

论题 2-1 作业

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1 [TC] Problem 2-1

- a.* For every sublist of length k , insertion sort can sort it in $\Theta(k^2)$ worst-case time, and there are n/k sublists, so these sublists can be sorted by insertion sort in $\Theta(k^2)n/k = \Theta(nk)$ worst-case time.
- b.* Apply the divide-and-conquer approach. Divide these sublists into two groups, each contains $n/(2k)$ sublists, and merge these sublists recursively, and finally merge the two groups. Let m denote the number of the sublists, i.e. n/k , and $T(m)$ denote the total running time of merging m sublists. The “divide”, “conquer” and “combine” steps take a running time of $\Theta(1)$, $2T(m/2)$, $\Theta(km)$, so the recurrence is

$$T(m) = \begin{cases} \Theta(1) & m = 1 \\ 2T(m/2) + \Theta(km) & m > 1 \end{cases}.$$

Solve this recurrence, we obtain $T(m) = \Theta(km \log(m)) = \Theta(m \log(k))$.

- c.* A standard merge sort takes a running time of $\Theta(n \log n)$. When $k = \Theta(\log(n))$, $\Theta(nk + n \log(n/k)) = \Theta(n \log n)$. For every $k = \omega(\log n)$, $\Theta(nk + n \log(n/k)) = \Theta(nk) = \omega(n \log n)$. Therefore, the largest value of k is $\Theta(\log n)$.
- d.* It mainly depends on the constant factors of merge sort and insertion sort. Let c_1 be the constant factor of merge sort, c_2 be the constant factor of insertion sort. We can rewrite the total running time as $T = c_1nk + c_2n \log(n/k)$. Minimize T with respect to k . Since $T'_k = nc_1 - nc_2/k$, $k = c_2/c_1$ is a minimum point of T . So we can choose $k = c_1/c_2$ in practice.

2 [TC] Problem 2-2

- a.* $\langle A'[1], A'[2], \dots, A'[n] \rangle$ is a permutation of $\langle A[1], A[2], \dots, A[n] \rangle$.
- b.* At the start of each iteration, the subarray $A[j..A.length]$ consists of the elements originally in $A[j..A.length]$, and $A[j]$ is the smallest item of $A[j..A.length]$.

Initialization Prior to the first iteration of this loop, we have $j = A.length$. Therefore, the subarray $A[j..A.length]$ consists only one element, and it is the original element in $A[j..A.length]$, and, of course, it is the smallest item of $A[j..A.length]$.

Maintenance In each iteration, we compare $A[j]$ with $A[j-1]$. If $A[j] \geq A[j-1]$, we do nothing. Because $A[j]$ is the smallest item of $A[j..A.length]$, $A[j-1]$ is the smallest item of $A[j-1..A.length]$. If $A[j] > A[j-1]$, we exchange $A[j]$ with $A[j-1]$. Because $A[j]$ is the smallest item of $A[j..A.length]$, $A[j-1]$ is

the smallest item of $A[j - 1..A.length]$ after exchanging, and $A[j - 1..A.length]$ consists of the elements originally from $A[j - 1..A.length]$. Summarizing, the invariant still holds after an iteration.

Termination Finally, we get $j = i$. Therefore, the subarray $A[i..A.length]$ consists of the elements originally in $A[i..A.length]$, and $A[i]$ is the smallest item of $A[i..A.length]$, and that is what we want.

- c. At the start of each iteration, $A[1..i]$ contains the i smallest elements of $A[1..A.length]$, in sorted order, and $A[i + 1..A.length]$ are the rest of the elements.

Initialization Prior to the first iteration, we have $i = 1$, and $A[1..i]$ contains only one element, in sorted order, trivially. Moreover, $A[i + 1..A.length]$ are the rest of the elements obviously.

Maintenance After each iteration, $A[i + 1..A.length]$ consists of the elements originally in $A[i + 1..A.length]$, and $A[i + 1]$ is the smallest item of $A[i + 1..A.length]$. Since $A[1..i]$ are the i smallest elements of $A[1..A.length]$, $A[i + 1]$ is greater than or equal to every element in $A[1..i]$, but less than or equal to every element in $A[i + 2..A.length]$. Therefore, $A[1..i + 1]$ contains the $i + 1$ smallest elements of $A[1..A.length]$ in sorted order, and $A[i + 2..A.length]$ is the rest of the elements, i.e. the loop invariant still holds.

