

Problem Solving: Homework 3.3

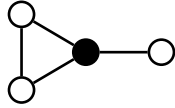
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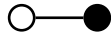
1 [GC] Problem 5.3

(a) No. Here is a counterexample.



Consider the solid vertex. It lies on some cycle, however, it is a cut-vertex too.

(b) No. Here is a counterexample.



Consider the solid vertex. It does not lie on any cycle, but it is not a cut-vertex as well.

(c) No. Here is a counterexample.



It has two end-vertices but only one cut-vertex.

(d) No. Consider the counterexample in (c), it has two bridges but only one cut-vertex.

2 [GC] Problem 5.4

Let a, b be any two distinct vertices in $G - v$ that are connected in \bar{G} . We are going to prove that a and b are still connected in $\bar{G} - v$.

Case 1: a and b are in different components of $G - v$. This means that $ab \notin E(G)$ and $ab \in E(\bar{G})$, and thus $ab \in E(\bar{G} - v)$. Therefore, a and b are still connected in $\bar{G} - v$.

Case 2: a and b are in the same component of $G - v$. Since $G - v$ has at least two components, let c be any vertex in the component of G other than the one a and b are in. Therefore, $ac, cb \notin E(G)$, and $ac, cb \in E(\bar{G})$, and thus $ac, cb \in E(\bar{G})$.

Therefore, removing v in \bar{G} does not make any two connected vertices $a, b (a, b \neq v)$ disconnected. This means that v is not a cut-vertex of \bar{G} .

3 [GC] Problem 5.6

The sufficiency is immediate by Corollary 5.2. We only have to prove the necessity.

Let v be any cut-vertex in G , and a, b, c be neighbors of v . a, b, c must in at least two different components of $G - v$, otherwise, v is not a cut-vertex. Furthermore, there exists a component of $G - v$ that contains exactly one of a, b, c , and without loss of generality, we assume the vertex is a . Suppose, to the contrary, that va is not a bridge of G , then va lies on some cycle by Theorem 4.1. Since a, b, c are only neighbors of v , either vb or vc lies on C . Without loss of generality, assume $vb \in C$. Therefore, $C - v$ is an $a - b$ path, which contradicts Theorem 5.3. Therefore, va is a bridge.

4 [GC] Problem 5.10

Note that a graph of size at least 2 contains at least 3 vertices, so Theorem 5.7 applies.

Sufficiency: let u, v be any two distinct vertices in G . Since G is connected, there exists a $u - v$ path, say $(u, w_1, w_2, \dots, w_n, v)$. Since edges uw_1, w_1w_2 , edges w_1w_2, w_2w_3, \dots , edges $w_{n-1}w_n, w_nv$ lie on common cycles respectively, by Theorem 5.8 uw_1, w_nv lie on common cycle, and thus u, v lie on same cycle. By Theorem 5.7 G is nonseparable.

Necessity: let uv, vw be any two adjacent edges in nonseparable graph G . By Theorem 5.7, u, w lie on common cycle C . If $v \in C$, then uv, vw lie on common cycle C ; otherwise, let P be either $u - w$ path in cycle C , then $P \cup \{uv, vw\}$ forms a cycle on which uv, vw lie.

5 [GC] Problem 5.11

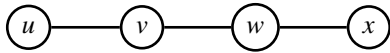
By Theorem 5.3, we only have to show that, for every three distinct vertices $u, v, w \in G$, there exists a $u - w$ path that does not contain v .

If $uw \in E(G)$, then the proof is done; otherwise, since $\deg u = \deg w \geq n/2$ and $|V(G - \{u, w\})| = n - 2$, by pigeonhole principle, there exist two vertices

a, b , such that a, b are neighbors of both u and w . At least one of a and b is not v , that vertex, along with u and v , forms a $u - w$ path that does not contain v .

6 [GC] Problem 5.14

- (a) Since G is connected, G_1 must be connected to some vertex in $G - G_1$. However, G_1 is not connected to $G - v - G_1$, because G_1 is a component of $G - v$. Therefore, only v is connected to G_1 , and $G[V(G_1) \cup \{v\}]$ is connected.
- (b) Consider the following graph G



v is a cut-vertex of G , and let G_1 be the component containing w and x . $G[V(G_1) \cup \{v\}]$ is not a block, because w is a cut-vertex.

7 [GC] Problem 5.15

Let B be a block of G .

(1) \rightarrow (2): since B is a nontrivial connected graph, it is induced by its edge set $E(B)$. Then we are going to prove $E(B)$ is an equivalent class resulting from the equivalence relation R defined in Theorem 5.8. Since B is connected, by Problem 5.10, for every two edges e, f , we have $e R f$. Let e be any edge in $E(B)$, if $e R f$, then $f \in E(B)$. Otherwise, let C be the common cycle where e, f lie on. $G \cup C$ is a nonseparable graph by Problem 5.10, and G is a proper subgraph of $G \cup C$, which violates interpretation (1). Hence, $E(B)$ is an equivalent class of R .

(2) \rightarrow (1): since $E(B)$ is an equivalent class of R , by Problem 5.10, B is nonseparable. B can't be a proper subgraph of any other nonseparable graph B' of G , otherwise, there exists $E(B)$ is a proper subset of $E(B')$ because B' contains no isolated vertex, and for every $e, f \in E(B')$, $e R f$, which means $E(B)$ is not an equivalent class.

