论题 1-3 作业

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1 [UD] Problem 6.12

Not always. Let $x = 1$ and $y = -2$, since x and y are both nonzero, they are in S. However, $x \# y = x + y + 1 = 1 - 2 + 1 = 0$ is zero, which means $x \# y$ is not in S.

2 [UD] Problem 6.14

(a) Let $a = 1$, $b = 0$, $c = -4$, where a, b, c are all integers with a is nonzero. When $x = 2$, the equality $ax^2 + bx + c = x^2 - 4 = 0$ holds, therefore $2 \in A$. $\square$

(b) Let $a = 1$, $b = 0$, $c = -2$, where a, b, c are all integers with a is nonzero. When $x = \sqrt{2}$, we have $x^2 - 2 = 0$, so $\sqrt{2} \in A$. $\square$

(c) $\sqrt[3]{2}$.

(d) Let $p = a$, $q = b$, $r = c$. When a, b, c are integers with at least one of them is nonzero, p, q, r are rational numbers with at least one of them is nonzero. For every real number x, if $ax^2 + bx + c = 0$, then $px^2 + qx + c = 0$, so we have $A \subseteq B$.
By the definition of rational numbers, there exists integers $m_p, n_p, m_q, n_q, m_r, n_r$ with $n_p, n_q, n_r \neq 0$ such that $p = \frac{m_p}{n_p}$, $q = \frac{m_q}{n_q}$ and $r = \frac{m_r}{n_r}$. Since at least one of p, q, r is nonzero, at least one of m_p, m_q, m_r is nonzero. Let $a = pn_pn_qn_r = m_pn_qn_r$, $b = qn_pn_qn_r = m_qn_pn_r$, $c = rn_pn_qn_r = m_rn_pn_q$, so a, b, c are all integers and with at least one of them is nonzero. For every real number x, if $px^2 + qx + r = 0$, then $n_pn_qn_r(px^2 + qx + r) = ax^2 + bx + c = 0$, so we have $B \subseteq A$.
By the definition of equality of sets, we get $A = B$. $\square$

(e) By the definition of rational numbers, for every number $x \in \mathbb{Q}$, there exists two integers m, n with $n \neq 0$ such that $x = \frac{m}{n}$. Let $a = 0$, $b = n$, $c = -m$ with c is nonzero, and we have $ax^2 + bx + c = 0$, so we get $\mathbb{Q} \subseteq A$. $\square$

3 [UD] Problem 6.15

(a) A is the graph of the inequality $y \neq 0$ with respect to x and y. In other words, A is a collection of points in a plane whose y-coordinate is nonzero.

(b) Since (x,y), (z,w) are elements of A, we have $y \neq 0$ and $w \neq 0$. Thus, $wy \neq 0$, which means $(xw+zy, wy)$ is again an object in A. $\square$

(c) $(a,b) \diamond (x,y) = (ay+bx, by) = (x,y)$ holds for every (x,y) in A. Compare the coefficients of x,y, we get $a = 0$ and $b = 1$, so the element is $(0,1)$.

(d) For every element (a,b) in A, if we regard a as the numerator, and b as the denominator, we can find that this "new" addition shows the addition of two fractions:

$$\frac{x}{y} + \frac{z}{w} = \frac{xw+zy}{wy}.$$

4 [UD] Problem 6.18

No. Consider $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, we have $\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1 \leq 1$ and $\left|\frac{\sqrt{2}}{2}\right| + \left|\frac{\sqrt{2}}{2}\right| = \sqrt{2} > 1$, so $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is an element of the first set and it is not an element of the second set. Therefore the second set is not a subset of the first set, and thus the two sets are not equal. $\square$

5 [UD] Problem 17.11

(a) Take $m = 0$ and $n = 1$, we have $g(1) = g(0)g(1)$. Cancel the positive real number $g(1) = a$ on both sides, we get $g(0) = 0$. $\square$

(b) Let $P(n)$ denote that $g(n) = a^n$. For the base step, we have to check $g(1) = 1$, this is obviously true because this is one of the properties of g.

For the induction step, assume that $P(n)$ holds for a positive integer n. That is $g(n) = a^n$. Apply the second property of g, we get $g(n+1) = g(n)g(1) = a^n \times a = a^{n+1}$. Thus, $P(n+1)$ holds.

