Problem Solving: Homework 3.6
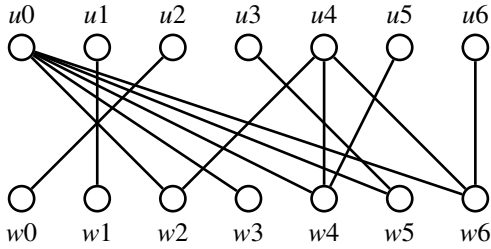
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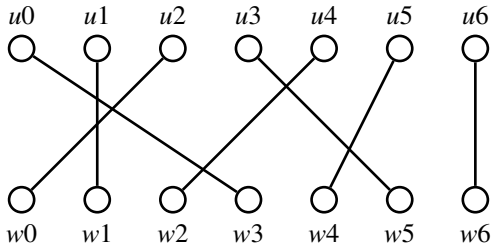
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1 [GC] Problem 8.1

(a) G:

(b) Yes. A perfect matching is

This means that there exists a permutation π of $\{0,1,2,3,4,5,6\}$, such that $w_{\pi(i)}$ is a correct response to u_i, for $0 \le i \le 6$.

2 [GC] Problem 8.3

For graph G_1, U can be matched to W. One possible matching is $\{(a,x),(b,w),(c,v),(d,z),(e,y)\}$.

For graph G_2, U can't be matched to W. Consider vertex set $\{b,d,e\}$, the cardinality of its neighborhood is only 2, which violates Hall's condition.

3 [GC] Problem 8.4

For all subset U' of U, since every two vertices in U' have distinct degrees, the maximum degree of all vertices of U' is at least $|U'|$, and thus the cardinality of the neighborhood of U' is at least $|U'|$. Therefore, the graph G satisfies Hall's condition, which means G contains a perfect matching.
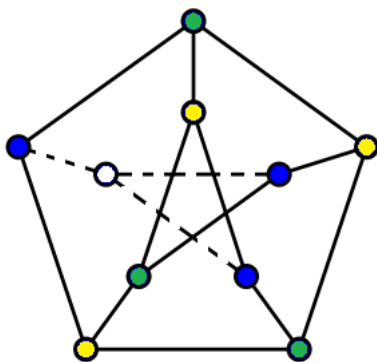
4 [GC] Problem 9.6

(a) True. It is obvious that if a graph does not contain a subdivision of K_5 or $K_{3,3}$, neither does its subgraph. Therefore, by Kuratowski's Theorem, every subgraph of a planar graph is planar.

(b) False. For any nonplanar graph, if we take one of its vertices as a trivial subgraph, then it is obviously a planar subgraph.

(c) False. Consider K_5, removing any of its edges or vertices will make the resulting graph not contain a subdivision of K_5 or $K_{3,3}$, so K_5 is a counterexample to this statement.

(d) False. If we insert an vertex to any edge of K_5, the resulting graph does not contain K_5 or $K_{3,3}$ as a subgraph, however, it is still nonplanar.

(e) False. Consider the union of K_5 and C_3, with order $n = 8$ and size $m = 13$, which satisfies $m \le 3n - 6$. However, one of its components is nonplanar, and thus the graph is nonplanar.

(f) False. Consider the union of $K_{3,3}$ and C_3, it has a triangle and contains no subdivision of K_5 as a subgraph, however, it is nonplanar.

5 [GC] Problem 9.13

(a) Since G contains no triangle, the boundary of every region has at least 4 edges. Because every edge belongs to at most two of the boundaries, we have $2m \ge 4r$, i.e. $2r \le m$. By Euler's Identity, we have $r = 2 + m - n$. Hence, $4 + 2m - 2n \le m$, i.e. $m \le 2n - 4$.

(b) For $K_{3,3}$, $n = 6$, $m = 9$, $m > 2n - 4$. Note that $K_{3,3}$ contains no triangle, so $K_{3,3}$ is nonplanar.

