

# 论题 1-10 作业

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## 1 [UD] Problem 13.3

- (a) No. Because both  $(1, \sqrt{3})$  and  $(1, -\sqrt{3})$  are elements of  $f$ , however,  $\sqrt{3} \neq -\sqrt{3}$ .
- (b) No. Because for  $x = 0$ , there does not exist  $y \in \mathbb{R}$ , such that  $y = 1/(x+1)$ .
- (c) Yes. Because for all  $(x, y) \in \mathbb{R}^2$ , there exists a unique real number  $z$  such that  $z = x + y$ .
- (d) Yes. Because for every closed interval of real numbers  $[a, b]$ , there exists a unique real number  $a$ , such that  $([a, b], a) \in f$ .
- (e) Yes. Because for every  $(n, m) \in \mathbb{N} \times \mathbb{N}$ , there exists a unique real number  $m$ , such that  $((n, m), m) \in f$ .
- (f) Yes. Because for every real number  $x$ , there exists a real number  $y$ , such that  $y = 0$  when  $x \geq 0$  or  $y = x$  when  $x < 0$ , i.e.  $(x, y) \in f$ .
- (g) No. Because both  $(6, 7)$  and  $(6, 5)$  are elements of  $f$ , however,  $7 \neq 5$ .
- (h) Yes. Because for every circle  $c$  in the plane  $\mathbb{R}^2$ , there exists a unique real number  $C$ , such that  $C$  is the circumference of  $c$ .
- (i) Yes. Because for every polynomial with real coefficients  $p$ ,  $p$  is differentiable, thus there exists a unique polynomial  $p'$ , such that  $p'$  is the derivative of  $p$ .
- (j) Yes. Because for every polynomial  $p$ ,  $p$  is integrable on  $[0, 1]$ , thus there exists a unique number  $I$  such that  $I = \int_0^1 p(x)dx$ .

## 2 [UD] Problem 13.4

We know that  $A \cap \mathbb{N}$  is either an empty or a nonempty set. In the case that  $A \cap \mathbb{N}$  is empty, there exists a unique integer  $-1$ , such that  $(A, -1) \in f$ . In the case that  $A \cap \mathbb{N}$  is nonempty,  $A \cap \mathbb{N}$  is a subset of  $\mathbb{N}$ . By well-ordering principle of  $\mathbb{N}$ ,  $\min(A \cap \mathbb{N})$  exists, so there exist a unique integer  $\min(A \cap \mathbb{N})$ , such that  $(A, \min(A \cap \mathbb{N})) \in f$ . Therefore  $f$  is a well-defined function.

### 3 [UD] Problem 13.5

- (a) For all  $x \in X$ , either  $x \in A$  or  $x \in X \setminus A$  holds, so there exists a unique number  $y$  ( $y = 1$  when  $x \in A$  and  $y = 0$  when  $x \in X \setminus A$ ), such that  $y = \chi_A$ . Therefore  $\chi_A$  is a function.
- (b) The domain is  $X$ . The range is  $\{0\}$  when  $A = \emptyset$ ,  $\{1\}$  when  $A = X$ , and  $\{0, 1\}$  when  $A \neq \emptyset$  and  $A \neq X$ .

### 4 [UD] Problem 13.7

For every real number  $y \neq 1/2$ , let  $(x - 5)/(2x - 3) = y$ , and we get  $x = (3y - 5)/(2y - 1) \neq 3/2$ , which is an element of the domain. So  $\text{ran}(f) = \mathbb{R} \setminus \{1/2\}$ .  $\square$

### 5 [UD] Problem 13.11

No. For every  $x \in A$ , there may not exist  $y$  such that  $(x, y) \in f$ . Even though for every  $x \in A$  there exists  $y$  such that  $(x, y) \in f$ , we cannot make sure that there only exists one  $y$  such that  $(x, y) \in f$ .

### 6 [UD] Problem 13.13

The only relation is  $\{(x, y) \in X^2 : x = y\}$ . By the reflexion of the equivalence, any relation on  $X$  is superset of  $\{(x, y) \in X^2 : x = y\}$ . Assume there exists relation  $X'$  such that  $X' \setminus X \neq \emptyset$ , let  $(a, b)$  be an element of  $X'$  such that  $a \neq b$ . However,  $(b, b)$  is an element of  $X'$  but  $a \neq b$ , so  $X'$  is not a function.

