# 论题 2-5 作业

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## 1 [CS] Problem 5.1-6

**a.** $\{(P,P),(P,N),(P,D),(N,P),(N,D),(D,P),(D,N)\}$
Weights:
$(P,P),(P,N),(P,D),(N,P),(D,P)$: 1/6;
$(N,D),(D,N)$: 1/12.

**b.** $P(\text{getting 11 cents}) = P(\{(P,D),(D,P)\}) = 1/3$.

## 2 [CS] Problem 5.1-10

(A standard deck of playing cards refers to a standard 52-card deck, excluding jokers; 10-J-Q-K-A, A-2-3-4-5 are both straights, but hands such as J-Q-K-A-2 are not, as is commonly acknowledged)

Using five-element sets as model. The sample space $S$ is all the five-element subsets of the 52 cards, and $|S| = \binom{52}{5}$. $P$ is the uniform probability measure defined on $S$. The event $E$ is the set of all the elements of $S$ which are straights, and we have $|E| = 4^5 \times 10$. Therefore, the probability that a five-card hand is a straight is

$$P(E) = \frac{|E|}{|S|} = \frac{10,240}{2,598,960} = \frac{128}{32,487} \approx 0.00394$$

Using five-element permutations as model. The sample space $S$ is all the five permutations of the 52 cards, and $|S| = 52^{\underline{5}}$. $P$ is the uniform probability measure defined on $S$. The event $E$ is the set of all the elements of $S$ which are straights, and we have $|E| = 4^5 \times 10 \times 5!$. Therefore, the probability that a five-card hand is a straight is

$$P(E) = \frac{|E|}{|S|} = \frac{1,228,800}{311,875,200} = \frac{128}{32,487} \approx 0.00394$$

The answers are same by using two different models.

## 3 [CS] Problem 5.1-11

We use $T$ or $F$ to represent whether or not he answers a single question correctly. The sample space $S$ is $\{T,F\}^{10}$, and we assume that the weights of all the outcomes are the same, i.e. $P$ is the uniform probability measure defined on $S$. Each problem is worth 10 points.

The event that the student gets a score of 80 or higher is $E_1$, and $|E_1| = 1 + 10 + 10 \times 9/2 = 56$. So the probability is

$$P(E_1) = \frac{|E_1|}{|S|} = \frac{56}{1024} = \frac{7}{128} \approx 0.0547$$

The event $E_2$ that the student gets a score of 70 or lower is the complement of $E_1$, therefore

$$P(E_2) = 1 - P(E_1) = 1 - \frac{7}{128} = \frac{121}{128} \approx 0.945$$

## 4 [CS] Problem 5.1-12

Let $S, C, T$ represent square, circle and triangle respectively. The sample space $S$ is $\{S, C, T\}^2$. The event $E$, is $\{SS, CC, TT\}$ ($SS$ stands for $(S, S)$, the same below). The probability that the two shapes on the top are the same is

$$P(E) = P(SS) + P(CC) + P(TT) = \frac{1}{36} + \frac{1}{9} + \frac{1}{4} = \frac{7}{18} \approx 0.389$$

## 5 [CS] Problem 5.1-13

For Event 1, the sample space $S_1$ is the set of all two-element subsets of the 13 spades. $P$ is the uniform probability measure defined on $S_1$. The event $E_1$ is $\{\{\spadesuit A, \spadesuit K\}\}$. The probability of Event 1 is

$$P(E_1) = \frac{|E_1|}{|S_1|} = \frac{1}{13 \times 12/2} = \frac{1}{78} \approx 0.0128$$

For Event 2, the sample space $S_2$ is the set of all two-element subsets of all 52 cards. $P$ is the uniform probability measure defined on $S_2$. The event $E_1$ is the sets of all two-element sets of an ace and a king. The probability of Event 2 is

$$P(E_2) = \frac{|E_2|}{|S_2|} = \frac{4 \times 4}{52 \times 51/2} = \frac{8}{663} \approx 0.0121$$

Therefore, the two events are not equally likely.

