论题 1-8 作业

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1 [UD] Problem 6.7

- $B \setminus A$;
- $(A \cup B) \setminus (A \cap B)$;
- $A \cap B \cap C$;
- $(B \cap C) \setminus A$;
- $((A \cap B) \cup (A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C)$.

2 [UD] Problem 6.16

(a) For every n in A, $n = x^2$ where x is an integer, therefore n is an integer, i.e. $n \in B$, so $A \subseteq B$. □

(b) For every t in A, t is a real number, there exists a real number $x = t/2$, such that $t = 2x$, so $t \in B$. Therefore $A \subseteq B$. □

(c) For every point (x, y) in A, we have $y = (5 - 3x)/2$, therefore $2y + 3x = 5$, which means that (x, y) is also an element of B. So $A \subseteq B$. □

3 [UD] Problem 6.17

(a) A is a proper subset of B. For every (x, y) in A, we have $xy > 0$, so both x and y are nonzero, thus $x^2 + y^2 > 0$, therefore A is a subset of B. However, $(1, -1)$ is an element of B, but not an element of A, so A is a proper subset of B. □

(b) A is a proper subset of B. By theorem 6.10, we have $A \subseteq B$. However, $(0, 0)$ is an element of B, but not an element of A, so A is a proper subset of B. □

4 [UD] Problem 7.1

(a) For every x in universe, by definition of complement, if $x \in A$, then $x \notin A^c$ and if $x \notin A^c$ then $x \in (A^c)^c$, therefore we have if $x \in A$, then $x \in (A^c)^c$, i.e. A is a subset of $(A^c)^c$. $(A^c)^c$ is a subset of A likewise. So $(A^c)^c = A$. □

(b) For every x in $A \cap (B \cup C)$, we have $x \in A$, and $x \in B$ or C, so $x \in A$ and B or $x \in A$ and C, thus $A \cap (B \cup C)$ is a subset of $(A \cap B) \cup (A \cap C)$. For every x in $(A \cap B) \cup (A \cap C)$, we have $x \in A$ and B or $x \in A$ and C, so $x \in A$, and $x \in B$ or C, thus $(A \cap B) \cup (A \cap C)$ is a subset of $A \cap (B \cup C)$. So $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. $\square$

(c) For every x in $X \setminus (A \cap B)$, we have $x \in X$ and, $x \notin A$ or $x \notin B$, thus $x \in X$ and $x \notin A$, or $x \in X$ and $x \notin B$, therefore $X \setminus (A \cap B)$ is a subset of $(X \setminus A) \cup (X \setminus B)$. For every x in $(X \setminus A) \cup (X \setminus B)$, we have $x \in X$ and $x \notin A$, or $x \in X$ and $x \notin B$, thus $x \in X$ and, $x \notin A$ or $x \notin B$, so $(X \setminus A) \cup (X \setminus B)$ is a subset of $X \setminus (A \cap B)$. Therefore $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$. $\square$

(d) Since A, B are subsets of X, for every $x \in X$, "if $x \in A$ then $x \in B$" and "if $x \notin B$ then $x \notin A$" are equivalent, so $A \subseteq B$ if and only if $(X \setminus B) \subseteq (X \setminus A)$. $\square$

(e) If $A \cap B = B$, then for every x, "$x \in B$" and "$x \in A$ and B" are equivalent, so $x \in B$ implies $x \in A$, i.e. A is a subset of B. If $B \subseteq A$, for every x, $x \in B$ implies $x \in A$, thus $x \in B$ and $x \in A$ and B are equivalent, so $A \cap B = B$. Therefore, $A \cap B = B$ if and only if $B \subseteq A$. $\square$

5 [UD] Problem 7.8

(a) (ii);

(b) (i), (ii), (iii), (iv), (v);

(c) For every x in $(A \cap B) \setminus C$, we have $x \in A, B$ and $x \notin C$, so $x \in A$ and $x \notin C$, and $x \in B$ and $x \notin C$, thus $(A \cap B) \setminus C$ is a subset of $(A \setminus C) \cap (B \setminus C)$. Likewise $(A \setminus C) \cap (B \setminus C)$ is a subset of $(A \cap B) \setminus C$. Therefore $(A \cap B) \setminus C = (A \setminus C) \cap (B \setminus C)$. $\square$

6 [UD] Problem 7.9

(a) For every x in $A \setminus B$, we have $x \in A$ and $x \notin B$, so $A \setminus B$ and B are disjoint. $\square$

(b) For every x in $A \cup B$, we have $x \in A$ or $x \in B$, so $x \in A$, or $x \in B$ and $x \notin A$, therefore $A \cup B$ is a subset of $(A \setminus B) \cup B$. For every x in $(A \setminus B) \cup B$, we have $x \in A$, or $x \in B$ and $x \notin A$, so $x \in A$ or $x \in B$, therefore $(A \setminus B) \cup B$ is a subset of $A \cup B$. So $A \cup B = (A \setminus B) \cup B$ $\square$

7 [UD] Problem 7.10

This statement is false. Here is a counterexample. Let $A = \{1, 2\}$, $B = \{1\}$ and $C = \{2\}$, then $A \cup B = A \cup C$, but $B \neq C$. $\square$

8 [UD] Problem 7.11

This statement is true. We know that for every x, $x \in S$ if and only if $\{x\} \cap S = \{x\}$. For every $x \in A$, let $Y = \{x\}$, then $B \cap Y = A \cap Y = \{x\}$, so $x \in B$, thus A is a subset of B. B is a subset of A likewise. So the statement is true. $\square$

9 [UD] Problem 8.1

(a) $\bigcup_{n=1}^{\infty} A_n = [0,1) \cup [0,1/2) \cup [0,1/3) \cdots = [0,1)$;

$\bigcup_{n=1}^{\infty} B_n = [0,1] \cup [0,1/2] \cup [0,1/3] \cdots = [0,1]$;

$\bigcup_{n=1}^{\infty} C_n = (0,1) \cup (0,1/2) \cup (0,1/3) \cdots = (0,1)$

(b) $\bigcap_{n=1}^{\infty} A_n = [0,1) \cap [0,1/2) \cap [0,1/3) \cdots = \{0\}$;

$\bigcap_{n=1}^{\infty} B_n = [0,1] \cap [0,1/2] \cap [0,1/3] \cdots = \{0\}$;

$\bigcap_{n=1}^{\infty} C_n = (0,1) \cap (0,1/2) \cap (0,1/3) \cdots = \varnothing$

(c) No. Because A_0 is undefined.

10 [UD] Problem 8.4

This statement is false. Here is a counterexample: let $A_n = (n, n+\frac{1}{2})$, $B_n = [n, n+\frac{1}{2}]$, for all positive integer n, $A_n \subset B_n$ holds, however,

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} B_n = \varnothing$$

which does not satisfy the definition of strict inclusion.

11 [UD] Problem 8.7

(a) Suppose, to the contrary, that $\bigcap_{\alpha \in I} A_\alpha \neq \varnothing$, then there exists $x \in \bigcap_{\alpha \in I} A_\alpha$, however, there exists $\alpha_0 \in I$ such that $A_{\alpha_0} = \varnothing$, so $x \in \varnothing$, which leads to a contradiction. Therefore $\bigcap_{\alpha \in I} A_\alpha = \varnothing$. □

(b) Let X be the universe. For all x in X, there exists $\alpha_0 \in I$, such that $A_{\alpha_0} = X$, thus $x \in X$, so $x \in \bigcup_{\alpha \in I} A_\alpha = X$. □

(c) For all $x \in B$, for all $\alpha \in I$, we have $x \in A_\alpha$, so $x \in \bigcap_{\alpha \in I} A_\alpha$, therefore $B \subseteq \bigcap_{\alpha \in I} A_\alpha$. □

12 [UD] Problem 8.8

$A = \mathbb{Z}$.

Proof: for every integer m, there exists $n = |m|$ such that $m \notin \mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, thus $m \notin \bigcap_{n \in \mathbb{Z}^+} \mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, i.e. $m \in A$, so $\mathbb{Z}$ is a subset of A. For every x in A, $x \notin \bigcap_{n \in \mathbb{Z}^+} \mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, that is, there exists a positive integer n, such that $x \notin \mathbb{R} \setminus \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, i.e. $x \in \{-n, -n+1, \cdots, 0, \cdots, n-1, n\}$, so $x \in \mathbb{Z}$, therefore $\mathbb{Z}$ is a subset of A. By the definition of equality of two sets, we get $\mathbb{Z} = A$. □

13 [UD] Problem 8.9

$A = \{n : n = 2m, m \in \mathbb{Z}\}$.

