论题 1-9 作业

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1　[UD] Problem 10.2

(a) $\{(1,1),(2,2),(3,3),(4,4),(5,5)\}$;

(b) $\{(1,1),(2,2),(2,3),(3,3),(3,4),(4,4),(5,5)\}$;

(c) $\{(1,2),(2,1)\}$

(d) $\{(1,2),(2,3),(3,4),(4,5),(5,1)\}$;

2　[UD] Problem 10.4

Yes. First, it is reflexive, because $x_1 - y_1 = x_2 - y_2 = 0$ are even when $(x_1,x_2) = (y_1,y_2)$. Second, it is symmetry because both $x_1 - y_1$ and $x_2 - y_2$ are even if and only if $y_1 - x_1$ and $y_2 - x_2$ are even. Third, it is transitive, because that both $x_1 - y_1$ and $y_1 - z_1$ are even implies $x_1 - z_1$ is even, and $x_2 - z_2$ is even likewise.

3　[UD] Problem 10.5

"If": for all $a \in E_x$, we have $a \sim x$, since $x \sim y$, we get $a \sim y$, therefore $a \in E_y$. Hence E_x is a subset of E_y. Likewise E_y is a subset of E_x. So $E_x = E_y$.

"Only if": by the definition of equivalence class, $E_x = \{a \in X : x \sim a\}$, since $y \in E_y$ and $E_x = E_y$, we have $y \in E_x$, that is, $x \sim y$. $\qquad\square$

4　[UD] Problem 10.8

(a) Yes. The equivalence class is $\{\sum_{i=0}^{n} a_i x^i : a_0 = 0\}$.

(b) Yes. E_r is the set of all the polynomials of degree 1.

(c) No. Because it is not reflexive.

5 [UD] Problem 11.3

(a) Yes. A_r represents a plane, on which the sum of the coordinates of a point is r.

(b) Yes. A_r represents a sphere whose center is the origin and its radius is $|r|$.

6 [UD] Problem 11.7

(a) Yes. Obviously, for all $m \in \mathbb{N}$, A_m is nonempty. Since every polynomial has a degree, so $\bigcup_{m \in \mathbb{N}} A_m = P$. Every polynomial has only one degree, so that for all $\alpha, \beta \in \mathbb{N}$, $A_\alpha = A_\beta$ (when $\alpha = \beta$) or $A_\alpha \cap A_\beta = \varnothing$ (when $\alpha \neq \beta$) holds. Therefore, A_m determine a partition.

(b) Yes. For all $c \in \mathbb{R}$, there exists a polynomial such that $p(0) = c$, so A_c is always nonempty. We have that $\bigcup_{c \in \mathbb{R}} A_c = P$. For every polynomial, $p(0)$ is a constant, so that for all $\alpha, \beta \in \mathbb{R}$, $A_\alpha = A_\beta$ (when $\alpha = \beta$) or $A_\alpha \cap A_\beta = \varnothing$ (when $\alpha \neq \beta$) holds.

(c) No. Consider A_x and A_{x^2}, x^2 is an element of both, however, $A_x \neq A_{x^2}$ because x is an element of the former one but not an element of the latter one.

(d) No. Consider A_0 and A_1, $x^2 - x$ is an element of both, however, $A_0 \neq A_1$ because x is an element of the former one but not an element of the latter one.

7 [UD] Problem 11.8

First, for all $\alpha \in I \cup J$, A_a is nonempty because $\{A_\alpha : \alpha \in I\}$ and $\{A_\alpha : \alpha \in J\}$ are both nonempty. Second, for every real number x, there exists $\alpha \in I \cup J$ ($\alpha \in I$ when $x > 0$ and $\alpha \in J$ when $x \leq 0$), therefore $\bigcup_{\alpha \in I \cup J} A_\alpha = \mathbb{R}$. Third, for all $\alpha, \beta \in I$ (or J), $A_\alpha = A_\beta$ or $A_\alpha \cap A_\beta$ holds, and for all $\alpha \in I$ and $\beta \in J$, $A_\alpha \cap A_\beta = \varnothing$, therefore for all $\alpha, \beta \in I \cup J$, $A_\alpha = A_\beta$ or $A_\alpha \cap A_\beta$ holds. Hence $\{A_\alpha : \alpha \in I \cup J\}$ is a partition of $\mathbb{R}$.

8 [UD] Problem 11.9

(a) No. Let $X = \{1,2,3\}$, and $\{A_\alpha : \alpha \in I\} = \{\{1\}, \{2,3\}\}$ be a partition of X. Let $B = \{1,2\} \subseteq X$ such that $B \cap \{1\} \neq \varnothing$ and $B \cap \{2,3\} \neq \varnothing$. However, $\{A_\alpha \cap B : \alpha \in I\} = \{\{1\}, \{2\}\}$ is not a partition of B, because $\bigcup_{\alpha \in I} A_\alpha \cap B = \{1,2\} \neq X$.

(b) No. Let $X = \{1,2,3\}$, and $\{A_\alpha : \alpha \in I\} = \{\{1\}, \{2\}, \{3\}\}$ be a partition of X. However, $\{X \setminus A_\alpha : \alpha \in I\} = \{\{1,2\}, \{1,3\}, \{2,3\}\}$ is not a partition of X because $\{1,2\} \neq \{1,3\}$ and $\{1,2\} \cap \{1,3\} \neq \varnothing$.