## 6 [CS] Problem 5.2-2

Let $E, F$ represent the events that $\spadesuit K$ or $\spadesuit Q$ is selected, respectively. The event that the king or queen of spades is among the cards selected is $E \cup F$. Therefore, the probability is

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{7}{28} + \frac{7}{28} - \frac{1}{28} = \frac{15}{28} \approx 0.536$$

## 7 [CS] Problem 5.2-9

The sample space $S$ is the set of all possible arrangements, and $|S| = n!$. Let $E_i$ denote the event that the $i$-th person who gets the letter intended for him or her. Then, by inclusion-exclusion principle, we have

$$\left| \bigcup_{i=1}^{n} E_i \right| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_1, i_2, \cdots i_k: \\ 1 \le i_1 < i_2 < \cdots < i_k \le n}} \left| \bigcap_{j=1}^{k} E_{i_j} \right|$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)!$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!}$$

Hence, the number of ways that nobody gets the correct letter is

$$n! - \left| \bigcup_{i=1}^{n} E_i \right| = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

And the probability $P(E)$ is

$$P(E) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} / n! = \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

# 8   [CS] Problem 5.2-10

The sample space $S = \{L_1, L_2, \cdots, L_k\}^n$, and $|S| = k^n$. Let $E_i$ represent the event that there is no key in the $i$-th location. For $i_1 < i_2 < \cdots < i_j$, we have

$$\left| \bigcap_{l=1}^{j} E_{i_l} \right| = (k-j)^n$$

By inclusion-exclusion principle, we have

$$\left| \bigcup_{i=1}^{k} E_i \right| = \sum_{j=1}^{k} (-1)^{j+1} \sum_{\substack{i_1,i_2,\cdots,i_j: \\ 1 \le i_1 < i_2 < \cdots < i_j \le k}} \left| \bigcap_{l=1}^{j} E_{i_l} \right| = \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} (k-j)^n$$

Hence, the number of cases that every location gets at least one key is

$$k^n - \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} (k-j)^n = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

And the probability $P(E)$ is

$$P(E) = \frac{\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n}{k^n}$$

# 9   [CS] Problem 5.2-14

Let the sample space $S$ be the set of all possible circular permutations, and $|S| = (2n-1)!$. Let $E_i$ represent the event that the $i$-th couple are side-by-side. For $i_1 < i_2 < \cdots < i_k$, we have

$$\left| \bigcap_{j=1}^{k} E_{i_j} \right| = (2n-k-1)!2^k$$

By inclusion-exclusion principle, we have

$$\left| \bigcup_{i=1}^{n} E_i \right| = \sum_{j=1}^{n} (-1)^{j+1} \sum_{\substack{i_1,i_2,\cdots,i_j: \\ 1 \le i_1 < i_2 < \cdots < i_j \le n}} \left| \bigcap_{l=1}^{j} E_{i_l} \right| = \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} (2n-i-1)!2^i$$

The number of circular permutations that no husband and wife are side-by-side is

$$(2n-1)! - \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} (2n-i-1)!2^i = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (2n-i-1)!2^i$$

The probability $P(E)$ is

$$P(E) = \frac{\sum_{i=0}^{n} (-1)^i \binom{n}{i} (2n-i-1)!2^i}{(2n-1)!}$$

# 10    [CS] Problem 5.2-15

Let $C$ be the set of all these $m$ objects. Let $E_i$ be the set of objects which have the $i$-th property. By inclusion-exclusion principle, we have

$$\left|\bigcup_{i=1}^{p} E_i\right| = \sum_{j=1}^{p}(-1)^{j+1}\sum_{\substack{i_1,i_2,\cdots,i_j:\\1\le i_1<i_2<\cdots<i_j\le p}}\left|\bigcap_{l=1}^{j} E_{i_l}\right| = \sum_{i=1}^{p}(-1)^{i+1}\sum_{K:K\subseteq P,|K|=i} N_a(K)$$

And we know that $C = N_a(\varnothing)$. Therefore,

$$N_e(\varnothing) = |C| - \left|\bigcup_{i=1}^{p} E_i\right| = \sum_{i=0}^{|P|}(-1)^i\sum_{K:K\subseteq P,|K|=i} N_a(K) = \sum_{K:K\subseteq P}(-1)^{|K|}N_a(K)$$

Explanation: For each element $b$ in $B$, we define a property $b$ for the functions. A function $f$ has the property $b$ if and only if for every $a \in A$, $(a,b) \notin f$. $N_a(K)$ means the number of the functions from $A$ to $B \setminus K$, i.e. $N_a(K) = (|B| - |K|)^{|A|}$. $N_e(\varnothing)$ means the number of the onto functions, and

$$N_e(\varnothing) = \sum_{K:K\subseteq P}(-1)^{|K|}N_a(K) = \sum_{i=0}^{|B|}(-1)^i\binom{|B|}{i}(|B|-i)^{|A|}$$