Proof: let $\mathbb{R}$ be the universe,

$$\begin{aligned}
A &= \mathbb{Q} \setminus \bigcap_{n\in\mathbb{Z}} (\mathbb{R} \setminus \{2n\}) \\
&= \mathbb{Q} \setminus (\mathbb{R} \setminus \bigcup_{n\in\mathbb{Z}} \{2n\}) && \text{(By De Morgan's laws)} \\
&= \mathbb{Q} \setminus (\bigcup_{n\in\mathbb{Z}} \{2n\})^c && \text{(By the definition of complement)} \\
&= \mathbb{Q} \setminus (\{n : n = 2m, m \in \mathbb{Z}\})^c \\
&= \mathbb{Q} \cap (\{n : n = 2m, m \in \mathbb{Z}\}^c)^c && \text{(By Theorem 7.4.17)} \\
&= \mathbb{Q} \cap \{n : n = 2m, m \in \mathbb{Z}\} && \text{(By Theorem 7.4.2)} \\
&= \{n : n = 2m, m \in \mathbb{Z}\} && \square
\end{aligned}$$

14 [UD] Problem 8.11

(a) $A_\alpha = \{\alpha\}$, $(\alpha \in \mathbb{Z})$;

(b) if $A_\alpha \neq A_\beta$, then $A_\alpha \cap A_\beta = \varnothing$;

(c) if $A_\alpha = A_\beta$, then $A_\alpha \cap A_\beta \neq \varnothing$;

(d) Yes.

(e) Yes.

(f) This assertion holds if and only if there is more than one element in I.

(g) No. Here is a counterexample: $\{A_\alpha : A_\alpha = \{1,2,3\} \setminus \{\alpha\}, \alpha \in I\}$ $(I = \{1,2,3\})$.

15 [UD] Problem 9.2

(a) By the definition of the power set, for every X in $\mathcal{P}(A) \cup \mathcal{P}(A)$, X is a subset of A or B, so X is a subset of $A \cup B$, therefore $\mathcal{P}(A) \cup \mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$.

(b) Let $A = \{1\}$, $B = \{2\}$, then $A \cup B = 1,2$, $\mathcal{P}(A) = \{\{1\}, \varnothing\}$, $\mathcal{P}(B) = \{\{2\}, \varnothing\}$, $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\{1\}, \{2\}, \varnothing\}$, $\mathcal{P}(A \cup B) = \{\{1\}, \{2\}, \{1,2\}, \varnothing\}$, $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

16 [UD] Problem 9.4

If $A \subseteq B$, then for all set X such that X is a subset of A, X is also a subset of B, so $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \in \mathcal{P}(A) \subseteq \mathcal{P}(B)$, thus A is a subset of B.

Therefore, $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. $\square$

17 [UD] Problem 9.12

(a) The sufficiency is obvious, and we only have to prove the necessity. Assume, to the contrary, that $A \neq C$ or $B \neq D$. Without loss of generality, suppose $A \neq C$. Therefore, there exists x such that $x \in A$ and $x \notin C$, or $x \notin A$ and $x \in C$. Suppose $x \in A$ and $x \notin C$ without loss of generality. Since B is a nonempty set, there exists y in B. Consider (x,y), by the definition of Cartesian product, it is an element of $A \times B$, however it isn't an element of $C \times D$, because x is not an element of C. Therefore, $A \times B = C \times D$ if and only if $A = C$ and $B = D$. $\square$

(b) When constructing the pair (x,y) (which leads to contradiction), we need to take an element y in B, and this requires the sets are nonempty.

18 [UD] Problem 9.13

No. Let $A = \{1\}, C = \{2\}, B = D = \varnothing$, then $A \times B = C \times D = \varnothing$, so $A \times B \subseteq C \times D$, however $A \nsubseteq C$.

19 [UD] Problem 9.14

(a) True. For all (x,y) in $A \times (B \cup C)$, we have $x \in A$ and $y \in B \cup C$, so $x \in A$ and $y \in B$, or $x \in A$ and $y \in C$, hence $x \in (A \times B) \cup (A \times C)$, therefore, $A \times (B \cup C)$ is a subset of $x \in (A \times B) \cup (A \times C)$. For all (x,y) in $x \in (A \times B) \cup (A \times C)$, we have $x \in A$ and $y \in B$, or $x \in A$ and $y \in C$, so $x \in A$ and $y \in B \cup C$, therefore $x \in (A \times B) \cup (A \times C)$ is a subset of $A \times (B \cup C)$. By the definition of the equality of two sets, we get $A \times (B \cup C) = x \in (A \times B) \cup (A \times C)$. $\square$

(b) True. For all (x,y) in $A \times (B \cap C)$, we have $x \in A$ and $y \in B \cap C$, so $x \in A$ and $y \in B$, and $x \in A$ and $y \in C$, hence $x \in (A \times B) \cap (A \times C)$, therefore, $A \times (B \cap C)$ is a subset of $x \in (A \times B) \cap (A \times C)$. For all (x,y) in $x \in (A \times B) \cap (A \times C)$, we have $x \in A$ and $y \in B$, and $x \in A$ and $y \in C$, so $x \in A$ and $y \in B \cap C$, therefore $x \in (A \times B) \cap (A \times C)$ is a subset of $A \times (B \cap C)$. By the definition of the equality of two sets, we get $A \times (B \cap C) = x \in (A \times B) \cap (A \times C)$. $\square$

20 [UD] Problem 9.16

(a) If $a = b$, then $(a,b) = \{\{a\},\{a,b\}\} = \{\{a\},\{a\}\} = \{\{a\}\}$, thus $\{\{x\},\{x,y\}\} = \{\{a\}\}$, so $\{x\} = \{x,y\} = \{a\}$, therefore $x = y = a$. Hence $a = x$ and $b = y$.

If $a \neq b$, then $\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\}$. Suppose $\{a\} = \{x,y\}$, then $a = x = y$, so $\{\{a\},\{a,b\}\} = \{\{x\},\{x,y\}\} = \{\{x\}\}$, thus $\{a\} = \{a,b\}$, and therefore $a = b$, which contradicts $a \neq b$. So $\{a\} = \{x\}$, therefore $a = x$ and $\{a,b\} = \{x,y\}$. If $a = y$, then $a = x = y$, thus $\{a,b\} = \{x,y\} = \{x\}$, therefore $a = b$, which contradicts $a \neq b$. So $a \neq y$, therefore $a = x$ and $b = y$. $\square$

(b) Since $\{a\}$ and $\{a,b\}$ are subsets of $A \cup B$, they are both the elements of $\mathcal{P}(A \cup B)$, therefore, $(a,b) = \{\{a\},\{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$. $\square$

(c) For all $x \in \mathcal{P}(\mathcal{P}(A \cup B))$ in $A \times B$, there exists $a \in A$ and $b \in B$, such that $x = (a,b)$. Therefore, $x \in \mathcal{P}(\mathcal{P}(C \cup D))$ (apply the conclusion in Problem 9.4 twice), and $a \in C$ and $b \in D$, so $x \in C \times D$. Therefore, $A \times B \subseteq C \times D$. $\square$

论题 1-9 作业

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1 [UD] Problem 10.2

(a) $\{(1,1),(2,2),(3,3),(4,4),(5,5)\}$;

(b) $\{(1,1),(2,2),(2,3),(3,3),(3,4),(4,4),(5,5)\}$;

(c) $\{(1,2),(2,1)\}$

(d) $\{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}$;

2 [UD] Problem 10.4

Yes. First, it is reflexive, because $x_1 - y_1 = x_2 - y_2 = 0$ are even when $(x_1, x_2) = (y_1, y_2)$. Second, it is symmetry because both $x_1 - y_1$ and $x_2 - y_2$ are even if and only if $y_1 - x_1$ and $y_2 - x_2$ are even. Third, it is transitive, because that both $x_1 - y_1$ and $y_1 - z_1$ are even implies $x_1 - z_1$ is even, and $x_2 - z_2$ is even likewise.

3 [UD] Problem 10.5

"If": for all $a \in E_x$, we have $a \sim x$, since $x \sim y$, we get $a \sim y$, therefore $a \in E_y$. Hence E_x is a subset of E_y. Likewise E_y is a subset of E_x. So $E_x = E_y$.

"Only if": by the definition of equivalence class, $E_x = \{a \in X : x \sim a\}$, since $y \in E_y$ and $E_x = E_y$, we have $y \in E_x$, that is, $x \sim y$. □

4 [UD] Problem 10.8

(a) Yes. The equivalence class given by $p(x) = x$ is $\{\sum_{i=0}^{n} a_i x^i : a_0 = 0\}$.

