论题 2-2 作业

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1 [CS] 1.1 Problem 9

Since

$$\binom{n}{2} = \frac{n(n-1)}{2},$$

we have

$$n\binom{n-1}{2} = \frac{n(n-1)(n-2)}{2}$$

and

$$\binom{n}{2}(n-1) = \frac{n(n-1)(n-2)}{2},$$

therefore,

$$n\binom{n-1}{2} = \binom{n}{2}(n-1).$$

Consider choosing one member as the president and two other members as a committee. If we choose the president first, then choose the committee, there are $n\binom{n-1}{2}$ different ways. If we choose the committee first, then choose the president, there are $\binom{n}{2}(n-1)$ different ways. Because the number of ways has nothing to do with the order we choose, we get $n\binom{n-1}{2} = \binom{n}{2}(n-1)$.

2 [CS] 1.1 Problem 13

Let P_i be the set of the pennies I receive on Day i. By supposition, $|P_1| = 1$ and $|P_{i+1}| = 2|P_i|$. Therefore $|P_i| = 2^{i-1}$. For every positive integer n, $P_1, P_2, \cdots, P_n$ are disjoint sets. By the sum principle, the number of pennies I have on Day 20 is

$$\left|\bigcup_{i=1}^{20} P_i\right| = 1 + 2 + \cdots + 2^{20} = 2^{21} - 1 = 2\,097\,151,$$

and the number of pennies I have on day n is

$$\left|\bigcup_{i=1}^{n} P_i\right| = 1 + 2 + \cdots + 2^{n} = 2^{n+1} - 1.$$

3 [CS] 1.2 Problem 15

First, numbering the members from 1 to $2n$. We can use a partition P of $\{1,2,\cdots,2n\}$ to represent how we pair up the members, where $|p|=2$ for every $p\in P$. Then we define a 'sorted partition' $[(a_1,b_1),(a_2,b_2),\cdots,(a_n,b_n)]$ of P, such that (1) $P=\{\{a_1,b_1\},\{a_2,b_2\},\cdots,\{a_n,b_n\}\}$, (2) $a_1<b_1,a_2<b_2,\cdots,a_n<b_n$ and (3) $a_1<a_2<\cdots<a_n$ hold.

We can prove that $a_i=\min\{1,2,\cdots,2n\}\setminus\{a_1,b_1,\cdots,a_{i-1},b_{i-1}\}$, because by (2) we have $a_i<a_{i+1}<\cdots<a_n$ and by (3) we have $b_{i+1}>a_{i+1}>a_i$, $b_{i+2}>a_{i+2}>a_i$, $\cdots$. However, b_i is an arbitrary element in $\{1,2,\cdots,2n\}\setminus\{a_1,b_1,\cdots,a_{i-1},b_{i-1},a_i\}$. Hence, by product principle, the number of the ways is

$$(2n-1)(2n-3)\cdots 1=(2n-1)!!.$$

If we have to determine who plays whom, just multiplying 2^n, because for each pair, we have two ways to determine who plays whom. Therefore, we can specify our pairs in $(2n-1)!!2^n$ ways.

4 [CS] 1.3 Problem 6

The coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$ in the expansion of $(x_1+x_2+\cdots+x_k)^n$ is $\binom{n}{n_1,n_2,\cdots,n_k}$.

Explanation: consider $x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$ in the expansion of $(x_1+x_2+\cdots+x_k)^n$. We have $\binom{n}{n_1}$ ways to choose n_1 x_1's, and then $\binom{n-n_1}{n_2}$ ways to choose n_2 x_2's, and so on. By the product principle, the coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$ is

$$\begin{aligned}&\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k}\\ =&\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!(n-n_1-\cdots-n_k)!}\\ =&\frac{n!}{n_1!n_2!\cdots n_k!}\end{aligned}$$

5 [CS] 1.3 Problem 9

$$(x+y)^n=\sum_{i=0}^{n}\binom{n}{i}x^iy^{n-i}$$

6 [CS] 1.3 Problem 14

Method 1: Since

$$\binom{n}{k}\binom{k}{j}=\frac{n!}{k!(n-k)!}\frac{k!}{j!(k-j)!}=\frac{n!}{j!(k-j)!(n-k)!}$$

and

$$\binom{n}{j}\binom{n-j}{k-j}=\frac{n!}{j!(n-j)!}\frac{(n-j)!}{(k-j)!(n-k)!}=\frac{n!}{j!(k-j)!(n-k)!},$$

we get

$$\binom{n}{k}\binom{n}{k} = \binom{n}{j}\binom{n-j}{k-j}.$$

Method 2: Consider choosing a k-element subset from an n-element set, then choosing a j-element subsubset from the k-element subset. By the product principle, there are $\binom{n}{k}\binom{k}{j}$ different ways.

If we choose the k-element subsubset from the n-element set first, then choose the $k-j$ elements that are in the subset but not in the subsubset. By the product principle, there are $\binom{n}{j}\binom{n-j}{k-j}$ different ways.

Therefore, $\binom{n}{k}\binom{n}{k} = \binom{n}{j}\binom{n-j}{k-j}$.

7 [CS] 1.5 Problem 8

a. Assume the checkers are different from each other, then we can place the checkers in $(k+n-1!)$ ways. We say two arrangements is equivalent, if and only if the checkers in the same place have the same color. Red checkers can be arbitrarily rearranged, and so are the black ones. Hence, the size of each equivalent class is $k!(n-1)!$. By quotient principle, we can place the checkers in a row in $\frac{(k-n-1)!}{k!(n-1)!}$ ways.

b. The $n-1$ black checkers divide the k red checkers into n groups. We define a function f from an arrangement of red and black checkers to a k-element multiset of $(\{1,2,\cdots,n\})$, that is, for any arrangement of checkers A, $f(A)$ is a k-element multiset of $\{1,2,\cdots,n\}$, in which, for every positive integer $i \leq n$, the number of element i exactly equals to the number of the red checkers in the i-th group. It can be verified that f is a bijection. Therefore, the number of ways of placing k red checkers and $n-1$ black checkers in a row equals to the number of k-element multisets of an n-element set.

c. Think about that we have $n+k-1$ items(checkers). Now, we have to color k of the checkers red, and color the rest of the checkers black. That is, we have to choose k checkers of $n+k-1$ to be colored red. This is the same as placing k red checkers and $n-1$ black checkers in a row. Therefore, the relation of the choice of k items out of $n-k+1$ items to the placement of red and black checkers is one-to-one correspondence.

8 [CS] 1.5 Problem 10

Substituting $x_1'+1, x_2'+1, \cdots, x_n'+1$ for $x_1, x_2, \cdots, x_n$ respectively, we obtain $x_1'+x_2'+\cdots+x_n' = k-n$, where $x_1', x_2', \cdots, x_n'$ are non-negative integers. If $k-n<0$, this equation has no solution. If $k-n>0$, every solution could be regarded as a $(k-n)$-element multiset chosen from an n-element set. Hence, the number of the solutions is $\binom{k-1}{k-n}$.

9 [CS] 1.5 Problem 15

a. n^k

b. $n^{\underline{k}}$

c. $\binom{n+k-1}{k}$

d. $\binom{n}{k}$

e. $n^{\underline{k}}$

f. n^k

g. $\binom{n}{k}$

h. $\binom{n+k-1}{k}$

i. $n^{\underline{k}}$

j. $\binom{n+k-1}{k}$

k. $n^{\underline{k}}$