论题 2-3 作业

姓名：陈劲源　　　　学号：161240004

1 [TC] Problem 4.1-5

```
FIND-MAXIMUM-SUBARRAY(A, n)
ans = 0
sum = 0
for i = 1 to n
    sum = sum + A[i]
    if sum > ans
        ans = sum
    elseif sum < 0
        sum = 0
return ans
```

2 [TC] Problem 4.3-3

We have to prove that $T(n) \geq cn\lg n$ for appropriate choice of constant $c > 0$. We use mathematical induction to prove it.

For the base step, we have $T(1) = 1$, $T(2) = 4$ and $T(3) = 5$. We should choose c such that $T(1) = 1 \geq 0$, $T(2) = 4 \geq 2c$, $T(3) = 5 \geq 3c\lg 3$. We can choose every $c < 1$.

For the induction step, assume that $T(m) \geq cm\lg m$ holds for every positive integer $m < n$, where $n \geq 4$. Substituting it into the recurrence, we obtain

$$\begin{aligned}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \\
&\geq 2(c\lfloor n/2 \rfloor \lg\lfloor n/2 \rfloor) + n \\
&\geq 2(c(n/2-1)\lg(n/2-1)) + n \\
&\geq c(n-2)\lg((n-2)/2) + n \\
&= cn\lg(n-2) - 2c\lg(n-2) + cn - 2c + n \\
&= cn\lg n - cn\lg\frac{n}{n-2} - 2c\lg(n-2) + cn - 2c + n \\
&= cn\lg n + (cn - cn\lg\frac{n}{n-2}) + (n/2 - 2c\lg(n-2)) + (n/2 - 2c) \\
&\geq cn\lg n
\end{aligned}$$

the last step holds if we choose $c < \frac{1}{2}$, because

(1) $n \geq 4 \Rightarrow \lg\frac{n}{n-2} \leq 1 \Rightarrow cn \geq cn\lg\frac{n}{n-2}$;

(2) $n \geq 4 \Rightarrow n/2 \geq \lg(n-2)$, $(n/2 \geq \lg(n-2)) \wedge (c < \frac{1}{2}) \Rightarrow n/2 < 2c\lg(n-2)$;

(3) $n \geq 4 \Rightarrow n/2 > 1$, $(n/2 \geq 1) \wedge (c < \frac{1}{2}) \Rightarrow n/2 > 2c$.

By mathematical induction, we conclude that for all positive constant $c < \frac{1}{2}$, $T(n) \geq cn\lg n$ holds. Hence the recurrence is also $\Omega(n\lg n)$. Therefore we conclude that the solution is $\Theta(n\lg n)$.

3 [TC] Problem 4.3-7

Assume $T(m) \leq cm^{\log_3 4}$ for sufficiently large $m < n$, especially for $m = n/3$ where n is large enough. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &= 4T(n/3) + n \\
&\leq 4c(\frac{n}{3})^{\log_3 4} + n \\
&= cn^{\log_3 4} + n \\
&\nleq cn^{\log_3 4} \qquad\qquad\qquad\qquad fails!
\end{aligned}$$

We guess $T(n) \leq c(n^{\log_3 4} - n)$ instead and prove it by mathematical induction.

For the base step, when $n = 3$, we have $T(n) = 7$ and $c(n^{\log_3 4} - n) = c$, therefore $T(n) \leq c(n^{\log_3 4} - n)$ holds for every $c > 7$.

For the induction step, assume $T(m) \leq c(m^{\log_3 4} - m)$ holds for every $m < n$, where $n > 3$. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &= 4T(n/3) + n \\
&\leq 4c((\frac{n}{3})^{\log_3 4} - \frac{n}{3}) + n \\
&= cn^{\log_3 4} + (1 - \frac{4c}{3})n \\
&\leq cn^{\log_3 4}
\end{aligned}$$

By mathematical induction, we conclude $T(n) \leq c(n^{\log_3 4} - n)$. Therefore $T(n) = O(n^{\log_3 4} - n) = O(n^{\log_3 4})$.

To complete the proof of $T(n) = \Omega(n^{\log_3 4})$, we assume $T(n) \geq cn^{\log_3 4}$.

For the base step, when $n = 1$, we have $T(n) = 1$ and $cn^{\log_3 4} = c$, therefore $T(n) \geq cn^{\log_3 4}$ holds for every positive constant $c < 1$.

