论题 2-5 作业

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1 [CS] Problem 5.1-6

a. $\{(P,P),(P,N),(P,D),(N,P),(N,D),(D,P),(D,N)\}$
Weights:
$(P,P),(P,N),(P,D),(N,P),(D,P)$: 1/6;
$(N,D),(D,N)$: 1/12.

b. $P(\text{getting 11 cents}) = P(\{(P,D),(D,P)\}) = 1/3$.

2 [CS] Problem 5.1-10

(A standard deck of playing cards refers to a standard 52-card deck, excluding jokers; 10-J-Q-K-A, A-2-3-4-5 are both straights, but hands such as J-Q-K-A-2 are not, as is commonly acknowledged)

Using five-element sets as model. The sample space S is all the five-element subsets of the 52 cards, and $|S| = \binom{52}{5}$. P is the uniform probability measure defined on S. The event E is the set of all the elements of S which are straights, and we have $|E| = 4^5 \times 10$. Therefore, the probability that a five-card hand is a straight is

$$P(E) = \frac{|E|}{|S|} = \frac{10,240}{2,598,960} = \frac{128}{32,487} \approx 0.00394$$

Using five-element permutations as model. The sample space S is all the five permutations of the 52 cards, and $|S| = 52^{\underline{5}}$. P is the uniform probability measure defined on S. The event E is the set of all the elements of S which are straights, and we have $|E| = 4^5 \times 10 \times 5!$. Therefore, the probability that a five-card hand is a straight is

$$P(E) = \frac{|E|}{|S|} = \frac{1,228,800}{311,875,200} = \frac{128}{32,487} \approx 0.00394$$

The answers are same by using two different models.

3 [CS] Problem 5.1-11

We use T or F to represent whether or not he answers a single question correctly. The sample space S is $\{T,F\}^{10}$, and we assume that the weights of all the outcomes are the same, i.e. P is the uniform probability measure defined on S. Each problem is worth 10 points.

The event that the student gets a score of 80 or higher is E_1, and $|E_1| = 1 + 10 + 10 \times 9/2 = 56$. So the probability is

$$P(E_1) = \frac{|E_1|}{|S|} = \frac{56}{1024} = \frac{7}{128} \approx 0.0547$$

The event E_2 that the student gets a score of 70 or lower is the complement of E_1, therefore

$$P(E_2) = 1 - P(E_1) = 1 - \frac{7}{128} = \frac{121}{128} \approx 0.945$$

4 [CS] Problem 5.1-12

Let S, C, T represent square, circle and triangle respectively. The sample space S is $\{S, C, T\}^2$. The event E, is $\{SS, CC, TT\}$ (SS stands for (S, S), the same below). The probability that the two shapes on the top are the same is

$$P(E) = P(SS) + P(CC) + P(TT) = \frac{1}{36} + \frac{1}{9} + \frac{1}{4} = \frac{7}{18} \approx 0.389$$

5 [CS] Problem 5.1-13

For Event 1, the sample space S_1 is the set of all two-element subsets of the 13 spades. P is the uniform probability measure defined on S_1. The event E_1 is $\{\{\spadesuit\text{A}, \spadesuit\text{K}\}\}$. The probability of Event 1 is

$$P(E_1) = \frac{|E_1|}{|S_1|} = \frac{1}{13 \times 12/2} = \frac{1}{78} \approx 0.0128$$

For Event 2, the sample space S_2 is the set of all two-element subsets of all 52 cards. P is the uniform probability measure defined on S_2. The event E_1 is the sets of all two-element sets of an ace and a king. The probability of Event 2 is

$$P(E_2) = \frac{|E_2|}{|S_2|} = \frac{4 \times 4}{52 \times 51/2} = \frac{8}{663} \approx 0.0121$$

Therefore, the two events are not equally likely.

