论题 2-15 作业

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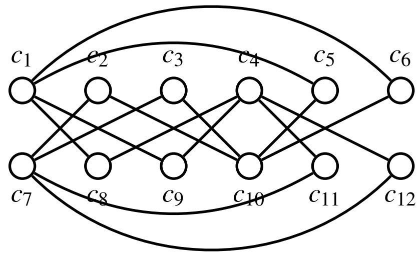
1 [CZ] Problem 1.6



2 [CZ] Problem 1.8

(a) The words in S_1 are (left to right): cat, cap, tap, top.

The words in S_2 are: map (center) , mop, tap, mat (surrounding).

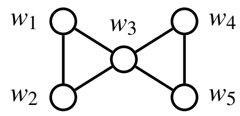
The words in S_3 are: run (top left), gun (top right), sun (center), son (bottom).

The words in S_4 are (clockwise): slit, slot, slop, slip.

The words in S_5 are (clockwise from top left): pot, put, pet, poet.

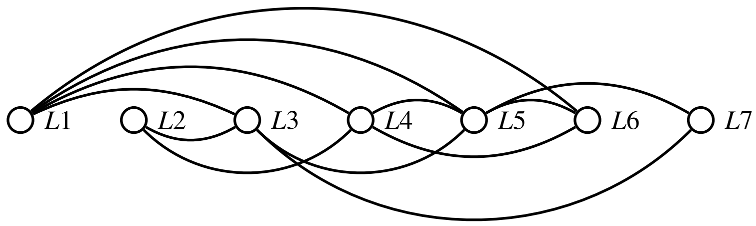
The words in S_6 are (clockwise): lake, sake, take, make.

(b) The graph H is:



It is a word graph of some set, and the corresponding words are: top, tap, tip, lip, dip.

3 [CZ] Problem 1.10



4 [CZ] Problem 1.14

Let C denote a component of G.

(1) $\to$ (2): Take any vertex v_0 in C. For any vertex v_i which is connected to v_0, it must in $V(C)$, for otherwise if we add v_i, along with the edges and vertices in the path from v_0 to v_i, to C, we get a proper connected supergraph of C, which leads to contradiction. Therefore, $V(C)$ is an equivalent class. Then we have to prove C is the subgraph induced by $V(C)$. If not, let C' be the subgraph induced by $V(C)$, and $E(C) \subset E(C')$, thus C is a proper subgraph of C', which leads to contradiction.

(2) $\to$ (1): Suppose, to the contrary that C is a proper subgraph of a connected subgraph of G, denoted by C'. If $V(C) \subset V(C')$, there exists some vertex connected to C but not in the equivalent class, which leads to contradiction. It is impossible that $V(C) = V(C')$, because the subgraph induced by $V(C)$ is the maximal subgraph whose vertex set is $V(C)$.

5 [CZ] Problem 1.16

For every i, we have a path from u to v_i: $(u = v_0, v_1, \cdots, v_i)$, whose length is i. Thus $d(u, v_i) \leq i$.

Suppose, to the contrary that $d(u, v_i) < i$, i.e. there exists path $(u_0 = v_0, u_1, \cdots, u_j = v_i)$, where $j < i$. Consider the walk $(u = u_0 = v_0, u_1, \cdots, u_j = v_i, v_{i+1}, \cdots, v_k = v)$, it's a $u - v$ walk shorter than the geodesic, which leads to contradiction.

Therefore, $d(u, v_i) = i$ for each integer i with $1 \leq i \leq k$.

6 [CZ] Problem 1.17

(a) Assume that P is an $x - z$ path and Q is a $u - w$ path, where $x \neq u, w$ and $y \neq u, w$, and they do not have common vertex. Let y be a vertex in P and v be a vertex in Q, then there exists a $y - v$ path $(p_0 = y, p_1, p_2, \cdots, p_n = v)$ by the connectivity. Let $k = \min_{P_i \in Q} i$, and $j = \max_{P_i \in P, i < k} i$, then the path $(p_j, p_{j+1}, \cdots, p_k)$ connects P and Q, and it has no vertex in P or Q except the first and the last one. Assume, without loss of generality, that $x - p_j$ is longer than $p_j - z$, and $u - p_k$ is longer than $p_k - w$. Consider the path $x - p_j - p_k - u$, it is longer than the $x - z$ path and the $u - v$ path, which leads to contradiction.

