# 论题 1-8 作业

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## 1 [UD] Problem 6.7

- $B \setminus A$;
- $(A \cup B) \setminus (A \cap B)$;
- $A \cup B \cup C$;
- $(B \cap C) \setminus A$;
- $((A \cap B) \cup (A \cap C) \cup (B \cap C)) \setminus (A \cup B \cup C)$.

## 2 [UD] Problem 6.16

(a) For every $n$ in $A$, $n = x^2$ where $x$ is an integer, therefore $n$ is an integer, i.e. $n \in B$, so $A \subseteq B$. $\square$

(b) For every $t$ in $A$, $t$ is a real number, there exists a real number $x = t/2$, such that $t = 2x$, so $t \in B$. Therefore $A \subseteq B$. $\square$

(c) For every point $(x, y)$ in $A$, we have $y = (5 - 3x)/2$, therefore $2y + 3x = 5$, which means that $(x, y)$ is also in $B$. So $A \subseteq B$. $\square$

## 3 [UD] Problem 6.17

(a) $A$ is a proper subset of $B$. For every $(x, y)$ in $A$, we have $xy > 0$, so both $x$ and $y$ are nonzero, thus $x^2 + y^2 > 0$, therefore $A$ is a subset of $B$. However, $(1, -1)$ is an element of $B$, but not an element of $A$, so $A$ is a proper subset of $B$. $\square$

(b) $A$ is a proper subset of $B$. By theorem 6.10, we have $A \subseteq B$. However, $(0, 0)$ is an element of $B$, but not an element of $A$, so $A$ is a proper subset of $B$. $\square$

## 4 [UD] Problem 7.1

(a) For every $x$ in universe, by definition of complement, if $x \in A$, then $x \notin A^c$ and if $x \notin A^c$ then $x \in (A^c)^c$, therefore we have if $x \in A$, then $x \in (A^c)^c$, i.e. $A$ is a subset of $(A^c)^c$. $(A^c)^c$ is a subset of $A$ likewise. So $(A^c)^c = A$. $\square$

(b) For every $x$ in $A \cap (B \cup C)$, we have $x \in A$, and $x \in B$ or $C$, so $x \in A$ and $B$ or $x \in A$ and $C$, thus $A \cap (B \cup C)$ is a subset of $(A \cap B) \cup (A \cap C)$. For every $x$ in $(A \cap B) \cup (A \cap C)$, we have $x \in A$ and $B$ or $x \in A$ and $C$, so $x \in A$, and $x \in B$ or $C$, thus $(A \cap B) \cup (A \cap C)$ is a subset of $A \cap (B \cup C)$. So $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. $\square$

(c) For every $x$ in $X \setminus (A \cap B)$, we have $x \in X$ and, $x \notin A$ or $x \notin B$, thus $x \in X$ and $x \notin A$, or $x \in X$ and $x \notin B$, therefore $X \setminus (A \cap B)$ is a subset of $(X \setminus A) \cup (X \setminus B)$. For every $x$ in $(X \setminus A) \cup (X \setminus B)$, we have $x \in X$ and $x \notin A$, or $x \in X$ and $x \notin B$, thus $x \in X$ and, $x \notin A$ or $x \notin B$, so $(X \setminus A) \cup (X \setminus B)$ is a subset of $X \setminus (A \cap B)$. Therefore $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. $\square$

(d) Since $A, B$ are subsets of $X$, for every $x \in X$, if $x \in A$ then $x \in B$ and if $x \notin B$ then $x \notin A$ are equivalent, so $A \subseteq B$ if and only if $(X \setminus B) \subseteq (X \setminus A)$. $\square$

(e) If $A \cap B = B$, then for every $x$, $x \in B$ and $x \in A$ and $B$ are equivalent, so $x \in B$ implies $x \in A$, i.e. $A$ is a subset of $B$. If $B \subseteq A$, for every $x$, $x \in B$ implies $x \in A$, thus $x \in B$ and $x \in A$ and $B$ are equivalent, so $A \cap B = B$. Therefore, $A \cap B = B$ if and only if $B \subseteq A$. $\square$

## 5 [UD] Problem 7.8

(a) (ii);

(b) (i), (ii), (iii), (iv), (v);