### 7 [UD] Problem 14.8

- (a) Not one-to-one.  $f(1) = f(-1) = 1/2$  but  $1 \neq -1$ .  
Not onto. The range is  $(0, 1]$ .
- (b) Not one-to-one.  $\sin 0 = \sin \pi = 0$  but  $0 \neq \pi$ .  
Not onto. The range is  $[-1, 1]$ .
- (c) Not one-to-one.  $f(1, 2) = f(2, 1) = 2$  but  $(1, 2) \neq (2, 1)$ .  
Onto.
- (d) Not one-to-one.  $f((1, 0), (0, 0)) = f((0, 0), (0, 0)) = 0$  but  $((1, 0), (0, 0)) \neq ((0, 0), (0, 0))$ .  
Onto.
- (e) Not one-to-one.  $f((0, 0), (0, 0)) = f((1, 1), (1, 1)) = 0$  but  $((0, 0), (0, 0)) \neq ((1, 1), (1, 1))$ .  
Not onto. The range is  $[0, +\infty)$ .

(f) One-to-one.

Not onto. The range is  $A \times \{b\}$ .

(g) One-to-one.

Onto.

(h) Not one-to-one.  $f(X) = f(B) = B$  but  $X \neq B$ .

Not onto. The range is  $\mathcal{P}(X \setminus B)$ .

(i) One-to-one.

Not onto. The range is  $(0, +\infty)$ .

## 8 [UD] Problem 14.12

$$f(x) = \frac{(d-c)x + cb - da}{b-a} (x \in [a, b]).$$

One-to-one: Let  $f(x_1) = f(x_2)$ , we have  $\frac{(d-c)x_1 + cb - da}{b-a} = \frac{(d-c)x_2 + cb - da}{b-a}$ . Multiplying  $b-a$  and cancelling on both sides, we have  $x_1 = x_2$ .

Onto: Let  $c \leq f(x) \leq d$ , that is  $c \leq \frac{(d-c)x + cb - da}{b-a} \leq d$ . Multiplying  $b-a$  and cancelling on both sides, we have  $a \leq x \leq b$ . It means, for every  $x \in [a, b]$ , there exists  $y$ , such that  $y = f(x)$ , thus  $f(x)$  is onto.

Since  $f(x)$  is both one-to-one and onto,  $f(x)$  is a bijection. □

## 9 [UD] Problem 14.13

$\phi$  is a function from  $F([0, 1])$  to  $\mathbb{R}$ . Because for all  $f \in F([0, 1])$ , there exists a unique real number  $y$ , such that  $y = f(0)$ .

$\phi$  is not one-to-one. Let  $f_1(x) = 0 \in F([0, 1])$ ,  $f_2(x) = x \in F([0, 1])$ , we have that  $\phi(f_1) = \phi(f_2)$ , however,  $f_1 \neq f_2$  because  $f_1(1) \neq f_2(1)$ .

$\phi$  is onto. For every real number  $a$ , there exists  $f_0(x) = a \in F([0, 1])$ , such that  $\phi(f_0) = a$ .

## 10 [UD] Problem 14.15

For all  $x \in \mathbb{R}$ , since  $f(x)$  is defined on  $\mathbb{R}$ , there exists a unique real number  $y = f(x) \cdot f(x)$ , such that  $y = (f \cdot f)(x)$ , therefore  $f \cdot f$  is a function. □

(a) Yes.  $f(x) = e^x$ .

(b) No.  $\text{ran}(f \cdot f) = \{a^2 : a \in \text{ran}(f)\}$ .