6 [CS] Problem 5.2-2

Let E, F represent the events that $\spadesuit$K or $\spadesuit$Q is selected, respectively. The event that the king or queen of spades is among the cards selected is $E \cup F$. Therefore, the probability is

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = \frac{7}{28} + \frac{7}{28} - \frac{1}{28} = \frac{15}{28} \approx 0.536$$

7 [CS] Problem 5.2-9

The sample space S is the set of all possible arrangements, and $|S| = n!$. Let E_i denote the event that the i-th person who gets the letter intended for him or her. Then, by inclusion-exclusion principle, we have

$$\begin{aligned}
\left|\bigcup_{i=1}^{n} E_i\right| &= \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{i_1, i_2, \cdots i_k: \\ 1 \le i_1 < i_2 < \cdots < i_k \le n}} \left|\bigcap_{j=1}^{k} E_{i_j}\right| \\
&= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (n-k)! \\
&= \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!}
\end{aligned}$$

Hence, the number of ways that nobody gets the correct letter is

$$n! - \left|\bigcup_{i=1}^{n} E_i\right| = \sum_{k=0}^{n}(-1)^k \frac{n!}{k!}$$

And the probability $P(E)$ is

$$P(E) = \sum_{k=0}^{n}(-1)^k \frac{n!}{k!}/n! = \sum_{k=0}^{n}(-1)^k \frac{1}{k!}$$

8 [CS] Problem 5.2-10

The sample space $S = \{L_1, L_2, \cdots, L_k\}^n$, and $|S| = k^n$. Let E_i represent the event that there is no key in the i-th location. For $i_1 < i_2 < \cdots < i_j$, we have

$$\left|\bigcap_{l=1}^{j} E_{i_l}\right| = (k-j)^n$$

By inclusion-exclusion principle, we have

$$\left|\bigcup_{i=1}^{k} E_i\right| = \sum_{j=1}^{k}(-1)^{j+1} \sum_{\substack{i_1,i_2,\cdots,i_j: \\ 1\le i_1<i_2<\cdots<i_j\le k}} \left|\bigcap_{l=1}^{j} E_{i_l}\right| = \sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}(k-j)^n$$

Hence, the number of cases that every location gets at least one key is

$$k^n - \sum_{j=1}^{k}(-1)^{j+1}\binom{k}{j}(k-j)^n = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n$$

And the probability $P(E)$ is

$$P(E) = \frac{\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n}{k^n}$$

9 [CS] Problem 5.2-14

Let the sample space S be the set of all possible circular permutations, and $|S| = (2n-1)!$. Let E_i represent the event that the i-th couple are side-by-side. For $i_1 < i_2 < \cdots < i_k$, we have

$$\left|\bigcap_{j=1}^{k} E_{i_j}\right| = (2n-k-1)!2^k$$

By inclusion-exclusion principle, we have

$$\left|\bigcup_{i=1}^{n} E_i\right| = \sum_{j=1}^{n}(-1)^{j+1} \sum_{\substack{i_1,i_2,\cdots,i_j: \\ 1\le i_1<i_2<\cdots<i_j\le n}} \left|\bigcap_{l=1}^{j} E_{i_l}\right| = \sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}(2n-i-1)!2^i$$

The number of circular permutations that no husband and wife are side-by-side is

$$(2n-1)! - \sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}(2n-i-1)!2^i = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(2n-i-1)!2^i$$

The probability $P(E)$ is

$$P(E) = \frac{\sum_{i=0}^{n}(-1)^i \binom{n}{i} (2n-i-1)!2^i}{(2n-1)!}$$

10 [CS] Problem 5.2-15

Let C be the set of all these m objects. Let E_i be the set of objects which have the i-th property. By inclusion-exclusion principle, we have

$$\left|\bigcup_{i=1}^{p} E_i\right| = \sum_{j=1}^{p}(-1)^{j+1} \sum_{\substack{i_1,i_2,\cdots,i_j:\\ 1\le i_1<i_2<\cdots<i_j\le p}} \left|\bigcap_{l=1}^{j} E_{i_l}\right| = \sum_{i=1}^{p}(-1)^{i+1} \sum_{K:K\subseteq P,|K|=i} N_a(K)$$

And we know that $C = N_a(\varnothing)$. Therefore,

$$N_e(\varnothing) = |C| - \left|\bigcup_{i=1}^{p} E_i\right| = \sum_{i=0}^{|P|}(-1)^i \sum_{K:K\subseteq P,|K|=i} N_a(K) = \sum_{K:K\subseteq P}(-1)^{|K|}N_a(K)$$

Explanation: For each element b in B, we define a property b for the functions. A function f has the property b if and only if for every $a \in A$, $(a,b) \notin f$. $N_a(K)$ means the number of the functions from A to $B \setminus K$, i.e. $N_a(K) = (|B|-|K|)^{|A|}$. $N_e(\varnothing)$ means the number of the onto functions, and

$$N_e(\varnothing) = \sum_{K:K\subseteq P}(-1)^{|K|}N_a(K) = \sum_{i=0}^{|B|}(-1)^i \binom{|B|}{i} (|B|-i)^{|A|}$$