(b) This is false. Consider $G = K_4$, whose diameter is 1, and the vertices are $\{v_1, v_2, v_3, v_4\}$. In this graph, (v_1, v_2) and (v_3, v_4) are two geodesics of length 1, but they have no common vertex.

7 [CZ] Problem 1.18

(a) The minimum size of such a subgraph is the distance from u to v. Any connected subgraph containing u and v must have a $u - v$ path, which is at least as long as the geodesic. So the geodesic has minimum size among all connected subgraphs of G.

(b) What is the maximum size of a subgraph of G where u and v are disconnected?

8 [CZ] Problem 1.22

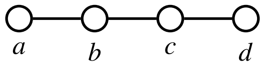
If u,v are in two different components of G, then $uv \in E(\overline{G})$, i.e. $d_{\overline{G}}(u,v) = 1$.

If u,v are in the same component of G, let w be a vertex in another component G. We have $uw, wv \in E(\overline{G})$, i.e. (u,w,v) is a path from u to v, and thus $d_{\overline{G}}(u,v) \le 2$.

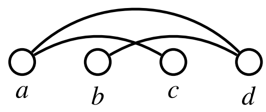
Therefore, $d_{\overline{G}}(u,v) = 1$ or $d_{\overline{G}}(u,v) = 2$.

9 [CZ] Problem 1.23

(a) For $k = 1$, the graph $G = (V,E)$ is:



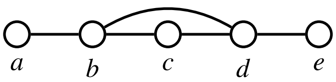
G



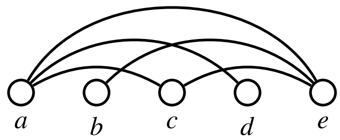
$\overline{G}$

and $d_G(b,c) = 1 = d_{\overline{G}}(a,d)$.

For $k = 2$, the graph $G = (V,E)$ is



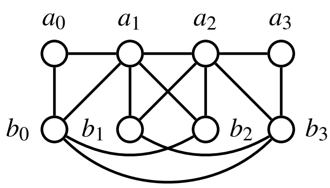
G



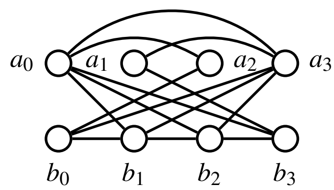
$\overline{G}$

and $d_G(a,d) = 2 = d_{\overline{G}}(b,c)$.

(b) The largest value of k is 3. Here is an example when $k = 3$, $d_G(a_0,a_3) = 3 = d_{\overline{G}}(b_0,b_3)$:



G



$\overline{G}$

Now we are going to prove that it is impossible that $k \ge 4$. Suppose, there exists distinct u,v,x,y, such that $d_G(u,v) = d_{\overline{G}}(x,y) = k \ge 4$. Let $(u = w_0, w_1, \cdots, w_k = v)$ be a $u-v$ geodesic, $(x = z_0, z_1, \cdots, z_k = y)$ be an $x-y$ geodesic. We claim that there exists $i \in \{0,k\}$, such that z_0w_i is not an edge in G, otherwise, (w_0, z_0, w_k) is a $u-v$ path shorter than the geodesic in G. Likewise there exists $j \in \{0,k\}$, such that z_kw_j is not an edge in G. Hence, z_0w_i and z_kw_j are edges in $\overline{G}$. If $i = j$, (z_0, w_i, z_k) is shorter than $x-y$ geodesic, which is impossible. If $i \ne j$, $w_iw_j = w_0w_k = uv$ is an edge in $\overline{G}$, otherwise $d_G(u,v) = 1$, contradicting the given condition. Note that $(x = z_0, w_i, w_j, z_k = y)$ is an $x-y$ path shorter than the geodesic, which is impossible.

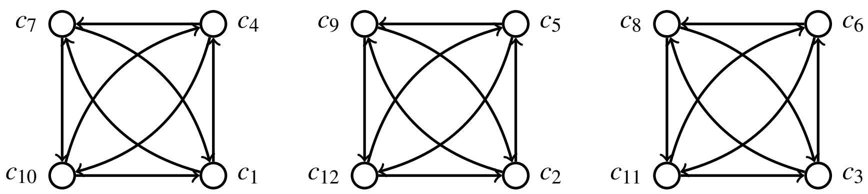
Therefore, $k \le 3$.

