(b) For every x in $A \cup B$, we have $x \in A$ or $x \in B$, so $x \in A$, or $x \in B$ and $x \notin A$, therefore $A \cup B$ is a subset of $(A \setminus B) \cup B$. For every x in $(A \setminus B) \cup B$, we have $x \in A$, or $x \in B$ and $x \notin A$, so $x \in A$ or $x \in B$, therefore $(A \setminus B) \cup B$ is a subset of $A \cup B$. So $A \cup B = (A \setminus B) \cup B$ $\square$

7 [UD] Problem 7.10

This statement is false. Here is a counterexample. Let $A = \{1, 2\}$, $B = \{1\}$ and $C = \{2\}$, then $A \cup B = A \cup C$, but $B \neq C$. $\square$

8 [UD] Problem 7.11

This statement is true. We know that for every x, $x \in S$ if and only if $\{x\} \cap S = \{x\}$. For every $x \in A$, let $Y = \{x\}$, then $B \cap Y = A \cap Y = \{x\}$, so $x \in B$, thus A is a subset of B. B is a subset of A likewise. So the statement is true. $\square$

9 [UD] Problem 8.1

(a) $\bigcup_{n=1}^{\infty} A_n = [0,1) \cup [0,1/2) \cup [0,1/3) \cdots = [0,1)$;

$\bigcup_{n=1}^{\infty} B_n = [0,1] \cup [0,1/2] \cup [0,1/3] \cdots = [0,1]$;

$\bigcup_{n=1}^{\infty} C_n = (0,1) \cup (0,1/2) \cup (0,1/3) \cdots = (0,1)$

(b) $\bigcap_{n=1}^{\infty} A_n = [0,1) \cap [0,1/2) \cap [0,1/3) \cdots = \{0\}$;

$\bigcap_{n=1}^{\infty} B_n = [0,1] \cap [0,1/2] \cap [0,1/3] \cdots = \{0\}$;

$\bigcap_{n=1}^{\infty} C_n = (0,1) \cap (0,1/2) \cap (0,1/3) \cdots = \varnothing$

(c) No. Because A_0 is undefined.

10 [UD] Problem 8.4

This statement is false. Here is a counterexample: let $A_n = (n, n+\frac{1}{2})$, $B_n = [n, n+\frac{1}{2}]$, for all positive integer n, $A_n \subset B_n$ holds, however,

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} B_n = \varnothing$$

which does not satisfy the definition of strict inclusion.

11 [UD] Problem 8.7

(a) Suppose, to the contrary, that $\bigcap_{\alpha \in I} A_\alpha \neq \varnothing$, then there exists $x \in \bigcap_{\alpha \in I} A_\alpha$, however, there exists $\alpha_0 \in I$ such that $A_{\alpha_0} = \varnothing$, so $x \in \varnothing$, which leads to a contradiction. Therefore $\bigcap_{\alpha \in I} A_\alpha = \varnothing$. □

(b) Let X be the universe. For all x in X, there exists $\alpha_0 \in I$, such that $A_{\alpha_0} = X$, thus $x \in X$, so $x \in \bigcup_{\alpha \in I} A_\alpha = X$. □

(c) For all $x \in B$, for all $\alpha \in I$, we have $x \in A_\alpha$, so $x \in \bigcap_{\alpha \in I} A_\alpha$, therefore $B \subseteq \bigcap_{\alpha \in I} A_\alpha$. □

12 [UD] Problem 8.8

$A = \mathbb{Z}$.

Proof: for every integer m, there exists $n = |m|$ such that $m \notin \mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, thus $m \notin \bigcap_{n \in \mathbb{Z}^+} \mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, i.e. $m \in A$, so $\mathbb{Z}$ is a subset of A. For every x in A, $x \notin \bigcap_{n \in \mathbb{Z}^+} \mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, that is, there exists a positive integer n, such that $x \notin \mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, i.e. $x \in \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, so $x \in \mathbb{Z}$, therefore $\mathbb{Z}$ is a subset of A. By the definition of equality of two sets, we get $\mathbb{Z} = A$. □

13 [UD] Problem 8.9

$A = \{n : n = 2m, m \in \mathbb{Z}\}$.
Proof: let $\mathbb{R}$ be the universe,

$$\begin{aligned}
A &= \mathbb{Q} \setminus \bigcap_{n\in\mathbb{Z}} (\mathbb{R} \setminus \{2n\}) \\
&= \mathbb{Q} \setminus (\mathbb{R} \setminus \bigcup_{n\in\mathbb{Z}} \{2n\}) && \text{(By De Morgan's laws)} \\
&= \mathbb{Q} \setminus (\bigcup_{n\in\mathbb{Z}} \{2n\})^c && \text{(By the definition of complement)} \\
&= \mathbb{Q} \setminus (\{n : n = 2m, m \in \mathbb{Z}\})^c \\
&= \mathbb{Q} \cap (\{n : n = 2m, m \in \mathbb{Z}\}^c)^c && \text{(By Theorem 7.4.17)} \\
&= \mathbb{Q} \cap \{n : n = 2m, m \in \mathbb{Z}\} && \text{(By Theorem 7.4.2)} \\
&= \{n : n = 2m, m \in \mathbb{Z}\} && \square
\end{aligned}$$

14 [UD] Problem 8.11

(a) $A_\alpha = \{\alpha\}$, $(\alpha \in \mathbb{Z})$;

(b) if $A_\alpha \neq A_\beta$, then $A_\alpha \cap A_\beta = \varnothing$;

(c) if $A_\alpha = A_\beta$, then $A_\alpha \cap A_\beta \neq \varnothing$;

(d) Yes.

