

论题 2-9 作业

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NOTE: “ $\lg x$ ” stands for “ $\log_2 x$ ”.

1 [TC] Problem 6.1-2

We use mathematical induction to prove this.

For base step, we have $n = 1$, and the height is 0, so the conclusion is obviously correct.

For induction step, assume that for $n = k$, the conclusion is correct. Consider a heap with $n = k + 1$ elements. If $k = 2^p - 1$ where p is a positive integer, then a k -element heap is a complete binary tree. Thus, a $(k + 1)$ -element heap has one more level than a k -element heap, and the height is $\lfloor \lg k \rfloor + 1 = \lfloor \lg(k + 1) \rfloor$. If $k \neq 2^p - 1$ for every positive integer p , then a k -element heap is nearly but not exactly a complete binary tree. Thus, a $(k + 1)$ -element heap has the same level as a k -element heap, and the height is $\lfloor \lg k \rfloor = \lfloor \lg(k + 1) \rfloor$. Therefore, the conclusion is correct for $n = k + 1$.

By mathematical induction, we conclude that an n -element heap has height $\lfloor \lg n \rfloor$.

2 [TC] Problem 6.1-4

The smallest element must reside in a leaf. If it does not reside in a leaf, then it is smaller than its child(ren), which is contradict to the max-heap property.

3 [TC] Problem 6.1-7

We know that a leaf has no child. For index $i \leq \lfloor n/2 \rfloor$, the index of its left child is $2i$, which is not out of range, i.e. the node indexed by i is not a leaf. However, for index $i > \lfloor n/2 \rfloor$, either the index of its left child $2i$ or the right child $2i + 1$, which is out of range, i.e. the node indexed by i is a leaf. Therefore, the leaves are nodes indexed by $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \dots, n$.

4 [TC] Problem 6.2-2

MIN-HEAPIFY(A, i)

```
1   $l = \text{LEFT}(i)$ 
2   $r = \text{RIGHT}(i)$ 
3  if  $l \leq A.\text{heap-size}$  and  $A[l] < A[i]$ 
4       $\text{smallest} = l$ 
5  else  $\text{smallest} = i$ 
6  if  $r \leq A.\text{heap-size}$  and  $A[r] < A[\text{smallest}]$ 
7       $\text{smallest} = r$ 
8  if  $\text{smallest} \neq i$ 
9      exchange  $A[i]$  with  $A[\text{smallest}]$ 
10     MIN-HEAPIFY( $A, \text{smallest}$ )
```

It has the same running time as MAX-HEAPIFY asymptotically.

5 [TC] Problem 6.2-5

MAX-HEAPIFY(A, i)

```
1  while TRUE
2       $l = \text{LEFT}(i)$ 
3       $r = \text{RIGHT}(i)$ 
4      if  $l \leq A.\text{heap-size}$  and  $A[l] > A[i]$ 
5           $\text{largest} = l$ 
6      else  $\text{largest} = i$ 
7      if  $r \leq A.\text{heap-size}$  and  $A[r] > A[\text{largest}]$ 
8           $\text{largest} = r$ 
9      if  $\text{largest} \neq i$ 
10         exchange  $A[i]$  with  $A[\text{largest}]$ 
11          $i = \text{largest}$ 
12     else return
```

6 [TC] Problem 6.2-6

When the value of the node which causes the MAX-HEAPIFY is the smallest in the whole tree, it must be swapped to a leaf. However, for each recursive call, it can only be swapped to its child. Therefore, it must be swapped k times, where $k = \lceil \lg n \rceil$ is the height of the tree. Hence, the worst-case running time is $\Omega(\lg n)$.

7 [TC] Problem 6.3-3

We have the following claims:

Claim 1: For any n -element heap $A[1..n]$, $A[\lfloor n/2 \rfloor + 1..n]$ exactly contains the elements of height 0, i.e. the leaves of the heap.

Claim 2: For any n -element heap $A[1..n]$, every left subarray of A is still a heap.

By **Claim 1** and **Claim 2**, we obtain the following lemma:

Lemma The height of element $A[i]$ in heap $A[1.. \lfloor n/2 \rfloor]$ is the height of element $A[i]$ in heap $A[1..n]$ minus 1.

Proof: assume that the height of $A[i]$ in heap $A[1..n]$ is h , that means, the length of the longest path from $A[i]$ to a leaf, say $A[l]$, is h . By **Claim 1**, $A[l]$ is in $A[\lfloor n/2 \rfloor + 1..n]$, but not in $A[1.. \lfloor n/2 \rfloor]$, and its parent, $A[\lfloor l/2 \rfloor]$, is in $A[1.. \lfloor n/2 \rfloor]$. That means, the longest path from $A[i]$ to a leaf of heap $A[1.. \lfloor n/2 \rfloor]$ is from $A[i]$ to $A[\lfloor l/2 \rfloor]$, whose length is $h - 1$, i.e. the height of $A[i]$ in $A[1.. \lfloor n/2 \rfloor]$ is $h - 1$. \square

Let $f(n, h)$ denote the number of nodes of height h in an n -element heap. By **Lemma** and **Claim 1** we get the following recurrence:

$$f(n, h) = \begin{cases} f(\lfloor n/2 \rfloor, h-1) & h > 0 \\ \lfloor n/2 \rfloor & h = 0 \end{cases}$$

Solve this recurrence by iteration, we obtain $f(n, h) \leq \lceil n/2^{h+1} \rceil$. Therefore, there are at most $\lceil n/2^{h+1} \rceil$ nodes of height h in any n -element heap.