## 11 [UD] Problem 15.1

	$(f \circ g)(x)$	$\text{dom}(f \circ g)$	$\text{ran}(f \circ g)$	$(g \circ f)(x)$	$\text{dom}(g \circ f)$	$\text{ran}(g \circ f)$
(a)	$1/(1+x^2)$	$\mathbb{R}$	$(0, 1]$	$1/(1+x)^2$	$\mathbb{R} \setminus \{-1\}$	$\mathbb{R}^+$
(b)	$x$	$\mathbb{R}^+$	$\mathbb{R}^+$	$ x $	$\mathbb{R}$	$[0, +\infty)$
(c)	$1/(x^2 + 1)$	$\mathbb{R}$	$(0, 1]$	$(1/x^2) + 1$	$\mathbb{R} \setminus \{0\}$	$(1, +\infty)$
(d)	$ x $	$\mathbb{R}$	$[0, +\infty)$	$ x $	$\mathbb{R}$	$[0, +\infty)$

## 12 [UD] Problem 15.6

$$(a) \quad (f \circ g)(x) = f(g(x)) = \frac{\frac{3+2x}{1-x} - 3}{\frac{3+2x}{1-x} + 2} = \frac{\frac{5x}{1-x}}{\frac{5}{1-x}} = x,$$

$$(g \circ f)(x) = g(f(x)) = \frac{3 + 2\frac{x-3}{x+2}}{1 - \frac{x-3}{x+2}} = \frac{\frac{5x}{x+2}}{\frac{5}{x+2}} = x.$$

- (b) (Theorem 15.4) Let  $f: A \rightarrow B$  be a bijective function, and  $f^{-1}$  be the inverse of  $f$ , then  $f \circ g = i_B$ , and  $g \circ f = i_A$ .

## 13 [UD] Problem 15.7

- (a) (i)  $f = \{(1, 4), (2, 5), (3, 5)\}$ ,  $g = \{(4, 1), (5, 2)\}$ ;  
(ii)  $f = \{(1, 4), (2, 5)\}$ ,  $g = \{(4, 1), (5, 2)\}$ ;  
(iii) Impossible.

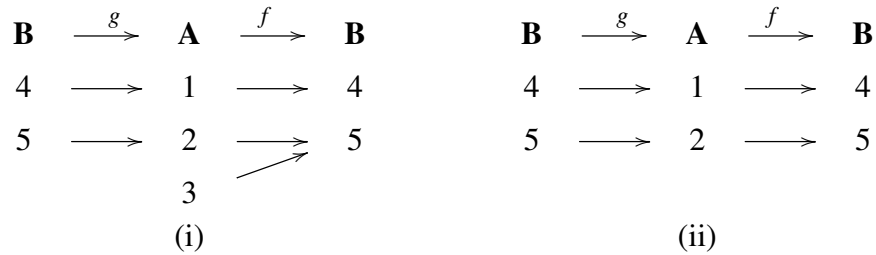


Figure 1: diagrams of  $A$  and  $B$

- (b) Let  $A = \{1, 2\}$ ,  $B = \{1\}$ ,  $f = \{(1, 1), (2, 1)\}$ ,  $g = \{(1, 1)\}$ , we have  $f \circ g = \{(1, 1)\} = i_B$ , but  $g \circ f = \{(1, 1), (2, 1)\} \neq i_A$ .

Because neither  $f$  nor  $g$  is a bijective function.

- (c) Let  $A = \{1\}$ ,  $B = \{1, 2\}$ ,  $f = \{(1, 1)\}$ ,  $g = \{(1, 1), (2, 1)\}$ , we have  $g \circ f = \{(1, 1)\} = i_A$ , but  $f \circ g = \{(1, 1), (2, 1)\} \neq i_B$ .

Because neither  $f$  nor  $g$  is a bijective function.

(d)  $f$  is not always one-to-one, but must be onto. For injectivity, we have a counterexample in (b). For surjectivity, suppose to the contrary that  $f$  is not onto. That means, there exists  $b \in B$ , for all  $a \in A$ ,  $f(a) \neq b$ . Therefore,  $(f \circ g)(b) = f(g(b)) \neq b$ , which is contradict to that  $f \circ g = i_B$ . Therefore  $f$  is onto.

(e) Guess whether the function has some property. If true, try to find the proof; if false, try to find a counterexample.