(e) Yes.

(f) This assertion holds if and only if there is more than one element in I.

(g) No. Here is a counterexample: $\{A_\alpha : A_\alpha = \{1,2,3\} \setminus \{\alpha\}, \alpha \in I\}$ $(I = \{1,2,3\})$.

15 [UD] Problem 9.2

(a) By the definition of the power set, for every X in $\mathcal{P}(A) \cup \mathcal{P}(A)$, X is a subset of A or B, so X is a subset of $A \cup B$, therefore $\mathcal{P}(A) \cup \mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$.

(b) Let $A = \{1\}$, $B = \{2\}$, then $A \cup B = 1,2$, $\mathcal{P}(A) = \{\{1\},\varnothing\}$, $\mathcal{P}(B) = \{\{2\},\varnothing\}$, $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\{1\},\{2\},\varnothing\}$, $\mathcal{P}(A \cup B) = \{\{1\},\{2\},\{1,2\},\varnothing\}$, $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

16 [UD] Problem 9.4

If $A \subseteq B$, then for all set X such that X is a subset of A, X is also a subset of B, so $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \in \mathcal{P}(A) \subseteq \mathcal{P}(B)$, thus A is a subset of B.
Therefore, $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. $\square$

17 [UD] Problem 9.12

(a) The sufficiency is obvious, and we only have to prove the necessity. Assume, to the contrary, that $A \neq C$ or $B \neq D$. Without loss of generality, suppose $A \neq C$. Therefore, there exists x such that $x \in A$ and $x \notin C$, or $x \notin A$ and $x \in C$. Suppose $x \in A$ and $x \notin C$ without loss of generality. Since B is a nonempty set, there exists y in B. Consider (x,y), by the definition of Cartesian product, it is an element of $A \times B$, however it isn't an element of $C \times D$, because x is not an element of C. Therefore, $A \times B = C \times D$ if and only if $A = C$ and $B = D$. $\square$

(b) When constructing the pair (x,y) (which leads to contradiction), we need to take an element y in B, and this requires the sets are nonempty.

18 [UD] Problem 9.13

No. Let $A = \{1\}, C = \{2\}, B = D = \varnothing$, then $A \times B = C \times D = \varnothing$, so $A \times B \subseteq C \times D$, however $A \nsubseteq C$.

19 [UD] Problem 9.14

(a) True. For all (x,y) in $A \times (B \cup C)$, we have $x \in A$ and $y \in B \cup C$, so $x \in A$ and $y \in B$, or $x \in A$ and $y \in C$, hence $x \in (A \times B) \cup (A \times C)$, therefore, $A \times (B \cup C)$ is a subset of $x \in (A \times B) \cup (A \times C)$. For all (x,y) in $x \in (A \times B) \cup (A \times C)$, we have $x \in A$ and $y \in B$, or $x \in A$ and $y \in C$, so $x \in A$ and $y \in B \cup C$, therefore $x \in (A \times B) \cup (A \times C)$ is a subset of $A \times (B \cup C)$. By the definition of the equality of two sets, we get $A \times (B \cup C) = x \in (A \times B) \cup (A \times C)$. $\square$

(b) True. For all (x,y) in $A \times (B \cap C)$, we have $x \in A$ and $y \in B \cap C$, so $x \in A$ and $y \in B$, and $x \in A$ and $y \in C$, hence $x \in (A \times B) \cap (A \times C)$, therefore, $A \times (B \cap C)$ is a subset of $x \in (A \times B) \cap (A \times C)$. For all (x,y) in $x \in (A \times B) \cap (A \times C)$, we have $x \in A$ and $y \in B$, and $x \in A$ and $y \in C$, so $x \in A$ and $y \in B \cap C$, therefore $x \in (A \times B) \cap (A \times C)$ is a subset of $A \times (B \cap C)$. By the definition of the equality of two sets, we get $A \times (B \cap C) = x \in (A \times B) \cap (A \times C)$. $\square$

20 [UD] Problem 9.16

(a) If $a = b$, then $(a,b) = \{\{a\},\{a,b\}\} = \{\{a\},\{a\}\} = \{\{a\}\}$, thus $\{\{x\},\{x,y\}\} = \{\{a\}\}$, so $\{x\} = \{x,y\} = \{a\}$, therefore $x = y = a$. Hence $a = x$ and $b = y$.

If $a \neq b$, then $\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}$. Suppose $\{a\} = \{x,y\}$, then $a = x = y$, so $\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\} = \{\{x\}\}$, thus $\{a\} = \{a,b\}$, and therefore $a = b$, which contradicts $a \neq b$. So $\{a\} = \{x\}$, therefore $a = x$ and $\{a,b\} = \{x,y\}$. If $a = y$, then $a = x = y$, thus $\{a,b\} = \{x,y\} = \{x\}$, therefore $a = b$, which contradicts $a \neq b$. So $a \neq y$, therefore $a = x$ and $b = y$. $\square$

(b) Since $\{a\}$ and $\{a,b\}$ are subsets of $A \cup B$, they are both the elements of $\mathcal{P}(A \cup B)$, therefore, $(a,b) = \{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$. $\square$

(c) For all $x \in \mathcal{P}(\mathcal{P}(A \cup B))$ in $A \times B$, there exists $a \in A$ and $b \in B$, such that $x = (a, b)$. Therefore, $x \in \mathcal{P}(\mathcal{P}(C \cup D))$ (apply the conclusion in Problem 9.4 twice), and $a \in C$ and $b \in D$, so $x \in C \times D$. Therefore, $A \times B \subseteq C \times D$. $\square$