Application to Problem 9: Define $n$ properties for the arrangements. The $i$-th property means that the $i$-th person gets the correct letter. The sample $S$ is the set of all possible arrangements, and $|S| = n!$. The event $E$ is that nobody gets the correct letter, and we have

$$|E| = N_e(\varnothing) = \sum_{K:K\in P}(-1)^{|K|}N_a(K) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)! = \sum_{i=0}^{n}(-1)^i\frac{n!}{i!}$$

Therefore, the probability is

$$P(E) = |E|/n! = \sum_{i=0}^{n}(-1)^i\frac{1}{i!}$$

# 11    [CS] Problem 5.3-3

Let the sample space $S = \{T,F\}^3$, and $|S| = 8$. $P$ is the uniform probability measure defined on $S$. Let $E_1$ denote the event of at most one tail, i.e. $E_1 = \{FFF, TFF, FTF, FFT\}$, $E_2$ denote the event that all not all flips are identical, i.e. $E_2 = \{TFF, FTF, FFT, TTF, TFT, FTT\}$. Calculate the probability of these events:

$$P(E_1) = \frac{|E_1|}{|S|} = \frac{1}{2} = 0.5$$

$$P(E_2) = \frac{|E_2|}{|S|} = \frac{3}{4} = 0.75$$

$$P(E_1 \cap E_2) = \frac{|E_1 \cap E_2|}{|S|} = \frac{3}{8} = 0.375$$

Since $P(E_1 \cap E_2) = P(E_1)P(E_2)$, by product principle for independent probabilities, the two events are independent.

# 12 [CS] Problem 5.3-4

The sample space $S = \{1,2,3,4,5,6\}^2$, and $P$ is the uniform probability measure defined on $S$.

Let $E_{1,i}$ denote the event that "$i$ dots are on top of the first die", i.e. $E_{1,i} = \{i1, i2, i3, i4, i5, i6\}$, and $E_{2,j}$ denote the event that "$j$ dots are on top of the second die" likewise. Therefore

$$P(E_{1,i}) = \frac{|E_{1,i}|}{|S|} = \frac{1}{6}$$

$$P(E_{2,j}) = \frac{|E_{2,j}|}{|S|} = \frac{1}{6}$$

$$P(E_{1,i} \cap E_{2,j}) = \frac{|E_{1,i} \cap E_{2,j}|}{|S|} = \frac{|\{ij\}|}{|S|} = \frac{1}{36}$$

Since, $P(E_{1,i})P(E_{2,j}) = P(E_{1,i} \cap E_{2,j})$, the two events are independent.

# 13 [CS] Problem 5.3-8

Let the sample space $S$ be the set of all 13-element subset of the 52 cards. Let $E_1$ denote the event that a bridge hand has four aces, $E_2$ denote the event that it has at least one ace, $E_3$ denote the event that it has the ace of spades. $P$ is the uniform probability measure defined on $S$. Calculate the probabilities:

$$P(E_1) = \frac{|E_1|}{|S|} = \binom{48}{9} / \binom{52}{13} = \frac{11}{4165} \approx 0.00264$$

$$P(E_2) = 1 - \frac{|S \setminus E_2|}{|S|} = 1 - \binom{48}{13} / \binom{52}{13} = \frac{14,498}{20,825} \approx 0.696$$

$$P(E_3) = 1 - \frac{|S \setminus E_3|}{|S|} = 1 - \binom{51}{13} / \binom{52}{13} = \frac{1}{4} = 0.25$$

Therefore

$$P(E_1|E_2) = \frac{|E_1 \cap E_2|}{|E_2|} = \frac{|E_1|}{|E_2|} = \frac{5}{1316} \approx 0.00379$$

$$P(E_1|E_3) = \frac{|E_1 \cap E_3|}{|E_3|} = \frac{|E_1|}{|E_3|} = \frac{44}{4165} \approx 0.01056$$

The latter one is larger.

# 14 [CS] Problem 5.3-11

$E$ and $F$ are independent if and only if $P(E \cap F) = P(E)P(F)$. Since $E \cap F = \varnothing$, $E$ and $F$ are independent if and only if $P(E) = 0$ or $P(F) = 0$.

