

## 论题 2-5 作业

姓名：陈劭源

学号：161240004

### 1 [CS] Problem 5.1-6

a.  $\{(P,P), (P,N), (P,D), (N,P), (N,D), (D,P), (D,N)\}$

Weights:

$(P,P), (P,N), (P,D), (N,P), (D,P)$ :  $1/6$ ;

$(N,D), (D,N)$ :  $1/12$ .

b.  $P(\text{getting 11 cents}) = P(\{(P,D), (D,P)\}) = 1/3$ .

### 2 [CS] Problem 5.1-10

(A standard deck of playing cards refers to a standard 52-card deck, excluding jokers; 10-J-Q-K-A, A-2-3-4-5 are both straights, but hands such as J-Q-K-A-2 are not, as is commonly acknowledged)

Using five-element sets as model. The sample space  $S$  is all the five-element subsets of the 52 cards, and  $|S| = \binom{52}{5}$ .  $P$  is the uniform probability measure defined on  $S$ . The event  $E$  is the set of all the elements of  $S$  which are straights, and we have  $|E| = 4^5 \times 10$ . Therefore, the probability that a five-card hand is a straight is

$$P(E) = \frac{|E|}{|S|} = \frac{10,240}{2,598,960} = \frac{128}{32,487} \approx 0.00394$$

Using five-element permutations as model. The sample space  $S$  is all the five permutations of the 52 cards, and  $|S| = 52^5$ .  $P$  is the uniform probability measure defined on  $S$ . The event  $E$  is the set of all the elements of  $S$  which are straights, and we have  $|E| = 4^5 \times 10 \times 5!$ . Therefore, the probability that a five-card hand is a straight is

$$P(E) = \frac{|E|}{|S|} = \frac{1,228,800}{311,875,200} = \frac{128}{32,487} \approx 0.00394$$

The answers are same by using two different models.

### 3 [CS] Problem 5.1-11

We use  $T$  or  $F$  to represent whether or not he answers a single question correctly. The sample space  $S$  is  $\{T, F\}^{10}$ , and we assume that the weights of all the outcomes are the same, i.e.  $P$  is the uniform probability measure defined on  $S$ . Each problem is worth 10 points.

The event that the student gets a score of 80 or higher is  $E_1$ , and  $|E_1| = 1 + 10 + 10 \times 9/2 = 56$ . So the probability is

$$P(E_1) = \frac{|E_1|}{|S|} = \frac{56}{1024} = \frac{7}{128} \approx 0.0547$$

The event  $E_2$  that the student gets a score of 70 or lower is the complement of  $E_1$ , therefore

$$P(E_2) = 1 - P(E_1) = 1 - \frac{7}{128} = \frac{121}{128} \approx 0.945$$

#### 4 [CS] Problem 5.1-12

Let  $S, C, T$  represent square, circle and triangle respectively. The sample space  $S$  is  $\{S, C, T\}^2$ . The event  $E$ , is  $\{(S, S), (C, C), (T, T)\}$ . The probability that the two shapes on the top are the same is

$$P(E) = P((S, S)) + P((C, C)) + P((T, T)) = \frac{1}{36} + \frac{1}{9} + \frac{1}{4} = \frac{7}{18} \approx 0.389$$

#### 5 [CS] Problem 5.1-13

For Event 1, the sample space  $S_1$  is the set of all two-element subsets of the 13 spades.  $P$  is the uniform probability measure defined on  $S_1$ . The event  $E_1$  is  $\{\{\spadesuit A, \spadesuit K\}\}$ . The probability of Event 1 is

$$P(E_1) = \frac{|E_1|}{|S_1|} = \frac{1}{13 \times 12/2} = \frac{1}{78} \approx 0.0128$$

For Event 2, the sample space  $S_2$  is the set of all two-element subsets of all 52 cards.  $P$  is the uniform probability measure defined on  $S_2$ . The event  $E_1$  is the sets of all two-element sets of an ace and a king. The probability of Event 2 is

$$P(E_2) = \frac{|E_2|}{|S_2|} = \frac{4 \times 4}{52 \times 51/2} = \frac{8}{663} \approx 0.0121$$

Therefore, the two events are not equally likely.

#### 6 [CS] Problem 5.2-2

Let  $E, F$  represent the events that  $\spadesuit K$  or  $\spadesuit Q$  is selected, respectively. The event that the king or queen of spades is among the cards selected is  $E \cup F$ . Therefore, the probability is

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{7}{28} + \frac{7}{28} - \frac{1}{28} = \frac{15}{28} \approx 0.536$$

#### 7 [CS] Problem 5.2-9

The sample space  $S$  is the set of all possible arrangements, and  $|S| = n!$ . Let  $E_i$  denote the event that the  $i$ -th person who gets the letter intended for him or her. Then, by inclusion-exclusion principle, we have

$$\begin{aligned} \left| \bigcup_{i=1}^n E_i \right| &= \sum_{k=1}^n (-1)^{k+1} \sum_{\substack{i_1, i_2, \dots, i_k: \\ 1 \leq i_1 < i_2 < \dots < i_k \leq n}} \left| \bigcap_{j=1}^k E_{i_j} \right| \\ &= \sum_{k=1}^n (-1)^{k+1} \binom{n}{k} (n-k)! \\ &= \sum_{k=1}^n (-1)^{k+1} \frac{n!}{k!} \end{aligned}$$