9   [UD] Problem 12.10

(a) Suppose $\sup(S \cup T) < \sup S$. Since $\sup(S \cup T)$ is an upper bound of $S \cup T$, $\sup(S \cup T)$ is also an upper bound of S. However, the least upper bound of S, $\sup S$, is greater than $\sup(S \cup T)$, another upper bound of S, which leads to contradiction. Therefore $\sup(S \cup T) \geq \sup S$, and likewise $\sup(S \cup T) \geq \sup T$. $\qquad\square$

(b) Without loss of generality, assume that $\sup S \geq \sup T$. Suppose to the contrary that $\sup(S \cup T) > \sup S$, take $M = \sup(S \cup T) + \sup S$, we have $\sup(S \cup T) > M > \sup S$, and M is the upper bound of S and T because $M > \sup S \geq \sup T$, therefore M is the upper bound of $S \cup T$, however, the least upper bound of $S \cup T$, $\sup(S \cup T)$, is greater than M, which leads to contradiction. Therefore $\sup(S \cup T) = \max\{\sup S, \sup T\}$. $\qquad\square$

(c) The supremum of the union of two sets is greater than or equal to the supremum of either set. In fact, the supremum of the union of two sets is the maximum of the suprema of the two sets.

10   [UD] Problem 12.13b

(Reflexive) For all $S \in \mathcal{P}(A)$, S is a subset of S, so $S \subseteq S$;

(Transitive) For all $A, B, C \in \mathcal{P}(A)$, if A is a subset of B and B is a subset of C, then for all $x \in A$, x is an element of B, thus x is an element of C, therefore, A is a subset of C;

(Antisymmetric) By the definition of the equality of two sets, the antisymmetric property holds for $(\mathcal{P}(A), \subseteq)$.

Let a, b be two distinct elements of A, then neither $\{a\} \subseteq \{b\}$ nor $\{b\} \subseteq \{a\}$ holds, therefore $(\mathcal{P}(A), \subseteq)$ is a partial order but not a total order. $\qquad\square$

11   [UD] Problem 12.16

(a) We can find that $\{1, 2, 5, 7, 8, 10\}$ is a least upper bound of $\mathcal{B}$. Therefore $\mathcal{B}$ is an upper bounded set. $\qquad\square$

(b) For every nonempty subset X of $\mathcal{P}(\mathbb{Z})$, for every element $x \in X$, x is an element of $\mathbb{Z}$, therefore $\mathbb{Z}$ is an upper set of X. Hence every nonempty subset of $\mathcal{P}(\mathbb{Z})$ is upper bounded. $\qquad\square$

(c) For every nonempty subset $\mathcal{A}$ of $\mathcal{P}(\mathbb{Z})$ we say that $L \in \mathcal{P}(\mathbb{Z})$ is an lower set of $\mathcal{A}$, if $L \subseteq X$ for all $X \in \mathcal{A}$. A nonempty set $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Z})$ will be called a lower bounded set if there exists a lower set of $\mathcal{A}$ in $\mathcal{P}(\mathbb{Z})$. We say $L_0 \in \mathcal{P}(\mathbb{Z})$ is a least upper set if (i) U_0 is a lower set of $\mathcal{A}$ and (ii) if L is another lower set of $\mathcal{A}$, then $L \subseteq L_0$.

(d) Least upper set of $\mathcal{A}$: $\displaystyle\bigcup_{X \in \mathcal{A}} X$;

Greatest lower set of $\mathcal{A}$: $\displaystyle\bigcap_{X \in \mathcal{A}} X$.

(e) For every $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Z})$, $\bigcup\limits_{X \in \mathcal{A}} X$ exists, and it is the least upper set of $\mathcal{A}$. $\qquad\square$

12 [UD] Problem 12.20

Suppose $\infty \in \mathbb{R}$, by Archimedean property of $\mathbb{R}$, there exists a positive integer such that $\infty < n$, which is contradictory to $a < \infty$ for all $a \in \mathbb{R}$. Therefore $\infty \notin \mathbb{R}$. $\qquad\square$

13 [UD] Problem 12.22

Let $b = a + \sqrt{2} > a$. Now we are going to prove that b is irrational. Suppose, to the contrary, that b is rational, that is, there exist integers $p, q (q \neq 0)$, such that $b = p/q$. And there exist integers $r, s (s \neq 0)$, such that $a = r/s$ because a is a rational number. We have that $\sqrt{2} = p/q - r/s = (ps - qr)/qs$, thus $\sqrt{2}$ is a rational number, however, we know from Theorem 5.2 that $\sqrt{2}$ is irrational, which leads to contradiction. Therefore, if a is a rational number, there exists an irrational number b such that $a < b$. $\qquad\square$

14 [UD] Problem 12.23

With out loss of generality, assume $a \geq 0$ (otherwise let $a' = 0$, let $b' = a + b$, and apply the following proof). By theorem 12.11, there exists a rational number c' such that $a/\sqrt{2} < c' < b/\sqrt{2}$, and we have $c' > 0$. Hence, $a < c'\sqrt{2} < b$. Let $c = c'\sqrt{2} > 0$, and now we are going to prove that c is irrational. Suppose, to the contrary, that $c > 0$ is rational, that is, there exist positive integers p, q, such that $c = p/q$. And there exist positive integers r, s, such that $c' = r/s$ because c' is a positive rational number. Now we have get that $\sqrt{2} = c/c' = ps/qr$ is a rational number, which is contradictory to Theorem 5.2. Therefore, there exists an irrational number c such that $a < c < b$ for two arbitrary real numbers a and b with $a < b$. $\qquad\square$