Problem Solving: Homework 3.10

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November 13, 2017

1 [TJ] Exercise 3-3

The Cayley table formed by symmetries of a rectangle is:

$\circ$	id	ρ	μ_h	μ_v
id	id	ρ	μ_h	μ_v
ρ	ρ	id	μ_v	μ_h
μ_h	μ_h	μ_v	id	ρ
μ_v	μ_v	μ_h	ρ	id

where ρ denotes 180° rotation, μ_h, μ_v denote reflection across the horizontal axis and the vertical axis, id denotes identity. The Cayley table for $\mathbb{Z}_4$ is:

$\cdot$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

There are 4 elements in each group.

The groups are not same, because they contain different elements.

2 [TJ] Exercise 3-6

The Cayley table for $\mathbb{Z}_4$ is:

$\cdot$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

3 [TJ] Exercise 3-7

First, we show that $*$ is a mapping from $S \times S$ to S. If $a, b \neq -1$, then $a * b = a + b + ab = (a+1)(b+1) - 1 \neq -1$. Hence the closure holds.

Second, the identity is 0, for every $a \in S$, $0 * a = 0 + a = a$. Likewise $a * 0 = a$.

Third, let's verify the associativity:

$$\begin{aligned}(a*b)*c &= (a+b+ab)*c \\ &= (a+b+ab)+c+(a+b+ab)c \\ &= a+b+c+ab+ac+bc+abc \\ a*(b*c) &= a*(b+c+bc) \\ &= a+b+c+bc+a(b+c+bc) \\ &= a+b+c+ab+ac+bc+abc\end{aligned}$$

So the associativity holds. Fourth, for every element $a \in S$, the inverse of a is $1/(a+1)-1$, for

$$\begin{aligned}a*(1/(a+1)-1) &= a+1/(a+1)-1+a(1/(a+1)-1) \\ &= a+1-1-a \\ &= 0\end{aligned}$$

Likewise $(1/(a+1)-1)*a = 0$.

Therefore, $(S, *)$ is a group. Also, it is easy to verify that $a*b = b*a$, hence it is abelian.

4 [TJ] Exercise 3-17

$$\begin{aligned}G_1 &= (\mathbb{Z}_8, +_8) \\ G_2 &= (\mathbb{Z}_8, \oplus) \\ G_3 &= (\{e^{2\pi i/8} : i \in \mathbb{Z}_8\}, \cdot)\end{aligned}$$

where $+_8$ means plus modulo 8, $\oplus$ means bitwise exclusive-or, $\cdot$ means the multiplication of two complex numbers.

These groups are different, because the sets of elements of the groups are different, or the binary operations differ.

5 [TJ] Exercise 3-28

1. If $m = 0$ or $n = 0$, then the $g^m = e$ or $g^n = e$, so the conclusion holds. If both m and n are positive, note that

   $$g^n = g^{n-1}g = (g^{n-2}g)g = g^{n-2}g^2 = \cdots = gg^{n-1}$$

   so

   $$g^m g^n = g^m(gg^{n-1}) = (g^m g)g^{n-1}$$

$$= g^{m+1}g^{n-1} = \cdots = g^{m+n}$$

Likewise, when both m and n are negative, the conclusion holds.

When one of m and n, assume with out loss of generality, n, is positive, and the other is negative, we have

$$g^m g^n = (g^{m+1}g^{-1})(gg^{n-1}) = g^{m+1}(g^{-1}g)g^{n-1}$$
$$= g^{m+1}g^{n-1} = \cdots = g^{m+n}$$

Therefore, the conclusion holds.

2. If n is non-negative, then $(g^m)^n = (g^m)^{n-1}g^m = (g^m)^{n-2}g^m g^m = (g^m)^{n-2}(g^{2m}) = \cdots = g^{mn}$.

   Otherwise, we have $(g^m)^n = ((g^m)^{-1})^{-n} = (g^{-m})^{-n} = g^{mn}$.

   So the conclusion holds.

3. $(gh)^n = ((gh)^{-1})^{-n} = (h^{-1}g^{-1})^{-n}$

   If G is abelian, then

   $$\begin{aligned}(gh)^n &= (gh)^{n-1}(gh) = (gh)^{n-2}(ghgh)\\ &= (gh)^{n-2}(gghh) = (gh)^{n-2}(g^2h^2)\\ &= \cdots = g^n h^n\end{aligned}$$

   if n is non-negative, otherwise

   $$(gh)^n = (h^{-1}g^{-1})^{-n} = h^n g^n = g^n h^n$$

   So the conclusion holds.

