论题 2-3 作业

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1 [TC] Problem 4.1-5

FIND-MAXIMUM-SUBARRAY(A, n)

$ans = 0$
$sum = 0$
for $i = 1$ **to** n
  $sum = sum + A[i]$
  if $sum > ans$
    $ans = sum$
  elseif $sum < 0$
    $sum = 0$
return ans

2 [TC] Problem 4.3-3

We have to prove that $T(n) \geq cn\lg n$ for appropriate choice of constant $c > 0$. We use mathematical induction to prove it.

For the base step, we have $T(1) = 1$, $T(2) = 4$ and $T(3) = 5$. We should choose c such that $T(1) = 1 \geq 0$, $T(2) = 4 \geq 2c$, $T(3) = 5 \geq 3c\lg 3$. We can choose every $c < 1$.

For the induction step, assume that $T(m) \geq cm\lg m$ holds for every positive integer $m < n$, where $n \geq 4$. Substituting it into the recurrence, we obtain

$$\begin{aligned}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \\
&\geq 2(c\lfloor n/2 \rfloor \lg\lfloor n/2 \rfloor) + n \\
&\geq 2(c(n/2-1)\lg(n/2-1)) + n \\
&\geq c(n-2)\lg((n-2)/2) + n \\
&= cn\lg(n-2) - 2c\lg(n-2) + cn - 2c + n \\
&= cn\lg n - cn\lg\frac{n}{n-2} - 2c\lg(n-2) + cn - 2c + n \\
&= cn\lg n + (cn - cn\lg\frac{n}{n-2}) + (n/2 - 2c\lg(n-2)) + (n/2 - 2c) \\
&\geq cn\lg n
\end{aligned}$$

The last step holds if we choose $c < \frac{1}{2}$, because

(1) $n \geq 4 \Rightarrow \lg\frac{n}{n-2} \leq 1 \Rightarrow cn \geq cn\lg\frac{n}{n-2}$;

(2) $n \geq 4 \Rightarrow n/2 \geq \lg(n-2)$, $(n/2 \geq \lg(n-2)) \wedge (c < \frac{1}{2}) \Rightarrow n/2 < 2c\lg(n-2)$;

(3) $n \geq 4 \Rightarrow n/2 > 1$, $(n/2 \geq 1) \wedge (c < \frac{1}{2}) \Rightarrow n/2 > 2c$.

By mathematical induction, we conclude that for all positive constant $c < \frac{1}{2}$, $T(n) \geq cn\lg n$ holds. Hence the recurrence is also $\Omega(n\lg n)$. Therefore we conclude that the solution is $\Theta(n\lg n)$.

3 [TC] Problem 4.3-7

Assume $T(m) \leq cm^{\log_3 4}$ for sufficiently large $m < n$, especially for $m = n/3$ where n is large enough. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 4T(n/3) + n \\ &\leq 4c(\frac{n}{3})^{\log_3 4} + n \\ &= cn^{\log_3 4} + n \\ &\nleq cn^{\log_3 4} \qquad\qquad fails! \end{aligned}$$

We guess $T(n) \leq c(n^{\log_3 4} - n)$ instead and prove it by mathematical induction.

For the base step, when $n = 3$, we have $T(n) = 7$ and $c(n^{\log_3 4} - n) = c$, therefore $T(n) \leq c(n^{\log_3 4} - n)$ holds for every $c > 7$.

For the induction step, assume $T(m) \leq c(m^{\log_3 4} - m)$ holds for every $m < n$, where $n > 3$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 4T(n/3) + n \\ &\leq 4c((\frac{n}{3})^{\log_3 4} - \frac{n}{3}) + n \\ &= cn^{\log_3 4} + (1 - \frac{4c}{3})n \\ &\leq cn^{\log_3 4} \end{aligned}$$

By mathematical induction, we conclude $T(n) \leq c(n^{\log_3 4} - n)$. Therefore $T(n) = O(n^{\log_3 4} - n) = O(n^{\log_3 4})$.

To complete the proof of $T(n) = \Omega(n^{\log_3 4})$, we assume $T(n) \geq cn^{\log_3 4}$.

For the base step, when $n = 1$, we have $T(1) = 1$ and $cn^{\log_3 4} = c$, therefore $T(n) \geq cn^{\log_3 4}$ holds for every positive constant $c < 1$.

For the induction step, assume $T(m) \geq cm^{\log_3 4}$ holds for every $m < n$, where $n > 3$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 4T(n/3) + n \\ &\geq 4c(\frac{n}{3})^{\log_3 4} + n \\ &= cn^{\log_3 4} + n \\ &\geq cn^{\log_3 4} \end{aligned}$$

By mathematical induction, we conclude that $T(n) \geq cn^{\log_3 4}$. Hence $T(n) = \Omega(n^{\log_3 4})$. Therefore $T(n) = \Theta(n^{\log_3 4})$.

4 [TC] Problem 4.4-2