Here,  $f$  is not always onto, but must be one-to-one. For surjectivity, we have a counterexample in (c). For injectivity, suppose to the contrary that  $f$  is not one-to-one. That means, there exists  $a$  and  $b$  in  $A$  such that  $f(a) = f(b)$ . However,  $(g \circ f)(a) = g(f(a)) = g(f(b)) = (g \circ f)(b)$ , which is contradict to that  $g \circ f = i_A$ . Therefore  $f$  is one-to-one.

## 14 [UD] Problem 15.11

By the definition of the inverse of a function, the inverse function of  $f$  exists because  $f$  is a bijection. Since  $f \circ g_1 = f \circ g_2$ , we have  $f^{-1} \circ (f \circ g_1) = f^{-1} \circ (f \circ g_2)$ , thus  $(f^{-1} \circ f) \circ g_1 = (f^{-1} \circ f) \circ g_2$  because the composition satisfies associative property, and by Theorem 15.4 (ii) we get  $g_1 = g_2$ .

□

If  $g_1 \circ f = g_2 \circ f$  and  $f$  is bijective,  $g_1 = g_2$  still holds. Just get  $g_1 \circ (f \circ f^{-1}) = g_2 \circ (f \circ f^{-1})$ , and prove in the similar way.

## 15 [UD] Problem 15.12

Yes.

The equivalence class of  $a \in A$  is  $\{x : f(x) = f(a)\}$ .

## 16 [UD] Problem 15.13

No.

Yes.  $f(x) = x$ .

## 17 [UD] Problem 15.14

(a) First, for all  $(a, c) \in A \times C$ , there exists a unique pair  $(f(a), g(c)) \in B \times D$ , such that  $H(a, c) = (f(a), g(c))$  because  $f : A \rightarrow B$  and  $g : C \rightarrow D$  are both functions. Therefore  $H$  is a function.

Second, let  $(f(a_1), g(c_1)) = (f(a_2), g(c_2))$ , by the definition of ordered pair, we have  $f(a_1) = f(a_2)$  and  $g(c_1) = g(c_2)$ , since  $f$  and  $g$  are both one-to-one, we get  $a_1 = a_2$  and  $c_1 = c_2$ , and this implies  $(a_1, c_1) = (a_2, c_2)$ . Therefore  $H$  is one-to-one. □

- (b) Since  $f$  and  $g$  are onto, for every  $(b, d) \in B \times D$ , there exist  $a$  and  $c$ , such that  $f(a) = b$  and  $g(c) = d$ , therefore  $H(a, c) = (b, d)$ . Hence  $H$  is also onto.  $\square$

## 18 [UD] Problem 15.15

$H$  is not a function:  $A = \{1, 2\}$ ,  $B = \{1, 2\}$ ,  $C = \{2, 3\}$ ,  $D = \{3, 4\}$ ,  $f = \{(1, 1), (2, 2)\}$ ,  $g = \{(2, 3), (3, 4)\}$ ,  $H = \{(1, 1), (2, 2), (2, 3), (3, 4)\}$ .

$H$  is a function:  $A = \{1\}$ ,  $B = \{1\}$ ,  $C = \{2\}$ ,  $D = \{2\}$ ,  $f = \{(1, 1)\}$ ,  $g = \{(2, 2)\}$ ,  $H = \{(1, 1), (2, 2)\}$ .

When  $A$  and  $C$  are disjoint, we are assured that  $H$  is a function. In fact,  $H$  is a function if and only if  $f \cap [(A \cap C) \times B] = g \cap [(A \cap C) \times D]$ .

## 19 [UD] Problem 15.20

- (a) Let  $f|_{A_1}(x) = f|_{A_1}(y)$ , and by the definition of the restriction function, we have  $f(x) = f(y)$ . Since  $f$  is one-to-one, we have  $x = y$ . Therefore  $f|_{A_1}$  is one-to-one.  $\square$
- (b) For every  $y \in B$ , there exists  $x \in A_1 \subset A$  such that  $f|_{A_1}(x) = f(x) = y$  because  $f|_{A_1}$  is onto. Therefore  $f$  is onto.  $\square$

## 20 [UD] Problem 16.19

For every  $a \in A$ , there exists  $b \in B$  such that  $b = f(a)$ , and we have that  $f^{-1}(\{b\}) \subseteq A$  because  $f$  is a function from  $A$  to  $B$ . Therefore,  $\bigcup_{b \in B} f^{-1}(\{b\}) = A$ .