(c) For every $x$ in $(A \cap B) \setminus C$, we have $x \in A$ and $B$ and $x \notin C$, so $x \in A$ and $x \notin C$, and $x \in B$ and $x \notin C$, thus $(A \cap B) \setminus C$ is a subset of $(A \setminus C) \cap (B \setminus C$. Likewise $(A \setminus C) \cap (B \setminus C$ is a subset of $(A \cap B) \setminus C$. Therefore $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$. $\square$

## 6 [UD] Problem 7.9

(a) For every $x$ in $A \setminus B$, we have $x \in A$ and $x \notin B$, so $A \setminus B$ and $B$ are disjoint. $\square$

(b) For every $x$ in $A \cup B$, we have $x \in A$ or $x \in B$, so $x \in A$, or $x \in B$ and $x \notin A$, therefore $A \cup B$ is a subset of $(A \setminus B) \cup B$. For every $x$ in $(A \setminus B) \cup B$, we have $x \in A$, or $x \in B$ and $x \notin A$, so $x \in A$ or $x \in B$, therefore $(A \setminus B) \cup B$ is a subset of $A \cup B$. So $A \cup B = (A \setminus B) \cup B$ $\square$

# 7 [UD] Problem 7.10

This statement is false. Here is a counterexample. Let $A = \{1,2\}$, $B = \{1\}$ and $C = \{2\}$, then $A \cup B = A \cup C$, but $B \neq C$. $\square$

# 8 [UD] Problem 7.11

This statement is true. We know that for every $x$, $x \in S$ if and only if $x \cap S = x$. For every $x \in A$, let $Y = x$, then $B \cap Y = A \cap Y = x$, so $x \in B$, thus $A$ is a subset of $B$. $B$ is a subset of $A$ likewise. So the statement is true. $\square$

# 9 [UD] Problem 8.1

(a) $\bigcup_{n=1}^{\infty} A_n = [0,1) \cup [0,1/2) \cup [0,1/3) \cdots = [0,1)$ ;

$\bigcup_{n=1}^{\infty} B_n = [0,1] \cup [0,1/2] \cup [0,1/3] \cdots = [0,1]$ ;

$\bigcup_{n=1}^{\infty} C_n = (0,1) \cup (0,1/2) \cup (0,1/3) \cdots = (0,1)$

(b) $\bigcap_{n=1}^{\infty} A_n = [0,1) \cap [0,1/2) \cap [0,1/3) \cdots = \{0\}$ ;

$\bigcap_{n=1}^{\infty} B_n = [0,1] \cap [0,1/2] \cap [0,1/3] \cdots = \{0\}$ ;

$\bigcap_{n=1}^{\infty} C_n = (0,1) \cap (0,1/2) \cap (0,1/3) \cdots = \varnothing$

(c) No. Because $A_0$ is undefined.

# 10 [UD] Problem 8.4

This statement is false. Here is a counterexample: let $A_n = (n, n + \frac{1}{2})$, $B_n = [n, n + \frac{1}{2}]$, for all positive integer $n$, $A_n \subset B_n$ holds, however,

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} B_n = \varnothing$$

which does not satisfy the definition of strict inclusion.

## 11 [UD] Problem 8.7

(a) Suppose, to the contrary, that $\bigcap_{\alpha\in I} A_\alpha \neq \varnothing$, then there exists $x \in \bigcap_{\alpha\in I} A_\alpha$, however, there exists $\alpha_0 \in I$ such that $A_{\alpha_0} = \varnothing$, so $x \in \varnothing$, which leads to a contradiction. Therefore $\bigcap_{\alpha\in I} A_\alpha = \varnothing$. $\square$

(b) Let $X$ be the universe. For all $x$ in $X$, there exists $\alpha_0 \in I$, such that $A_{\alpha_0} = X$, so $x \in \bigcup_{\alpha\in I} A_\alpha = X$. $\square$

(c) For all $x \in B$, for all $\alpha \in I$, we have $x \in A_\alpha$, so $x \in \bigcap_{\alpha\in I} A_\alpha$, therefore $B \subseteq \bigcap_{\alpha\in I} A_\alpha$. $\square$

## 12 [UD] Problem 8.8

$A = \mathbb{Z}$.

