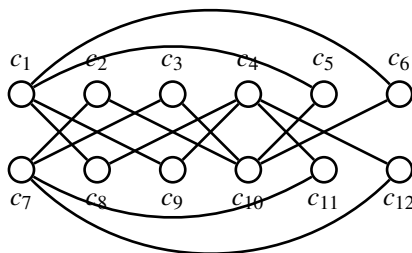


# 论题 2-15 作业

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## 1 [CZ] Problem 1.6



## 2 [CZ] Problem 1.8

(a) The words in  $S_1$  are (presented from left to right): cat, cap, tap, top.

The words in  $S_2$  are: map (center) , mop, tap, mat (surrounding).

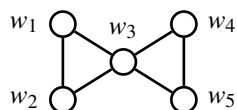
The words in  $S_3$  are: run (top left), gun (top right), sun (center), son (bottom).

The words in  $S_4$  are (presented clockwise): slit, slot, slop, slip.

The words in  $S_5$  are (presented clockwise from top left): pot, put, pet, poet.

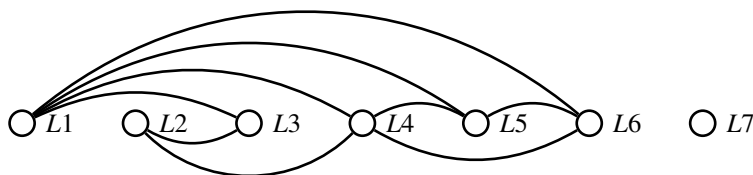
The words in  $S_6$  are (presented clockwise): lake, sake, take, make.

(b) The graph  $H$  is:



It is a word graph of some set, and the corresponding words are: top, tap, tip, lip, dip.

## 3 [CZ] Problem 1.10



## 4 [CZ] Problem 1.14

Let  $C$  denote a component of  $G$ .

(1)  $\rightarrow$  (2): Take any vertex  $v_0$  in  $C$ . For any vertex  $v_i$  which is connected to  $v_0$ , it must be in  $V(C)$ , otherwise if we add  $v_i$ , along with the edges in the path from  $v_0$  to  $v_i$ , to  $C$ , we get a proper connected supergraph of  $C$ , which leads to contradiction. Therefore,  $V(C)$  is an equivalent class. Then we have to prove  $C$  is the subgraph induced by  $V(C)$ . If not, let  $C'$  be the subgraph induced by  $V(C)$ , and  $E(C) \subset E(C')$ , thus  $C$  is a proper subgraph of  $C'$ , which leads to contradiction.

(2)  $\rightarrow$  (1): Suppose, to the contrary that  $C$  is a proper subgraph of a connected subgraph of  $G$ , denoted by  $C'$ . If  $V(C) \subset V(C')$ , there exists some vertex connected to  $C$  but not in the equivalent class, which leads to contradiction. It is impossible that  $V(C) = V(C')$ , because the subgraph induced by  $V(C)$  is the maximal subgraph whose vertex set is  $V(C)$ .

## 5 [CZ] Problem 1.16

For every  $i$ , we have a path from  $u$  to  $v_i$ :  $(u = v_0, v_1, \dots, v_i)$ , whose length is  $i$ . Thus  $d(u, v_i) \leq i$ .

Suppose, to the contrary that  $d(u, v_i) < i$ , i.e. there exists path  $(u_0 = v_0, u_1, \dots, u_j = v_i)$ , where  $j < i$ . Consider the walk  $(u = u_0 = v_0, u_1, \dots, u_j = v_i, v_{i+1}, \dots, v_k = v)$ , it's a  $u - v$  walk shorter than the geodesic, which leads to contradiction.

Therefore,  $d(u, v_i) = i$  for each integer  $i$  with  $1 \leq i \leq k$ .

## 6 [CZ] Problem 1.17

(a) Assume that  $P$  is an  $x - z$  path and  $Q$  is a  $u - w$  path, where  $x \neq u, v$  and  $y \neq u, v$ , and they do not have common vertex. Let  $y$  be a vertex in  $P$  and  $v$  be a vertex in  $Q$ , then there exists a  $y - v$  path  $(p_0 = y, p_1, p_2, \dots, p_n = v)$ . If there exists  $p_i$  such that  $p_i$  ( $0 < i < n$ ) is in  $P$  or  $Q$ , since  $P \cap Q = \emptyset$ , there exists a segment of the path, from any vertex in  $P$  (let it be  $y$ ), to any vertex in  $Q$  (let it be  $v$ ), such that the vertices in the segment are not in  $P$  or  $Q$ , except the first and the last one. Assume  $x - y$  is longer than  $y - z$ , and  $u - v$  is longer than  $v - w$ , consider the path  $x - y - v - u$ , it is longer than the  $x - z$  path and the  $u - v$  path, which leads to contradiction.