(b) Yes. E_r is the set of all the polynomials of degree 1.

(c) No. Because it is not symmetric.

5 [UD] Problem 11.3

(a) Yes. A_r represents a plane, on which the sum of the coordinates of a point is r.

(b) Yes. A_r represents a sphere whose center is the origin and its radius is $|r|$.

6 [UD] Problem 11.7

(a) Yes. Obviously, for all $m \in \mathbb{N}$, A_m is nonempty. Since every polynomial has a degree, so $\bigcup_{m\in\mathbb{N}} A_m = P$. Every polynomial has only one degree, so that for all $\alpha, \beta \in \mathbb{N}$, $A_\alpha = A_\beta$(when $\alpha = \beta$) or $A_\alpha \cap A_\beta = \varnothing$ (when $\alpha \neq \beta$) holds. Therefore, A_m determine a partition of P.

(b) Yes. For all $c \in \mathbb{R}$, there exists a polynomial such that $p(0) = c$, so A_c is always nonempty. We have that $\bigcup_{c\in\mathbb{R}} A_c = P$. For every polynomial, $p(0)$ is a constant, so that for all $\alpha, \beta \in \mathbb{R}$, $A_\alpha = A_\beta$(when $\alpha = \beta$) or $A_\alpha \cap A_\beta = \varnothing$ (when $\alpha \neq \beta$) holds. Therefore, A_c determine a partition of P.

(c) No. Consider A_x and A_{x^2}, x^2 is an element of both, however, $A_x \neq A_{x^2}$ because x is an element of the former one but not an element of the latter one.

(d) No. Consider A_0 and A_1, $x^2 - x$ is an element of both, however, $A_0 \neq A_1$ because x is an element of the former one but not an element of the latter one.

7 [UD] Problem 11.8

First, for all $\alpha \in I \cup J$, A_a is nonempty because $\{A_\alpha : \alpha \in I\}$ and $\{A_\alpha : \alpha \in J\}$ are both nonempty. Second, for every real number x, there exists $\alpha \in I \cup J$ ($\alpha \in I$ when $x > 0$ and $\alpha \in J$ when $x \leq 0$), therefore $\bigcup_{\alpha\in I\cup J} A_\alpha = \mathbb{R}$. Third, for all $\alpha, \beta \in I$ (or J), $A_\alpha = A_\beta$ or $A_\alpha \cap A_\beta$ holds, and for all $\alpha \in I$ and $\beta \in J$, $A_\alpha \cap A_\beta = \varnothing$, therefore for all $\alpha, \beta \in I \cup J$, $A_\alpha = A_\beta$ or $A_\alpha \cap A_\beta$ holds. Hence $\{A_\alpha : \alpha \in I \cup J\}$ is a partition of $\mathbb{R}$.

8 [UD] Problem 11.9

(a) No. Let $X = \{1,2,3\}$, and $\{A_\alpha : \alpha \in I\} = \{\{1\},\{2,3\}\}$ be a partition of X. Let $B = \{1,2\} \subseteq X$ such that $B \cap \{1\} \neq \varnothing$ and $B \cap \{2,3\} \neq \varnothing$. However, $\{A_\alpha \cap B : \alpha \in I\} = \{\{1\},\{2\}\}$ is not a partition of B, because $\bigcup_{\alpha\in I} A_\alpha \cap B = \{1,2\} \neq X$.

(b) No. Let $X = \{1,2,3\}$, and $\{A_\alpha : \alpha \in I\} = \{\{1\},\{2\},\{3\}\}$ be a partition of X. However, $\{X \setminus A_\alpha : \alpha \in I\} = \{\{1,2\},\{1,3\},\{2,3\}\}$ is not a partition of X because $\{1,2\} \neq \{1,3\}$ and $\{1,2\} \cap \{1,3\} \neq \varnothing$.

9 [UD] Problem 12.10

(a) Suppose $\sup(S \cup T) < \sup S$. Since $\sup(S \cup T)$ is an upper bound of $S \cup T$, $\sup(S \cup T)$ is also an upper bound of S. However, the least upper bound of S, $\sup S$, is greater than $\sup(S \cup T)$, another upper bound of S, which leads to contradiction. Therefore $\sup(S \cup T) \geq \sup S$, and likewise $\sup(S \cup T) \geq \sup T$. $\square$

(b) Without loss of generality, assume that $\sup S \geq \sup T$. Suppose to the contrary that $\sup(S \cup T) > \sup S$, take $M = (\sup(S \cup T) + \sup S)/2$, we have $\sup(S \cup T) > M > \sup S$, and M is the upper bound of S and T because $M > \sup S \geq \sup T$, therefore M is the upper bound of $S \cup T$, however, the least upper bound of $S \cup T$, $\sup(S \cup T)$, is greater than M, which leads to contradiction. Therefore $\sup(S \cup T) = \max\{\sup S, \sup T\}$. $\square$

(c) The supremum of the union of two sets is greater than or equal to the supremum of either set. In fact, the supremum of the union of two sets is the maximum of the suprema of the two sets.

10 [UD] Problem 12.13b

(Reflexive) For all $S \in \mathcal{P}(A)$, S is a subset of S, so $S \subseteq S$;

(Transitive) For all $A, B, C \in \mathcal{P}(A)$, if A is a subset of B and B is a subset of C, then for all $x \in A$, x is an element of B, thus x is an element of C, therefore, A is a subset of C;

(Antisymmetric) By the definition of the equality of two sets, the antisymmetric property holds for $(\mathcal{P}(A), \subseteq)$.

Let a, b be two distinct elements of A, then neither $\{a\} \subseteq \{b\}$ nor $\{b\} \subseteq \{a\}$ holds, therefore $(\mathcal{P}(A), \subseteq)$ is a partial order but not a total order. $\square$

11 [UD] Problem 12.16

(a) We can find that $\{1, 2, 5, 7, 8, 10\}$ is a least upper bound of $\mathcal{B}$. Therefore $\mathcal{B}$ is an upper bounded set. $\square$

(b) For every nonempty subset X of $\mathcal{P}(\mathbb{Z})$, for every element $x \in X$, x is an element of $\mathbb{Z}$, therefore $\mathbb{Z}$ is an upper set of X. Hence every nonempty subset of $\mathcal{P}(\mathbb{Z})$ is upper bounded. $\square$

(c) For every nonempty subset $\mathcal{A}$ of $\mathcal{P}(\mathbb{Z})$ we say that $L \in \mathcal{P}(\mathbb{Z})$ is an lower set of $\mathcal{A}$, if $L \subseteq X$ for all $X \in \mathcal{A}$. A nonempty set $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Z})$ will be called a lower bounded set if there exists a lower set of $\mathcal{A}$ in $\mathcal{P}(\mathbb{Z})$. We say $L_0 \in \mathcal{P}(\mathbb{Z})$ is a least upper set if (i) U_0 is a lower set of $\mathcal{A}$ and (ii) if L is another lower set of $\mathcal{A}$, then $L \subseteq L_0$.

(d) Least upper set of $\mathcal{A}$: $\bigcup\limits_{X \in \mathcal{A}} X$;
Greatest lower set of $\mathcal{A}$: $\bigcap\limits_{X \in \mathcal{A}} X$.

(e) For every $\mathcal{A} \subseteq \mathcal{P}(\mathbb{Z})$, $\bigcup\limits_{X \in \mathcal{A}} X$ exists, and it is the least upper set of $\mathcal{A}$. $\square$

12 [UD] Problem 12.20

Suppose $\infty \in \mathbb{R}$, by Archimedean property of $\mathbb{R}$, there exists a positive integer n such that $\infty < n$, which is contradictory to $a < \infty$ for all $a \in \mathbb{R}$. Therefore $\infty \notin \mathbb{R}$. $\square$

13 [UD] Problem 12.22

Let $b = a + \sqrt{2} > a$. Now we are going to prove that b is irrational. Suppose, to the contrary, that b is rational, that is, there exist integers $p, q (q \neq 0)$, such that $b = p/q$. And there exist integers $r, s (s \neq 0)$, such that $a = r/s$ because a is a rational number. We have that $\sqrt{2} = p/q - r/s = (ps - qr)/qs$, thus $\sqrt{2}$ is a rational number, however, by Theorem 5.2 we know that $\sqrt{2}$ is irrational, which leads to contradiction. Therefore, if a is a rational number, there exists an irrational number b such that $a < b$. $\square$

14 [UD] Problem 12.23

With out loss of generality, assume $a \geq 0$ (otherwise let $a' = 0$ and $b' = a + b$. By theorem 12.11, there exists a rational number c' such that $a/\sqrt{2} < c' < b/\sqrt{2}$, and we have $c' > 0$. Hence, $a < \sqrt{2}c' < b$. Let

$c = \sqrt{2}c' > 0$, and now we are going to prove that c is irrational. Suppose, to the contrary, that $c > 0$ is rational, that is, there exist positive integers p, q, such that $c = p/q$. And there exist positive integers r, s, such that $c' = r/s$ because c' is a positive rational number. Now we have get that $\sqrt{2} = c/c' = ps/qr$ is a rational number, which is contradictory to Theorem 5.2. Therefore, there exists an irrational number c such that $a < c < b$ for two arbitrary real numbers a and b with $a < b$. $\square$

论题 1-10 作业

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1 [UD] Problem 13.3

(a) No. Because both $(1,\sqrt{3})$ and $(1,-\sqrt{3})$ are elements of f, however, $\sqrt{3} \neq -\sqrt{3}$.