Proof: for every integer $m$, there exists $n = |m|$ such that $m \notin \mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, thus $n \notin \bigcap_{n\in\mathbb{Z}^+} \mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, i.e. $m \in A$, so $\mathbb{Z}$ is a subset of $A$. For every $x$ in $A$, $x \notin \bigcap_{n\in\mathbb{Z}^+} \mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, that is, there exists a positive integer $n$, such that $x \notin \mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, i.e. $x \in \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, so $x \in \mathbb{Z}$, therefore $\mathbb{Z}$ is a subset of $A$. By the definition of equality of two sets, we get $\mathbb{Z} = A$. $\square$

## 13 [UD] Problem 8.9

$A = \mathbb{Q} \setminus \{n : n \neq 2m, m \in \mathbb{Z}\}$.

Proof: for every real number $x$ in $\bigcap_{n\in\mathbb{Z}} (\mathbb{R} \setminus \{2n\})$, for every integer $n$, we have $x \notin \mathbb{R} \setminus 2n$, so $\bigcap_{n\in\mathbb{Z}} (\mathbb{R} \setminus \{2n\})$ is a subset of $\{n : n \neq 2m, m \in \mathbb{Z}\}$. For every real number $x$ in $\{n : n \neq 2m, m \in \mathbb{Z}\}$, we have that for all integer $n$, $x \in \mathbb{R} \setminus \{2n\}$, so $x \in \bigcap_{n\in\mathbb{Z}} (\mathbb{R} \setminus \{2n\})$, therefore $\{n : n \neq 2m, m \in \mathbb{Z}\}$ is a subset of $x \in \bigcap_{n\in\mathbb{Z}} (\mathbb{R} \setminus \{2n\})$. By the definition of equality of two sets, we have $\{n : n \neq 2m, m \in \mathbb{Z}\} = \bigcap_{n\in\mathbb{Z}} (\mathbb{R} \setminus \{2n\})$, and this completes the proof. $\square$

## 14 [UD] Problem 8.11

(a) $A_\alpha = \{\alpha\}$, $(I = \mathbb{Z})$;

(b) if $A_\alpha \neq A_\beta$, then $A_\alpha \cap A_\beta = \varnothing$;

(c) if $A_\alpha = A_\beta$, then $A_\alpha \cap A_\beta \neq \varnothing$;

(d) Yes.

(e) Yes.

(f) Yes, except the trivial case: there is only zero or one element in $I$.

(g) No. Here is a counterexample: $\{A_\alpha : A_\alpha = \{1,2,3\} \setminus \{\alpha\}, \alpha \in I\}$, $(I = \{1,2,3\})$.

# 15 [UD] Problem 9.2

(a) By the definition of the power set, for every $X$ in $\mathcal{P}(A) \cup \mathcal{P}(A)$, $X$ is a subset of $A$ or $B$, so $X$ is a subset of $A \cup B$, therefore $\mathcal{P}(A) \cup \mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$.

(b) Let $A = \{1\}$, $B = \{2\}$, then $A \cup B = 1,2$, $\mathcal{P}(A) = \{\{1\}, \varnothing\}$, $\mathcal{P}(B) = \{\{2\}, \varnothing\}$, $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\{1\}, \{2\}, \varnothing\}$, $\mathcal{P}(A \cup B) = \{\{1\}, \{2\}, \{1,2\}, \varnothing\}$, $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

# 16 [UD] Problem 9.4

If $A \subseteq B$, then for all set $X$ such that $X$ is a subset of $A$, $X$ is also a subset of $B$, so $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \in \mathcal{P}(A) \subseteq \mathcal{P}(B)$, thus $A$ is a subset of $B$.
Therefore, $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. □

# 17 [UD] Problem 9.12

(a) The sufficiency is obvious, and we only have to prove the necessity. Assume, to the contrary, that $A \neq C$ or $B \neq D$. Without loss of generality, suppose $A \neq C$. Therefore, there exists $x$ such that $x \in A$ and $x \notin C$, or $x \notin A$ and $x \in C$. Suppose $x \in A$ and $x \notin C$ without loss of generality. Since $B$ is a nonempty set, there exists $y$ in $B$. Consider $(x, y)$, by the definition of Cartesian product, it is an element of $A \times B$, however it isn't an element of $C \times D$, because $x$ is not an element of $C$. Therefore, $A \times B = C \times D$ if and only if $A = C$ and $B = D$. □

(b) In the construction of $(x, y)$, we need to take an element $y$ in $B$, and this requires the sets are nonempty.

