论题 1-10 作业

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1 [UD] Problem 13.3

(a) No. Because both $(1,\sqrt{3})$ and $(1,-\sqrt{3})$ are elements of f, however, $\sqrt{3}\neq-\sqrt{3}$.

(b) No. Because for $x=-1$, there does not exist $y\in\mathbb{R}$, such that $y=1/(x+1)$.

(c) Yes. Because for all $(x,y)\in\mathbb{R}^2$, there exists a unique real number z such that $z=x+y$.

(d) Yes. Because for every closed interval of real numbers $[a,b]$, there exists a unique real number a, such that $([a,b],a)\in f$.

(e) Yes. Because for every $(n,m)\in\mathbb{N}\times\mathbb{N}$, there exists a unique real number m, such that $((n,m),m)\in f$.

(f) Yes. Because for every real number x, there exists a real number y, such that $y=0$ when $x\geq 0$ or $y=x$ when $x<0$, i.e. $(x,y)\in f$.

(g) No. Because both $(6,7)$ and $(6,5)$ are elements of f, however, $7\neq 5$.

(h) Yes. Because for every circle c in the plane $\mathbb{R}^2$, there exists a unique real number C, such that C is the circumference of c.

(i) Yes. Because for every polynomial with real coefficients p, p is differentiable, thus there exists a unique polynomial p', such that p' is the derivative of p.

(j) Yes. Because for every polynomial p, p is integrable on $[0,1]$, thus there exists a unique number I such that $I=\int_0^1 p(x)dx$.

2 [UD] Problem 13.4

We know that $A\cap\mathbb{N}$ is either an empty or a nonempty set. In the case that $A\cap\mathbb{N}$ is empty, there exists a unique integer -1, such that $(A,-1)\in f$. In the case that $A\cap\mathbb{N}$ is nonempty, $A\cap\mathbb{N}$ is a subset of $\mathbb{N}$. By well-ordering principle of $\mathbb{N}$, $\min(A\cap\mathbb{N})$ exists, so there exists a unique integer $\min(A\cap\mathbb{N})$, such that $(A,\min(A\cap\mathbb{N}))\in f$. Therefore f is a well-defined function.

3 [UD] Problem 13.5

(a) For all $x\in X$, either $x\in A$ or $x\in X\setminus A$ holds, so there exists a unique number y ($y=1$ when $x\in A$ and $y=0$ when $x\in X\setminus A$), such that $y=\chi_A$. Therefore χ_A is a function.

(b) The domain is X. The range is $\{0\}$ when $A=\varnothing$, $\{1\}$ when $A=X$, and $\{0,1\}$ when $A\neq\varnothing$ and $A\neq X$.

4 [UD] Problem 13.7

For every real number $y \neq 1/2$, let $(x-5)/(2x-3) = y$, and we get $x = (3y-5)/(2y-1) \neq 3/2$, which is an element of the domain. So $\operatorname{ran}(f) = \mathbb{R} \setminus \{1/2\}$. $\square$

5 [UD] Problem 13.11

No. For every $x \in A$, there may not exist y such that $(x,y) \in f$. Even though for every $x \in A$ there exists y such that $(x,y) \in f$, we cannot make sure that there only exists one y such that $(x,y) \in f$.

6 [UD] Problem 13.13

The only possible relation is $\{(x,y) \in X^2 : x = y\}$. By the reflexion of the equivalence, any relation on X is a superset of $\{(x,y) \in X^2 : x = y\}$. Assume there exists relation X' such that $X' \setminus X \neq \varnothing$, let (a,b) be an element of X' such that $a \neq b$. However, (b,b) is an element of X' but $a \neq b$, so X' is not a function.

7 [UD] Problem 14.8

(a) Not one-to-one. $f(1) = f(-1) = 1/2$ but $1 \neq -1$.
Not onto. The range is $(0,1]$.

(b) Not one-to-one. $\sin 0 = \sin \pi = 0$ but $0 \neq \pi$.
Not onto. The range is $[-1,1]$.

(c) Not one-to-one. $f(1,2) = f(2,1) = 2$ but $(1,2) \neq (2,1)$.
Onto.

(d) Not one-to-one. $f((1,0),(0,0)) = f((0,0),(0,0)) = 0$ but $((1,0),(0,0)) \neq ((0,0),(0,0))$.
Onto.

(e) Not one-to-one. $f((0,0),(0,0)) = f((1,1),(1,1)) = 0$ but $((0,0),(0,0)) \neq ((1,1),(1,1))$.
Not onto. The range is $[0,+\infty)$.