10 [CZ] Problem 1.25

Assume that G is bipartite, with partite sets U and W. Since $|V(G)| = |U| + |W| \geq 5$, at least one of $|U|$ and $|G|$ is greater than 2. Assume, without loss of generality, that $|U| \geq 3$. Let u, v, w be three distinct vertices in U. They are mutually adjacent in $\overline{G}$, which forms an odd cycle. By Theorem 1.12, $\overline{G}$ is not bipartite.

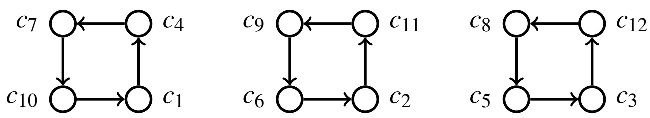
Therefore, if $|V(G)| \geq 5$, at most one of G and $\overline{G}$ is bipartite.

11 [CZ] Problem 1.30



12 [CZ] Problem 1.31

Define that, (c_i, c_j) is a directed edge, if and only if c_j can be obtained from c_i by rotating the configuration 90° clockwise, and then interchanging the two coins. The graph is:



13 [CZ] Problem 2.6

Consider the sum of the degrees over all vertices:

$$\sum_{v \in V(G)} \deg v = n(n-1) + n^2 + n(n+1) = 3n^2$$

By Theorem 2.1, $3n^2$ is even, therefore n is even.

14 [CZ] Problem 2.7

(a) For every $v \in V(G)$, by the definition of bipartite graph, one of its incident vertices is in U, the other is in W. Therefore $m = \sum_{u \in U} \deg u = \sum_{w \in W} \deg w$.

(b) Let x be the number of vertices of G having degree 2. Then we have the following equation

$$\sum_{u \in U} \deg u = \sum_{w \in W} \deg w$$
$$3 \times |U| = 2 \times n + 4 \times (|W| - n)$$

After some algebra we get $n = 2$.

15 [CZ] Problem 2.9

Suppose that these odd vertices are not in the same component. Let C be the component containing only one odd vertex. Component is an induced subgraph of the original graph, so the degrees of its vertices do not change. The sum of the degrees of all vertices in C is odd, which contradicts Theorem 2.1.

16 [CZ] Problem 2.10

(a) Let K_a, K_b be two complete graph with degrees a and b, where $a, b \geq 2$ and $a+b=n$. Let u, v be any vertices in K_a and K_b, respectively. Connect the two graphs by edge uv, we get a new connected graph G. For every nonadjacent vertices x, y, they are in two different complete graphs and $\{x, y\} \neq \{u, v\}$, otherwise they are adjacent. Assume that $x \in K_a$ and $y \in K_b$. If $x = u$ or $y = v$, $\deg x + \deg y = n-1$, otherwise $\deg x + \deg y = n-2$.

(b) If the graph has more than one components, remove any one of them. The remaining part of the graph has at most $n-1$ vertices, with $\deg u + \deg v \geq n-2$ still holds. By Theorem 2.4, it is connected. Therefore, G has at most two components.

(c) No. Instead of moving any component, we remove the one with greater order. The remaining part of the graph has at most $\lfloor n/2 \rfloor$ vertices. If $\deg u + \deg v \geq \lfloor n/2 \rfloor - 1$ for all nonadjacent u, v, the remaining part is connected. Thus $\lfloor n/2 \rfloor - 1$ is a shaper bound.

17 [CZ] Problem 2.13

(a) Suppose, to the contrary that G contains more than two components. The component containing minimum number of vertices contains at most $\lfloor n/3 \rfloor$ vertices. The degree of every vertex in this component is at most $\lfloor n/3 \rfloor - 1$, less than $(n-2)/3$, which leads to contradiction.

(b) If $\deg v \geq (n-3)/3 = n/3 - 1$ for every vertex v of G, G might contain more than two components. For example, let n be a multiple of 3, consider the graph $G = 3K_{n/3}$, for every $v \in V(G)$, $\deg v = n/3 - 1$, however, it contains three components. Hence the bound in (a) is sharp.