6 [TJ] Exercise 3-36

H is a subset of $\mathbb{Q}^*$, and the identity $1 \in H$. For every $g = 2^{k_1}$, $h = 2^{k_2}$, we have $gh = 2^{k_1+k_2} \in H$, and $g^{-1} = 2^{-k_1} \in H$, hence H is a subgroup of $\mathbb{Q}^*$.

7 [TJ] Exercise 3-38

$\mathbb{T}$ is a subset of $\mathbb{C}^*$, and the identity $1 \in \mathbb{T}$. For every $g, h \in \mathbb{T}$, we have $|gh| = 1$ and $|g^{-1}| = 1$, so $gh, g^{-1} \in \mathbb{T}$, hence $\mathbb{T}$ is a subgroup of $\mathbb{C}^*$.

8 [TJ] Exercise 3-41

H is a subset of G, and the identity $0_{2\times 2} \in H$. For every $A, B \in H$, we have $(A+B)_{11} = (A+B)_{22} = 0$, $-A_{11} = -A_{22} = 0$, so $A+B, -B \in H$, hence H is a subgroup of G.

9 [TJ] Exercise 3-41

$ba = a^4b = (a^3a)b = (ea)b = ab$

10 [TJ] Exercise 3-52

Let $x = e$, we have $y^2 = y$, therefore $y = e$, so the group G contains only the identity, i.e. G is trivial, and of course G is abelian.

11 [TJ] Exercise 4-1

(a) False. $U(8) = \{1,3,5,7\}$, while none of them is a generator of $U(8)$.

(b) False. 49 is relatively prime to 60, so it is a generator of $\mathbb{Z}_{60}$, while 49 is not prime.

(c) False. Assume, to the contrary that $\mathbb{Q}$ has a generator a, then for all $q \in \mathbb{Q}$, there exists an integer n, such that $q = na$. However, if $q = a/2$, such n does not exist. So $\mathbb{Q}$ is not cyclic.

(d) False. Consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (Klein 4-group), all of its subgroups are cyclic, however, the group itself is not cyclic.

(e) True. If G contains any infinite order element a, then G contains infinitely many different subgroups: $\langle a \rangle$, $\langle a^2 \rangle$, $\langle a^3 \rangle$, $\cdots$. Therefore, all elements of G have finite order. Since G contains finite number of subgroups, it contains finitely many cyclic subgroups, each of which is finite. Because every element belongs to at least one cyclic subgroup, the group G is exactly the union of all its cyclic subgraphs, which is still finite.

12 [TJ] Exercise 4-12

The trivial group is a cyclic group with exactly one generator.

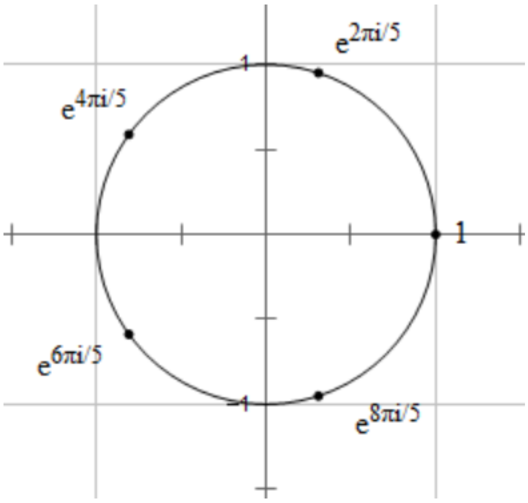
The cyclic group $\mathbb{Z}$ has exactly two generators, 1 and -1.

The cyclic group $\mathbb{Z}_8$ has exactly four generators, 1, 3, 5 and 7.

For arbitrary n, since every cyclic group is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_n$, we only have to consider $\mathbb{Z}_n$. So, whether there exists a cyclic group with n generators depends on whether there exists positive integer m such that $\phi(m) = n$, where ϕ is the Euler ϕ-function. For example, $\phi(3) = 2$, so $\mathbb{Z}_3$ has 2 generators. However, it can be proved in number theory that there does not exist positive integer m such that $\phi(m) = 3$, hence there does not exist cyclic group that contains exactly 3 generators.