(f) One-to-one.
Onto when $B = \{b\}$. Not onto when $B \neq \{b\}$. The range is $A \times \{b\}$.

(g) One-to-one.
Onto.

(h) Not one-to-one. $f(X) = f(B) = B$ but $X \neq B$.
Not onto. The range is $\mathcal{P}(B)$.

(i) One-to-one.
Not onto. The range is $(0,+\infty)$.

8 [UD] Problem 14.12

$f(x) = \dfrac{(d-c)x+cb-da}{b-a} (x \in [a,b])$.

One-to-one: Let $f(x_1) = f(x_2)$, we have $\dfrac{(d-c)x_1+cb-da}{b-a} = \dfrac{(d-c)x_2+cb-da}{b-a}$. Multiplying $b-a$ and cancelling on both sides, we have $x_1 = x_2$.

Onto: Let $c \le f(x) \le d$, that is $c \le \dfrac{(d-c)x+cb-da}{b-a} \le d$. Multiplying $b-a$ and cancelling on both sides, we have $a \le x \le b$. It means, for every $x \in [a,b]$, there exists y, such that $y = f(x)$, thus $f(x)$ is onto.

Since $f(x)$ is both one-to-one and onto, $f(x)$ is a bijection. □

9 [UD] Problem 14.13

ϕ is a function from $F([0,1])$ to $\mathbb{R}$. Because for all $f \in F([0,1])$, there exists a unique real number y, such that $y = f(0)$.

ϕ is not one-to-one. Let $f_1(x) = 0 \in F([0,1])$, $f_2(x) = x \in F([0,1])$, we have that $\phi(f_1) = \phi(f_2)$, however, $f_1 \ne f_2$ because $f_1(1) \ne f_2(1)$.

ϕ is onto. For every real number a, there exists $f_0(x) = a \in F([0,1])$, such that $\phi(f_0) = a$.

10 [UD] Problem 14.15

For all $x \in \mathbb{R}$, since $f(x)$ is defined on $\mathbb{R}$, there exists a unique real number $y = f(x) \cdot f(x)$, such that $y = (f \cdot f)(x)$, therefore $f \cdot f$ is a function. □

(a) Yes. $f(x) = e^x$.

(b) No. $\mathrm{ran}(f \cdot f) = \{a^2 : a \in \mathrm{ran}(f)\}$.

11 [UD] Problem 15.1

	$(f \circ g)(x)$	$\mathrm{dom}(f \circ g)$	$\mathrm{ran}(f \circ g)$	$(g \circ f)(x)$	$\mathrm{dom}(g \circ f)$	$\mathrm{ran}(g \circ f)$
(a)	$1/(1+x^2)$	$\mathbb{R}$	$(0,1]$	$1/(1+x)^2$	$\mathbb{R} \setminus \{-1\}$	$\mathbb{R}^+$
(b)	x	$[0,+\infty)$	$[0,+\infty)$	$\lvert x\rvert$	$\mathbb{R}$	$[0,+\infty)$
(c)	$1/(x^2+1)$	$\mathbb{R}$	$(0,1]$	$(1/x^2)+1$	$\mathbb{R} \setminus \{0\}$	$(1,+\infty)$
(d)	$\lvert x\rvert$	$\mathbb{R}$	$[0,+\infty)$	$\lvert x\rvert$	$\mathbb{R}$	$[0,+\infty)$

12 [UD] Problem 15.6

(a) $(f \circ g)(x) = f(g(x)) = \dfrac{\dfrac{3+2x}{1-x} - 3}{\dfrac{3+2x}{1-x} + 2} = \dfrac{\dfrac{5x}{1-x}}{\dfrac{5}{1-x}} = x \ (x \ne 1)$,

$$(g \circ f)(x) = g(f(x)) = \frac{3 + 2\dfrac{x-3}{x+2}}{1 - \dfrac{x-3}{x+2}} = \frac{\dfrac{5x}{x+2}}{\dfrac{5}{x+2}} = x \ (x \ne -2).$$

(b) (Theorem 15.4) Let $f : A \to B$ be a bijective function, and f^{-1} be the inverse of f, then $f \circ g = i_B$, and $g \circ f = i_A$.