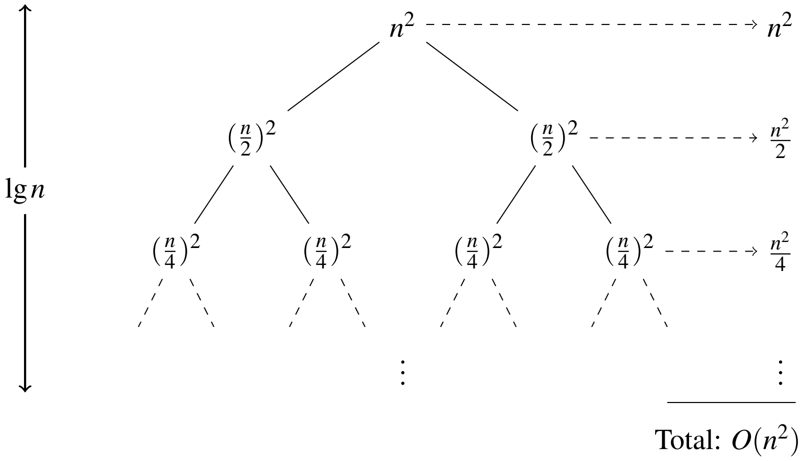
For the induction step, assume $T(m) \geq cm^{\log_3 4}$ holds for every $m < n$, where $n > 3$. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &= 4T(n/3) + n \\
&\geq 4c(\frac{n}{3})^{\log_3 4} + n \\
&= cn^{\log_3 4} + n \\
&\geq cn^{\log_3 4}
\end{aligned}$$

By mathematical induction, we conclude that $T(n) \geq cn^{\log_3 4}$. Hence $T(n) = \Omega(n^{\log_3 4})$. Therefore $T(n) = \Theta(n^{\log_3 4})$.

4 [TC] Problem 4.4-2

Recursion tree:



We guess that $T(n) \leq cn^2$ for an appropriate choice of positive constant c. For the base step, we have $T(1) = 1$ and $cn^2 = c$, so $T(n) \leq cn^2$ holds for $c > 1$ when $n = 1$.

For the induction step, assume that $T(m) \leq cm^2$ holds for all positive integer $m < n$, where $n > 1$. Substituting into the recurrence yields

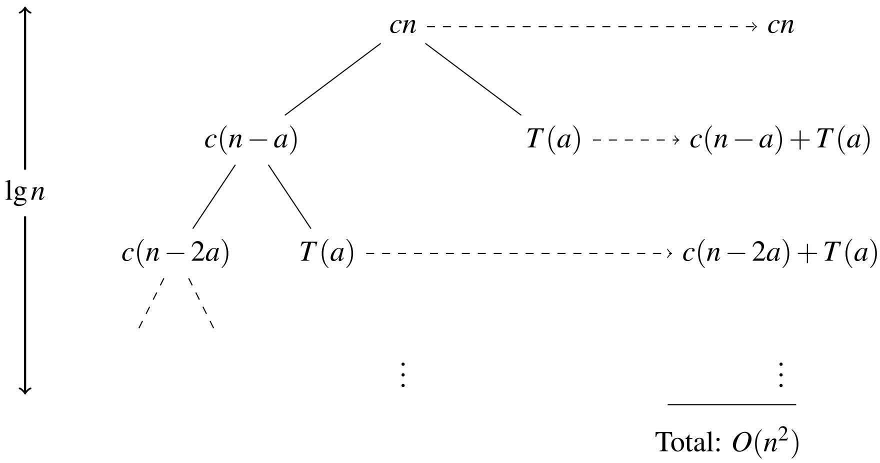
$$\begin{aligned}
T(n) &= T(n/2) + n^2 \\
&\leq c^2\frac{n^2}{4} + n^2 \\
&= (1 + \frac{c^2}{4})n^2 \\
&= cn^2
\end{aligned}$$

where the last step holds as long as $c = 2$.

By mathematical induction, we conclude that $T(n) \leq 2n^2$, i.e. $T(n) = O(n^2)$.

5 [TC] Problem 4.4-8

Recursion tree:



Adding up all levels of the tree, we get

$$
\begin{aligned}
T(n) &= cn + \sum_{i=1}^{n/a} (c(n-ia) + T(a)) \\
&= \frac{n}{a}T(a) + (\frac{n}{a}+1)cn - ca(1+\frac{n}{a})(\frac{n}{a})/2 \\
&= \frac{T(a)}{a}n + cn - cn/2 + \frac{c}{a}n^2 - \frac{c}{2a}n^2 \\
&= \left(\frac{T(a)}{a} + \frac{c}{2}\right)n + \frac{c}{2a}n^2 \\
&= O(n^2)
\end{aligned}
$$

6 [TC] Problem 4-1

a. $T(n) = O(n^4) = \Omega(n^4)$

For this recurrence, we have $a = 2$, $b = 2$, $f(n) = n^4$, and thus we have $n^{\log_a b} = n^{\log_2 2} = n$. Hence $f(n) = \Omega(n^{\log_2 2+\varepsilon})$ where $\varepsilon = 3$. The regularity condition holds for $f(n)$ because $2f(n/2) = 2(n/2)^4 = n^4/8 \leq cn^4$ where $c = 1/8 < 1$. Therefore, the master theorem, case 3 applies and we obtain $T(n) = \Theta(n^4)$.