18 [CZ] Problem 2.15

For every cycle in graph G, let u, v be two distinct vertices in the cycle, and thus the cycle can be divided into two $u-v$ path, whose lengths have the same parity. Therefore the cycle is an even cycle. By Theorem 1.12 G is bipartite.

19 [CZ] Problem 2.20

Suppose, to the contrary that for every adjacent vertices u and v, $\deg u = \deg v$. Since the graph is connected, for every distinct vertices x and y, there exists an $x-y$ path. By transitivity of "=", we get $\deg x = \deg y$, i.e. G is regular, which leads to contradiction.

20 [CZ] Problem 2.25

(a) By Theorem 2.1, $\sum_{u \in V(G)} \deg u$ is even, thus $G - v$ has even order, therefore G has odd order.

(b) Suppose, to the contrary, that there exists some component of odd order, denoted by C. Consider

$$\sum_{v \in V(C)} \deg v = r|V(C)|$$

it is odd, which contradicts Theorem 2.1. Therefore, G does not contain any component of odd order.

21 [CZ] Problem 2.27

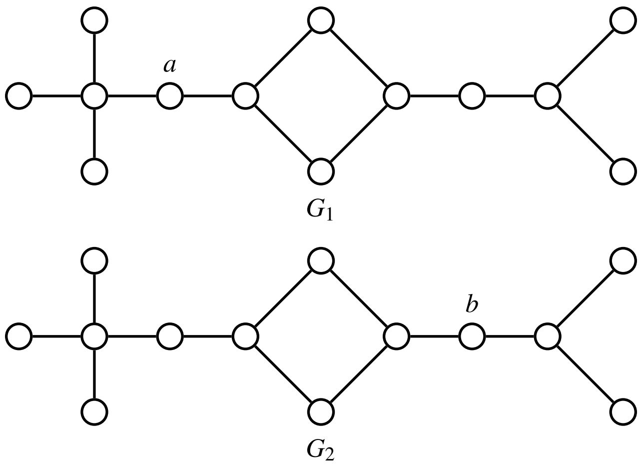
For bipartite graph, we have $\sum_{u \in U} \deg u = \sum_{w \in W} \deg w$, so $r|U| = r|W|$. Eliminating r yields $|U| = |W|$.

22 [CZ] Problem 2.28

Yes. When G itself is an r-regular graph, the graph H in Theorem 2.7 is equal to G, which of course has the smallest order.

23 [CZ] Problem 3.6

No. Consider the two graphs



G_1 and G_2 have the same degree sequence. G_1 contains vertex a of degree 2 that is adjacent to a vertex of 3 and a vertex of 4. G_2 contains vertex b of degree 2 that is adjacent to two vertices of degree 3. However, $G_1 \cong G_2$.

24 [CZ] Problem 3.9

They are not isomorphic. Both G_1 and G_2 contain exactly two vertices of degree 2. In G_1, the two vertices of degree 2 (the leftmost one and the rightmost one) are adjacent to two vertices, however, in G_2, the two vertices of degree 2 (the top left one and the top right one) are adjacent to three vertices. Hence G_1 and G_2 are not isomorphic.

25 [CZ] Problem 3.11

Let $X = \{v \in V(G) : \deg_G v = n/2\}$, $Y = \{v \in V(G) : \deg_G v < n/2\}$. We have

$$U = X \cup Y = X \cup \{v \in V(G) : \deg_G v < n/2\} = X \cup \{v \in V(\overline{G}) : \deg_{\overline{G}} v \geq n/2\}$$

Since G is self-complementary, we have

$$|W| = |\{v \in V(G) : \deg_G v \geq n/2\}| = |\{v \in V(\overline{G}) : \deg_{\overline{G}} v \geq n/2\}|$$

$|U| = |W|$ implies $|X| = 0$, i.e. G contains no vertex v of degree $n/2$.

26 [CZ] Problem 3.13

This statement is true. Note that $d(i,j) = 1$ if and only if i and j are adjacent. Therefore, for every edge $uv \in E(G)$, $d_G(u,v) = 1$, thus $d_H(\phi(u),\phi(v)) = 1$, i.e $\phi(u)\phi(v) \in E(H)$. Likewise, if $\phi(u)\phi(v) \in E(H)$, then $uv \in E(G)$. By the definition of isomorphism, H and G are isomorphic.