(b) No. Because for $x=-1$, there does not exist $y \in \mathbb{R}$, such that $y = 1/(x+1)$.

(c) Yes. Because for all $(x,y) \in \mathbb{R}^2$, there exists a unique real number z such that $z = x+y$.

(d) Yes. Because for every closed interval of real numbers $[a,b]$, there exists a unique real number a, such that $([a,b],a) \in f$.

(e) Yes. Because for every $(n,m) \in \mathbb{N} \times \mathbb{N}$, there exists a unique real number m, such that $((n,m),m) \in f$.

(f) Yes. Because for every real number x, there exists a real number y, such that $y=0$ when $x \geq 0$ or $y=x$ when $x<0$, i.e. $(x,y) \in f$.

(g) No. Because both $(6,7)$ and $(6,5)$ are elements of f, however, $7 \neq 5$.

(h) Yes. Because for every circle c in the plane $\mathbb{R}^2$, there exists a unique real number C, such that C is the circumference of c.

(i) Yes. Because for every polynomial with real coefficients p, p is differentiable, thus there exists a unique polynomial p', such that p' is the derivative of p.

(j) Yes. Because for every polynomial p, p is integrable on $[0,1]$, thus there exists a unique number I such that $I = \int_0^1 p(x)dx$.

2 [UD] Problem 13.4

We know that $A \cap \mathbb{N}$ is either an empty or a nonempty set. In the case that $A \cap \mathbb{N}$ is empty, there exists a unique integer -1, such that $(A,-1) \in f$. In the case that $A \cap \mathbb{N}$ is nonempty, $A \cap \mathbb{N}$ is a subset of $\mathbb{N}$. By well-ordering principle of $\mathbb{N}$, $\min(A \cap \mathbb{N})$ exists, so there exist a unique integer $\min(A \cap \mathbb{N})$, such that $(A, \min(A \cap \mathbb{N})) \in f$. Therefore f is a well-defined function.

3 [UD] Problem 13.5

(a) For all $x \in X$, either $x \in A$ or $x \in X \setminus A$ holds, so there exists a unique number y ($y=1$ when $x \in A$ and $y=0$ when $x \in X \setminus A$), such that $y = \chi_A$. Therefore χ_A is a function.

(b) The domain is X. The range is $\{0\}$ when $A = \varnothing$, $\{1\}$ when $A = X$, and $\{0,1\}$ when $A \neq \varnothing$ and $A \neq X$.

4 [UD] Problem 13.7

For every real number $y \neq 1/2$, let $(x-5)/(2x-3) = y$, and we get $x = (3y-5)/(2y-1) \neq 3/2$, which is an element of the domain. So $\operatorname{ran}(f) = \mathbb{R} \setminus \{1/2\}$. $\square$

5 [UD] Problem 13.11

No. For every $x \in A$, there may not exist y such that $(x,y) \in f$. Even though for every $x \in A$ there exists y such that $(x,y) \in f$, we cannot make sure that there only exists one y such that $(x,y) \in f$.

6 [UD] Problem 13.13

The only possible relation is $\{(x,y) \in X^2 : x = y\}$. By the reflexion of the equivalence, any relation on X is a superset of $\{(x,y) \in X^2 : x = y\}$. Assume there exists relation X' such that $X' \setminus X \neq \varnothing$, let (a,b) be an element of X' such that $a \neq b$. However, (b,b) is an element of X' but $a \neq b$, so X' is not a function.

7 [UD] Problem 14.8

(a) Not one-to-one. $f(1) = f(-1) = 1/2$ but $1 \neq -1$.
Not onto. The range is $(0,1]$.

(b) Not one-to-one. $\sin 0 = \sin \pi = 0$ but $0 \neq \pi$.
Not onto. The range is $[-1,1]$.

(c) Not one-to-one. $f(1,2) = f(2,1) = 2$ but $(1,2) \neq (2,1)$.
Onto.

(d) Not one-to-one. $f((1,0),(0,0)) = f((0,0),(0,0)) = 0$ but $((1,0),(0,0)) \neq ((0,0),(0,0))$.
Onto.

(e) Not one-to-one. $f((0,0),(0,0)) = f((1,1),(1,1)) = 0$ but $((0,0),(0,0)) \neq ((1,1),(1,1))$.
Not onto. The range is $[0,+\infty)$.

(f) One-to-one.
Not onto. The range is $A \times \{b\}$.

(g) One-to-one.
Onto.

(h) Not one-to-one. $f(X) = f(B) = B$ but $X \neq B$.
Not onto. The range is $\mathcal{P}(B)$.

(i) One-to-one.
Not onto. The range is $(0,+\infty)$.

8 [UD] Problem 14.12

$f(x) = \dfrac{(d-c)x + cb - da}{b-a} (x \in [a,b])$.

One-to-one: Let $f(x_1) = f(x_2)$, we have $\dfrac{(d-c)x_1 + cb - da}{b-a} = \dfrac{(d-c)x_2 + cb - da}{b-a}$. Multiplying $b-a$ and cancelling on both sides, we have $x_1 = x_2$.

Onto: Let $c \le f(x) \le d$, that is $c \le \dfrac{(d-c)x + cb - da}{b-a} \le d$. Multiplying $b-a$ and cancelling on both sides, we have $a \le x \le b$. It means, for every $x \in [a,b]$, there exists y, such that $y = f(x)$, thus $f(x)$ is onto.

Since $f(x)$ is both one-to-one and onto, $f(x)$ is a bijection. $\square$

9 [UD] Problem 14.13

ϕ is a function from $F([0,1])$ to $\mathbb{R}$. Because for all $f \in F([0,1])$, there exists a unique real number y, such that $y = f(0)$.

ϕ is not one-to-one. Let $f_1(x) = 0 \in F([0,1])$, $f_2(x) = x \in F([0,1])$, we have that $\phi(f_1) = \phi(f_2)$, however, $f_1 \neq f_2$ because $f_1(1) \neq f_2(1)$.

ϕ is onto. For every real number a, there exists $f_0(x) = a \in F([0,1])$, such that $\phi(f_0) = a$.

10 [UD] Problem 14.15

For all $x \in \mathbb{R}$, since $f(x)$ is defined on $\mathbb{R}$, there exists a unique real number $y = f(x) \cdot f(x)$, such that $y = (f \cdot f)(x)$, therefore $f \cdot f$ is a function. $\square$

(a) Yes. $f(x) = e^x$.

(b) No. $\operatorname{ran}(f \cdot f) = \{a^2 : a \in \operatorname{ran}(f)\}$.

11 [UD] Problem 15.1

	$(f \circ g)(x)$	$\operatorname{dom}(f \circ g)$	$\operatorname{ran}(f \circ g)$	$(g \circ f)(x)$	$\operatorname{dom}(g \circ f)$	$\operatorname{ran}(g \circ f)$
(a)	$1/(1+x^2)$	$\mathbb{R}$	$(0,1]$	$1/(1+x)^2$	$\mathbb{R} \setminus \{-1\}$	$\mathbb{R}^+$
(b)	x	$[0,+\infty)$	$[0,+\infty)$	$\lvert x \rvert$	$\mathbb{R}$	$[0,+\infty)$
(c)	$1/(x^2+1)$	$\mathbb{R}$	$(0,1]$	$(1/x^2)+1$	$\mathbb{R} \setminus \{0\}$	$(1,+\infty)$
(d)	$\lvert x \rvert$	$\mathbb{R}$	$[0,+\infty)$	$\lvert x \rvert$	$\mathbb{R}$	$[0,+\infty)$

12 [UD] Problem 15.6

(a) $(f \circ g)(x) = f(g(x)) = \dfrac{\dfrac{3+2x}{1-x} - 3}{\dfrac{3+2x}{1-x} + 2} = \dfrac{\dfrac{5x}{1-x}}{\dfrac{5}{1-x}} = x \ (x \neq 1)$,

$(g \circ f)(x) = g(f(x)) = \dfrac{3 + 2\dfrac{x-3}{x+2}}{1 - \dfrac{x-3}{x+2}} = \dfrac{\dfrac{5x}{x+2}}{\dfrac{5}{x+2}} = x \ (x \neq -2)$.