13 [TJ] Exercise 4-21

The 5th roots of unity are: 1, $e^{2\pi i/5}$, $e^{4\pi i/5}$, $e^{6\pi i/5}$, $e^{8\pi i/5}$.



The generators of this group are $e^{2\pi i/5}$, $e^{4\pi i/5}$, $e^{6\pi i/5}$, $e^{8\pi i/5}$.

The primitive 5-th roots of unity are $e^{2\pi i/5}$, $e^{4\pi i/5}$, $e^{6\pi i/5}$, $e^{8\pi i/5}$.

14 [TJ] Exercise 4-24

$\mathbb{Z}_{pq}$ has $\phi(pq)$ generators, since p and q are distinct primes, we have $\phi(pq) = \phi(p)\phi(q) = (p-1)(q-1)$, because ϕ is a multiplicative function. So $\mathbb{Z}_{pq}$ has $(p-1)(q-1)$ generators.

15 [TJ] Exercise 4-32

The order of y is $n/\gcd(k,n) = n$, which means $\langle y\rangle$ contains as many elements as G has, i.e. y is a generator of G.

16 [TJ] Exercise 5-3

(a) (16)(15)(13)(14), even

(b) (16)(15)(24)(23), even

(c) (16)(12)(14)(12)(14), odd

(d) (14)(15)(12)(17)(13)(12)(14)(12)(13)(16)(14)(15), even

(e) (17)(13)(16)(12)(14), odd

17 [TJ] Exercise 5-5

The subgroups of S_4 are

- the trivial group: $\{\text{id}\}$;
- subgroups of order 2: $\{\text{id},(12)\}$, $\{\text{id},(13)\}$, $\{\text{id},(14)\}$, $\{\text{id},(23)\}$, $\{\text{id},(24)\}$, $\{\text{id},(34)\}$, $\{\text{id},(12)(34)\}$, $\{\text{id},(13)(24)\}$, $\{\text{id},(14)(23)\}$;
- cyclic subgroups of order 3: $\langle(123)\rangle$, $\langle(124)\rangle$, $\langle(134)\rangle$ and $\langle(234)\rangle$;
- cyclic subgroups of order 4: $\langle(1234)\rangle$, $\langle(1324)\rangle$, $\langle(1243)\rangle$;
- Klein 4-groups: {id, (12), (34), (12)(34)}, {id, (13), (24), (13)(24)}, {id, (14), (23), (14)(23)}, {id, (12)(34), (13)(24), (14)(23)};
- S_3 subgroups: permutations of $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$ and $\{2,3,4\}$;
- "rectangle" subgroups: $\{\text{id},(13),(24),(13)(24),(12)(34),(14)(23),(1234),(1432)\}$, $\{\text{id},(14),(23),(14)(23),(12)(34),(13)(24),(1243),(1342)\}$ and $\{\text{id},(12),(34),(14)(23),(12)(34),(13)(24),(1324),(1423)\}$;
- the alternating group: A_4;
- S_4 itself.

The sets are

(a) $\{\sigma\in S_4 : \sigma(1)=3\} = \{(13),(13)(24),(132),(134),(1324),(1342)\}$;

(b) $\{\sigma\in S_4 : \sigma(2)=2\} = \{\text{id},(13),(14),(34),(134),(143)\}$;

(c) $\{\sigma\in S_4 : \sigma(1)=3 \text{ and } \sigma(2)=2\} = \{(13),(134)\}$;

18 [TJ] Exercise 5-16

Let 1, 2, 3, 4 denote the four vertices of a tetrahedron, respectively. Then all its rigid motions can be represented as a permutation of 1, 2, 3, 4, and they are:

- eight 120° rotation operations: (123), (321), (124), (421), (134), (431), (234), (432);
- three 180° rotation operations: (12)(34), (13)(24), (14)(23);
- and, the identity: id.

Note that these permutations are exactly all even permutations on 4 letters, so it is the same as A_4.

19 [TJ] Exercise 5-27

We only have to prove that λ_g is one-to-one and onto.

For every $b \in G$, there exists $a = g^{-1}b$, such that

$$\lambda_g(a) = gg^{-1}b = b$$

so λ_g is onto.

For every $a, b \in G$, if $\lambda_g(a) = \lambda_g(b)$, i.e. $ga = gb$, then we have $a = b$, so λ_g is one-to-one.

Therefore, λ_g is one-to-one and onto, so λ_g is a permutation of G.