(b) This is true. The geodesics are as well the longest paths in  $G$ , otherwise  $\text{diam}(G) > k$ . Apply the conclusion we've proved in (1), we obtain that  $P$  and  $Q$  must have at least one common vertex.

## 7 [CZ] Problem 1.18

(a) The minimum size of such a subgraph contains only the vertices and edges in a  $u - v$  geodesic. Any connected subgraph containing  $u$  and  $v$  must have a  $u - v$  path, which is at least as long as the geodesic. So a subgraph contains only the vertices and edges in a  $u - v$  geodesic has less edges or vertices than other graphs.

(b) What is the maximum size of a connected subgraph of  $G$  containing  $u$  and  $v$ ? It is  $G$ .

## 8 [CZ] Problem 1.22

If  $u, v$  are in two different components of  $G$ , then  $uv \in E(\overline{G})$ , i.e.  $d_{\overline{G}}(u, v) = 1$ .

If  $u, v$  are in the same component of  $G$ , let  $w$  be a vertex in another component  $G$ . We have  $uw, wv \in E(\overline{G})$ , i.e.  $(u, w, v)$  is a path from  $u$  to  $v$ , and thus  $d_{\overline{G}}(u, v) \leq 2$ .

Therefore,  $d_{\overline{G}}(u, v) = 1$  or  $d_{\overline{G}}(u, v) = 2$ .

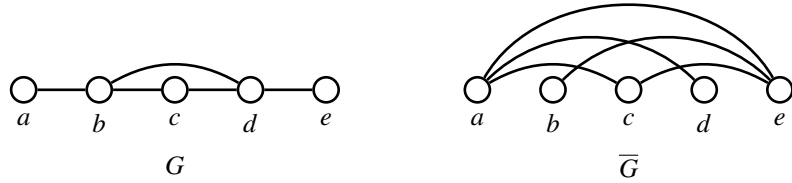
## 9 [CZ] Problem 1.23

(a) For  $k = 1$ , the graph  $G = (V, E)$  is:



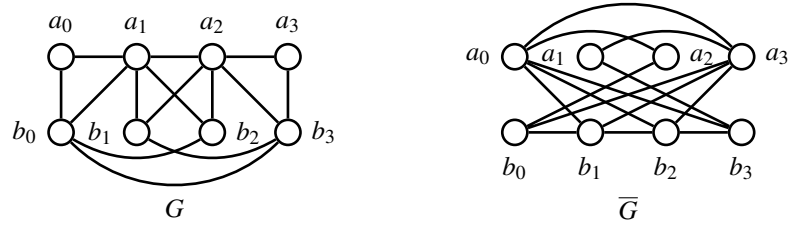
and  $d_G(b, c) = 1 = d_{\overline{G}}(a, d)$ .

For  $k = 2$ , the graph  $G = (V, E)$  is



and  $d_G(a, d) = 2 = d_{\overline{G}}(b, c)$ .

(b) The largest value of  $k$  is 3. Here is an example when  $k = 3$ ,  $d_G(a_0, a_3) = 3 = d_{\overline{G}}(b_0, b_3)$ :



Now we are going to prove that it is impossible that  $k \geq 4$ . Suppose, there exists distinct  $u, v, x, y$ , such that  $d_G(u, v) = d_{\overline{G}}(x, y) = k \geq 4$ . Let  $(u = w_0, w_1, \dots, w_k = v)$  be a  $u - v$  geodesic,  $(x = z_0, z_1, \dots, z_k = y)$  be an  $x - y$  geodesic. We claim that there exists  $i \in \{0, k\}$ , such that  $z_0 w_i$  is not an edge in  $G$ , otherwise,  $(w_0, z_0, w_k)$  is a  $u - v$  path shorter than the geodesic in  $G$ . Likewise there exists  $j \in \{0, k\}$ , such that  $z_k w_j$  is not an edge in  $G$ . Hence,  $z_0 w_i$  and  $z_k w_j$  are edges in  $\overline{G}$ . If  $i = j$ ,  $(z_0, w_i, z_k)$  is shorter than  $x - y$  geodesic, which is impossible. If  $i \neq j$ ,  $w_i w_j = w_0 w_k = uv$  is an edge in  $\overline{G}$ , otherwise  $d_G(u, v) = 1$ , contradicting the provided condition. Note that  $(x = z_0, w_i, w_j, z_k = y)$  is an  $x - y$  path shorter than the geodesic, which is impossible.

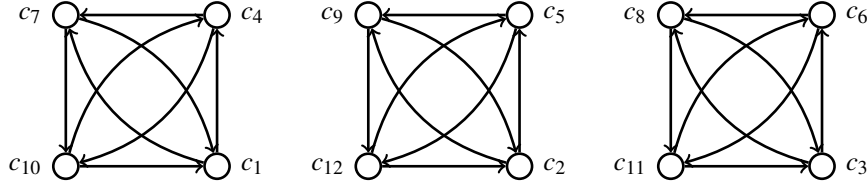
Therefore,  $k \leq 3$ .

