Problem Solving: Homework 3.1

Name: Chen Shaoyuan Student ID: 161240004

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1 [TC] Problem 24.1-2

We claim that, after k passes of relaxation, $v.d < \infty$ holds if $d(s,v) \leq k$. This can be proved by mathematical induction: for the base step, only the source s satisfies $s.d = 0 < \infty$, and the conclusion is true; for the induction step, assume that before the i-th pass of relaxation, for every vertex v such that $d(s,v) < i$, $v.d < \infty$. When performing relaxation, an estimate $v.d$ is changed from infinity to a finite number, if and only if there exists a directed edge (u,v), such that $u.d < \infty$ and $v.d = \infty$. Therefore, for all vertices v such that $d(s,v) = i$, $v.d < \infty$. By mathematical induction, we prove the correctness of the claim.

Note that, if there exists a path from s to v, then $d(s,v) < |V|$ always holds, and Bellman-Ford algorithm performs $|V|-1$ passes of relaxation in total. If for all vertices v, there exists a path from s to v, then the algorithms must terminate with $v.d < \infty$. Otherwise, if there exists a vertex v, such that v is unreachable from s, then it remains $v.d = \infty$ when algorithm terminates. This is because, $v.d$ is always an upper bound of $\delta(s,v)$ which is infinity, so every relaxation attempting to tighten $v.d$ will not succeed.

2 [TC] Problem 24.1-3

We use a flag to record whether a relaxation succeeds in a pass of relaxation. When no relaxation succeeds in a pass, we terminate the algorithm. The total number of passes executed is $m+1$.

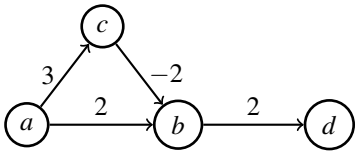
3 [TC] Problem 24.1-4

```
BELLMAN-FORD-MODIFIED(G, w, s)
 1  INITIALIZE-SINGLE-SOURCE(G, S)
 2  for i = 1 to |G.V| − 1
 3      for each edge (u, v) ∈ G.E
 4          RELAX(u, v, w)
 5  Let Q be a new queue of vertices
 6  for each vertex v ∈ G.V
 7      v.flag = FALSE
 8  for each edge (u, v) ∈ G.E
 9      if v.d > u.d + w(u, v)
10          Q.enqueue(v)
11          v.flag = TRUE
12  while Q is not empty
13      u = Q.dequeue()
14      for each v ∈ u.adj
15          v.d = −∞
16          if v.flag == FALSE
17              Q.enqueue(v)
18              v.flag = TRUE
```

4 [TC] Problem 24.2-2

After changing, only the last vertex in topological order is not taken. However, the last vertex does not have successor, therefore, the procedure remains correct.

5 [TC] Problem 24.3-2



Consider the graph shown above, if we take a as the source and run Dijkstra's algorithm, the procedure adds a, b, c, d to set S and relaxes edges $(a,b),(a,c)$; (b,d); (c,b). It finally terminates with $a.d = 0$, $b.d = 1$, $c.d = 3$, $d.d = 4$. Note that for vertex d, we have path $\langle a,c,b,d\rangle$ whose length is 3, which means Dijkstra's algorithm gives wrong answer.

In the **maintenance** part of the proof, formula (24.2) holds because all edges are non-negative. If negative edges are allowed, the formula no longer holds and the $u.d = \delta(s,u)$ might not hold for vertex u added to set S.

6 [TC] Problem 24.3-4

DIJKSTRA-CHECKER(G,w,s)

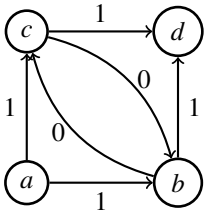
```
1  if s.d ≠ 0 or π.≠NIL
2      return FALSE
3  for each vertex v ∈ G.V
4      v.d' = ∞
5  s.d' = 0
6  for each edge (u,v) ∈ G.E
7      v.d' = min(v.d, u.d + w(u,v))
8  for each vertex v ∈ G.V
9      if v ≠ s
10         if v.d ≠ v.d'
11             return FALSE
12         if v.π ≠ NIL
13             if v.d ≠ v.π.d + w(v.π, v)
14                 return FALSE
15         elseif v.d ≠ ∞
16             return FALSE
17 return TRUE
```

7 [TC] Problem 24.3-7

G' has $|V| + \sum_{e \in E}[w(e) - 1]$ vertices.

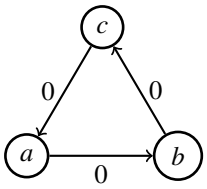
In breadth first-first search, a vertices v with less $d_{G'}(s,v)$ are colored black before one with larger $d_{G'}(s,v)$. For vertices $v \in V$, $d_{G'}(s,v) = \delta_G(s,v)$. In Dijkstra's algorithm, a vertex v with smallest $v.d$ is extracted from priority queue, and we've proved that $v.d = \delta_G(s,v)$. So, both breadth-first search and Dijkstra's algorithm repeat coloring or extracting vertex $v \in V$ with least $\delta_G(s,v)$, and since $\delta_G(s,v)$ are distinct, the two orders are same.