8 [TC] Problem 6.4-2

Initialization: Prior to the first iteration, $i = n$, $A[1..i]$ is a max-heap because BUILD-MAX-HEAP has been executed, and of course it contains the i smallest elements of $A[1..n]$. $A[i+1..n]$ is an empty array, so it contains the $n - i$ largest elements of $A[1..n]$ sorted trivially. Therefore, the loop invariant holds before the loop.

Maintenance: By the property of the max-heap, $A[i]$ is the largest element in $A[1..i]$, but it is smaller than every element in $A[i+1..n]$ by the loop invariant. So after swapping, $A[i..n]$ contains the $n - i + 1$ largest elements $A[1..n]$, sorted, and after MAX-HEAPIFY executed, $A[1..i]$ is a max-heap containing the $i - 1$ smallest elements. So the loop invariant holds after each iteration.

Termination: By the loop invariant, we know that after the loop $A[1..n]$ contains all the elements sorted. Hence the procedure is partially correct.

The **for** loop will be exactly executed for $A.length - 1$ times, thus the procedure can terminate. Therefore, HEAPSORT could sort a given array totally correctly.

9 [TC] Problem 6.4-4

Consider a case, that the input array $A[1..n]$ stores a monotonously decreasing sequence, then it is already a max-heap, and BUILD-MAX-HEAP does not change the array. For each iteration of **for** loop, the smallest element is swapped to the root, and it takes at least $\lfloor \lg(i-1) \rfloor - 1$ swaps to max-heapify the array. Therefore, the total running time is

$$\Omega\left(\sum_{i=2}^n \lfloor \lg(i-1) \rfloor - 1\right) = \Omega\left(-n + 1 + \sum_{i=(n-1)/2}^{n-1} \lfloor \lg i \rfloor\right)$$

$$\begin{aligned}
&= \Omega\left(-n+1 + \lfloor \frac{n-1}{2} \rfloor \lceil \lg \lceil \frac{n-1}{2} \rceil \rceil\right) \\
&= \Omega(n \lg n)
\end{aligned}$$

Therefore, the worst-case running time of HEAPSORT is $\Omega(n \lg n)$.

10 [TC] Problem 6.5-5

Initialization: Before the **while** loop, only $A[i]$ is increased, so it still satisfies the max-heap property, except that “ $A[i] \leq A[\text{PARENT}(i)]$ ” may be violated. Therefore the loop invariant holds before the loop.

Maintenance: Before each iteration to execute, we have $A[\text{PARENT}(i)] < A[i]$, and the tree rooted at $A[\text{PARENT}(i)]$ is the largest element except $A[i]$, according to the loop invariant. After exchanging $A[i]$ with $A[\text{PARENT}(i)]$, the tree rooted at $A[\text{PARENT}(i)]$ is a heap. Then, i is assigned $A[\text{PARENT}(i)]$, and after the assignment “ $A[\text{PARENT}(i)] < A[i]$ ” may be violated. Therefore, the loop invariant holds after each iteration.

Maintenance: After the loop, $i == 1$ or $A[\text{PARENT}(i)] > A[i]$. In the former case, $\text{PARENT}(i)$ does not exist, thus the whole array is a heap. In the latter case, “ $A[\text{PARENT}(i)] < A[i]$ ” is in fact not violated, therefore the whole array is also a heap. Hence, the procedure is partially correct.

For each iteration, the depth of $A[i]$ is decremented by 1, so the procedure can terminate. Therefore, the procedure is totally correct.

11 [TC] Problem 6.5-7

Let H be a max-priority queue. $H.\text{insert}(k, x)$ inserts the element x associated with key k into the priority queue, $H.\text{extract}()$ returns the element with the maximum key in H , and deletes it from H . Let I be a global integer variable with initial value 0.

Implementation of queue:

ENQUEUE(H, x)

1 $H.\text{insert}(I, x)$

2 $I = I + 1$

DEQUEUE(H)

1 **return** $H.\text{extract}()$

Implementation of stack:

PUSH(H, x)

1 $H.\text{insert}(I, x)$

2 $I = I + 1$

POP(H)

1 **return** $H.\text{extract}()$

12 [TC] Problem 6.5-9

MULTI-WAY-MERGE(*lists*)

```
1  Let  $C[1 \dots \text{lists.count}]$  be a new array
2  Let  $H$  be a min-priority queue of length  $\text{lists.count}$ 
3   $n = 0$ 
4  for  $i = 1$  to  $\text{lists.count}$ 
5       $C[i] = \text{lists}[i].\text{length}$ 
6       $n = n + C[i]$ 
7      if  $C[i] > 0$ 
8           $H.\text{insert}(\text{lists}[i][C[i]], i)$ 
9           $C[i] = C[i] - 1$ 
10 Let  $A[1 \dots n]$  be a new array
11  $j = 0$ 
12 while  $j < n$ 
13      $j = j + 1$ 
14      $A[j] = H.\text{min-key}$ 
15      $i = H.\text{extract}()$ 
16     if  $C[i] > 0$ 
17          $H.\text{insert}(\text{lists}[i][C[i]], i)$ 
18          $C[i] = C[i] - 1$ 
19 return  $A$ 
```

During the whole procedure, the size of the priority queue is at most k , and each element in all the input lists is inserted into and removed from the priority queue once, and the running time of each operation is $O(\lg k)$. Therefore, the algorithm merges k sorted lists into one sorted list in a running time of $O(n \lg k)$.