Termination Finally we get $i = A.length$, therefore $A[1..A.length]$ contains the $A.length$ smallest elements of $A[1..A.length]$ in sorted order. Hence, the algorithm is correct.

- d. Whatever the original sequence is, lines 3-4 will always be executed $n(n - 1)/2$ times, where n is the number of the elements of the original sequence. Therefore the worst-case running time of bubble sort is $\Theta(n^2)$, as much as the worst-case running time of insertion sort.

3 [TC] Problem 2-3

- a. $\Theta(n)$.

- b. Given the coefficients a_0, a_1, \dots, a_n and a value for x :

```

1  y = 0
2  for i = 0 to n
3      t = 1
4      for j = 1 to i
5          t = t * x
6      y = y + a_i * t

```

- c. **Initialization** Prior to the first iteration, we have $y = 0$ and $i = n$, so the loop invariant trivially holds.

Maintenance Assume, prior the t th iteration, we have $i = i_t$ and $y = y_t$. After the iteration and incrementing i , we get $i_{t+1} = i_t - 1$ and

$$\begin{aligned}
 y_{t+1} &= a_{i_t} + x * y_t = a_{i_t} + x * \sum_{k=0}^{n-(i_t+1)} a_{k+i_t+1} x^k \\
 &= a_{i_t} + \sum_{k=0}^{n-(i_{t+1}+2)} a_{k+i_{t+1}+2} x^{k+1} = a_{i_{t+1}+1} + \sum_{k=1}^{n-(i_{t+1}+1)} a_{k+i_{t+1}+1} x^k \\
 &= \sum_{k=0}^{n-(i_{t+1}+1)} a_{k+i_{t+1}+1} x^k,
 \end{aligned}$$

therefore the invariant still holds.

Termination At termination, we have $i = -1$. Substituting i for -1 in invariant, we obtain

$$y = \sum_{k=0}^n a_k x^k,$$

so the algorithm is correct.

4 [TC] Problem 2-3

a. $(2, 1), (3, 1), (8, 6), (8, 1), (6, 1)$.

b. $\{n, n-1, \dots, 1\}$.
 $n(n-1)/2$.

c. Assume there are $I(n)$ inversions in the input array, then the running time of insertion sort is $\Theta(n + I(n))$.

Proof: In page 26, t_j stands for the number of times the **while** loop test is executed for that value of j . Every time we execute the **while** loop, we insert $A[j]$ into the correct position, and exactly $t_j - 1$ inversions are eliminated. Finally, all the inversions are eliminated, and the array is sorted in order. Substituting $\sum_{j=2}^n (t_j - 1)$ for $I(n)$ and rewrite the formula, we get

$$T(n) = an + bI(n) + c$$

, therefore the running time of insertion sort is $\Theta(n + I(n))$.

d. Let c be an integer which stores the number of inversions.

1 $c = 0$

2 MODIFIED-MERGE-SORT($A, 1, A.length$)

MODIFIED-MERGE-SORT(A, p, r)

1 **if** $p < r$

2 $q = \lfloor (p + r) / 2 \rfloor$

3 MODIFIED-MERGE-SORT(A, p, q)

4 MODIFIED-MERGE-SORT($A, q + 1, r$)

5 MODIFIED-MERGE(A, p, q, r)

MODIFIED-MERGE(A, p, q, r)

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1   $n_1 = q - p + 1$ 
2   $n_2 = r - q$ 
3  Let  $L[1..n_1 + 1]$  and  $R[1..n_2 + 1]$  be new arrays
4  for  $i = 1$  to  $n_1$ 
5       $L[i] = A[p + i - 1]$ 
6  for  $j = 1$  to  $n_2$ 
7       $R[j] = A[q + j]$ 
8   $L[n_1 + 1] = \infty$ 
9   $R[n_2 + 1] = \infty$ 
10  $i = 1$ 
11  $j = 1$ 
12 for  $k = p$  to  $r$ 
13     if  $L[i] \leq R[j]$ 
14          $A[k] = L[i]$ 
15          $i = i + 1$ 
16     else
17          $A[k] = R[j]$ 
18         // Add the number of inversions  $(A[i], R[j]), (A[i + 1], A[j]) \dots (A[q], A[j])$  to  $c$ 
19          $c = c + (q - i + 1)$ 
20      $j = j + 1$ 

