论题 2-15 作业

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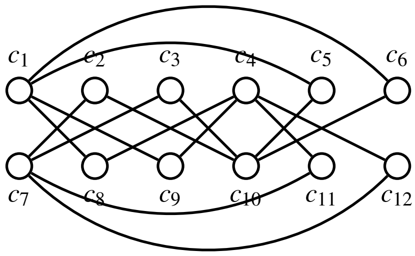
1 [CZ] Problem 1.6



2 [CZ] Problem 1.8

(a) The words in S_1 are (presented from left to right): cat, cap, tap, top.

The words in S_2 are: map (center) , mop, tap, mat (surrounding).

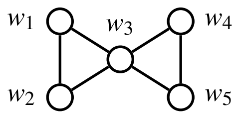
The words in S_3 are: run (top left), gun (top right), sun (center), son (bottom).

The words in S_4 are (presented clockwise): slit, slot, slop, slip.

The words in S_5 are (presented clockwise from top left): pot, put, pet, poet.

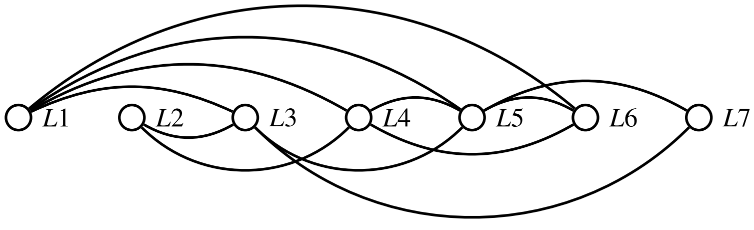
The words in S_6 are (presented clockwise): lake, sake, take, make.

(b) The graph H is:



It is a word graph of some set, and the corresponding words are: top, tap, tip, lip, dip.

3 [CZ] Problem 1.10



4 [CZ] Problem 1.14

Let C denote a component of G.

(1) $\to$ (2): Take any vertex v_0 in C. For any vertex v_i which is connected to v_0, it must in $V(C)$, otherwise if we add v_i, along with the edges in the path from v_0 to v_i, to C, we get a proper connected supergraph of C, which leads to contradiction. Therefore, $V(C)$ is an equivalent class. Then we have to prove C is the subgraph induced by $V(C)$. If not, let C' be the subgraph induced by $V(C)$, and $E(C) \subset E(C')$, thus C is a proper subgraph of C', which leads to contradiction.

(2) $\to$ (1): Suppose, to the contrary that C is a proper subgraph of a connected subgraph of G, denoted by C'. If $V(C) \subset V(C')$, there exists some vertex connected to C but not in the equivalent class, which leads to contradiction. It is impossible that $V(C) = V(C')$, because the subgraph induced by $V(C)$ is the maximal subgraph whose vertex set is $V(C)$.

5 [CZ] Problem 1.16

For every i, we have a path from u to v_i: $(u = v_0, v_1, \cdots, v_i)$, whose length is i. Thus $d(u, v_i) \leq i$.

Suppose, to the contrary that $d(u, v_i) < i$, i.e. there exists path $(u_0 = v_0, u_1, \cdots, u_j = v_i)$, where $j < i$. Consider the walk $(u = u_0 = v_0, u_1, \cdots, u_j = v_i, v_{i+1}, \cdots, v_k = v)$, it's a $u - v$ walk shorter than the geodesic, which leads to contradiction.

Therefore, $d(u, v_i) = i$ for each integer i with $1 \leq i \leq k$.

6 [CZ] Problem 1.17

(a) Assume that P is an $x - z$ path and Q is a $u - w$ path, where $x \neq u, v$ and $y \neq u, v$, and they do not have common vertex. Let y be a vertex in P and v be a vertex in Q, then there exists a $y - v$ path $(p_0 = y, p_1, p_2, \cdots, p_n = v)$. If there exists p_i such that p_i $(0 < i < n)$ is in P or Q, since $P \cap Q = \varnothing$, there exists a segment of the path, from any vertex in P (let it be y), to any vertex in Q (let it be v), such that the vertices in the segment are not in P or Q, except the first and the last one. Assume $x - y$ is longer than $y - z$, and $u - v$ is longer than $v - w$, consider the path $x - y - v - u$, it is longer than the $x - z$ path and the $u - v$ path, which leads to contradiction.