(b) (Theorem 15.4) Let $f : A \to B$ be a bijective function, and f^{-1} be the inverse of f, then $f \circ g = i_B$, and $g \circ f = i_A$.

13 [UD] Problem 15.7

(a) (i) $f = \{(1,4),(2,5),(3,5)\},\ g = \{(4,1),(5,2)\}$;

(ii) $f = \{(1,4),(2,5)\},\ g = \{(4,1),(5,2)\}$;

(iii) Impossible.

$$\begin{tikzcd} \mathbf{B} \arrow[r, "g"] & \mathbf{A} \arrow[r, "f"] & \mathbf{B} \\ 4 \arrow[r] & 1 \arrow[r] & 4 \\ 5 \arrow[r] & 2 \arrow[r] & 5 \\ & 3 \arrow[ur] & \end{tikzcd}$$

(i)

$$\begin{tikzcd} \mathbf{B} \arrow[r, "g"] & \mathbf{A} \arrow[r, "f"] & \mathbf{B} \\ 4 \arrow[r] & 1 \arrow[r] & 4 \\ 5 \arrow[r] & 2 \arrow[r] & 5 \end{tikzcd}$$

(ii)

Figure 1: diagrams of A and B

(b) Let $A = \{1,2\}$, $B = \{1\}$, $f = \{(1,1),(2,1)\}$, $g = \{(1,1)\}$, we have $f \circ g = \{(1,1)\} = i_B$, but $g \circ f = \{(1,1),(2,1)\} \neq i_A$.

Because neither f nor g is a bijective function.

(c) Let $A = \{1\}$, $B = \{1,2\}$, $f = \{(1,1)\}$, $g = \{(1,1),(2,1)\}$, we have $g \circ f = \{(1,1)\} = i_A$, but $f \circ g = \{(1,1),(2,1)\} \neq i_B$.

Because neither f nor g is a bijective function.

(d) f is not always one-to-one, but must be onto. For injectivity, we have a counterexample in (b). For surjectivity, suppose to the contrary that f is not onto. That means, there exists $b \in B$, for all $a \in A$, $f(a) \neq b$. Therefore, $(f \circ g)(b) = f(g(b)) \neq b$, which is contradict to that $f \circ g = i_B$. Therefore f is onto.

(e) Guess whether the function has some property. If true, try to find the proof; if false, try to find a counterexample.

Here, f is not always onto, but must be one-to-one. For surjectivity, we have a counterexample in (c). For injectivity, suppose to the contrary that f is not one-to-one. That means, there exists a and b in A such that $f(a) = f(b)$. However, $(g \circ f)(a) = g(f(a)) = g(f(b)) = (g \circ b)(b)$, which is contradict to that $g \circ f = i_A$. Therefore f is one-to-one.

14 [UD] Problem 15.11

By the definition of the inverse of a function, the inverse function of f exists because f is a bijection. Since $f \circ g_1 = f \circ g_2$, we have $f^{-1} \circ (f \circ g_1) = f^{-1} \circ (f \circ g_2)$, thus $(f^{-1} \circ f) \circ g_1 = (f^{-1} \circ f) \circ g_2$ by associative property, and by Theorem 15.4 (ii) we get $g_1 = g_2$. $\square$

If $g_1 \circ f = g_2 \circ f$ and f is bijective, $g_1 = g_2$ still holds. Just get $g_1 \circ (f \circ f^{-1}) = g_2 \circ (f \circ f^{-1})$, and prove in the similar way.

15 [UD] Problem 15.12

Yes.

The equivalence class of $a \in A$ is $\{a\}$.

16 [UD] Problem 15.13

No.

Yes. $f(x) = x$.

17 [UD] Problem 15.14

(a) First, for all $(a,c) \in A \times C$, there exists a unique pair $(f(a), g(c)) \in B \times D$, such that $H(a,c) = (f(a), g(c))$ because $f : A \to B$ and $g : C \to D$ are both functions. Therefore H is a function.

Second, let $(f(a_1), g(c_1)) = (f(a_2), g(c_2))$, by the definition of ordered pair, we have $f(a_1) = f(a_2)$ and $g(c_1) = g(c_2)$, since f and g are both one-to-one, we get $a_1 = a_2$ and $c_1 = c_2$, and this implies $(a_1, c_1) = (a_2, c_2)$. Therefore H is one-to-one. □

(b) Since f and g are onto, for every $(b,d) \in B \times D$, there exist a and c, such that $f(a) = b$ and $g(c) = d$, therefore $H(a,c) = (b,d)$. Hence H is also onto. □

18 [UD] Problem 15.15

H is not a function: $A = \{1,2\}$, $B = \{1,2\}$, $C = \{2,3\}$, $D = \{3,4\}$, $f = \{(1,1),(2,2)\}$, $g = \{(2,3),(3,4)\}$, $H = \{(1,1),(2,2),(2,3),(3,4)\}$.

H is a function: $A = \{1\}$, $B = \{1\}$, $C = \{2\}$, $D = \{2\}$, $f = \{(1,1)\}$, $g = \{(2,2)\}$, $H = \{(1,1),(2,2)\}$.

When A and C are disjoint, we are assured that H is a function. In fact, H is a function if and only if $f \cap [(A \cap C) \times B] = g \cap [(A \cap C) \times D]$.

19 [UD] Problem 15.20

(a) Let $f|_{A_1}(x) = f|_{A_1}(y)$, and by the definition of the restriction function, we have $f(x) = f(y)$. Since f is one-to-one, we have $x = y$. Therefore $f|_{A_1}$ is one-to-one. □

(b) For every $y \in B$, there exists $x \in A_1 \subset A$ such that $f|_{A_1}(x) = f(x) = y$ because $f|_{A_1}$ is onto. Therefore f is onto. □

20 [UD] Problem 16.19

For every $a \in A$, there exists $b \in B$ such that $b = f(a)$, and we have that $f^{-1}(\{b\}) \subseteq A$ because f is a function from A to B. Therefore, $\bigcup\limits_{b \in B} f^{-1}(\{b\}) = A$.

Since f is onto, for every $b \in B$, there exists $a \in A$, such that $f(a) = b$, thus $f^{-1}(\{b\})$ is always nonempty.

If $f^{-1}(\{b_1\}) \cap f^{-1}(\{b_2\})$ is nonempty, there exists a, such that $f(a) = b_1$ and $f(a) = b_2$, thus $b_1 = b_2$, therefore $f^{-1}(\{b_1\}) = f^{-1}(\{b_2\})$.

Summarizing, we conclude that $\{f^{-1}(\{b\}) : b \in B\}$ is a partition of A. $\square$

21 [UD] Problem 16.20

(a) No.

(b) For every $a \in A_1$, we have $f(a) \in f(A_1) = f(A_2)$, thus there exists $a' \in A_2$ such that $f(a') = f(a)$. Since f is **one-to-one**, we have that $a' = a$, therefore $a \in A_2$. Hence $A_1 \subseteq A_2$, and $A_2 \subseteq A_1$ likewise. Therefore $A_1 = A_2$. $\square$

I used only one-to-one.

22 [UD] Problem 16.21

(a) No.

(b) For every $b \in B_1 \subseteq Y$, there exists $a \in X$ such that $f(a) = b$ because f is **onto**. Hence, a is an element of $f^{-1}(B_1) = f^{-1}(B_2)$, therefore there exists $b' \in B_2$ such that $f(a) = b'$, thus $b = b'$, and b is an element of B_2. Therefore B_1 is a subset of B_2, and B_2 is a subset of B_1 likewise. So $B_1 = B_2$. $\square$

I used only onto.

23 [UD] Problem 16.22

(a) Yes.

(b) For all $x \in A_1 \cap A_2$, both $\chi_{A_1}(x)$ and $\chi_{A_2}(x) = 1$, therefore $\chi_{A_1 \cap A_2}(x) = \chi_{A_1}(x) \cdot \chi_{A_2}(x) = 1$.

