论题 2-3 作业

姓名：陈劲源　　　　学号：161240004

1 [TC] Problem 4.1-5

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FIND-MAXIMUM-SUBARRAY(A, n)
ans = 0
sum = 0
for i = 1 to n
    sum = sum + A[i]
    if sum > ans
        ans = sum
    elseif sum < 0
        sum = 0
return ans
```

2 [TC] Problem 4.3-3

We have to prove that $T(n) \geq cn\lg n$ for appropriate choice of constant $c > 0$. We use mathematical induction to prove it.

For the base step, we have $T(1) = 1$, $T(2) = 4$ and $T(3) = 5$. We should choose c such that $T(1) = 1 \geq 0$, $T(2) = 4 \geq 2c$, $T(3) = 5 \geq 3c\lg 3$. We can choose every $c < 1$.

For the induction step, assume that $T(m) \geq cm\lg m$ holds for every positive integer $m < n$, where $n \geq 4$. Substituting it into the recurrence, we obtain

$$\begin{aligned}
T(n) &= 2T(\lfloor n/2 \rfloor) + n \\
&\geq 2(c\lfloor n/2 \rfloor \lg \lfloor n/2 \rfloor) + n \\
&\geq 2(c(n/2-1)\lg(n/2-1)) + n \\
&\geq c(n-2)\lg((n-2)/2) + n \\
&= cn\lg(n-2) - 2c\lg(n-2) + cn - 2c + n \\
&= cn\lg n - cn\lg\frac{n}{n-2} - 2c\lg(n-2) + cn - 2c + n \\
&= cn\lg n + (cn - cn\lg\frac{n}{n-2}) + (n/2 - 2c\lg(n-2)) + (n/2 - 2c) \\
&\geq cn\lg n
\end{aligned}$$

the last step holds if we choose $c < \frac{1}{2}$, because

(1) $n \geq 4 \Rightarrow \lg\frac{n}{n-2} \leq 1 \Rightarrow cn \geq cn\lg\frac{n}{n-2}$;

(2) $n \geq 4 \Rightarrow n/2 \geq \lg(n-2)$, $(n/2 \geq \lg(n-2)) \wedge (c < \frac{1}{2}) \Rightarrow n/2 < 2c\lg(n-2)$;

(3) $n \geq 4 \Rightarrow n/2 > 1$, $(n/2 \geq 1) \wedge (c < \frac{1}{2}) \Rightarrow n/2 > 2c$.

By mathematical induction, we conclude that for all positive constant $c < \frac{1}{2}$, $T(n) \geq cn\lg n$ holds. Hence the recurrence is also $\Omega(n\lg n)$. Therefore we conclude that the solution is $\Theta(n\lg n)$.

3 [TC] Problem 4.3-7

Assume $T(m) \leq cm^{\log_3 4}$ for sufficiently large $m < n$, especially for $m = n/3$ where n is large enough. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 4T(n/3) + n \\ &\leq 4c(\frac{n}{3})^{\log_3 4} + n \\ &= cn^{\log_3 4} + n \\ &\not\leq cn^{\log_3 4} \qquad\qquad fails! \end{aligned}$$

We guess $T(n) \leq c(n^{\log_3 4} - n)$ instead and prove it by mathematical induction.

For the base step, when $n = 3$, we have $T(n) = 7$ and $c(n^{\log_3 4} - n) = c$, therefore $T(n) \leq c(n^{\log_3 4} - n)$ holds for every $c > 7$.

For the induction step, assume $T(m) \leq c(m^{\log_3 4} - m)$ holds for every $m < n$, where $n > 3$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 4T(n/3) + n \\ &\leq 4c((\frac{n}{3})^{\log_3 4} - \frac{n}{3}) + n \\ &= cn^{\log_3 4} + (1 - \frac{4c}{3})n \\ &\leq cn^{\log_3 4} \end{aligned}$$

By mathematical induction, we conclude $T(n) \leq c(n^{\log_3 4} - n)$. Therefore $T(n) = O(n^{\log_3 4} - n) = O(n^{\log_3 4})$.

To complete the proof of $T(n) = \Omega(n^{\log_3 4})$, we assume $T(n) \geq cn^{\log_3 4}$.

For the base step, when $n = 1$, we have $T(n) = 1$ and $cn^{\log_3 4} = c$, therefore $T(n) \geq cn^{\log_3 4}$ holds for every positive constant $c < 1$.