(b) This is true. The geodesics are as well the longest paths in G, otherwise $\text{diam}(G) > k$. Apply the conclusion we've proved in (1), we obtain that P and Q must have at least one common vertex.

7 [CZ] Problem 1.18

(a) The minimum size of such a subgraph contains only the vertices and edges in a $u - v$ geodesic. Any connected subgraph containing u and v must have a $u - v$ path, which is at least as long as the geodesic. So a subgraph contains only the vertices and edges in a $u - v$ geodesic has less edges or vertices than other graphs.

(b) What is the maximum size of a connected subgraph of G containing u and v? It is G.

8 [CZ] Problem 1.22

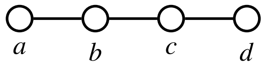
If u, v are in two different components of G, then $uv \in E(\overline{G})$, i.e. $d_{\overline{G}}(u,v) = 1$.

If u, v are in the same component of G, let w be a vertex in another component G. We have $uw, wv \in E(\overline{G})$, i.e. (u,w,v) is a path from u to v, and thus $d_{\overline{G}}(u,v) \leq 2$.

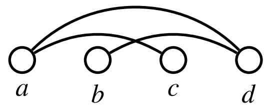
Therefore, $d_{\overline{G}}(u,v) = 1$ or $d_{\overline{G}}(u,v) = 2$.

9 [CZ] Problem 1.23

(a) For $k = 1$, the graph $G = (V,E)$ is:



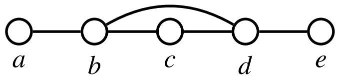
G



$\overline{G}$

and $d_G(b,c) = 1 = d_{\overline{G}}(a,d)$.

For $k = 2$, the graph $G = (V,E)$ is



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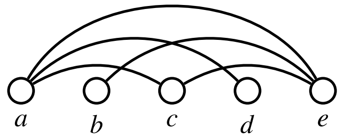
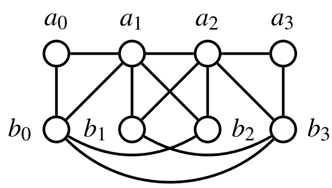


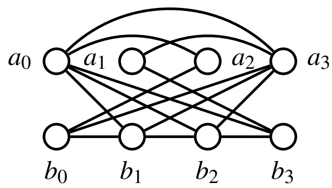
$\overline{G}$

and $d_G(a,d) = 2 = d_{\overline{G}}(b,c)$.

(b) The largest value of k is 3. Here is an example when $k = 3$, $d_G(a_0,a_3) = 3 = d_{\overline{G}}(b_0,b_3)$:



G



$\overline{G}$

Now we are going to prove that it is impossible that $k \geq 4$. Suppose, there exists distinct u, v, x, y, such that $d_G(u,v) = d_{\overline{G}}(x,y) = k \geq 4$. Let $(u = w_0, w_1, \cdots, w_k = v)$ be a $u-v$ geodesic, $(x = z_0, z_1, \cdots, z_k = y)$ be an $x-y$ geodesic. We claim that there exists $i \in \{0,k\}$, such that $z_0 w_i$ is not an edge in G, otherwise, (w_0, z_0, w_k) is a $u-v$ path shorter than the geodesic in G. Likewise there exists $j \in \{0,k\}$, such that $z_k w_j$ is not an edge in G. Hence, $z_0 w_i$ and $z_k w_j$ are edges in $\overline{G}$. If $i = j$, (z_0, w_i, z_k) is shorter than $x-y$ geodesic, which is impossible. If $i \neq j$, $w_i w_j = w_0 w_k = uv$ is an edge in $\overline{G}$, otherwise $d_G(u,v) = 1$, contradicting the provided condition. Note that $(x = z_0, w_i, w_j, z_k = y)$ is an $x-y$ path shorter than the geodesic, which is impossible.

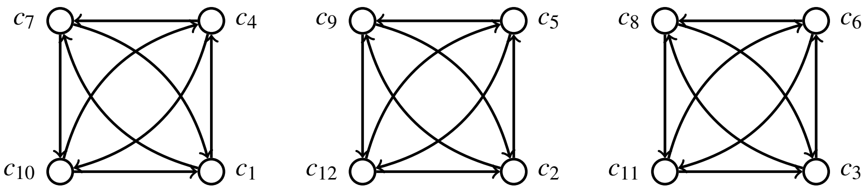
Therefore, $k \leq 3$.