# 18 [UD] Problem 9.13

No. Let $A = \{1\}, C = \{2\}, B = D = \varnothing$, then $A \times B = C \times D = \varnothing$, so $A \times B \subseteq C \times D$, however $A \nsubseteq C$.

## 19 [UD] Problem 9.14

(a) True. For all $(x,y)$ in $A\times(B\cup C)$, we have $x\in A$ and $y\in B\cup C$, so $x\in A$ and $y\in B$, or $x\in A$ and $y\in C$, hence $x\in(A\times B)\cup(A\times C)$, therefore, $A\times(B\cup C)$ is a subset of $x\in(A\times B)\cup(A\times C)$. For all $(x,y)$ in $x\in(A\times B)\cup(A\times C)$, we have $x\in A$ and $y\in B$, or $x\in A$ and $y\in C$, so $x\in A$ and $y\in B\cup C$, therefore $x\in(A\times B)\cup(A\times C)$ is a subset of $A\times(B\cup C)$. By the definition of the equality of two sets, we get $A\times(B\cup C)=x\in(A\times B)\cup(A\times C)$. $\square$

(b) True. For all $(x,y)$ in $A\times(B\cap C)$, we have $x\in A$ and $y\in B\cap C$, so $x\in A$ and $y\in B$, and $x\in A$ and $y\in C$, hence $x\in(A\times B)\cap(A\times C)$, therefore, $A\times(B\cap C)$ is a subset of $x\in(A\times B)\cap(A\times C)$. For all $(x,y)$ in $x\in(A\times B)\cap(A\times C)$, we have $x\in A$ and $y\in B$, and $x\in A$ and $y\in C$, so $x\in A$ and $y\in B\cap C$, therefore $x\in(A\times B)\cap(A\times C)$ is a subset of $A\times(B\cap C)$. By the definition of the equality of two sets, we get $A\times(B\cap C)=x\in(A\times B)\cap(A\times C)$. $\square$

## 20 [UD] Problem 9.16

(a) If $a=b$, then $(a,b)=\{\{a\},\{a,b\}\}=\{\{a\},\{a\}\}=\{\{a\}\}$, thus $\{\{x\},\{x,y\}\}=\{\{a\}\}$, so $\{x\}=\{x,y\}=\{a\}$, therefore $x=y=a$. Hence $a=x$ and $b=y$.

If $a\neq b$, then $\{\{a\},\{a,b\}\}=\{\{x\},\{x,y\}\}$. Suppose $\{a\}=\{x,y\}$, then $a=x=y$, so $\{\{a\},\{a,b\}\}=\{\{x\},\{x,y\}\}=\{\{x\}\}$, thus $\{a\}=\{a,b\}$, and therefore $a=b$, which contradicts $a\neq b$. So $\{a\}=\{x\}$, therefore $a=x$ and $\{a,b\}=\{x,y\}$. If $a=y$, then $a=x=y$, thus $\{a,b\}=\{x,y\}=\{x\}$, therefore $a=b$, which contradicts $a\neq b$. So $a\neq y$, therefore $a=x$ and $b=y$. $\square$

(b) Since $\{a\}$ and $\{a,b\}$ are subsets of $A\cup B$, they are both the elements of $\mathcal{P}(A\cup B)$, therefore, $(a,b)=\{\{a\},\{a,b\}\}\in\mathcal{P}(\mathcal{P}(A\cup B))$. $\square$

(c) For all $x\in\mathcal{P}(\mathcal{P}(A\cup B))$ in $A\times B$, there exists $a\in A$ and $b\in B$, such that $x=(a,b)$. Therefore, $x\in\mathcal{P}(\mathcal{P}(C\cup D))$ (apply the conclusion in Problem 9.4 twice), and $a\in C$ and $b\in D$, so $x\in C\times D$. Therefore, $A\times B\subseteq C\times D$. $\square$