Since  $f$  is onto, for every  $b \in B$ , there exists  $a \in A$ , such that  $f(a) = b$ , thus  $f^{-1}(\{b\})$  is always nonempty.

If  $f^{-1}(\{b_1\}) \cap f^{-1}(\{b_2\})$  is nonempty, there exists  $a$ , such that  $f(a) = b_1$  and  $f(a) = b_2$ , thus  $b_1 = b_2$ , therefore  $f^{-1}(\{b_1\}) = f^{-1}(\{b_2\})$ .

Summarizing, we conclude that  $\{f^{-1}(\{b\}) : b \in B\}$  is a partition of  $A$ .  $\square$

## 21 [UD] Problem 16.20

- (a) No.
- (b) For every  $a \in A_1$ , we have  $f(a) \in f(A_1) = f(A_2)$ , thus there exists  $a' \in A_2$  such that  $f(a') = f(a)$ . Since  $f$  is **one-to-one**, we have that  $a' = a$ , therefore  $a \in A_2$ . Hence  $A_1 \subseteq A_2$ , and  $A_2 \subseteq A_1$  likewise. Therefore  $A_1 = A_2$ .  $\square$

I used only one-to-one.

## 22 [UD] Problem 16.21

(a) No.

(b) For every  $b \in B_1 \subseteq Y$ , there exists  $a \in X$  such that  $f(a) = b$  because  $f$  is **onto**. Hence,  $a$  is an element of  $f^{-1}(B_1) = f^{-1}(B_2)$ , therefore there exists  $b' \in B_2$  such that  $f(a) = b'$ , thus  $b = b'$ , and  $b$  is an element of  $B_2$ . Therefore  $B_1$  is a subset of  $B_2$ , and  $B_2$  is a subset of  $B_1$  likewise. So  $B_1 = B_2$ .  $\square$

I used only onto.

## 23 [UD] Problem 16.22

(a) Yes.

(b) For all  $x \in A_1 \cap A_2$ , both  $\chi_{A_1}(x)$  and  $\chi_{A_2}(x) = 1$ , therefore  $\chi_{A_1 \cap A_2}(x) = \chi_{A_1}(x) \cdot \chi_{A_2}(x) = 1$ .

For all  $x \notin A_1 \cap A_2$ , either  $\chi_{A_1}(x)$  or  $\chi_{A_2}(x) = 0$ , therefore  $\chi_{A_1 \cap A_2}(x) = \chi_{A_1}(x) \cdot \chi_{A_2}(x) = 0$ .

Summarizing, we have  $\chi_{A_1} \cdot \chi_{A_2} = \chi_{A_1 \cap A_2}$ .  $\square$

(c) For all  $x$  s.t.  $x \in A_1$  and  $x \in A_2$ ,  $\chi_{A_1}(x) = \chi_{A_2}(x) = 1$ ,  $\chi_{A_1 \cap A_2}(x) = 1$ , therefore  $\chi_{A_1 \cup A_2}(x) = \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = 1$

For all  $x$  s.t.  $x \in A_1$  and  $x \notin A_2$ ,  $\chi_{A_1}(x) = 1, \chi_{A_2}(x) = 0$ ,  $\chi_{A_1 \cap A_2}(x) = 0$ , therefore  $\chi_{A_1 \cup A_2}(x) = \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = 1$

For all  $x$  s.t.  $x \notin A_1$  and  $x \in A_2$ ,  $\chi_{A_1 \cup A_2}(x) = \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = 1$  holds likewise.

For all  $x$  s.t.  $x \notin A_1$  and  $x \notin A_2$ ,  $\chi_{A_1}(x) = \chi_{A_2}(x) = 0$ ,  $\chi_{A_1 \cap A_2}(x) = 0$ , therefore  $\chi_{A_1 \cup A_2}(x) = \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = 0$

Summarizing, we have  $\chi_{A_1 \cup A_2} = \chi_{A_1} + \chi_{A_2} - \chi_{A_1 \cap A_2}$ .  $\square$

(d)  $\chi_{X \setminus A_1} = 1 - \chi_{A_1}$ .