10 [CZ] Problem 1.25

Assume that G is bipartite, with partite sets U and W. Since $|V(G)| = |U| + |W| \geq 5$, at least one of $|U|$ and $|G|$ is greater than 2. Assume, without loss of generality, that $|U| \geq 3$. Let u, v, w be three distinct vertices in U. They are mutually adjacent in $\overline{G}$, which forms an odd cycle. By Theorem 1.12, $\overline{G}$ is not bipartite.

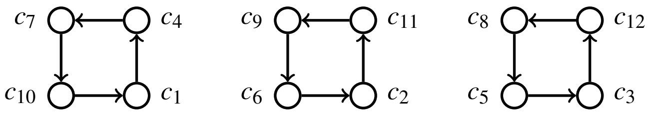
Therefore, if $|V(G)| \geq 5$, at most one of G and $\overline{G}$ is bipartite.

11 [CZ] Problem 1.30



12 [CZ] Problem 1.31

Define that, (c_i, c_j) is a directed edge, if and only if c_j can be obtained from c_i by rotating the configuration 90° clockwise, and then interchanging the two coins. The graph is:



13 [CZ] Problem 2.6

Consider the sum of the degrees over all vertices:

$$\sum_{v \in V(G)} \deg v = n(n-1) + n^2 + n(n+1) = 3n^2$$

By Theorem 2.1, $3n^2$ is even, thus n is even.

14 [CZ] Problem 2.7

(a) For every $v \in V(G)$, by the definition of bipartite graph, one of its incident vertices is in U, the other is in W. Therefore $m = \sum_{u \in U} \deg u = \sum_{w \in W} \deg w$.

(b) Let x be the number of vertices of G having degree 2. Then we have the following equation

$$\sum_{u \in U} \deg u = \sum_{w \in W} \deg w$$
$$3 \times |U| = 2 \times n + 4 \times (|W| - n)$$

After some algebra we get $n = 2$.

15 [CZ] Problem 2.9

Suppose that these odd vertices are not in the same component. Let C be the component containing only one odd vertex. Component is an induced subgraph, so the degrees of its vertices do not change. The sum of the degrees over all vertices in C is odd, which contradicts Theorem 2.1.

16 [CZ] Problem 2.10

(a) Let K_a, K_b be two complete graph with degrees a and b, where $a, b \geq 2$ and $a+b=n$. Let u, v be any vertices in K_a and K_b, respectively. Connect the two graphs by edge uv, we get a new connected graph G. For every nonadjacent vertices x, y, they are in two different complete graphs and $\{x,y\} \neq \{u,v\}$, otherwise they are adjacent. Assume that $x \in K_a$ and $y \in K_b$. If $x=u$ or $y=v$, $\deg x + \deg y = n-1$, otherwise $\deg x + \deg y = n-2$.

(b) If the graph has more than one components, remove any one of them. The remaining part of the graph has at most $n-1$ vertices, with $\deg u + \deg v \geq n-2$ still holds. By Theorem 2.4, it is connected. Therefore, G has at most two components.

(c) No. Instead of moving any component, we remove the one with greater order. The remaining part of the graph has at most $\lfloor n/2 \rfloor$ vertices. If $\deg u + \deg v \geq \lfloor n/2 \rfloor - 1$ for all nonadjacent u, v, the remaining part is connected. Thus $\lfloor n/2 \rfloor - 1$ is a shaper bound.

17 [CZ] Problem 2.13

(a) Suppose, to the contrary that G contains more than two components. The component containing minimum number of vertices contains at most $\lfloor n/3 \rfloor$ vertices. The degree of every vertex in this component is at most $\lfloor n/3 \rfloor - 1$, less than $(n-2)/3$, which leads to contradiction.

(b) If $\deg v \geq (n-3)/3 = n/3 - 1$ for every vertex v of G, G might contain more than two components. For example, let n be a multiple of 3, consider the graph $G = 3K_{n/3}$, for every $v \in V(G)$, $\deg v = n/3 - 1$, however, it contains three components.