## 10 [CZ] Problem 1.25

Assume that  $G$  is bipartite, with partite sets  $U$  and  $W$ . Since  $|V(G)| = |U| + |W| \geq 5$ , at least one of  $|U|$  and  $|W|$  is greater than 2. Assume, without loss of generality, that  $|U| \geq 3$ . Let  $u, v, w$  be three distinct vertices in  $U$ . They are mutually adjacent in  $\overline{G}$ , which forms an odd cycle. By Theorem 1.12,  $\overline{G}$  is not bipartite.

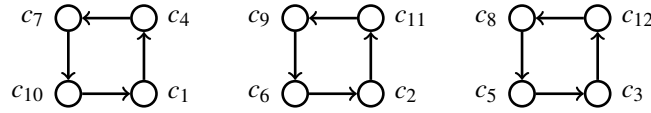
Therefore, if  $|V(G)| \geq 5$ , at most one of  $G$  and  $\overline{G}$  is bipartite.

## 11 [CZ] Problem 1.30



## 12 [CZ] Problem 1.31

Define that,  $(c_i, c_j)$  is a directed edge, if and only if  $c_j$  can be obtained from  $c_i$  by rotating the configuration  $90^\circ$  clockwise, and then interchanging the two coins. The graph is:



## 13 [CZ] Problem 2.6

Consider the sum of the degrees over all vertices:

$$\sum_{v \in V(G)} \deg v = n(n-1) + n^2 + n(n+1) = 3n^2$$

By Theorem 2.1,  $3n^2$  is even, thus  $n$  is even.

## 14 [CZ] Problem 2.7

(a) For every  $v \in V(G)$ , by the definition of bipartite graph, one of its incident vertices is in  $U$ , the other is in  $W$ . Therefore  $m = \sum_{u \in U} \deg u = \sum_{w \in W} \deg w$ .

(b) Let  $x$  be the number of vertices of  $G$  having degree 2. Then we have the following equation

$$\begin{aligned} \sum_{u \in U} \deg u &= \sum_{w \in W} \deg w \\ 3 \times |U| &= 2 \times n + 4 \times (|W| - n) \end{aligned}$$

After some algebra we get  $n = 2$ .

## 15 [CZ] Problem 2.9

Suppose that these odd vertices are not in the same component. Let  $C$  be the component containing only one odd vertex. Component is an induced subgraph, so the degrees of its vertices do not change. The sum of the degrees over all vertices in  $C$  is odd, which contradicts Theorem 2.1.

## 16 [CZ] Problem 2.10

- (a) Let  $K_a, K_b$  be two complete graph with degrees  $a$  and  $b$ , where  $a, b \geq 2$  and  $a + b = n$ . Let  $u, v$  be any vertices in  $K_a$  and  $K_b$ , respectively. Connect the two graphs by edge  $uv$ , we get a new connected graph  $G$ . For every nonadjacent vertices  $x, y$ , they are in two different complete graphs and  $\{x, y\} \neq \{u, v\}$ , otherwise they are adjacent. Assume that  $x \in K_a$  and  $y \in K_b$ . If  $x = u$  or  $y = v$ ,  $\deg x + \deg y = n - 1$ , otherwise  $\deg x + \deg y = n - 2$ .
- (b) If the graph has more than one components, remove any one of them. The remaining part of the graph has at most  $n - 1$  vertices, with  $\deg u + \deg v \geq n - 2$  still holds. By Theorem 2.4, it is connected. Therefore,  $G$  has at most two components.
- (c) No. Instead of moving any component, we remove the one with greater order. The remaining part of the graph has at most  $\lfloor n/2 \rfloor$  vertices. If  $\deg u + \deg v \geq \lfloor n/2 \rfloor - 1$  for all nonadjacent  $u, v$ , the remaining part is connected. Thus  $\lfloor n/2 \rfloor - 1$  is a shaper bound.

## 17 [CZ] Problem 2.13

- (a) Suppose, to the contrary that  $G$  contains more than two components. The component containing minimum number of vertices contains at most  $\lfloor n/3 \rfloor$  vertices. The degree of every vertex in this component is at most  $\lfloor n/3 \rfloor - 1$ , less than  $(n - 2)/3$ , which leads to contradiction.
- (b) If  $\deg v \geq (n - 3)/3 = n/3 - 1$  for every vertex  $v$  of  $G$ ,  $G$  might contain more than two components. For example, let  $n$  be a multiple of 3, consider the graph  $G = 3K_{n/3}$ , for every  $v \in V(G)$ ,  $\deg v = n/3 - 1$ , however, it contains three components.