8 [TC] Problem 24.5-2



Consider the graph above, take a as the source, then trees consisting of edges $(a,b),(b,d),(b,c)$ and $(a,c),(c,d),(c,b)$ are both shortest-paths trees rooted at a, and for every edge $e \in E$, exactly one of the two trees contains e.

9 [TC] Problem 24.5-5



Consider the graph above, let a be the source vertex. If we take $a.\pi = c$, $b.\pi = a$, $c.\pi = b$, then they form a cycle.

10 [TC] Problem 24-2

a. If box A with dimensions $(x_1, x_2, \cdots, x_d)$ nests in box B with dimensions $(y_1, y_2, \cdots, y_d)$, box B nests in box C with dimensions $(z_1, z_2, \cdots, z_d)$, then there exist permutations π_1, π_2 on $\{1, 2, \cdots, d\}$, such that

$$x_{\pi_1(1)} < y_1, x_{\pi_1(2)} < y_2, \cdots, x_{\pi_1(d)} < y_d$$

$$y_{\pi_2(1)} < z_1, y_{\pi_2(2)} < z_2, \cdots, y_{\pi_2(d)} < z_d$$

therefore

$$x_{\pi_1(\pi_2(1))} < z_1, x_{\pi_1(\pi_2(2))} < z_2, \cdots, x_{\pi_1(\pi_2(d))} < z_d$$

note that $\pi_1 \circ \pi_2$ is still a permutation, so box A nests in box C, i.e. the nesting relation is transitive.

b. Given boxes A with dimensions $(x_1, x_2, \cdots, x_d)$ and B with dimensions $(y_1, y_2, \cdots, y_d)$. Sort $(x_1, x_2, \cdots, x_d)$ and $(y_1, y_2, \cdots, y_d)$ in increasing order respectively, then check whether there exists $i (1 \le i \le d)$, such that $x_i \ge y_i$. If exists, then A does not nest in B; otherwise, A nests in B.

Assume that array $(y_1, y_2, \cdots, y_n)$ is sorted in increasing order. For array $(x_1, x_2, \cdots, x_n)$ which is not sorted in increasing order, there must exist an inversion pair (x_i, x_j) $(x_i > x_j, i < j)$. If $x_i < y_i$ and $x_j < y_j$, since $x_i > x_j$, $y_i < y_j$, we have $x_j < y_i$ and $x_i < y_j$, so after exchanging x_i with x_j, the condition still holds. Since any finite permutation can be generated by swapping, greedy algorithm applies here and the method mentioned before is correct.

c. The pseudocode is shown in the next page.

```
LONGEST-NESTED-BOXES(B, n)
 1  maxd = 0
 2  for i = 1 to n
 3      sort dimensions of B_i in increasing order
 4      B_i.d = 0
 5      B_i.π = NIL
 6  for i = 1 to n
 7      for j = 1 to n
 8          if B_i is nested in B_j
 9              B_j.adj.insert(i)
10  for every B_i in B in topological order
11      for every j in B_i.adj
12      if B_j.d < B_i.d + 1
13          B_j.d = B_i.d + 1
14          B_j.π = B_i
15          if B_j.d > maxd
16              maxd = B_j.d
17              r = B_j
18  while r ≠ NIL
19      print r
20      r = r.π
```

This algorithm first builds a graph G, in which edge (u,v) means that v is nested in u. Then it calculates the longest nested boxes ending in v in topological order. Since we presorted the dimensions of all the boxes in $O(nd\log d)$ time, determining whether one box nests in another takes only $O(d)$ time, and thus building the graph takes $O(n^2d)$ in all. Calculating the longest nested boxes takes $O(|V|+|E|) = O(n^2)$ time. Therefore, the total running time is $O(nd\log d + n^2d)$.

11 [TC] Problem 24-3

a. First, build a graph $G = (V,E)$, where $V = \{c_1, c_2, \cdots, c_n\}$, $E = \{(i,j)|i,j \in V\}$, and $w(i,j) = \ln R[i,j]$. Then we run Bellman-Ford algorithm on G to determine whether negative-weight cycle. Such sequence exists if and only if G contains negative-weight cycle. The running time is $O(|V||E|) = O(n^3)$.

b. In the following procedure, assume that Bellman-Ford algorithm has been performed on G.

```
FIND-SEQUENCE(G)
 1  for every vertex v ∈ G.V
 2      v.flag = 0
 3  cnt = 0
 4  for every vertex v ∈ G.V
 5      u = v
 6      cnt = cnt + 1
 7      while u ≠ NIL and u.cnt == 0
 8          u.flag = cnt
 9          u = u.π
10          if u ≠ NIL and u.flag == cnt
11              print u
12              PRINT-SEQUENCE(u.π, u)
13              return TRUE
14  return FALSE
```

```
PRINT-SEQUENCE(X, u)
1  if X == u
2      return
3  PRINT-SEQUENCE(X.π, u)
4  print X
```

The total running time is $O(|V|+|E|) = O(n^2)$.