b. $T(n) = O(n) = \Omega(n)$

For this recurrence, we have $a = 1$, $b = 10/7$, $f(n) = n$, and thus we have $n^{\log_a b} = n^{\log_{10/7} 1} = n^0$. Hence $f(n) = \Omega(n^{\log_{10/7} 1+\varepsilon})$ where $\varepsilon = 1$. The regularity condition holds for $f(n)$ because $f(7n/10) = 7n/10 \leq cn$ where $c = 7/10 < 1$. Therefore, the master theorem, case 3 applies and we obtain $T(n) = \Theta(n)$.

c. $T(n) = O(n^2 \lg n) = \Omega(n^2 \lg n)$

For this recurrence, we have $a = 16$, $b = 4$, $f(n) = n^2$, and thus we have $n^{\log_a b} = n^{\log_4 16} = n^2$. Hence $f(n) = \Theta(n^2)$. Therefore, the master theorem, case 2 applies and we obtain $T(n) = \Theta(n^2 \lg n)$.

d. $T(n) = O(n^2) = \Omega(n^2)$

For this recurrence, we have $a = 7$, $b = 3$, $f(n) = n^2$, and thus we have $n^{\log_a b} = n^{\log_3 7}$. Hence $f(n) = \Omega(n^{\log_3 7+\varepsilon})$ where $\varepsilon = 2 - \log_3 7 > 0$. The regularity condition holds for $f(n)$ because $7f(n/3) = 7(n/3)^2 = 7n/9 = cn$ where $c = 7/9 < 1$. Therefore, the master theorem, case 3 applies and we obtain $T(n) = \Theta(n^2)$.

e. $T(n) = O(n^{\log_2 7}) = \Omega(n^{\log_2 7})$

For this recurrence, we have $a = 7$, $b = 2$, $f(n) = n^2$, and thus we have $n^{\log_a b} = n^{\log_2 7}$. Hence $f(n) = O(n^{\log_2 7-\varepsilon})$ where $\varepsilon = \log_2 7 - 2 > 0$. Therefore, the master theorem, case 1 applies and we obtain $T(n) = \Theta(n^{\log_2 7})$.

f. $T(n) = O(\sqrt{n}\lg n) = \Omega(\sqrt{n}\lg n)$

For this recurrence, we have $a = 2$, $b = 4$, $f(n) = \sqrt{n}$, and thus we have $n^{\log_a b} = n^{\log_4 2} = \sqrt{n}$. Hence $f(n) = \Theta(\sqrt{n})$. Therefore, the master theorem, case 2 applies and we obtain $T(n) = \Theta(\sqrt{(n)}\lg n)$.

g. $T(n) = O(n^3) = \Omega(n^3)$.

We use mathematical induction to prove the upper bounds and the lower bounds.

First, we have to prove $T(n) = O(n^3)$, i.e, $T(n) \le cn^3$ for an appropriate choice of positive constant c.

For the base step, $T(1) \le c$, $T(2) \le 8c$, $T(3) \le 27c$, $T(4) \le 64c$, $T(5) = 125c$, so $T(n) \le cn^3$ holds for all $c > \max\{T(1), T(2)/8, T(3)/27, T(4)/64, T(5)/125\}$ when $n = 1,2,3,4,5$.

For the induction step, assume $T(m) \le cm^3$ holds for all $m < n$ where $n > 5$. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &= T(n-2) + n^2 \\
&\le c(n-2)^3 + n^2 \\
&= cn^3 + (1-6c)n^2 + 24cn - 8c \\
&\le cn^3 + ((1-6c)n + 24c)n \\
&\le cn^3
\end{aligned}$$

the last step holds if we choose $c > 1$ because $n > 5$.

By mathematical induction, we conclude that $T(n) \le cn^3$ holds for all positive integer n when $c > \max\{1, T(1), T(2)/8, T(3)/27, T(4)/64, T(5)/125\}$, i.e. $T(n) = O(n^3)$.

Second, we have to prove that $f(n) \ge cn^3$ for an appropriate choice of positive constant c.

For the base step, $T(1) \ge c$, $T(2) \ge 8c$, so $T(n) \ge cn^3$ holds for all $c < \min\{T(1), T(2)/8\}$ when $n = 1,2$.