## 18 [CZ] Problem 2.15

For every cycle in graph  $G$ , let  $u, v$  be two distinct vertices in the cycle, and thus the cycle consists two  $u - v$  path, whose lengths have the same parity. Therefore the cycle is an even cycle. By Theorem 1.12  $G$  is bipartite.

## 19 [CZ] Problem 2.20

Suppose, to the contrary that for every adjacent vertices  $u$  and  $v$ ,  $\deg u = \deg v$ . Since the graph is connected, for every distinct vertices  $x$  and  $y$ , there exists an  $x - y$  path. By transitivity of “=”, we get  $\deg x = \deg y$ , i.e.  $G$  is regular, which leads to contradiction.

## 20 [CZ] Problem 2.25

- (a) By Theorem 2.1,  $\sum_{u \in V(G)} \deg u$  is even, thus  $G - v$  has even order, therefore  $G$  has odd order.
- (b) Suppose, to the contrary, that there exists some component of odd order, denoted by  $C$ . Consider

$$\sum_{v \in V(C)} \deg v = r|V(C)|$$

it is odd, which contradicts Theorem 2.1. Therefore,  $G$  does not contain any component of odd order.

## 21 [CZ] Problem 2.27

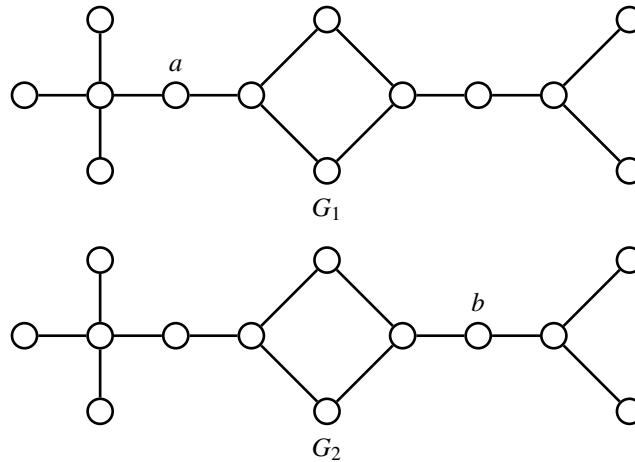
For bipartite graph, we have  $\sum_{u \in U} \deg u = \sum_{w \in W} \deg w$ , so  $r|U| = r|W|$ . Eliminating  $r$  yields  $|U| = |W|$ .

## 22 [CZ] Problem 2.28

Yes. When  $G$  itself is an  $r$ -regular graph, the graph  $H$  in Theorem 2.7 is equal to  $G$ , which of course has the smallest order.

## 23 [CZ] Problem 3.6

No. Consider the two graphs



$G_1$  and  $G_2$  have the same degree sequence.  $G_1$  contains vertex  $a$  of degree 2 that is adjacent to a vertex of 3 and a vertex of 4.  $G_2$  contains vertex  $b$  of degree 2 that is adjacent to two vertices of degree 3. However,  $G_1 \not\cong G_2$ .

## 24 [CZ] Problem 3.9

They are not isomorphic. Both  $G_1$  and  $G_2$  contain exactly two vertices of degree 2. In  $G_1$ , the two vertices of degree 2 (the leftmost one and the rightmost one) are adjacent to two vertices, however, in  $G_2$ , the two vertices of degree 2 (the top left one and the top right one) are adjacent to three vertices. Hence  $G_1$  and  $G_2$  are not isomorphic.

## 25 [CZ] Problem 3.11

Let  $X = \{v \in V(G) : \deg_G v = n/2\}$ ,  $Y = \{v \in V(G) : \deg_G v < n/2\}$ . We have

$$U = X \cup Y = X \cup \{v \in V(G) : \deg_G v < n/2\} = X \cup \{v \in V(\overline{G}) : \deg_{\overline{G}} v \geq n/2\}$$

Since  $G$  is self-complementary, we have

$$|W| = |\{v \in V(G) : \deg_G v \geq n/2\}| = |\{v \in V(\overline{G}) : \deg_{\overline{G}} v \geq n/2\}|$$

$|U| = |W|$  implies  $|X| = 0$ , i.e.  $G$  contains no vertex  $v$  of degree  $n/2$ .

## 26 [CZ] Problem 3.13

This statement is true. Note that  $d(i, j) = 1$  if and only if  $i$  and  $j$  are adjacent. Therefore, for every edge  $uv \in V(G)$ ,  $d_G(u, v) = 1$ , thus  $d_H(\phi(u), \phi(v)) = 1$ , i.e.  $\phi(u)\phi(v) \in V(H)$ . Likewise, if  $\phi(u)\phi(v) \in V(H)$ , then  $uv \in V(G)$ . By the definition of isomorphism,  $H$  and  $G$  are isomorphic.