Problem Solving: Homework 3.7

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1 [GC] Problem 28.1-2

By using formula (28.8) repeatedly, we can get the LU decomposition of a matrix.

$$\begin{aligned}
A &= \begin{pmatrix} 4 & -5 & 6 \\ 8 & -6 & 7 \\ 12 & -7 & 12 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & -5 & 6 \\ 0 & 4 & -5 \\ 0 & 8 & -6 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & -5 & 6 \\ 0 & 4 & -5 \\ 0 & 0 & 4 \end{pmatrix}
\end{aligned}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}, U = \begin{pmatrix} 4 & -5 & 6 \\ 0 & 4 & -5 \\ 0 & 0 & 4 \end{pmatrix}$$

2 [GC] Problem 28.1-3

We use the algorithm described in LUP-DECOMPOSITION to calculate the LUP decomposition.

For the first iteration, $p = 3$, $\pi = (3, 2, 1)$

$$A \to \begin{pmatrix} 5 & 8 & 2 \\ 2 & 0 & 3 \\ 1 & 5 & 4 \end{pmatrix} \to \begin{pmatrix} 5 & 8 & 2 \\ 2/5 & -16/5 & 11/5 \\ 1/5 & 17/5 & 18/5 \end{pmatrix}$$

For the second iteration, $p = 3$, $\pi = (3, 1, 2)$

$$A \to \begin{pmatrix} 5 & 8 & 2 \\ 1/5 & 17/5 & 18/5 \\ 2/5 & -16/5 & 11/5 \end{pmatrix} \to \begin{pmatrix} 5 & 8 & 2 \\ 1/5 & 17/5 & 18/5 \\ 2/5 & -16/17 & 95/17 \end{pmatrix}$$

Therefore, $\pi = (3, 1, 2)$,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/5 & 1 & 0 \\ 2/5 & -16/17 & 1 \end{pmatrix}, U = \begin{pmatrix} 5 & 8 & 2 \\ 0 & 17/5 & 18/5 \\ 0 & 0 & 95/17 \end{pmatrix}$$

Multiplying P on both sides and substitute PA with LU, we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/5 & 1 & 0 \\ 2/5 & -16/17 & 1 \end{pmatrix} \begin{pmatrix} 5 & 8 & 2 \\ 0 & 17/5 & 18/5 \\ 0 & 0 & 95/17 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \\ 9 \end{pmatrix}$$

Eliminating L,

$$\begin{pmatrix} 5 & 8 & 2 \\ 0 & 17/5 & 18/5 \\ 0 & 0 & 95/17 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ 295/17 \end{pmatrix}$$

Eliminating U,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3/19 \\ -1/19 \\ 59/19 \end{pmatrix}.$$

3 [GC] Problem 28.1-6

Since L is a unit lower-triangular matrix, it is non-singular. Therefore, if a matrix A has an LU decomposition is singular if and only if U is singular. Hence, let L be any $n \times n$ unit lower-triangular matrix, e.g. I_n, and U be any $n \times n$ singular upper-triangular matrix, e.g. $0_{n\times n}$, then $A = LU = 0_{n\times n}$ is an $n \times n$ singular matrix that has LU decomposition.

4 [GC] Problem 28.1-7

It is necessary in LU-DECOMPOSITION, because a_{nn} needs be assigned to u_{nn}. It is unnecessary in LUP-DECOMPOSITION, because when executing the **for** iteration with $k = n$, k' must be k, thus the statements in line 13–15 have no effect, and the statements in line 16–19 will not be executed at all.

5 [GC] Problem 28.2-1

The first half, an $M(n)$-time matrix-multiplication algorithm implies an $O(M(n))$-time squaring algorithm, is obvious, because to square a matrix is just to multiply two identical matrices.

For the second half, if $S(n) = \Omega(n^3)$, we can use $\Theta(n^3) = O(S(n))$ brute-force matrix-multiplication algorithm to compute the product of two $n \times n$ matrices. Otherwise, we can compute the product of two $n \times n$ matrices AB by squaring a $2n \times 2n$ matrix:

$$\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} A^2 & AB \\ 0 & 0 \end{pmatrix}.$$

Since $S(n) = O(n^3)$ is polynomially bounded, we have the regularity property $S(2n) = O(S(n))$, and the total running times is $O(S(2n)) = O(S(n))$

6 [GC] Problem 28.2-2

(to be done)

7 [GC] Problem 28.2-3

If the LUP decomposition of matrix A is $PA = LU$, since $\det(L) = 1$ and U is triangular, the determinant of A is the product of diagonal entries of U divided by $\det(P) = (-1)^s$, where s is the number of inversion pairs in P, which is easy to find. We have proved that computing LUP decomposition and multiplying two matrices have essentially the same difficulty, so we can compute the determinant in $O(M(n))$-time if $M(n)$-time matrix-multiplication algorithm exists.