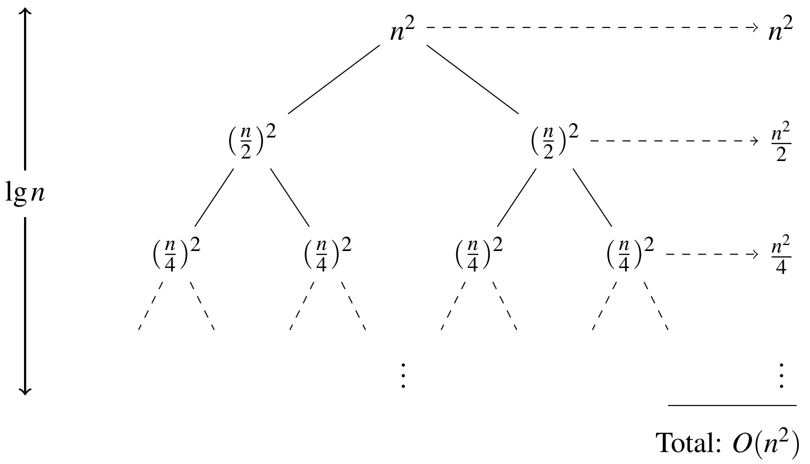
For the induction step, assume $T(m) \geq cm^{\log_3 4}$ holds for every $m < n$, where $n > 3$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 4T(n/3) + n \\ &\geq 4c(\frac{n}{3})^{\log_3 4} + n \\ &= cn^{\log_3 4} + n \\ &\geq cn^{\log_3 4} \end{aligned}$$

By mathematical induction, we conclude that $T(n) \geq cn^{\log_3 4}$. Hence $T(n) = \Omega(n^{\log_3 4})$. Therefore $T(n) = \Theta(n^{\log_3 4})$.

4 [TC] Problem 4.4-2

Recursion tree:



We guess that $T(n) \le cn^2$ for an appropriate choice of positive constant c. For the base step, we have $T(1) = 1$ and $cn^2 = c$, so $T(n) \le cn^2$ holds for $c > 1$ when $n = 1$.

For the induction step, assume that $T(m) \le cm^2$ holds for all positive integer $m < n$, where $n > 1$. Substituting into the recurrence yields

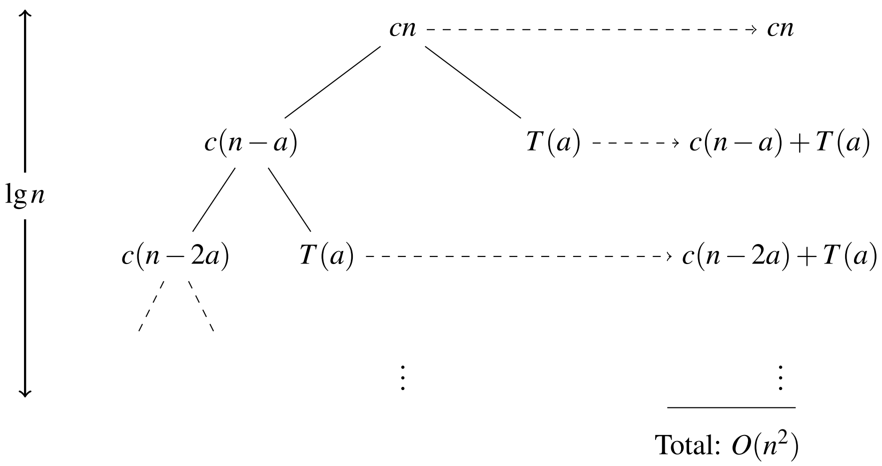
$$
\begin{aligned}
T(n) &= T(n/2) + n^2 \\
&\le c^2 \frac{n^2}{4} + n^2 \\
&= (1 + \frac{c^2}{4})n^2 \\
&= cn^2
\end{aligned}
$$

where the last step holds as long as $c = 2$.

By mathematical induction, we conclude that $T(n) \le 2n^2$, i.e. $T(n) = O(n^2)$.

5 [TC] Problem 4.4-8

Recursion tree:



Adding up all levels of the tree, we get

$$
\begin{aligned}
T(n) &= cn + \sum_{i=1}^{n/a}(c(n-ia)+T(a)) \\
&= \frac{n}{a}T(a) + (\frac{n}{a}+1)cn - ca(1+\frac{n}{a})(\frac{n}{a})/2 \\
&= \frac{T(a)}{a}n + cn - cn/2 + \frac{c}{a}n^2 - \frac{c}{2a}n^2 \\
&= \left(\frac{T(a)}{a} + \frac{c}{2}\right)n + \frac{c}{2a}n^2 \\
&= O(n^2)
\end{aligned}
$$

6 [TC] Problem 4-1

a. $T(n) = O(n^4) = \Omega(n^4)$

First, we have to prove that $T(n) = O(n^4)$, i.e. $T(n) \le cn^4$ for appropriate choice of positive constant c.

For the base step, $T(n) = 1$, $n^4 = 1$, so for all $c > 1$, $T(n) \le n^4$ holds when $n = 1$.

