论题 2-6 作业

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1 [CS] Problem 5.6-4

Let X denote the amount of money one wins by playing this game once, and X is a random variable. The expectation of X is

$$E(X) = \frac{1}{4}(1 + E(X)) + \frac{1}{4} \times 2 + \frac{1}{4} \times 3 + \frac{1}{4} \times 4$$

Solve this equation, we obtain

$$E(X) = \frac{10}{3} \approx 3.33$$

Therefore, the maximum amount of money a rational person would pay to play this game is \$3.33 .

2 [CS] Problem 5.6-8

$$\begin{aligned}
\sum_{i=1}^{n} E(X|F_i)P(F_i) &= \sum_{i=1}^{n} P(F_i) \sum_{s:s\in F_i} X(s)\frac{P(s)}{P(F_i)} \\
&= \sum_{i=1}^{n} \sum_{s:s\in F_i} X(s)P(s) \\
&= \sum_{s:s\in S} X(s)P(s) \\
&= E(X)
\end{aligned}$$

3 [CS] Problem 5.7-1

X follows the binomial distribution with parameters $n = 5$ and $p = 0.6$. The expectation and variance is

$$E(X) = 5 \times 0.6 = 3$$

$$V(X) = 5 \times 0.6 \times 0.4 = 1.2$$

Therefore,

$$E(X-3) = E(X) - E(3) = 3 - 3 = 0$$

$$E((X-3)^2) = E((X - E(X))^2) = V(X) = 1.2$$

4 [CS] Problem 5.7-2

Every question is one Bernoulli trial with probability 0.6 of success, therefore $E(X_i) = p = 0.6$ $V(X_1) = p(1-p) = 0.24$ The sum of the variances of X_1 through X_5 is 1.2, which equals to the variance of X, because random variables X_1 through X_5 are independent.

5 [CS] Problem 5.7-4

Let random variable X be the number of right answers. X follows the binomial distribution with parameters $n = 100$ and $p = 0.6$. Therefore

$$E(X) = np = 100 \times 0.6 = 60$$

$$V(X) = np(1-p) = 100 \times 0.6 \times 0.4 = 24$$

$$\sigma(X) = \sqrt{V(X)} = 2\sqrt{6} \approx 4.90$$

6 [CS] Problem 5.7-6

Let random variable X_i be the number of right answers in an i-question quiz. X_i follows the binomial distribution with parameters $n = i$ and $p = 0.8$. Therefore

$$V(X_{25}) = 25p(1-p) = 4$$

$$V(X_{100}) = 100p(1-p) = 16$$

$$V(X_{400}) = 400p(1-p) = 64$$

To "correct" these variances, we can use the standard deviation, i.e. the square root of the variance, instead of variance.

7 [CS] Problem 5.7-12

Assume there are n questions on a short-answer test. Let random variable X denote the number of questions a student who knows 80% of the course material answers correctly. X follows the binomial distribution with parameter n and $p = 0.8$.

By the central limit theorem, the distribution of X converges to the normal distribution with expectation np and variance $np(1-p)$, as n grows large. Hence, if we are 95% sure that such student gets a grade between 75% and 85%, 2 standard deviations should not be greater than 5% of n approximately, i.e.

$$2\sqrt{np(1-p)} \leq 0.05n$$

Solve this inequality and we get

$$n \geq 1600p(1-p) = 256$$

Therefore, about 256 questions are needed. (Actually, the exact minimum of n is 245)

8 [CS] Problem 5.7-16

a.

$$\begin{aligned} V(X) &= E((X - E(X))^2) \\ &= \sum_{i=1}^{n} (X(x_i) - E(X))^2 P(x_i) \end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{i=1}^{k}(X(x_i)-E(X))^2P(x_i)\\
&> \sum_{i=1}^{k}P(x_i)r^2\\
&= P(E)r^2
\end{aligned}
$$

b. Dividing by r^2 on both sides yields

$$P(E) < V(X)/r^2$$

That means, the probability of $|X(x)-E(X)| \geq r$ is no more than $V(X)/r^2$.

9 [CS] Problem 5.7-18

a. X follows the binomial distribution with parameter n and p. The expectation of X is np. By Chebyshev's law, we get

$$P(|X(x)-np| \geq sn) = P(|X(x)-E(X)| \geq sn) \leq V(X)/(s^2n^2) = np(1-p)/(s^2n^2) = p(1-p)/(s^2n)$$

b. For every positive ε (arbitrarily small), if we take $n > \frac{p(1-p)}{s^2\varepsilon}$, $P(|X(x)-np| < ns) > 1-\varepsilon$ holds, because

$$
\begin{aligned}
P(|X(x)-np| < ns) &= 1-P(|X(x)-E(X)| \geq sn)\\
&> 1-p(1-p)/(s^2n)\\
&\geq 1-\frac{p(1-p)}{s^2}\frac{s^2\varepsilon}{p(1-p)}\\
&= 1-\varepsilon
\end{aligned}
$$

That means, the probability of $X(x)$ being between $np-sn$ and $np+sn$ can be arbitrarily close to 1 if n is sufficiently large.

10 [TC] Problem 5.2-1

When hiring exactly once, the first candidate is the best candidate among the n candidates. Therefore, the probability of hiring exactly once is $1/n$.

When hiring exactly n times, the candidates are in increasing order of quality. Since there are $n!$ possible permutations of these candidates, the probability of hiring exactly n times is $1/n!$.

11 [TC] Problem 5.2-2

The first candidate must be hired. Despite the first candidate, exactly one of the remaining candidates is hired. Let p_i denote the probability that the $i-$th candidate is hired, and we have $p_i = 1/i$. Then the probability of hiring twice, denoted as P, is

$$P = \sum_{i=2}^{n}(1-p_2)\cdots p_i \cdots (1-p_n)$$

$$
\begin{aligned}
&= \sum_{i=2}^{n} (1-p_2)\cdots(1-p_n)\frac{p_i}{1-p_i} \\
&= \sum_{i=2}^{n} \left(\prod_{j=2}^{n} (1-p_j) \right) \frac{p_i}{1-p_i} \\
&= \sum_{i=2}^{n} \left(\prod_{j=2}^{n} \frac{j-1}{j} \right) \frac{1/i}{1-1/i} \\
&= \frac{1}{n} \sum_{i=2}^{n} \frac{1}{i-1} \\
&= \frac{\ln n}{n} + O(1/n)
\end{aligned}
$$

12 [TC] Problem 5.2-4

Let X_i be the indicator random variable associated with the event that the $i-$th customer gets his own hat back. We have $X = X_1 + X_2 + \cdots + X_n$, and by the linearity of expectation, we get

$$
E[X] = E\left[\sum_{i=1}^{n} X_i \right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 1/n = 1
$$