20 [TJ] Exercise 5-29

The centers of D_8, D_{10} are $\{\text{id}, i\}$, where i denotes inversion through the center of the polygon, or, equivalently, 180° rotation.

For arbitrary n, the center of D_n is $\{\text{id}\}$ if n is odd, or $\{\text{id}, i\}$ is even. Note, that only when n is even D_n contains i.

It is obvious that id and i are central. For any element other than id, it is either a reflection or a rotation. Let ρ be any rotation and μ be any reflection. It is easy to verify that $\rho\mu = \mu\rho^{-1}$. Note that $\rho^{-1} = \rho$ if and only if $\rho = i$. So any element other than id and i is not central.

21 [TJ] Exercise 6-11

(a) $\to$ (b): for every $h \in H$, there exists $h' \in H$, such that $g_1 h = g_2 h'$. Hence we have $hg_2^{-1} = h(g_1 hh'^{-1})^{-1} = (hh'^{-1}h^{-1})g_1^{-1} \in Hg_1^{-1}$, so $Hg_2^{-1} \subseteq Hg_1^{-1}$. Likewise we have $Hg_1^{-1} \subseteq Hg_2^{-1}$, so $Hg_1^{-1} = Hg_2^{-1}$.

(b) $\to$ (a): for every $h \in H$, there exists $h' \in H$, such that $hg_1^{-1} = h'g_2^{-1}$. Hence we have $g_2 h = (g_1 h^{-1}h')h = g_1(h^{-1}h'h) \in g_1 H$, so $g_2 H \subseteq g_1 H$. Likewise $g_1 H \subseteq g_2 H$, so $g_1 H = g_2 H$.

(a) $\to$ (c) is immediate.

(c) $\to$ (d): since $g_1 H \subseteq g_2 H$ and $g_1 = g_1 e \in g_1 H$, we have $g_1 \in g_2 H$, so there exists $h \in H$, such that $g_1 = g_2 h$, i.e. $g_2 = g_1 h^{-1}$, so $g_2 \in g_1 H$.

(d) $\to$ (e): since $g_2 \in g_1 H$, there exists $h \in H$, such that $g_2 = g_1 h$, so $g_1^{-1} g_2 = h \in H$.

(e) $\to$ (a): for every $h \in H$, we have $g_1 h = g_1(g_1^{-1}g_2)(g_1^{-1}g_2)^{-1}h = g_2((g_1^{-1}g_2)^{-1}h) \in g_2 H$ and $g_2 h = g_2(g_1^{-1}g_2)^{-1}(g_1^{-1}g_2)h = g_1(g_1^{-1}g_2 h) \in g_1 H$, so every element of $g_1 H$ is an element of $g_2 H$, and vice versa. Hence $g_1 H = g_2 H$.

Therefore, these 5 conditions are equivalent.

22 [TJ] Exercise 6-12

Consider the left coset gH and right coset Hg for arbitrary $g \in G$. For every $h \in H$, $gh = gh(g^{-1}g) = (ghg^{-1})g \in Hg$, which means $gH \subseteq Hg$; $hg = (gg^{-1})hg = g(g^{-1}h(g^{-1})^{-1}) \in gH$, which means $Hg \subseteq gH$, so $gH = Hg$. Therefore, the right cosets are identical to left cosets.

23 [TJ] Exercise 6-16

Since G is finite, every element of G has finite order. Let's consider the elements whose orders are not 2. G contains exactly one element of order 1, the identity. For every $a \in G$ that the order or a are greater than 2, a^{-1} is also an element whose order is greater than 2. Furthermore, $a^{-1} \neq a$, for otherwise $a^2 = 1$, which leads to contradiction. This means that the elements whose orders are greater than 2 can be paired up. Also, we have $|G| = 2n$. Therefore, the number of elements of order 2 is odd.

The conclusion above shows that G contains at least one element of order 2. The cyclic graph generated by such an element is a subgroup of G of order 2.

24 [TJ] Exercise 6-21

For arbitrary element $a \in G$ ($a \neq e$), the order of a is p^k, where $0 < k \leq n$. Then, $a^{p^{k-1}}$ is an element of order p, which means $\langle a^{p^{k-1}} \rangle$ is a proper subgroup of order p.