For all $x \notin A_1 \cap A_2$, either $\chi_{A_1}(x)$ or $\chi_{A_2}(x) = 0$, therefore $\chi_{A_1 \cap A_2}(x) = \chi_{A_1}(x) \cdot \chi_{A_2}(x) = 0$.

Summarizing, we have $\chi_{A_1} \cdot \chi_{A_2} = \chi_{A_1 \cap A_2}$. $\square$

(c) For all x s.t. $x \in A_1$ and $x \in A_2$, $\chi_{A_1}(x) = \chi_{A_2}(x) = 1$, $\chi_{A_1 \cap A_2}(x) = 1$, therefore $\chi_{A_1 \cup A_2}(x) = \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = 1$

For all x s.t. $x \in A_1$ and $x \notin A_2$, $\chi_{A_1}(x) = 1$,$\chi_{A_2}(x) = 0$, $\chi_{A_1 \cap A_2}(x) = 0$, therefore $\chi_{A_1 \cup A_2}(x) = \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = 1$

For all x s.t. $x \notin A_1$ and $x \in A_2$, $\chi_{A_1 \cup A_2}(x) = \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = 1$ holds likewise.

For all x s.t. $x \notin A_1$ and $x \notin A_2$, $\chi_{A_1}(x) = \chi_{A_2}(x) = 0$, $\chi_{A_1 \cap A_2}(x) = 0$, therefore $\chi_{A_1 \cup A_2}(x) = \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = 0$

Summarizing, we have $\chi_{A_1 \cup A_2} = \chi_{A_1} + \chi_{A_2} - \chi_{A_1 \cap A_2}$. $\square$

(d) $\chi_{X \setminus A_1} = 1 - \chi_{A_1}$.

论题 1-11 作业

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1 [DH] Problem 4.1

(a) $S \leftarrow 0$;
for i going from 1 to N do the following:
    if $A[i,1] > A[A[i,2],1]$ then $S \leftarrow S + A[i,1]$;
output S.

(b) Suppose the root of the binary tree is R.

$S \leftarrow 0$;
$P \leftarrow R$;
$N \leftarrow$ the content of the first offspring of R;
if the content of $R >$ **get the salary of Nth employee** then $S \leftarrow S$ + the content of R;
while P has a second offspring do the following:
    $P \leftarrow$ the second offspring of P;
    $N \leftarrow$ the content of the first offspring of S;
    if the content of $P >$ **get the salary of Nth employee** then $S \leftarrow S$ + the content of P;
output S.

subroutine **get the salary of Nth employee**
$T \leftarrow R$;
do the following $N-1$ times:
    $T \leftarrow$ the second offspring of T;
return the content of T;

2 [DH] Problem 4.2

(a) $S \leftarrow 0$;
call **add**$(T, 0)$;
output S.

subroutine **add**(P, x)
    $S \leftarrow S + x$;
    $N \leftarrow 1$;

```
    while P has an Nth offspring do the following:
        call add(the Nth offspring of P, x+1);
        N ← N+1;
    return.
```

(b)
```
S ← 0;
call count(T, 0);
output S;

subroutine count(P, x)
    if x = K then do the following:
        S ← S+1;
        return;
    N ← 1;
    while P has an Nth offspring do the following:
        call count(the Nth offspring of P, x+1);
        N ← N+1;
    return.
```

(c)
```
R ← false;
call check(T, 0);
output R.

subroutine check(P, x)
    if x is even then do the following:
        if P doesn't have a first offspring then do the following:
            R ← true;
            return;
    N ← 1;
    while P has an Nth offspring do the following:
        call check(the Nth offspring of P, x+1);
        N ← N+1;
    return.
```

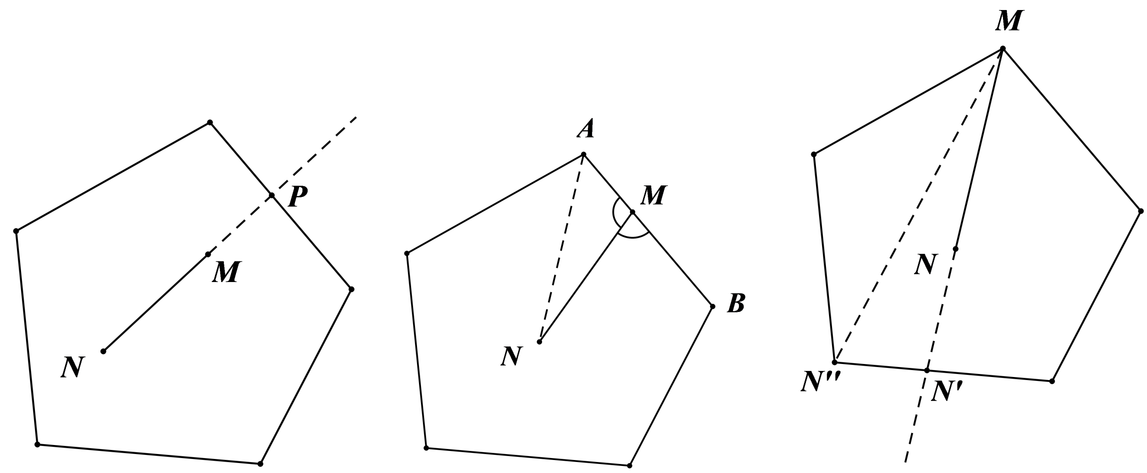
3 [DH] Problem 4.8

Suppose that the maximal distance between any two points on a polygon occurs between M and N. First, regard N as an arbitrary fixed point, and consider point M.

Case 1: M is in the polygon. Extend NM cutting the polygon at E (Figure 2(a)). NP is longer than than NM.

Case 2: M is on one edge of the polygon, but M is not a vertex (Figure 2(b)). Let the edge where M is on be AB. At least one of $\angle NMA$ and $\angle NMB$ is not less than 90 degrees. Assume, WLOG, that $\angle NMA \geq 90°$. By the law of sines, we get $NA > NM$.

Now, we have proved that for arbitrary N, the length of NM is maximal when M is a vertex of the polygon. Consider point N, we can prove that the length of MN is maximal when N is a vertex of the polygon likewise (Figure 2(c)). Hence, the maximal distance between any two points on a polygon occurs between two of the vertices. □



(a) (b) (c)

Figure 2: the distance of two points on a polygon

4 [DH] Problem 4.9

Language: C++

The first line of the input contains a positive integer n, giving the number of the vertices of the polygon. The following $n+1$ lines of the input contains the coordinates of the vertices (in clockwise or counterclockwise order). The x-coordinate and y-coordinate are separated by a space.

The output of the program is a number, the maximal distance of two points on the polygon.

```
#include <iostream>
#include <cmath>
#include <algorithm>
using namespace std;

int n;
double x[1000], y[1000];

double dist(int i1, int i2)
{
    return hypot(x[i1 % n] - x[i2 % n], y[i1 % n] - y[i2 % n]);
}

int main()
```

```
{
    double ans = 0;
    int j, k;
    cin >> n;
    for (int i = 0; i < n; i++)
        cin >> x[i] >> y[i];
    j = n;
    k = n + 1;
    while (k < 2*n)
    {
        while (!(dist(j, k) > dist(j, k - 1) && dist(j, k) > dist(j, k + 1)))
            k++;
        ans = max(ans, dist(j, k));
        j++;
    }
    cout << ans << endl;
    return 0;
}
```

5 [DH] Problem 4.11

Suppose the vector is named V.

(a) $M_1 \leftarrow$ **find maximum of first** N **elements**;

$I \leftarrow 1$;

do the following while $V[I] \neq M_1$:

$I \leftarrow I+1$;

for I going from I to $N-1$ do the following:

$V[I] \leftarrow V[I+1]$;

$M_2 \leftarrow$ **find maximum of first** $N-1$ **elements**;

output M_1, M_2.

subroutine **find maximum of first** n **elements**

$A \leftarrow V[1]$;

for i going from 2 to n do the following:

if $V[i] > A$ then $A \leftarrow V[i]$;

return A.

(b) $M_1 \leftarrow$ **find maximum from** 1**th to** N**th element**;

$I \leftarrow 1$;

do the following while $V[I] \neq M_1$:

$I \leftarrow I+1$;

for I going from I to $N-1$ do the following:

$V[I] \leftarrow V[I+1]$;

$M_2 \leftarrow$ **find maximum from** 1**th to** $(N-1)$**th element**;

output M_1, M_2.

subroutine **find maximum from** m**th to** n**th element**;

if $m = n$ then then return $V[m]$;

$p \leftarrow \lfloor (m+n)/2 \rfloor$;

$T_1 \leftarrow$ **find maximum from** m**th to** p**th element**;

$T_2 \leftarrow$ **find maximum from** $(p+1)$**th to** n**th element**;

if $T_1 > T_2$ then return T_1;

otherwise return T_2.