For the induction step, assume that $T(m) \le cm^4$ for all $m < n$ where $n > 1$. Substituting into the recurrence yields

$$
\begin{aligned}
T(n) &= 2T(n/2) + n^4 \\
&\le 2c(n/2)^4 + n^4 \\
&= (c/8+1)n^4 \\
&\le cn^4
\end{aligned}
$$

the last step holds if we take $c = 2$.

By mathematical induction, we conclude that $T(n) \le 2n^4$ for all positive integer n. Hence $T(n) = O(n^4)$.

Second, we have to prove that $T(n) = \Omega(n^4)$, i.e. $T(n) \ge cn^4$ for appropriate choice of positive constant c.

For the base step, $T(n) = 1$, $n^4 = 1$, so $T(n) \ge cn^4$ when $n = 1$ if we choose $c < 1$.

For the induction step, assume that $T(m) \ge cm^4$ for all positive integer $m < n$ where $n > 1$. Substituting into the recurrence yields

$$
\begin{aligned}
T(n) &= 2T(n/2) + n^4 \\
&\ge 2c(n/2)^4 + n^4 \\
&= (c/8+1)n^4 \\
&\ge cn^4
\end{aligned}
$$

the last step holds because $c < 1$.

By mathematical induction, we conclude that $T(n) \ge cn^4$ for positive constant $c < 1$. Therefore $T(n) = \Omega(n^4)$.

b. $T(n) = O(n) = \Omega(n)$

First, we have to prove $T(n) = O(n)$, i.e. $T(n) \leq cn$ for appropriate choice of positive constant c.

For the base step, when $n = 1$, $T(n) = 1$, $n = 1$, so $T(n) \leq cn$ holds for all $c > 1$.

For the induction step, assume that $T(m) \leq cm$ for all $m < n$ where $n > 2$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= T(7n/10) + n \\ &\leq (\frac{7c}{10} + 1)n \\ &\leq cn \end{aligned}$$

the last step holds for all $c > 10/3$.

By mathematical induction, we conclude that $T(n) \leq cn$ for all positive integer n when $c > 10/3$. Hence $T(n) = O(n)$.

Second, we have to prove $T(n) = \Omega(n)$, i.e. $T(n) \geq cn$ for appropriate choice of positive constant c.

For the base step, when $n = 1$, we have $T(n) = 1$, $n^4 = 1$, so $T(n) \geq cn$ for all $c < 1$.

For the induction step, assume that $T(m) \geq cm$ for all $m < n$ where $n > 1$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= T(7n/10) + n \\ &\geq \left(\frac{7c}{10} + 1\right) n \\ &\geq cn \end{aligned}$$

the last step holds because $c < 1$.

By mathematical induction, we conclude that $T(n) \geq cn$ for $c < 1$. Therefore $T(n) = \Omega(n)$.

c. $T(n) = O(n^2 \lg n) = \Omega(n^2 \lg n)$

First, we have to prove that $T(n) = O(n^2 \lg n)$, i.e. $T(n) \leq cn^2 \lg n$ for appropriate choice of positive constant c.

For the base step, when $n = 4$, $T(n) = 32$, $n^2 \lg n = 32$, so for all $c > 1$, $T(n) \leq cn^2 \lg n$ holds when $n = 4$.

For the induction step, assume that $T(m) \leq cm^2 \lg m$ for all $m < n (m \geq 4)$ where $n > 4$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 16T(n/4) + n^2 \\ &\leq 16c(n/4)^2 \lg(n/4) + n^2 \\ &= cn^2 \lg n - 2cn^2 + n^2 \\ &\leq cn^2 \lg n \end{aligned}$$

the last step holds because $c > 1$.

By mathematical induction, we conclude that $T(n) \le cn^2 \lg n$ for all positive integer n when $c > 1$.

Second, we have to prove that $T(n) = \Omega(n^2 \lg n)$, i.e. $T(n) \ge cn^2 \lg n$ for appropriate choice of positive constant c.

For the base step, when $n = 1$, we have $T(n) = 1$, $n^2 \lg n = 0$, so $T(n) \ge cn^2 \lg n$ holds.

For the second step, assume that $T(m) \ge cn^2 \lg n$ for all $m < n$ where $n > 1$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 16T(n/4) + n^2 \\ &\ge 16c(n/4)^2 \lg(n/4) + n^2 \\ &= cn^2 \lg n - 2cn^2 + n^2 \\ &\ge cn^2 \lg n \end{aligned}$$

the last step holds for all $c < 1/2$.

By mathematical induction, we conclude that $T(n) \ge cn^2 \lg n$ for $c < 1/2$. Therefore $T(n) = \Omega(n^2 \lg n)$.

d. $T(n) = O(n^{\log_3 7}) = \Omega(n^{\log_3 7})$