论题 2-2 作业

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1 [CS] 1.1 Problem 9

Since

$$\binom{n}{2} = \frac{n(n-1)}{2},$$

we have

$$n\binom{n-1}{2} = \frac{n(n-1)(n-2)}{2}$$

and

$$\binom{n}{2}(n-1) = \frac{n(n-1)(n-2)}{2},$$

therefore,

$$n\binom{n-1}{2} = \binom{n}{2}(n-1).$$

Consider choosing one member as the president and two other members as a committee. If we choose the president first, then choose the committee, there are $n\binom{n-1}{2}$ different ways. If we choose the committee first, then choose the president, there are $\binom{n}{2}(n-1)$ different ways. Because the number of ways has nothing to do with the order we choose, we get $n\binom{n-1}{2} = \binom{n}{2}(n-1)$.

2 [CS] 1.1 Problem 13

Let P_i be the set of the pennies I receive on Day i. By supposition, $|P_1| = 1$ and $|P_{i+1}| = 2|P_i|$. Therefore $|P_i| = 2^{i-1}$. For every positive integer n, $P_1, P_2, \cdots, P_n$ are disjoint sets. By the sum principle, the number of pennies I have on Day 20 is

$$\left|\bigcup_{i=1}^{20} P_i\right| = 1 + 2 + \cdots + 2^{20} = 2^{21} - 1 = 2\,097\,151,$$

and the number of pennies I have on day n is

$$\left|\bigcup_{i=1}^{n} P_i\right| = 1 + 2 + \cdots + 2^{n} = 2^{n+1} - 1.$$

3 [CS] 1.2 Problem 15

First, numbering the members from 1 to $2n$. We can use a partition P of $\{1,2,\cdots,2n\}$ to represent how we pair up the members, where $|p|=2$ for every $p \in P$. Then we define a 'sorted partition' $[(a_1,b_1),(a_2,b_2),\cdots,(a_n,b_n)]$ of P, such that (1) $P=\{\{a_1,b_1\},\{a_2,b_2\},\cdots,\{a_n,b_n\}\}$, (2) $a_1<b_1, a_2<b_2,\cdots,a_n<b_n$ and (3) $a_1<a_2<\cdots<a_n$ hold.

We can prove that $a_i=\min\{1,2,\cdots,2n\}\setminus\{a_1,b_1,\cdots,a_{i-1},b_{i-1}\}$, because by (2) we have $a_i<a_{i+1}<\cdots<a_n$ and by (3) we have $b_{i+1}>a_{i+1}>a_i$, $b_{i+2}>a_{i+2}>a_i$, $\cdots$. However, b_i is an arbitrary element in $\{1,2,\cdots,2n\}\setminus\{a_1,b_1,\cdots,a_{i-1},b_{i-1},a_i\}$. Hence, by product principle, the number of the ways is

$$(2n-1)(2n-3)\cdots 1=(2n-1)!!.$$

If we have to determine who plays whom, just multiplying 2^n, because for each pair, we have two ways to determine who plays whom. Therefore, we can specify our pairs in $(2n-1)!!2^n$ ways.

4 [CS] 1.3 Problem 6

The coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$ in the expansion of $(x_1+x_2+\cdots+x_k)^n$ is $\binom{n}{n_1,n_2,\cdots,n_k}$.

Explanation: consider $x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$ in the expansion of $(x_1+x_2+\cdots+x_k)^n$. We have $\binom{n}{n_1}$ ways to choose n_1 x_1's, and then $\binom{n-n_1}{n_2}$ ways to choose n_2 x_2's, and so on. By the product principle, the coefficient of $x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}$ is

$$\begin{aligned}
&\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k}\\
=&\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!(n-n_1-\cdots-n_k)!}\\
=&\frac{n!}{n_1!n_2!\cdots n_k!}
\end{aligned}$$

5 [CS] 1.3 Problem 9

$$(x+y)^n=\sum_{i=0}^{n}\binom{n}{i}x^iy^{n-i}$$

6 [CS] 1.3 Problem 14

Method 1: Since

$$\binom{n}{k}\binom{k}{j}=\frac{n!}{k!(n-k)!}\frac{k!}{j!(k-j)!}=\frac{n!}{j!(k-j)!(n-k)!}$$

and

$$\binom{n}{j}\binom{n-j}{k-j}=\frac{n!}{j!(n-j)!}\frac{(n-j)!}{(k-j)!(n-k)!}=\frac{n!}{j!(k-j)!(n-k)!},$$

we get

$$\binom{n}{k}\binom{n}{k}=\binom{n}{j}\binom{n-j}{k-j}.$$

Method 2: Consider choosing a k-element subset from an n-element set, then choosing a j-element subsubset from the k-element subset. By the product principle, there are $\binom{n}{k}\binom{k}{j}$ different ways.

If we choose the k-element subsubset from the n-element set first, then choose the $k-j$ elements that are in the subset but not in the subsubset. By the product principle, there are $\binom{n}{j}\binom{n-j}{k-j}$ different ways.

Therefore, $\binom{n}{k}\binom{n}{k}=\binom{n}{j}\binom{n-j}{k-j}$.