(c) This is true. First, G contains no triangle because it is bipartite. Suppose that every vertex has degree 4 or more, then $2m \ge 4n$, i.e. $m \ge 2n$, which violates the inequality we proved in (a). Therefore, G has a vertex of degree 3 or less.

6 [GC] Problem 9.14

(a) Since the length of a smallest cycle in G is 5, the boundary of every region has at least 5 edges. Because every edge belongs to at most two of the boundaries, we have $2m \geq 5r$. By Euler's Identity, we have $r = 2 + m - n$. Hence $2 + m - n \leq \frac{2}{5}m$, i.e. $m \leq \frac{5}{3}(n-2)$.

(b) Petersen graph has 15 edges and 10 vertices, so $m > \frac{5}{3}(n-2)$. Since the length of a smallest cycle in Petersen graph is 5, it is nonplanar.

(c) Removing any vertex of the Petersen graph yields a subdivision of $K_{3,3}$, so the Peterson graph is nonplanar.

(d) Suppose, to the contrary that every vertex of G has a degree of 3 or more, then $2m \geq 3n$. Therefore, $\frac{3}{2}n \leq m \leq \frac{5}{3}(n-2)$. After some algebra we get $n \geq 20$, which leads to contradiction.

7 [GC] Problem 9.15

Suppose that every vertex of G has degree 5 or more, then we have $2m \geq 5n$. Since $m \leq 3n - 6$, we have $n \geq 12$, which contradicts $n \leq 11$.

8 [GC] Problem 10.1

$\chi(G_1) = 3$; $\chi(G_2) = 4$; $\chi(G_3) = 4$; $\chi(G_4) = 3$; $\chi(G_5) = 4$.
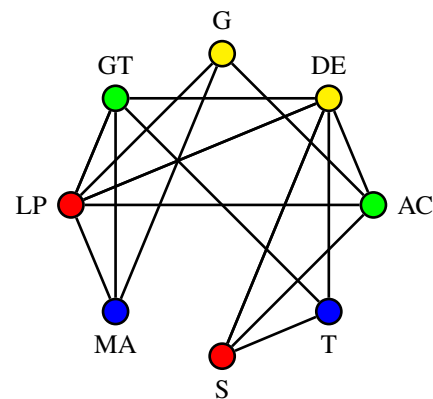
9 [GC] Problem 10.4

(a) False. K_4 is a planar graph that contains a triangle, however, its chromatic number is 4.

(b) False. Consider C_4, it has a 4-coloring, however, it also has a 2-coloring, which means $\chi(C_4) = 2$.

(c) False. Consider K_5, it has no 3-coloring, but $\chi(K_5) = 5$.

(d) False. Consider $K_{3,3}$, since it is bipartite, its chromatic number is 2, but it is not planar.

10 [GC] Problem 10.10

Consider a graph whose vertex set are these subjects, and two vertex i, j are joined by edge if and only if there exists some student who takes both the two subjects. The chromatic number is the minimum number of time periods needed for arranging all these subjects.

In this case, the chromatic number of the graph is 4, and a possible 4-coloring is:


so the earliest time that they can finish all exams is 4:45.

11 [GC] Problem 10.11

Consider a graph whose vertex set are these chemicals, and two vertex i, j are joined by edge if and only if they react with each other. The chromatic number of this graph is the minimum number of containers needed for shipping.

In this case, the chromatic number of the graph is 4, and a possible 4-coloring is:
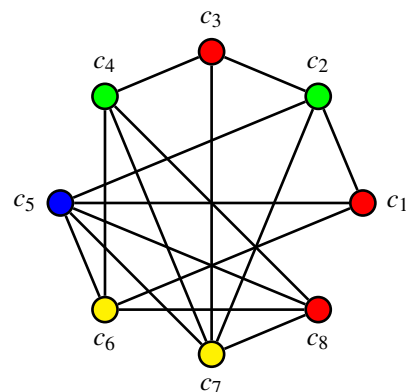
so at least 4 containers are needed, and thus the minimum cost is \$380.