Application to Problem 9: Define n properties for the arrangements. The i-th property means that the i-th person gets the correct letter. The sample S is the set of all possible arrangements, and $|S| = n!$. The event E is that nobody gets the correct letter, and we have

$$|E| = N_e(\varnothing) = \sum_{K:K\in P}(-1)^{|K|}N_a(K) = \sum_{i=0}^{n}(-1)^i \binom{n}{i} (n-i)! = \sum_{i=0}^{n}(-1)^i \frac{n!}{i!}$$

Therefore, the probability is

$$P(E) = |E|/n! = \sum_{i=0}^{n}(-1)^i \frac{1}{i!}$$

11 [CS] Problem 5.3-3

Let the sample space $S = \{T,F\}^3$, and $|S| = 8$. P is the uniform probability measure defined on S. Let E_1 denote the event of at most one tail, i.e. $E_1 = \{FFF, TFF, FTF, FFT\}$, E_2 denote the event that all not all flips are identical, i.e. $E_2 = \{TFF, FTF, FFT, TTF, TFT, FTT\}$. Calculate the probability of these events:

$$P(E_1) = \frac{|E_1|}{|S|} = \frac{1}{2} = 0.5$$

$$P(E_2) = \frac{|E_2|}{|S|} = \frac{3}{4} = 0.75$$

$$P(E_1 \cap E_2) = \frac{|E_1 \cap E_2|}{|S|} = \frac{3}{8} = 0.375$$

Since $P(E_1 \cap E_2) = P(E_1)P(E_2)$, by product principle for independent probabilities, the two events are independent.

12 [CS] Problem 5.3-4

The sample space $S = \{1,2,3,4,5,6\}^2$, and P is the uniform probability measure defined on S.

Let $E_{1,i}$ denote the event that "i dots are on top of the first die", i.e. $E_{1,i} = \{i1, i2, i3, i4, i5, i6\}$, and $E_{2,j}$ denote the event that "j dots are on top of the second die" likewise. Therefore

$$P(E_{1,i}) = \frac{|E_{1,i}|}{|S|} = \frac{1}{6}$$

$$P(E_{2,j}) = \frac{|E_{2,j}|}{|S|} = \frac{1}{6}$$

$$P(E_{1,i} \cap E_{2,j}) = \frac{|E_{1,i} \cap E_{2,j}|}{|S|} = \frac{|\{ij\}|}{|S|} = \frac{1}{36}$$

Since, $P(E_{1,i})P(E_{2,j}) = P(E_{1,i} \cap E_{2,j})$, the two events are independent.

13 [CS] Problem 5.3-8

Let the sample space S be the set of all 13-element subset of the 52 cards. Let E_1 denote the event that a bridge hand has four aces, E_2 denote the event that it has at least one ace, E_3 denote the event that it has the ace of spades. P is the uniform probability measure defined on S. Calculate the probabilities:

$$P(E_1) = \frac{|E_1|}{|S|} = \binom{48}{9} \Big/ \binom{52}{13} = \frac{11}{4165} \approx 0.00264$$

$$P(E_2) = 1 - \frac{|S \setminus E_2|}{|S|} = 1 - \binom{48}{13} \Big/ \binom{52}{13} = \frac{14,498}{20,825} \approx 0.696$$

$$P(E_3) = 1 - \frac{|S \setminus E_3|}{|S|} = 1 - \binom{51}{13} \Big/ \binom{52}{13} = \frac{1}{4} = 0.25$$

Therefore

$$P(E_1|E_2) = \frac{|E_1 \cap E_2|}{|E_2|} = \frac{|E_1|}{|E_2|} = \frac{5}{1316} \approx 0.00379$$

$$P(E_1|E_3) = \frac{|E_1 \cap E_3|}{|E_3|} = \frac{|E_1|}{|E_3|} = \frac{44}{4165} \approx 0.01056$$

The latter one is larger.

14 [CS] Problem 5.3-11

E and F are independent if and only if $P(E \cap F) = P(E)P(F)$. Since $E \cap F = \varnothing$, E and F are independent if and only if $P(E) = 0$ or $P(F) = 0$.