For the induction step, assume $T(m) \ge cm^3$ holds for all $m < n$ where $n > 2$. Substituting into the recurrence yields

$$\begin{aligned}
T(n) &= T(n-2) + n^2 \\
&\ge c(n-2)^3 + n^2 \\
&= cn^3 + (1-6c)n^2 + 24cn - 8c \\
&\ge cn^3
\end{aligned}$$

the last step holds if we choose $c < 1/6$.

By mathematical induction, we conclude that $T(n) \ge cn^3$ holds for all positive integer n when $c < \min\{1/6, T(1), T(2)/8\}$, i.e. $T(n) = \Omega(n^3)$.

7 [TC] Problem 4-2

a. **Strategy 1** $T(n) = T(n/2) + \Theta(1)$, $T(n) = O(\lg n)$.

Strategy 2 $T(n) = T(n/2) + \Theta(N)$, $T(n) = O(N \lg n)$.

Strategy 3 $T(n) = T(n/2) + \Theta(n)$, $T(n) = O(n)$.

b. **Strategy 1** $T(n) = 2T(n/2) + \Theta(n)$, $T(n) = O(n \lg n)$.

Strategy 2 $T(n) = 2T(n/2) + \Theta(N)$, $T(n) = O(Nn)$.

Strategy 3 $T(n) = 2T(n/2) + \Theta(n)$, $T(n) = O(n \lg n)$.

8 [TC] Problem 4-4

a.

$$
\begin{aligned}
z + z\mathscr{F}(z) + z^2\mathscr{F}(z) &= z + z\sum_{i=0}^{\infty} F_i z^i + z^2 \sum_{i=0}^{\infty} F_i z^i \\
&= z + \sum_{i=1}^{\infty} F_{i-1} z^i + \sum_{i=2}^{\infty} F_{i-2} z^i \\
&= z + F_0 z^1 + \sum_{i=2}^{\infty} (F_{i-2} + F_{i-1}) z^i \\
&= 0 + z + \sum_{i=2}^{\infty} F_i z^i \\
&= \sum_{i=0}^{\infty} F_i z^i \\
&= \mathscr{F}(z)
\end{aligned}
$$

b.

$$
\begin{aligned}
&\mathscr{F}(z) = z + z\mathscr{F}(z) + z^2\mathscr{F}(z) \\
\Rightarrow &\mathscr{F}(z)(1 - z - z^2) = z \\
\Rightarrow &\mathscr{F}(z) = \frac{z}{1 - z - z^2}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z} - \frac{1}{1-\hat{\phi} z}\right) \\
=&\frac{1}{\sqrt{5}} \frac{\phi z - \hat{\phi} z}{(1-\phi z)(1-\hat{\phi} z)} \\
=&\frac{1}{\sqrt{5}} \frac{\sqrt{5} z}{1 - (\phi + \hat{\phi}) z + (\phi \cdot \hat{\phi}) z^2} \\
=&\frac{z}{1 - z - z^2} = \mathscr{F}(z)
\end{aligned}
$$

c. For every constant k, define $f_k(x) := \frac{1}{1-kx}$, and we have

$$
\begin{aligned}
f_k(x) &= \frac{1}{1-kx} & f_k(0) &= 1 \\
f_k'(x) &= \frac{k}{(1-kx)^2} & f_k'(0) &= k \\
f_k''(x) &= \frac{k^2}{2(1-kx)^2} & f_k'(0) &= \frac{k^2}{2} \\
&\cdots & &\cdots \\
f_k^{(n)}(x) &= \frac{k^n}{n!(1-kx)^n} & f_k^{(n)}(0) &= \frac{k^n}{n!}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathscr{F}(z) &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi z}-\frac{1}{1-\hat{\phi}z}\right) \\
&= \frac{1}{\sqrt{5}}\left(f_{\phi}(z)-f_{\hat{\phi}}(z)\right) \\
&= \frac{1}{\sqrt{5}}\left(\sum_{i=0}^{\infty} i!f_{\phi}^{(i)}(0)z^i+\sum_{i=0}^{\infty} i!f_{\hat{\phi}}^{(i)}(0)z^i\right) \\
&= \frac{1}{\sqrt{5}}\sum_{i=0}^{\infty} i!\left(\frac{\phi^i}{i!}-\frac{\hat{\phi}^i}{i!}\right)z^i \\
&= \sum_{i=0}^{\infty}\frac{1}{\sqrt{5}}\left(\phi^i-\hat{\phi}^i\right)z^i
\end{aligned}
$$

d. Since $|\hat{\phi}|<1$, we have $|\hat{\phi}^i|<1$ and thus $|\hat{\phi}^i|/\sqrt{5}<0.5$ for all positive integer i. Therefore, $\left|F_i-\phi^i/\sqrt{5}\right| = |\hat{\phi}^i|/\sqrt{5}<0.5$, i.e. $F_i=\phi^i/\sqrt{5}$ rounded to the nearest integer.