# 15 [CS] Problem 5.3-12

Let $B$ denote a boy, $G$ denote a girl, then the sample space $S = \{BB, BG, GB, GG\}$. Assume every outcome has the same weight, i.e. $P$ is the uniform probability measure.

Let $E_1$ be the event that the family has two girls, $E_2$ be the event that one of the children is a girl. We have $P(E_1) = 0.25$, $P(E_2) = 0.75$, therefore

$$P(E_1|E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(E_1)}{P(E_2)} = \frac{1}{3}$$

Let $E_3$ be the event that the children are both boys, $E_4$ be the event that the older child is a boy. We have $P(E_3) = 0.25$, $P(E_4) = 0.5$, therefore

$$P(E_3|E_4) = \frac{P(E_3 \cap E_4)}{P(E_4)} = \frac{P(E_3)}{P(E_4)} = 0.5$$

## 16 [CS] Problem 5.3-13

Let $A$, $B$, $C$ denote the three curtains. The first step of this process is to determine a curtain behind which is a new car randomly. The second step is that you pick a curtain randomly. The third step is that the emcee rules out a curtain that you didn't pick randomly. We use a tree diagram to illustrate the whole process.
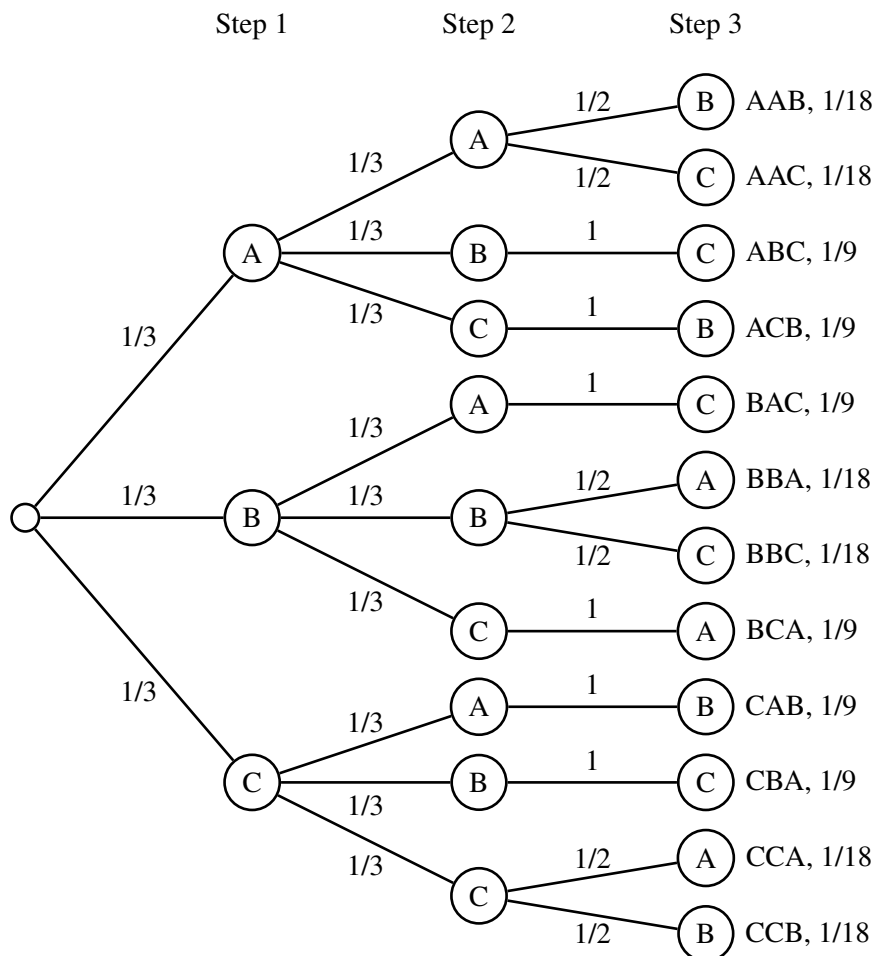


The sample space $S = \{AAB, AAC, ABC, ACB, BAC, BBA, BBC, BCA, CAB, CBA, CCA, CCB\}$. The weight of each outcome is shown in the diagram. Let $E_1$ be the event that the car is behind the curtain you first chose, i.e. $E_1 = \{AAB, AAC, BBA, BBC, CCA, CCB\}$, $E_2$ be the event that the car is behind the curtain you switch. $E_2$ is the complement of $E_1$. Calculate the probabilities:

$$P(E_1) = 6 \times \frac{1}{18} = \frac{1}{3}, \quad P(E_2) = 1 - P(E_2) = \frac{2}{3}$$

So you'd better switch.