6 [DH] Problem 4.12

Suppose there are M nodes and N edges in the graph, the nodes are numbered from 1 to M and the edges are stored in vector V. Every edge T support three operations: get the number of the first node it connects (T.first), get the number of the second node it connects (T.second) and get the length of the node (T.length). Let U be an empty vector of integers. The output is the edges forming the minimal spanning tree.

call **initialize**;

call **quicksort from** 1 **to** N;

$m \leftarrow 0$;

$i \leftarrow 1$;

while $m < M-1$ do the following:

if **find** $V[i]$.first $\neq$ **find** $V[i]$.second then do the following:

call **union** $V[i]$.first **and** y.first;

output $V[i]$;

$m \leftarrow m+1$;

$i \leftarrow i+1$.

subroutine **initialize**

for i going from 1 to N do the following:

$U[i] = i$;

subroutine **find** x

if $U[x] = x$ then return x;

$t \leftarrow$ **find** $U[x]$;

$U[x] \leftarrow t$;

return t.

subroutine **union** x **and** y

$p \leftarrow$ **find** x;

$q \leftarrow$ **find** y;

$U[p] \leftarrow q$.

subroutine **quicksort from** a **to** b

if $a \geq b$ then return;

$p \leftarrow$ **partition from** a **to** b;

call **quicksort from** a **to** $p-1$;

call **quicksort from** $p+1$ **to** b.

subroutine **partition from** a **to** b

call **swap** $\lfloor (a+b)/2 \rfloor$ **and** b;

$L \leftarrow a$;

for i going from a to $b-1$ do the following:

if $V[i]$.length $< V[b]$.length do the following:

call **swap** i **and** L;

$L \leftarrow L+1$;

call **swap** b **and** L;

return L.

subroutine **swap** a **and** b

$t \leftarrow V[a]$;

$V[a] \leftarrow V[b]$;

$V[b] \leftarrow t$;

return.

7 [DH] Problem 4.13

(a) Let R be an empty vector of integers, S be an empty two-dimensional array of integers.

for i going from 0 to C do the following:

$R[i] \leftarrow 0$;

for j going from 1 to N do the following:

$S[i][j] = 0$;

for i going from 1 to N do the following:

for j going from 1 to $Q[i]$ do the following:

for k going down from C to $W[i]$ do the following:

if $R[j-W[i]]+P[i] > R[j]$ do the following:

$R[j] \leftarrow R[j-W[i]]+P[i]$;

for l going from 1 to i do the following:

$S[j][l] \leftarrow S[j-W[i]][l]$;

$S[j][i] \leftarrow S[j][i]+1$;

output $S[C]$.

(b) The output is [0, 1, 3, 2, 1]. The total profit of the knapsack is 194.

8 [DH] Problem 4.14

(a) Let S be an empty vector of real numbers.

while $C \neq 0$ do the following:

$t \leftarrow$ **find best material**;

if $W[t] \times Q[t] < C$ then do the following:

$C \leftarrow C - W[t] \times Q[t]$;

$Q[t] \leftarrow 0$;

$S[t] \leftarrow Q[t]$;

otherwise do the following:

$Q[t] \leftarrow Q[t] - C/W[t]$;

$S[t] \leftarrow C/W[t]$;

$C \leftarrow 0$;

output S.

subroutine **find best material**

$i \leftarrow 1$;

while $Q[i] = 0$ do the following:

$i \leftarrow i+1$;

$t \leftarrow i$;

for i going from $i+1$ to N do the following:

if $Q[i] > 0$ and $P[i]/W[i] > P[t]/W[t]$ then $t \leftarrow i$;

return t.

(b) The output is [0, 1, 1.8, 5, 1]. The total profit of the knapsack is 200.

论题 1-12 作业

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1 [DH] Problem 5.4

(a)

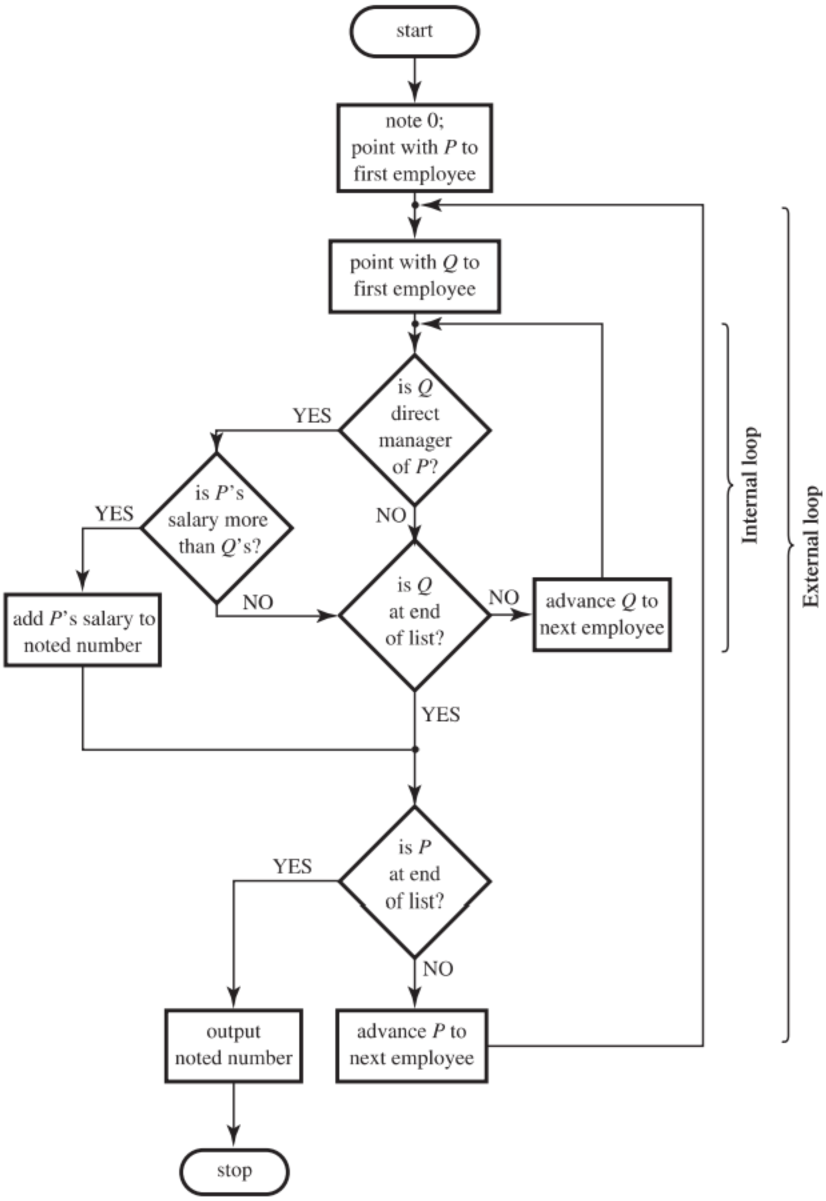

Figure 3: modified flowchart

(b) 部分正确性的证明：在外层循环前断言（记为 A）"当前的数字是前 M 位员工中，比至少一个直接上级工资高的那些员工的工资总和"，其中 M 是外层循环已经执行的次数；在内层循环前断言

（记为 B）“前 N 位雇员要么不是 P 的直接上级，要么虽然是 P 的直接上级，但工资不比 P 低”。断言 A 在第一次循环执行前是正确的，因为当前数字被初始化为 0；断言 B 在第一次循环执行前是显然正确的。当内层循环执行一次后，如果没有退出循环，即已确认 Q 不是 P 的直接上级，或者 Q 虽然是 P 的直接上级，但工资不比 P 低，断言 B 依旧成立；如果退出了循环，则说明 Q 是 P 的直接上级且工资比 P 低，并且 P 的工资已加入当前数字之中，断言 A 依旧成立；如果因 Q 是最后一个员工而结束了循环，则说明所有员工都不是比 P 工资低的直接上级，并且当前数字无变化，断言 A 依旧成立。当外层循环执行结束后，断言 A 指出，输出的数字是所有员工中，比至少一个直接上级工资高的那些员工的工资总和，这是符合要求的正确输出。

可终止性的证明：对于内层循环，指针 Q 未指向过的员工数量每次循环之后会减少，且最少为 0，因此内层循环能够终止；对于外层循环，指针 P 未指向过的员工数量也有类似的性质，因此外层循环也能够终止。总之，对于任意合法的输入，该算法总是能够终止。

这一算法既是部分正确的又是可终止的，因而是完全正确的。

(c) Yes.