By mathematical induction, we conclude that $g(n) = a^n$ for all $n \in \mathbb{N}$. $\square$

6 [UD] Problem 17.13

Let $P(n)$ denote that $p(c) = 0$ implies $c = 0$ for every polynomial $p(x)$ of order n satisfies the conditions in the nontheorem. The implication $P(1) \to P(2)$ doesn't hold. When $n = 2$, $p(c)$ has only two factors, so if we remove one factor to form $q(x)$, $q(x)$ must be a 1-order polynomial and $q(c)$ doesn't have the factor $ac(a_1c+b_1)$.

7 [UD] Problem 17.14

Let $P(n)$ denote that "$Q(1), \dots, Q(n)$" are all true. For the base step, let $n = 1$. $P(1)$ is certainly true because $Q(1)$ is true.

For the induction step, assume that $P(n)$ is true where n is a positive integer, i.e. $Q(1),\dots,Q(n)$ are all true. By supposition (ii), we have that $Q(n+1)$ is true. $Q(n+1)$, along with $Q(1),\dots,Q(n)$ are all true, thus $P(n+1)$ is true.

By mathematical induction, we conclude that $P(n)$ holds for all positive integers n. Therefore, $Q(n)$ holds for all positive integers n. $\square$

8 [UD] Problem 17.16

For the base step, we have to prove that the sum of all the interior angles of a triangle is 180°. Let $\triangle ABC$ be a triangle. Draw line l through point A and parallel to BC. By the property of parallel lines, we have that $\angle 1 = \angle C$, $\angle 3 = \angle B$, so the sum of all the interior angles of a triangle is $\angle A + \angle B + \angle C = \angle 2 + \angle 3 + \angle 1 = 180^\circ$.

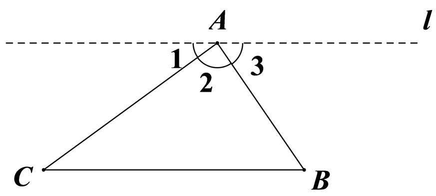

Figure 1: the sum of all the interior angles of a triangle is 180°

For the induction step, assume that the sum of all the interior angles of a convex polygon with n $(n \geq 3, n \in \mathbb{N})$ vertices is $(n-2)180^\circ$. Let $A_1A_2 \cdots A_{n-1}A_nA_{n+1}$ be a convex polygon with $n+1$ vertices. Draw a line through A_1A_n, then $A_1A_nA_{n+1}$ is a triangle, of which the sum of all the interior angles is 180°, and a convex polygon with n vertices, of which the sum of all the interior angles is $(n-2)180^\circ$ by induction hypothesis. So the sum of all the interior angles of $A_1A_2 \cdots A_{n-1}A_nA_{n+1}$ is $\angle A_1 + \angle A_2 + \cdots + \angle A_{n+1} = (\angle A_{n+1} + \angle A_{n+1}A_nA_1 + \angle A_{n+1}A_1A_n) + (\angle A_{n-1}A_nA_1 + \angle A_nA_1A_2 + \angle A_1 + \cdots + \angle A_{n-1}) = 180^\circ + (n-2)180^\circ = [(n+1)-2]180^\circ$.

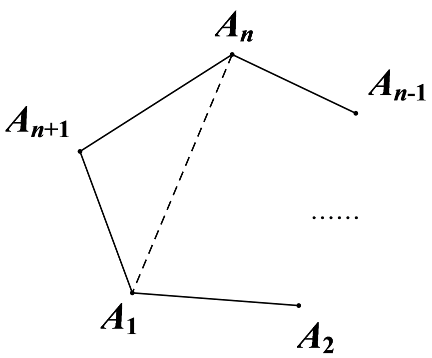

Figure 2: a convex polygon with $n+1$ vertices

By mathematical induction, we conclude that for all integers n where $n \geq 3$, the sum of all the interior angles of a convex polygon with n vertices is $(n-2)180^\circ$. $\square$

9 [UD] Problem 17.18

(a) $T_n = \dfrac{n(n+1)}{2}$.

Proof: The validity of the base step is obvious. Now assume that $T_n = \dfrac{n(n+1)}{2}$. We can find that $T_{n+1} = T_n + (n+1)$, so $T_{n+1} = \dfrac{n(n+1)}{2} + (n+1) = \dfrac{n(n+1)+2n+2}{2} = \dfrac{(n+1)(n+2)}{2}$.