13 [UD] Problem 15.7

(a) (i) $f = \{(1,4),(2,5),(3,5)\}$, $g = \{(4,1),(5,2)\}$;

(ii) $f = \{(1,4),(2,5)\}$, $g = \{(4,1),(5,2)\}$;

(iii) Impossible.

$$\begin{tikzcd}
\mathbf{B} \arrow[r, "g"] & \mathbf{A} \arrow[r, "f"] & \mathbf{B} \\
4 \arrow[r] & 1 \arrow[r] & 4 \\
5 \arrow[r] & 2 \arrow[r] & 5 \\
 & 3 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{B} \arrow[r, "g"] & \mathbf{A} \arrow[r, "f"] & \mathbf{B} \\
4 \arrow[r] & 1 \arrow[r] & 4 \\
5 \arrow[r] & 2 \arrow[r] & 5
\end{tikzcd}$$

(i) (ii)

Figure 1: diagrams of A and B

(b) Let $A = \{1,2\}$, $B = \{1\}$, $f = \{(1,1),(2,1)\}$, $g = \{(1,1)\}$, we have $f \circ g = \{(1,1)\} = i_B$, but $g \circ f = \{(1,1),(2,1)\} \neq i_A$.

Because neither f nor g is a bijective function.

(c) Let $A = \{1\}$, $B = \{1,2\}$, $f = \{(1,1)\}$, $g = \{(1,1),(2,1)\}$, we have $g \circ f = \{(1,1)\} = i_A$, but $f \circ g = \{(1,1),(2,1)\} \neq i_B$.

Because neither f nor g is a bijective function.

(d) f is not always one-to-one, but must be onto. For injectivity, we have a counterexample in (b). For surjectivity, suppose to the contrary that f is not onto. That means, there exists $b \in B$, for all $a \in A$, $f(a) \neq b$. Therefore, $(f \circ g)(b) = f(g(b)) \neq b$, which is contradict to that $f \circ g = i_B$. Therefore f is onto.

(e) Guess whether the function has some property. If true, try to find the proof; if false, try to find a counterexample.

Here, f is not always onto, but must be one-to-one. For surjectivity, we have a counterexample in (c). For injectivity, suppose to the contrary that f is not one-to-one. That means, there exists a and b in A such that $f(a) = f(b)$. However, $(g \circ f)(a) = g(f(a)) = g(f(b)) = (g \circ b)(b)$, which is contradict to that $g \circ f = i_A$. Therefore f is one-to-one.

14 [UD] Problem 15.11

By the definition of the inverse of a function, the inverse function of f exists because f is a bijection. Since $f \circ g_1 = f \circ g_2$, we have $f^{-1} \circ (f \circ g_1) = f^{-1} \circ (f \circ g_2)$, thus $(f^{-1} \circ f) \circ g_1 = (f^{-1} \circ f) \circ g_2$ by associative property, and by Theorem 15.4 (ii) we get $g_1 = g_2$. $\square$

If $g_1 \circ f = g_2 \circ f$ and f is bijective, $g_1 = g_2$ still holds. Just get $g_1 \circ (f \circ f^{-1}) = g_2 \circ (f \circ f^{-1})$, and prove in the similar way.

15 [UD] Problem 15.12

Yes.

The equivalence class of $a \in A$ is $\{a\}$.

16 [UD] Problem 15.13

No.

Yes. $f(x) = x$.

17 [UD] Problem 15.14

(a) First, for all $(a,c) \in A \times C$, there exists a unique pair $(f(a), g(c)) \in B \times D$, such that $H(a,c) = (f(a), g(c))$ because $f : A \to B$ and $g : C \to D$ are both functions. Therefore H is a function.

Second, let $(f(a_1), g(c_1)) = (f(a_2), g(c_2))$, by the definition of ordered pair, we have $f(a_1) = f(a_2)$ and $g(c_1) = g(c_2)$, since f and g are both one-to-one, we get $a_1 = a_2$ and $c_1 = c_2$, and this implies $(a_1, c_1) = (a_2, c_2)$. Therefore H is one-to-one. $\square$

(b) Since f and g are onto, for every $(b,d) \in B \times D$, there exist a and c, such that $f(a) = b$ and $g(c) = d$, therefore $H(a,c) = (b,d)$. Hence H is also onto. $\square$

18 [UD] Problem 15.15

H is not a function: $A = \{1,2\}$, $B = \{1,2\}$, $C = \{2,3\}$, $D = \{3,4\}$, $f = \{(1,1),(2,2)\}$, $g = \{(2,3),(3,4)\}$, $H = \{(1,1),(2,2),(2,3),(3,4)\}$.

H is a function: $A = \{1\}$, $B = \{1\}$, $C = \{2\}$, $D = \{2\}$, $f = \{(1,1)\}$, $g = \{(2,2)\}$, $H = \{(1,1),(2,2)\}$.

When A and C are disjoint, we are assured that H is a function. In fact, H is a function if and only if $f \cap [(A \cap C) \times B] = g \cap [(A \cap C) \times D]$.