(d) No.

2 [DH] Problem 5.6

(a) Only three well-placed invariants are sufficient for the proof of the partial correctness of an algorithm which contains only one loop.

(b) The flowchart that doesn't contain any loop.

(c) 4.

(d) 4.

3 [DH] Problem 5.8

function **rev**(X)
    $T \leftarrow X$;
    $Y \leftarrow \Lambda$;
    while $T \neq \Lambda$ do the following:
        $Y \leftarrow Y \cdot$ **last**(T);
        $T \leftarrow$ **all-but-last**(T);
    return Y.

部分正确性的证明：在每次循环执行前断言“$X = T \cdot \mathbf{rev}(Y)$”。第一次循环执行之前，$T$ 被设置为 X，Y 被设置为空串，因而断言成立；循环每执行一次后，T 少了最后一个字符，而这一字符被加到 Y 的最后，因而断言仍然成立。循环结束后，T 为空串，并且断言依然成立，从而有 $Y = \mathbf{rev}(X)$，这是正确的输出。

可终止性的证明：每次执行循环后，T 的字符数会少 1，并且当字符数为 0 时循环终止；而对于合法的输入，T 的字符数在第一次循环执行之前是有限的，因此算法总是能够终止。

综上，该算法是完全正确的。

4 [DH] Problem 5.9

function **equal**(X, Y)

  $P \leftarrow X$;

  $Q \leftarrow Y$;

  while $P \neq \Lambda$ and $Q \neq \Lambda$ do the following:

    if **eq** (**head**(P), **head**(Q)) is false then return false;

    $P \leftarrow$ **tail**(P);

    $Q \leftarrow$ **tail**(Q);

  if $P = \Lambda$ and $Q = \Lambda$ then return true;

  otherwise return false.

部分正确性的证明：在每次循环执行前断言“X 和 Y 的前 N 个字符相同，并且 P, Q 分别是 X, Y 除去 N 个字符后的剩余部分”，其中 N 是循环已经执行的次数。当第一次循环开始执行前，断言显然是成立的。当循环执行了一次后，若循环没有退出，X 和 Y 的第 $N+1$ 个字符相同，并且 P, Q 的首字符被去掉了，因而断言依然成立；若循环退出，则 P 和 Q 的首字母不同，也就是说，X 和 Y 在某个位置处的字母不同，因此整个字符串也不同，算法给出了正确的输出。当循环结束时，P 和 Q 其中有一个是空串，若两个都是空串，则这两个字符串相等，算法给出了正确的结果；若有一个不是空串，根据断言可知，X 和 Y 的字符个数不相等，字符串本身当然不相等，算法的输出也是正确的。

可终止性的证明：每次循环执行后，P 和 Q 的字母个数少了 1，当 P 和 Q 的字母个数有一个为 0 时，整个算法终止。而对于合法的输入，字符串 P 和 Q 的字母个数都是有限的，因而算法是可终止的。

综上，该算法是完全正确的。

5 [DH] Problem 5.10

(a) 对于任意合法的输入 S，根据回文串的定义，当它自身和它的逆序串相等时，S 是回文串，否则不是，因此该算法是部分正确的。该算法不含循环结构，并且上面已经证明了，**rev** 和 **equal** 都是可终止的，因而该算法是可终止的。综上，该算法是完全正确的。

(b) 如果程序 A 中不含循环结构，程序 A 用合法的参数调用了程序 B，并且程序 B 是可终止的，那么程序 A 也是可终止的。

6 [DH] Problem 5.11

(a) `"abcdefghijklmnopqrstuvwxyz"`.

(b) 3（**eq**, **head**, **last** 各 1 次）

7 [DH] Problem 5.12

(a) 正确。定义 **left(**S**,** i**)** 为字符串 S 最开始的 i 个字符构成的字符串，**right(**S**,** i**)** 为字符串 S 最后的 i 个字符构成的字符串，**len(**S**)** 为字符串 S 中字符的个数。在每次执行循环前，断言

“$\mathbf{left}(S,i)=\mathbf{rev}(\mathbf{right}(S,i))$，其中 i 为语句‘$X \leftarrow \mathbf{all\text{-}but\text{-}last}(\mathbf{tail}(X))$’执行的次数”。在第一次循环执行之前，断言显然是成立的。当循环执行了一次之后，如果该次循环执行时 $\mathbf{eq}(\mathbf{head}(X), \mathbf{last}(X))$ 成立，那么对该次循环执行之前的 X（记为 X'），有 $\mathbf{left}(S\cdot\mathbf{head}(X'),i)=\mathbf{rev}(\mathbf{right}(S\cdot\mathbf{tail}(X'), i))$，因此断言依然成立；反之，断言未发生变化，依然是成立的。当循环结束后，根据断言有 $\mathbf{left}(S, \lceil\mathbf{len}(S)/2\rceil)=\mathbf{rev}(\mathbf{right}(S, \lceil\mathbf{len}(S)/2\rceil))$，从而有 $S=\mathbf{rev}(S)$。在这种情况下，循环时 $\mathbf{eq}(\mathbf{head}(X), \mathbf{last}(X))$ 总是成立的，因而输出总是 true，这是正确答案。

(b) 错误。对于输入 `"ab"`，每次执行循环时，$\mathbf{eq}(\mathbf{head}(X), \mathbf{last}(X))$ 总不成立，因而 X 的长度不会发生变化，循环不会终止。

8 [DH] Problem 5.13

(a) 错误。对于输入 `"a"`，第一次执行循环时，$X=$“a”，$Y=\Lambda$，$\mathbf{eq}(\mathbf{head}(X), \mathbf{last}(Y))$ 不成立，循环结束，输出 false. 但是 `"a"` 是一个回文串。

(b) 正确。每次执行循环后，如果 $\mathbf{eq}(\mathbf{head}(X), \mathbf{last}(Y))$ 成立，X 的长度减少，循环继续执行；否则退出推出循环。对于合法的输入，字符串 X 的长度总是有限的，因而算法总是能够结束。

9 [DH] Problem 5.14

(a) $X \leftarrow S$;
   $E \leftarrow$ true;
   while $X \neq \Lambda$ and E is true do the following:
      if $\mathbf{eq}(\mathbf{head}(X), \mathbf{last}(X))$ then $X \leftarrow \mathbf{all\text{-}but\text{-}last}(\mathbf{tail}(X))$;
      otherwise $E \leftarrow$ false;
   return E.

(b) 部分正确性的证明：在每次执行循环前，断言“如果 E 为 true，那么 $\mathbf{left}(S,i)=\mathbf{rev}(\mathbf{right}(S,i))$，其中 i 为语句‘$X \leftarrow \mathbf{all\text{-}but\text{-}last}(\mathbf{tail}(X))$’执行的次数；否则 S 不是回文串”。在第一次循环执行之前，可以验证断言成立；在循环被执行了一次之后，如果该次循环执行时 $\mathbf{eq}(\mathbf{head}(X), \mathbf{last}(X))$ 成立，那么对该次循环执行之前的 X（记为 X'），有 $\mathbf{left}(S\cdot\mathbf{head}(X'),i)=\mathbf{rev}(\mathbf{right}(S\cdot\mathbf{tail}(X'), i))$，因此断言依然成立；否则，即可判定 S 不是回文串，断言也成立，并且下一次循环不会被执行。当循环结束后，根据断言，当 E 为 true 时有 $\mathbf{left}(S, \lceil\mathbf{len}(S)/2\rceil)=\mathbf{rev}(\mathbf{right}(S, \lceil\mathbf{len}(S)/2\rceil))$，从而有 $S=\mathbf{rev}(S)$，即 S 是回文串；当 E 为 false 时 S 不是回文串，因此算法能够给出正确的输出。

   可终止性的证明：每次执行循环后，如果 $\mathbf{eq}(\mathbf{head}(X), \mathbf{last}(Y))$ 成立，X 的长度减少，循环继续执行；否则退出循环。对于合法的输入，字符串 X 的长度总是有限的，因而算法是可终止的。

   综上，该算法是完全正确的。

(c) 使用 *Pal*1 算法判断时，需要将整个字符串反转后再和原字符串比较，消耗的时间和字符串的长度成正比；而使用 *Pal*4 算法判断时，只需判断首尾两个字符即可断定字符串不是回文串，而无需继续比较，因此效率较高。