18 [CZ] Problem 2.15

For every cycle in graph G, let u, v be two distinct vertices in the cycle, and thus the cycle consists two $u-v$ path, whose lengths have the same parity. Therefore the cycle is an even cycle. By Theorem 1.12 G is bipartite.

19 [CZ] Problem 2.20

Suppose, to the contrary that for every adjacent vertices u and v, $\deg u = \deg v$. Since the graph is connected, for every distinct vertices x and y, there exists an $x-y$ path. By transitivity of "=", we get $\deg x = \deg y$, i.e. G is regular, which leads to contradiction.

20 [CZ] Problem 2.25

(a) By Theorem 2.1, $\sum_{u\in V(G)} \deg u$ is even, thus $G-v$ has even order, therefore G has odd order.

(b) Suppose, to the contrary, that there exists some component of odd order, denoted by C. Consider

$$\sum_{v\in V(C)} \deg v = r|V(C)|$$

it is odd, which contradicts Theorem 2.1. Therefore, G does not contain any component of odd order.

21 [CZ] Problem 2.27

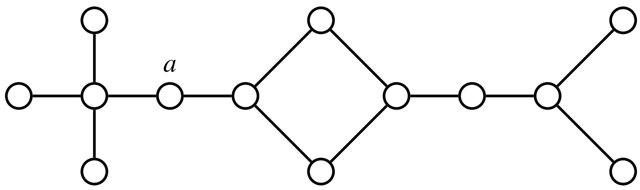
For bipartite graph, we have $\sum_{u\in U} \deg u = \sum_{w\in W} \deg w$, so $r|U| = r|W|$. Eliminating r yields $|U| = |W|$.

22 [CZ] Problem 2.28

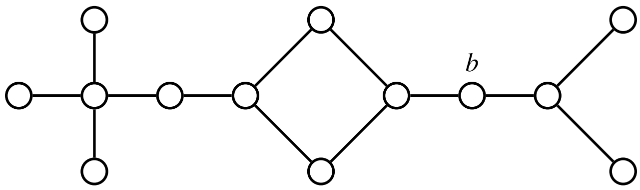
Yes. When G itself is an r-regular graph, the graph H in Theorem 2.7 is equal to G, which of course has the smallest order.

23 [CZ] Problem 3.6

No. Consider the two graphs



G_1



G_2

G_1 and G_2 have the same degree sequence. G_1 contains vertex a of degree 2 that is adjacent to a vertex of 3 and a vertex of 4. G_2 contains vertex b of degree 2 that is adjacent to two vertices of degree 3. However, $G_1 \cong G_2$.

24 [CZ] Problem 3.9

They are not isomorphic. Both G_1 and G_2 contain exactly two vertices of degree 2. In G_1, the two vertices of degree 2 (the leftmost one and the rightmost one) are adjacent to two vertices, however, in G_2, the two vertices of degree 2 (the top left one and the top right one) are adjacent to three vertices. Hence G_1 and G_2 are not isomorphic.

25 [CZ] Problem 3.11

Let $X = \{v \in V(G) : \deg_G v = n/2\}$, $Y = \{v \in V(G) : \deg_G yv < n/2\}$. We have

$$U = X \cup Y = X \cup \{v \in V(G) : \deg_G v < n/2\} = X \cup \{v \in V(\overline{G}) : \deg_{\overline{G}} v \geq n/2\}$$

Since G is self-complementary, we have

$$|W| = |\{v \in V(G) : \deg_G v \geq n/2\}| = |\{v \in V(\overline{G}) : \deg_{\overline{G}} v \geq n/2\}|$$

$|U| = |W|$ implies $|X| = 0$, i.e. G contains no vertex v of degree $n/2$.

26 [CZ] Problem 3.13

This statement is true. Note that $d(i, j) = 1$ if and only if i and j are adjacent. Therefore, for every edge $uv \in E(G)$, $d_G(u,v) = 1$, thus $d_H(\phi(u), \phi(v)) = 1$, i.e $\phi(u)\phi(v) \in E(H)$. Likewise, if $\phi(u)\phi(v) \in E(H)$, then $uv \in E(G)$. By the definition of isomorphism, H and G are isomorphic.