By mathematical induction, we conclude that $T_n = \dfrac{n(n+1)}{2}$ holds for all positive integers n. $\square$

(b)

10 [UD] Problem 17.19

(a) $5! = 120$, $\binom{8}{3} = 56$, $\binom{8}{5} = 56$, $\binom{5}{2} = 10$, $\binom{5}{3} = 10$, $\binom{7}{0} = 1$, $\binom{7}{7} = 1$.

(b) $(m+1)^2$ equally spaced dots form a square with sides built of $m+1$ equally spaced dots. Divide these dots into four parts as the picture shows. There are m^2 dots in the upper-left part, m dots in the upper-right part, m dots in the lower-left part and one dot in the lower-right part. Thus $(m+1)^2 = m^2 + 2m + 1$.

(c) By the definition of binomial coefficient, we have

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!} \\
&= \frac{(n+1)!}{k!(n-k+1)!} \\
&= \frac{(n+1)!}{k!(n-k+1)!} \\
&= \binom{n+1}{k}
\end{aligned}$$

$\square$

(d) We use mathematical induction. For the base step, we have to check that $a+b = \binom{1}{0} b + \binom{1}{1} a$, which is certainly true.

For the induction step, assume that

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

holds for every a and b. Then,

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b)\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k+1} \\
&= \sum_{k=1}^{n+1}\binom{n}{k-1}a^k b^{n-k+1} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k+1} \\
&= a^{n+1} + b^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k-1}+\binom{n}{k}\right]a^k b^{n-k+1} \\
&= a^{n+1} + b^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}a^k b^{n-k+1} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}a^k b^{n-k+1}
\end{aligned}$$

By mathematical induction, we conclude that $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$. □

(e) Take $a=-1, b=1$ and apply binomial theorem, the left side is identically zero, the right side is $\sum_{k=0}^{n}\binom{n}{k}(-1)^k$, so $\sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0$. □

11 [ES] Problem 24.4

Divide the square into four identical small squares as Figure 3 shows. Each small square includes its border. By the Pigeonhole Principle, there exists one small square who has at least two points. In one small square, the maximum of the distance of two points is $\sqrt{2}/2$, the length of the diagonal. So there always exist two points whose distance is at most $\sqrt{2}/2$. □

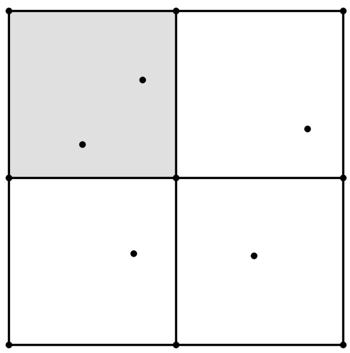

Figure 3: the square is divided into four small squares

12 [ES] Problem 24.6

Proposition Give nine distinct lattice points in three-dimensional space, at least one of the segments determined by these points has a lattice point as its midpoint.

Proof Classify the nine points into eight types by the parity of their coordinates:

$$\begin{array}{cccc} (even, even, even) & (even, even, odd) & (even, odd, even) & (even, odd, odd) \\ (odd, even, even) & (odd, even, odd) & (odd, odd, even) & (odd, odd, odd) \end{array}$$

By the Pigeonhole Principle, there exist two points who have the same parity. Let (a,b,c) and (d,e,f) be the two points, recall the midpoint formula, the coordinates of their midpoint are

$$\left(\frac{a+d}{2}, \frac{b+e}{2}, \frac{c+f}{2}\right).$$

$\frac{a+d}{2}$ is an integer because a and d share the same parity. $\frac{b+e}{2}$ and $\frac{c+f}{2}$ are integers likewise. This proves that the midpoint is a lattice point. $\square$

13 [ES] Problem 24.8

```
 Language:  C++
#include <iostream>
#include <algorithm>
using namespace std;

int arr[1050];
int n = 0;
int s[1050];
int top = 0;

int main()
{
    int ans;
    int temp;
    while (cin >> arr[n]) n++;
    for (int i = 0; i < n; i++)
    {
        temp = upper_bound(s, s + top, arr[i]) - s;
        s[temp] = arr[i];
        top = max(temp + 1, top);
    }
```

```
    ans = top;
    top = 0;
    for (int i = 0; i < n; i++)
    {
        arr[i] = -arr[i];
        temp = upper_bound(s, s + top, arr[i]) - s;
        s[temp] = arr[i];
        top = max(temp + 1, top);
    }
    cout << max(ans, top) << endl;
    return 0;
}
```