## 17 [CS] Problem 5.4-5

Let $X$ be the number of dots on the top, i.e. the amount of money one wins. The values of $X$ are $\{1,2,3,4,5,6\}$, and for each possible value of $X$, $P(X = x_i) = 1/6$. The expected value of $X$ is

$$E(X) = \sum_{i=1}^{6} i/6 = 7/2 = 3.5$$

Therefore, the minimum amount of money a rational person might be willing to pay is $3.5.

## 18 [CS] Problem 5.4-6

Let $X_i$ be random variables, where $X_i$ denotes the number of dots on the top of the $i$-th dice, and we have $E(X_i) = 3.5$. Therefore, the expected sum of the tops of $n$ dice is

$$E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = 3.5n$$

## 19 [CS] Problem 5.4-8

Let $E_i$ denote the event of having a king on the $i$-th draw. For every $i$, we have $P(E_i) = 4/52 = 1/13$.

If $i = j$, $P(E_i \cap E_j) = 1/13)$. Since $P(E_i \cap E_j) \neq P(E_i)P(E_j)$, $E_i$ and $E_j$ are not independent.

If $i \neq j$, $P(E_i \cap E_j) = (4/52) \times (3/51) = 1/221$. Since $P(E_i \cap E_j) \neq P(E_i)P(E_j)$, $E_i$ and $E_j$ are not independent.

Therefore, whatever $i$ and $j$ are, these two events are not independent.

Let $X$ be the number of kings you have, the values of $X$ is $\{0,1,2,3,4\}$. The distribution function of $X$ is

$$P(X = k) = \frac{\binom{4}{k}\binom{48}{26-k}}{\binom{52}{26}}$$

The expectation of $X$ is

$$\begin{aligned}
E(X) &= \sum_{i=0}^{4} iP(X = i) \\
&= \sum_{i=0}^{4} i\frac{\binom{4}{i}\binom{48}{26-i}}{\binom{52}{26}} \\
&= 0 \times \frac{46}{833} + 1 \times \frac{208}{833} + 2 \times \frac{325}{833} + 3 \times \frac{208}{83} + 4 \times \frac{46}{833} \\
&= 2
\end{aligned}$$

Therefore, 2 kings are expected to see.

## 20 [CS] Problem 5.4-10

Let $S$ be the sample space.

$$E(c) = \sum_{s \in S} cP(s) = c \sum_{s \in S} P(S) = c$$

# 21 [CS] Problem 5.4-17

By the binomial theorem,

$$(.1+x)^{10} = \sum_{i=0}^{10} \binom{10}{i}(.1)^{10-i}x^i$$

Differentiate with respect to $x$ on both side and we have

$$10(.1+x)^9 = \sum_{i=0}^{10} i\binom{10}{i}(.1)^{10-i}x^{i-1}$$

Substituting $x = 0.9$ and multiplying 0.9 on both sides, we get

$$10(.1+.9) \times .9 = \sum_{i=0}^{10} i\binom{10}{i}(.9)^i(.1)^{10-i}$$

Therefore

$$\sum_{i=0}^{10} i\binom{10}{i}(.9)^i(.1)^{10-i} = 9$$

# 22 [CS] Problem 5.4-20

By the sum of a geometric series, we have

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

Differentiate termwise with respect to $x$ and we have

$$\sum_{j=0}^{\infty} jx^{j-1} = \frac{1}{(1-x)^2}$$

Multiplying $x$ on both sides, we get

$$\sum_{j=0}^{\infty} jx^j = \frac{x}{(1-x)^2}$$

# 23 [CS] Problem 5.4-21

Assume that $F^iS$ follows the geometric distribution with parameter $p$, i.e. $P(F^iS) = (1-p)^i p$. Let the random variable $X = (1-p)^{-i}$, and the expectation of $X$ is

$$E(X) = \sum_{i=0}^{\infty} x_i P(X = x_i)$$

$$= \sum_{i=0}^{\infty} (1-p)^{-i}(1-p)^i p$$

$$= \sum_{i=0}^{\infty} p$$

$$= \infty$$