8 [GC] Problem 5.20

- (a) Since G is not k -connected ($k \leq n - 2$), $\kappa(G) \leq k - 1$, and there exists a vertex cut U_0 with $|U_0| = \kappa(G)$. Let C_1, C_2 be any two components of $G - U_0$, and let u, v be two vertices such that $u \in C_1$ and $v \in C_2$. Take any $k - 1 - \kappa(G)$ (possibly zero) vertices from $G - U_0 - \{u, v\}$ and add them to U_0 , we obtain a vertex-cut U with $|U| = k - 1$.

- (b) Since G is not k -edge-connected, $\lambda(G) \leq k - 1$, and there exists an edge cut X_0 with $|X_0| = \lambda(G)$. Since removing arbitrary number of edges does not decrease the number of components, we can take any $k - 1 - \lambda(G)$ (possibly zero) edges from $E(G) \setminus X_0$ and add them to X_0 , and thus we obtain an edge cut X with $|X| = k - 1$.

9 [GC] Problem 5.22

- (a) When $k = 1$, the proof is done. Otherwise, let $e = uv$. Consider every vertex subset U of $G - e$ with $|U| = k - 2$. Since G is k -connected, if $u \in |U|$, $G - e - |U|$ is still connected; if $u \notin |U|$, $G - e - |U \cup \{u\}| = G - |U \cup \{u\}|$ is still connected. Therefore, $G - e$ is $(k - 1)$ -connected.
- (b) When $k = 1$, the proof is done. Otherwise, consider every edge subset X of $G - e$ with $|X| = k - 2$, since $G - e - X = iG - (\{e\} \cup X)$ and $\{e\} \cup X$ is not an edge cut of G because G is k -edge-connected, X is neither an edge cut of $G - e$. Therefore, $G - e$ is $(k - 1)$ -edge-connected.

10 [GC] Problem 5.30

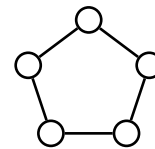
$$\bar{\kappa}(G) \geq \kappa(G), \bar{\lambda}(G) \geq \lambda(G), \bar{\kappa}(G) \leq \bar{\lambda}(G).$$

The first two inequalities are what the definitions of $\bar{\kappa}$ and $\bar{\lambda}$ imply. Let's prove the third one.

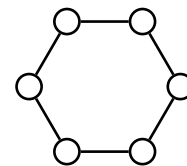
Let H' be the subgraph of G such that $\kappa(H') = \bar{\kappa}(G)$. By Theorem 5.11, $\bar{\kappa}(G) = \kappa(H') \leq \lambda(H')$. Since $\lambda(H') \leq \bar{\lambda}(G)$, we have $\bar{\kappa}(G) \leq \bar{\lambda}(G)$.

11 [GC] Problem 6.4

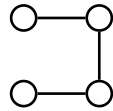
- (a) C_5



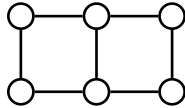
- (b) C_6



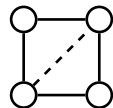
(c) P_4



(d) $P_2 \times P_3$



(e) (the dashed edge is e)



12 [GC] Problem 6.5

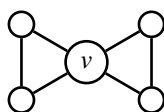
It is K_5 with any of its edges removed.

13 [GC] Problem 6.6

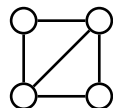
Assume G is r -regular. Since G is not Eulerian, by Theorem 6.1, r is odd, and thus n is even. Therefore, \bar{G} is $(n-1-r)$ -regular graph, where $n-1-r$ is even. By Theorem 6.1, \bar{G} is Eulerian.

14 [GC] Problem 6.13

(a) The following graph is not Hamiltonian, because it violates a necessary condition for a graph to be Hamiltonian (Theorem 6.5), if we take $S = \{v\}$.

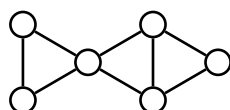


(b) The following graph is not Eulerian, because it has two odd vertices.



(c) The same one as (b).

(d)



15 [GC] Problem 6.16

(a) We have proved in Problem 6.6 that if G is not Eulerian, then \bar{G} is Eulerian. If G is Eulerian, then r is even. Hence, \bar{G} is $(n-1-r)$ -regular, where $n-1-r$ is odd, which means \bar{G} is not Eulerian. Therefore, either G or \bar{G} is Eulerian.

(b) If G is not Hamiltonian, by Corollary 6.7, there exists a vertices u , such that

$$\deg u = r < n/2.$$

Therefore, for each vertex u' in \bar{G} ,

$$\deg u' = n-1-r > n/2-1 \geq n/2$$

by Corollary 6.7, \bar{G} is Hamiltonian.

It is possible that both G and \bar{G} are Hamiltonian. (consider C_6)

16 [GC] Problem 6.21

Add a new vertex u along with edges $uv (v \in G)$ to G , and let G' denote the the resulting graph. It is obvious that for every pair of non- adjacent vertices $u, v \in G'$, $\deg_{G'} u + \deg_{G'} v \geq n+1$. By Theorem 6.6, G' contains a Hamiltonian cycle C . Removing v and edges incident with v from C gives a Hamiltonian path of G .