论题 2-15 作业

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1 [CZ] Problem 1.6

$F = (V, E)$

$V = \{c_1, c_2, \cdots, c_{12}\}$

$E = \{\{c_1, c_5\}, \{c_1, c_6\}, \{c_1, c_8\}, \{c_1, c_9\}, \{c_2, c_7\}, \{c_2, c_{10}\}, \{c_3, c_7\}, \{c_3, c_{10}\}, \{c_4, c_8\}, \{c_4, c_9\}, \{c_4, c_{10}\}, \{c_4, c_{11}\}, \{c_4, c_{12}\}, \{c_5, c_{10}\}, \{c_6, c_{10}\}, \{c_7, c_{11}\}, \{c_7, c_{12}\}\}$

2 [CZ] Problem 1.8

(a) The words in S_1 are (presented from left to right): cat, cap, tap, top.

The words in S_2 are: map (center) , mop, tap, mat (surrounding).

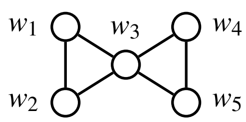
The words in S_3 are: run (top left), gun (top right), sun (center), son (bottom).

The words in S_4 are (presented clockwise): slit, slot, slop, slip.

The words in S_5 are (presented clockwise from top left): pot, put, pet, poet.

The words in S_6 are (presented clockwise): lake, sake, take, make.

(b) The graph H is:



It is a word graph of some set, and the corresponding words are: top, tap, tip, lip, dip.

3 [CZ] Problem 1.10

$F = (V, E)$

$V = \{L1, L2, \cdots, L7\}$

$E = \{\{L1, L3\}, \{L1, L4\}, \{L1, L5\}, \{L1, L6\}, \{L2, L3\}, \{L2, L4\}, \{L4, L5\}, \{L4, L6\}, \{L5, L6\}\}$

4 [CZ] Problem 1.14

Let C denote a component of G.

(1) $\rightarrow$ (2): Take any vertex v_0 in C. For any vertex v_i which is connected to v_0, it must in $V(C)$, otherwise if we add v_i, along with the edges in the path from v_0 to v_i, to C, we get a proper connected supergraph of C,

which leads to contradiction. Therefore, $V(C)$ is an equivalent class. Then we have to prove C is the subgraph induced by $V(C)$. If not, let C' be the subgraph induced by $V(C)$, and $E(C) \subset E(C')$, thus C is a proper subgraph of C', which leads to contradiction.

(2) $\to$ (1): Suppose, to the contrary that C is a proper subgraph of a connected subgraph of G, denoted by C'. If $V(C) \subset V(C')$, there exists some vertex connected to C but not in the equivalent class, which leads to contradiction. It is impossible that $V(C) = V(C')$, because the subgraph induced by $V(C)$ is the maximal subgraph whose vertex set is $V(C)$.

5 [CZ] Problem 1.16

For every i, we have a path from u to v_i: $(u = v_0, v_1, \cdots, v_i)$, whose length is i. Thus $d(u, v_i) \leq i$.

Suppose, to the contrary that $d(u, v_i) < i$, i.e. there exists path $(u_0 = v_0, u_1, \cdots, u_j = v_i)$, where $j < i$. Consider the walk $(u = u_0 = v_0, u_1, \cdots, u_j = v_i, v_{i+1}, \cdots, v_k = v)$, it's a $u - v$ walk shorter than the geodesic, which leads to contradiction.

Therefore, $d(u, v_i) = i$ for each integer i with $1 \leq i \leq k$.

6 [CZ] Problem 1.17

(a) Assume that P is an $x - z$ path and Q is a $u - w$ path, where $x \neq u, v$ and $y \neq u, v$, and they do not have common vertex. Let y be a vertex in P and v be a vertex in Q, then there exists a $y - v$ path $(p_0 = y, p_1, p_2, \cdots, p_n = v)$. If there exists p_i such that p_i $(0 < i < n)$ is in P or Q, since $P \cap Q = \varnothing$, there exists a segment of the path, from any vertex in P (let it be y), to any vertex in Q (let it be v), such that the vertices in the segment are not in P or Q, except the first and the last one. Assume $x - y$ is longer than $y - z$, and $u - v$ is longer than $v - w$, consider the path $x - y - v - u$, it is longer than the $x - z$ path and the $u - v$ path, which leads to contradiction.

(b) This is true. The geodesics are as well the longest paths in G, otherwise $\text{diam}(G) > k$. Apply the conclusion we've proved in (1), we obtain that P and Q must have at least one common vertex.

7 [CZ] Problem 1.18

(a) The minimum size of such a subgraph contains only the vertices and edges in a $u - v$ geodesic. Any connected subgraph containing u and v must have a $u - v$ path, which is at least as long as the geodesic. So a subgraph contains only the vertices and edges in a $u - v$ geodesic has less edges or vertices than other graphs.

(b) What is the maximum size of a connected subgraph of G containing u and v? It is G.