19 [UD] Problem 15.20

(a) Let $f|_{A_1}(x) = f|_{A_1}(y)$, and by the definition of the restriction function, we have $f(x) = f(y)$. Since f is one-to-one, we have $x = y$. Therefore $f|_{A_1}$ is one-to-one. $\square$

(b) For every $y \in B$, there exists $x \in A_1 \subset A$ such that $f|_{A_1}(x) = f(x) = y$ because $f|_{A_1}$ is onto. Therefore f is onto. $\square$

20 [UD] Problem 16.19

For every $a \in A$, there exists $b \in B$ such that $b = f(a)$, and we have that $f^{-1}(\{b\}) \subseteq A$ because f is a function from A to B. Therefore, $\bigcup_{b \in B} f^{-1}(\{b\}) = A$.

Since f is onto, for every $b \in B$, there exists $a \in A$, such that $f(a) = b$, thus $f^{-1}(\{b\})$ is always nonempty.

If $f^{-1}(\{b_1\}) \cap f^{-1}(\{b_2\})$ is nonempty, there exists a, such that $f(a) = b_1$ and $f(a) = b_2$, thus $b_1 = b_2$, therefore $f^{-1}(\{b_1\}) = f^{-1}(\{b_2\})$.

Summarizing, we conclude that $\{f^{-1}(\{b\}) : b \in B\}$ is a partition of A. $\square$

21 [UD] Problem 16.20

(a) No.

(b) For every $a \in A_1$, we have $f(a) \in f(A_1) = f(A_2)$, thus there exists $a' \in A_2$ such that $f(a') = f(a)$. Since f is **one-to-one**, we have that $a' = a$, therefore $a \in A_2$. Hence $A_1 \subseteq A_2$, and $A_2 \subseteq A_1$ likewise. Therefore $A_1 = A_2$. $\square$

I used only one-to-one.

22 [UD] Problem 16.21

(a) No.

(b) For every $b \in B_1 \subseteq Y$, there exists $a \in X$ such that $f(a) = b$ because f is **onto**. Hence, a is an element of $f^{-1}(B_1) = f^{-1}(B_2)$, therefore there exists $b' \in B_2$ such that $f(a) = b'$, thus $b = b'$, and b is an element of B_2. Therefore B_1 is a subset of B_2, and B_2 is a subset of B_1 likewise. So $B_1 = B_2$. $\square$

I used only onto.

23 [UD] Problem 16.22

(a) Yes.

(b) For all $x \in A_1 \cap A_2$, both $\chi_{A_1}(x)$ and $\chi_{A_2}(x) = 1$, therefore $\chi_{A_1 \cap A_2}(x) = \chi_{A_1}(x) \cdot \chi_{A_2}(x) = 1$.

For all $x \notin A_1 \cap A_2$, either $\chi_{A_1}(x)$ or $\chi_{A_2}(x) = 0$, therefore $\chi_{A_1 \cap A_2}(x) = \chi_{A_1}(x) \cdot \chi_{A_2}(x) = 0$.

Summarizing, we have $\chi_{A_1} \cdot \chi_{A_2} = \chi_{A_1 \cap A_2}$. $\square$

(c) For all x s.t. $x \in A_1$ and $x \in A_2$, $\chi_{A_1}(x) = \chi_{A_2}(x) = 1$, $\chi_{A_1 \cap A_2}(x) = 1$, therefore $\chi_{A_1 \cup A_2}(x) = \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = 1$

For all x s.t. $x \in A_1$ and $x \notin A_2$, $\chi_{A_1}(x) = 1$,$\chi_{A_2}(x) = 0$, $\chi_{A_1 \cap A_2}(x) = 0$, therefore $\chi_{A_1 \cup A_2}(x) = \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = 1$

For all x s.t. $x \notin A_1$ and $x \in A_2$, $\chi_{A_1 \cup A_2}(x) = \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = 1$ holds likewise.

For all x s.t. $x \notin A_1$ and $x \notin A_2$, $\chi_{A_1}(x) = \chi_{A_2}(x) = 0$, $\chi_{A_1 \cap A_2}(x) = 0$, therefore $\chi_{A_1 \cup A_2}(x) = \chi_{A_1}(x) + \chi_{A_2}(x) - \chi_{A_1 \cap A_2}(x) = 0$

Summarizing, we have $\chi_{A_1 \cup A_2} = \chi_{A_1} + \chi_{A_2} - \chi_{A_1 \cap A_2}$. $\square$

(d) $\chi_{X \setminus A_1} = 1 - \chi_{A_1}$.