(to be done)

8 [GC] Problem 28.3-1

Let A be a symmetric positive-definite matrix. Suppose, to the contrary, that there exists i, such that $a_{ii} \le 0$. Let $x_j = \delta_{ij}$ (δ_{ij} is the Kronecker delta), then vector $x \ne 0$, and $x^{\mathrm{T}}Ax = a_{ii} \le 0$, which contradicts the definition of positive-definite matrix.

9 [GC] Problem 28.3-3

Let A be a symmetric positive-definite matrix. Suppose, to the contrary that a_{ij} ($i \ne j$) is the maximum element. Since $a_{ii} > 0$, let $x_k = -(a_{ij}/a_{ii})\delta_{ik} + \delta_{jk}$, then vector $x \ne 0$. Consider $x^{\mathrm{T}}Ax$, we have

$$\begin{aligned} x^{\mathrm{T}}Ax &= a_{ii}\left(-\frac{a_{ij}}{a_{ii}}\right)^2 + 2a_{ij}\left(-\frac{a_{ij}}{a_{ii}}\right) + a_{jj} \\ &= \frac{a_{ii}a_{jj} - a_{ij}^2}{a_{ii}}. \end{aligned}$$

Since a_{ii} is positive and $a_{ij} \ge a_{ii} > 0$, $a_{ij} \ge a_{ii} > 0$, we have $x^{\mathrm{T}}Ax \le 0$, which leads to contradiction.

10 [GC] Problem 28-1

a. By using formula (28.8) repeatedly,

$$\begin{aligned} A &= \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix} \end{aligned}$$

$$L = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix}.$$

b. Let $Ux = y$, then $Ly = (1,1,1,1,1)^{\mathrm{T}}$. By applying forward substitution, we get $y_1 = 1$, $y_2 = y_1 + 1 = 2$, $y_3 = y_2 + 1 = 3$, $y_4 = y_3 + 1 = 4$, $y_5 = y_4 + 1 = 5$. Hence $Ux = y = (1,2,3,4,5)^{\mathrm{T}}$. By applying backward substitution, we get $x_5 = 5$, $x_4 = x_5 + 4 = 9$, $x_3 = x_4 + 3 = 12$, $x_2 = x_3 + 2 = 14$, $x_1 = x_2 + 1 = 15$, i.e. $x = (15,14,12,9,5)^{\mathrm{T}}$.

c. The inverses of L and U are easy to find:

$$L^{-1} = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, U^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix},$$

and the inverse of A is

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

d. Recall formula (28.8):

$$A = \begin{pmatrix} a_{11} & w^{\mathrm{T}} \\ v & A' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v/a_{11} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{11} & w^{\mathrm{T}} \\ 0 & A' - vw^{\mathrm{T}}/a_{11} \end{pmatrix}$$

Note that all elements except the first of v and w^{T} are zero, and all entries except the one on the top left corner of vw^{T}/a_{11} are zero, thus $A' - vw^{\mathrm{T}}/a_{11}$ is still a tridiagonal matrix, and by Schur complement lemma (Lemma 28.5), it is still symmetric and positive-definite, which means that we can transform an $n \times n$ symmetric positive-definite tridiagonal matrix LU decomposition problem to an $(n-1) \times (n-1)$ one in constant time, and thus computing LU decomposition in $O(n)$ time. Furthermore, we can easily conclude by mathematical induction, that L and U are lower bidiagonal and upper bidiagonal, respectively.

For a lower bidiagonal matrix L, we can solve $Ly = b$ in $O(n)$ time:

$$y_1 = \frac{b_1}{l_{11}}, \quad y_2 = \frac{b_2 - l_{21}b_1}{l_{22}}, \quad \cdots, \quad y_n = \frac{b_n - l_{n,n-1}b_{n-1}}{l_{nn}}.$$

Likewise, for upper bidiagonal matrix U, we can solve $Uy = b$ in $O(n)$ time. Therefore, we can solve $Ax = b$ in $O(n)$ time by performing LU decomposition if A is tridiagonal.

Since forming A^{-1} takes $\Omega(n^2)$ to compute the entries, any method based on the inverse is asymptotically more expensive in the worst case.

e. We try to apply formula (28.8) repeatedly, if the entry on the top left corner is nonzero. In case that a zero lies on the top left corner, we swap the first two rows, or equivalently, left multiply a permutation matrix P_{21} on both sides, and we only have to compute the LUP decomposition of tridiagonal matrix A'.

$$A = \begin{pmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix}$$

$$P_{21}A = \begin{pmatrix} a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{12} & 0 & 0 \\ 0 & a_{32} & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix} = I \begin{pmatrix} a_{21} & w^{\mathrm{T}} \\ 0 & A' \end{pmatrix} = LU'$$

Therefore, we obtain $PA = LU$ in $O(n)$ time. The only subtlety here, is that the matrix U might be upper tridiagonal, instead of bidiagonal, due to row switching. But this do not affect the result, for, every unknown x_i in a linear equation system $Ux = b$ whose coefficient matrix U is upper tridiagonal only depends on b_i and x_{i+1}, x_{i+2}, if exist, which means we can still solve such system in $O(n)$ time.