Hence, the number of ways that nobody gets the correct letter is

$$n! - \left| \bigcup_{i=1}^n E_i \right| = \sum_{k=0}^n (-1)^k \frac{n!}{k!}$$

And the probability  $P(E)$  is

$$P(E) = \sum_{k=0}^n (-1)^k \frac{n!}{k!} / n! = \sum_{k=0}^n (-1)^k \frac{1}{k!}$$

## 8 [CS] Problem 5.2-10

The sample space  $S = \{L_1, L_2, \dots, L_k\}^n$ , and  $|S| = k^n$ . Let  $E_i$  represent the event that there is no key in the  $i$ -th location. For  $i_1 < i_2 < \dots < i_j$ , we have

$$\left| \bigcap_{l=1}^j E_{i_l} \right| = (k-j)^n$$

By inclusion-exclusion principle, we have

$$\left| \bigcup_{i=1}^k E_i \right| = \sum_{j=1}^k (-1)^{j+1} \sum_{\substack{i_1, i_2, \dots, i_j: \\ 1 \leq i_1 < i_2 < \dots < i_j \leq k}} \left| \bigcap_{l=1}^j E_{i_l} \right| = \sum_{j=1}^k (-1)^{j+1} \binom{k}{j} (k-j)^n$$

Hence, the number of cases that every location gets at least one key is

$$k^n - \sum_{j=1}^k (-1)^{j+1} \binom{k}{j} (k-j)^n = \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n$$

And the probability  $P(E)$  is

$$P(E) = \frac{\sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n}{k^n}$$

## 9 [CS] Problem 5.2-14

Let the sample space  $S$  be the set of all possible circular permutations, and  $|S| = (2n-1)!$ . Let  $E_i$  represent the event that the  $i$ -th couple are side-by-side. For  $i_1 < i_2 < \dots < i_k$ , we have

$$\left| \bigcap_{j=1}^k E_{i_j} \right| = (2n-k-1)! 2^k$$

By inclusion-exclusion principle, we have

$$\left| \bigcup_{i=1}^n E_i \right| = \sum_{j=1}^n (-1)^{j+1} \sum_{\substack{i_1, i_2, \dots, i_j: \\ 1 \leq i_1 < i_2 < \dots < i_j \leq n}} \left| \bigcap_{l=1}^j E_{i_l} \right| = \sum_{i=1}^n (-1)^{i+1} \binom{n}{i} (2n-i-1)! 2^i$$

The number of circular permutations that no husband and wife are side-by-side is

$$(2n-1)! - \sum_{i=1}^n (-1)^{i+1} \binom{n}{i} (2n-i-1)! 2^i = \sum_{i=0}^n (-1)^i \binom{n}{i} (2n-i-1)! 2^i$$

The probability  $P(E)$  is

$$P(E) = \frac{\sum_{i=0}^n (-1)^i \binom{n}{i} (2n-i-1)! 2^i}{(2n-1)!}$$

## 10 [CS] Problem 5.2-15

Let  $C$  be the set of all these  $m$  objects. Let  $E_i$  be the set of objects which have the  $i$ -th property. By inclusion-exclusion principle, we have

$$\left| \bigcup_{i=1}^p E_i \right| = \sum_{j=1}^p (-1)^{j+1} \sum_{\substack{i_1, i_2, \dots, i_j: \\ 1 \leq i_1 < i_2 < \dots < i_j \leq p}} \left| \bigcap_{l=1}^j E_{i_l} \right| = \sum_{i=1}^p (-1)^{i+1} \sum_{K: K \subseteq P, |K|=i} N_a(K)$$

And we know that  $C = N_a(\emptyset)$ . Therefore,

$$N_e(\emptyset) = |C| - \left| \bigcup_{i=1}^p E_i \right| = \sum_{i=0}^{|P|} (-1)^i \sum_{K: K \subseteq P, |K|=i} N_a(K) = \sum_{K: K \subseteq P} (-1)^{|K|} N_a(K)$$

Explanation: For each element  $b$  in  $B$ , we define a property  $b$  for the functions. A function  $f$  has the property  $b$  if and only if for every  $a \in A$ ,  $(a, b) \notin f$ .  $N_a(K)$  means the number of the functions from  $A$  to  $B \setminus K$ , i.e.  $N_a(K) = (|B| - |K|)^{|A|}$ .  $N_e(\emptyset)$  means the number of the onto functions, and

$$N_e(\emptyset) = \sum_{K: K \subseteq P} (-1)^{|K|} N_a(K) = \sum_{i=0}^{|B|} (-1)^i \binom{|B|}{i} (|B| - i)^{|A|}$$

Application to Problem 9: Define  $n$  properties for the arrangements. The  $i$ -th property means that the  $i$ -th person gets the correct letter. The sample  $S$  is the set of all possible arrangements, and  $|S| = n!$ . The event  $E$  is that nobody gets the correct letter, and we have

$$|E| = N_e(\emptyset) = \sum_{K: K \subseteq P} (-1)^{|K|} N_a(K) = \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)! = \sum_{i=0}^n (-1)^i \frac{n!}{i!}$$

Therefore, the probability is

$$P(E) = |E|/n! = \sum_{i=0}^n (-1)^i \frac{1}{i!}$$