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5 [TC] Problem 3-2

| A | B | O | o | Ω | ω | Θ |
|-------------|--------------|-----|-----|----------|----------|----------|
| $\lg^k n$ | n^ϵ | yes | yes | no | no | no |
| n^k | c^n | yes | yes | no | no | no |
| \sqrt{n} | $n^s \ln n$ | no | no | no | no | no |
| 2^n | $2^{n/2}$ | no | no | yes | yes | no |
| $n^{\lg c}$ | $c^{\lg n}$ | yes | no | yes | no | yes |
| $\lg(n!)$ | $\lg(n^n)$ | yes | no | yes | no | yes |

6 [TC] Problem 3-3

a. List:

| | | | | | |
|--------------------|----------------|-------------------|-----------------|-------------------|----------------------|
| 1 | $n^{1/\lg n}$ | $\lg(\lg^* n)$ | $\lg^*(\lg n)$ | $\lg^* n$ | $2^{\lg^* n}$ |
| $\ln \ln n$ | $\sqrt{\lg n}$ | $\ln n$ | $\lg(n!)$ | $\lg^2 n$ | $2^{\sqrt{2 \lg n}}$ |
| n | $2^{\lg n}$ | $n \lg n$ | n^2 | $4^{\lg n}$ | n^3 |
| $\sqrt{2}^{\lg n}$ | $(\lg n)!$ | $(\lg n)^{\lg n}$ | $n^{\lg \lg n}$ | $(\frac{3}{2})^n$ | 2^n |
| e^n | $n 2^n$ | $n!$ | $(n + 1)!$ | 2^{2^n} | $2^{2^{n+1}}$ |

Equivalence classes: $\{1, n^{1/\lg n}\}$, $\{\lg(\lg^* n)\}$, $\{\lg^*(\lg n), \lg^* n\}$, $\{2^{\lg^* n}\}$, $\{\ln \ln n\}$, $\{\sqrt{\lg n}\}$, $\{\ln n, \lg(n!)\}$, $\{\lg^2 n\}$, $\{2^{\sqrt{2 \lg n}}\}$, $\{n, 2^{\lg n}\}$, $\{n \lg n\}$, $\{n^2, 4^{\lg n}\}$, n^3 , $\{\sqrt{2}^{\lg n}\}$, $\{(\lg n)!\}$, $\{(\lg n)^{\lg n}, n^{\lg \lg n}\}$, $\{(\frac{3}{2})^n\}$, $\{2^n\}$, $\{e^n\}$, $\{n2^n\}$, $\{n!\}$, $\{(n+1)!\}$, $\{2^{2^n}\}$, $\{2^{2^{n+1}}\}$.

b. $f(n) = (2^{2^{n+2}})^{\sin n}$

7 [TC] Problem 3-4

a. False. Take $f(n) = n$, $g(n) = n^2$, $f(n) = O(g(n))$, but $g(n) \neq O(f(n))$.

b. False. Take $f(n) = n$, $g(n) = n^2$, $f(n) + g(n) = n + n^2 = \Theta(n^2) \neq \Theta(\min(f(n), g(n))) = \Theta(n)$.

c. True. Since $f(n) = O(g(n))$, there exists positive constant $c > 1$ such that for all sufficiently large n , $1 \leq f(n) \leq cg(n)$ holds. Therefore, $\lg(f(n)) \leq \lg c + \lg(g(n))$. Take $c' = \lg c + 1$, then $0 \leq \lg(f(n)) \leq c' \lg(g(n))$ for all sufficiently large n . Hence, $\lg(f(n)) = O(\lg(g(n)))$.

d. False. Take $f(n) = n \lg n$, $g(n) = \lg(n!)$, $f(n) = O(g(n))$, however, $2^{f(n)} = n^n \neq O(2^{g(n)}) = O(n!)$.

e. False. Take $f(n) = 1/n$, then $(f(n))^2 = 1/n^2$, however, $\lim_{n \rightarrow \infty} f(n)/(f(n))^2 = +\infty$, that means, $f(n)$ could not be asymptotically upper-bounded by $(f(n))^2$.

f. True. By transpose symmetry we know this is true.

g. False. Take $f(n) = 4^n$, then $\Theta(f(n/2)) = \Theta(2^n)$, however $4^n \neq \Theta(2^n)$.

h. True. By the definition of o -notation, for any positive constant c , there exists a positive integer n_0 , for any integer $n > n_0$, $0 \leq o(f(n)) \leq cf(n)$ holds. Take $c = 1$, for sufficiently large positive integer n , $f(n) \leq f(n) + o(f(n)) \leq 2f(n)$, therefore $f(n) + o(f(n)) = \Theta(f(n))$.