If $n \geq 3$, it is true that G must have proper subgroup of order p^2. If G contains element of order p^k with $k \geq 2$, the proof is done. So we only have to consider the case that the orders of all non-identity elements are p. To prove this statement, let us introduce some concepts. In group G,

- *Conjugacy*: if $yg = gx$, then x and y are said to be conjugate, and one is the other's conjugate. It is easy to verify that conjugacy is an equivalence relation, and let $CI(x)$ denote the equivalence class that contains x, i.e. the set of all conjugates of x.
- *Centralizer*: for $x \in G$, the set $C(x) = \{g \in G : xg = gx\}$ is called the centralizer of x. It is easy to verify that $C(x)$ is a subgroup of G.
- *Central element*: if $x \in G$ commutates with all elements, i.e. $xg = gx$ for all $g \in G$, then x is called a central element of S. It is immediate that x is a central of G if and only if x itself is the only conjugate of x.

- *Center*: the set of all central elements of G is called the center of G and denoted as $Z(G)$.

The conjugacy is an equivalence relation means that all equivalence classes partitions G. Since all equivalence classes of size 1 contains only the central elements, we replace these classes with $Z(G)$. Hence we obtain:

$$|G| = |Z(G)| + \sum |CI(x_i)|$$

where $CI(x_i)$ are equivalence classes with more than one elements. For every $CI(x_i)$, consider every two conjugate elements $x, y \in CI(x_i)$, which satisfies $yg = gx$. If there exists a, such that $ya = ax$, then $x(g^{-1}a) = g^{-1}gxg^{-1}a = g^{-1}ygg^{-1}a = g^{-1}ya = (g^{-1}a)x$, i.e. $g^{-1}a \in C(x)$, or equivalently $a \in gC(x)$. Also, we can prove for every $b \in gC(x)$, $g^{-1}b \in C(x)$, $yb = ygg^{-1}b = gxg^{-1}b = gg^{-1}b = bx$. So the number of elements in $CI(x_i)$ equals to the number of left cosets of $C(x_i)$, i.e. $|CI(x_i)| = |G|/|C(x_i)| = p^i, i \geq 1$.

Since $|G|$ and $\sum |CI(x_i)|$ are multiples of p, $|Z(G)|$ is also. This means $Z(G)$ contains non-identity elements, and let a denote one of them. Let b be any element that is not in $\langle a \rangle$, then $\{x : x = a^m b^n, m, n \in \mathbb{Z}\}$ is an abelian subgroup of G with p^2 elements, which completes the proof.

25 [TJ] Exercise 9-6

Suppose $f : \{\omega_n{}^i\} \to \mathbb{Z}_n$ is defined as

$$f(\omega_n{}^i) = i$$

then f is one-to-one and onto. And we have

$$\begin{aligned} &f(\omega_n{}^i \cdot \omega_n{}^j) = f(\omega_n{}^{(i+j) \bmod n}) = (i+j) \bmod n \\ =&[f(\omega_n{}^i) + f(\omega_n{}^j)] \bmod n = (i+j) \bmod n \end{aligned}$$

So the nth roots of unity are isomorphic to $\mathbb{Z}_n$.

26 [TJ] Exercise 9-7

Let $\langle a \rangle = \{e, a, a^2, \cdots, a^{n-1}\}$ denote the cyclic group of order n. Suppose $f : \langle a \rangle \to \mathbb{Z}_n$ is defined as

$$f(a^n) = n$$

then f is one-to-one and onto. And we have

$$\begin{aligned} &f(a^i \cdot a^j) = f(a^{(i+j) \bmod n}) = (i+j) \bmod n \\ =&[f(a^i) + f(a^j)] \bmod n = (i+j) \bmod n \end{aligned}$$

So $\langle a \rangle$ is isomorphic to $\mathbb{Z}_n$.

27 [TJ] Exercise 9-8

Suppose, to the contrary, that $\mathbb{Q}$ is isomorphic to $\mathbb{Z}$. Since $\mathbb{Z}$ is cyclic, $\mathbb{Q}$ is also cyclic. However, we have already proved in Exercise 4-1(c) that $\mathbb{Q}$ is not cyclic, which leads to contradiction.

28 [TJ] Exercise 9-9

We have proved in Exercise 3-7 that G is a group.

We define a map f from G to $\mathbb{R}^*$ as

$$f(a) = a + 1$$

then f is one-to-one and onto. Also, we have

$$\begin{aligned} &f(a * b) = f(a + b + ab) = a + b + ab + 1 \\ =&f(a)f(b) = (a+1)(b+1) = a + b + ab + 1 \end{aligned}$$

so $(G, *)$ is isomorphic to $\mathbb{R}^*$.