

# Problem Solving: Homework 3.16

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## 1 [TJ] Problem 12.2

$O(n)$  is a subset of  $GL_n(\mathbb{R})$ . Hence,  $O(n)$  is a group only if it is a subgroup of  $GL_n(\mathbb{R})$ . For every two orthogonal matrices  $M, N$ , we have  $(MN)^{-1} = N^{-1}M^{-1} = N^t M^t = (MN)^t$  and  $(M^{-1})^{-1} = M = (M^{-1})^t$ , so  $MN$  and  $M^{-1}$  are also orthogonal matrices. Therefore,  $O(n)$  is a subgroup of  $GL_n(\mathbb{R})$ , and thus  $O(n)$  is a group.

## 2 [TJ] Problem 12.3

(a)

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

(b)

$$\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

(c)

$$\begin{pmatrix} 4/\sqrt{5} & 0 & 3/\sqrt{5} \\ -3/\sqrt{5} & 0 & 4/\sqrt{5} \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 4/\sqrt{5} & -3/\sqrt{5} & 0 \\ 0 & 0 & -1 \\ 3/\sqrt{5} & 4/\sqrt{5} & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq I$$

(d)

$$\begin{pmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -4/9 \\ 0 & 1 & 8/9 \\ -4/9 & 8/9 & 1 \end{pmatrix} \neq I$$

So the first two matrices are orthogonal. Both of their determinants are 1, so they are in  $SO(n)$ .

## 3 [TJ] Problem 12.6

For every two elements of  $E(n)$ , say  $(A, \mathbf{x})$  and  $(B, \mathbf{y})$ , their product  $(A, \mathbf{x})(B, \mathbf{y}) = (AB, A\mathbf{y} + \mathbf{x})$  is still an element of  $E(n)$ . Also, we have

**Identity**

$$(I, \mathbf{0})(A, \mathbf{x}) = (A, \mathbf{x})(I, \mathbf{0}) = (A, \mathbf{x})$$

**Invertibility**

$$(A, \mathbf{x})(A^{-1}, -A^{-1}\mathbf{x}) = (A^{-1}, -A^{-1}\mathbf{x})(A, \mathbf{x}) = (I, \mathbf{0})$$

**Associativity**

$$\begin{aligned} & ((A, \mathbf{x})(B, \mathbf{y}))(C, \mathbf{z}) \\ &= (AB, A\mathbf{y} + \mathbf{x})(C, \mathbf{z}) \\ &= (ABC, AB\mathbf{z} + A\mathbf{y} + \mathbf{x}) \\ &= (A, \mathbf{x})((B, \mathbf{y})(C, \mathbf{z})) \\ &= (A, \mathbf{x})(BC, B\mathbf{z} + \mathbf{y}) \\ &= (ABC, AB\mathbf{z} + A\mathbf{y} + \mathbf{z}) \end{aligned}$$

$$\therefore ((A, \mathbf{x})(B, \mathbf{y}))(C, \mathbf{z}) = (A, \mathbf{x})((B, \mathbf{y})(C, \mathbf{z}))$$

So  $E(n)$  is indeed a group.

## 4 [TJ] Problem 12.11

For  $x, y \in \mathbb{R}^n$ , if  $f(x) = f(y)$ , then  $\|f(x) - f(y)\| = 0$ , hence  $\|x - y\| = 0$ , i.e.  $x = y$ , so  $f$  is one-to-one.

## 5 [TJ] Problem 14.2

(a)  $X_{(1)} = X$ ,  $X_{(12)} = \{3\}$ ,  $X_{(13)} = \{2\}$ ,  $X_{(23)} = \{1\}$ ,  $X_{(123)} = X_{(132)} = \emptyset$ ;

$G_1 = \{(1), (23)\}$ ,  $G_2 = \{(1), (13)\}$ ,  $G_3 = \{(1), (12)\}$ .

(b)  $X_{(1)} = X$ ,  $X_{(12)} = \{3, 4, 5, 6\}$ ,  $X_{(345)} = X_{(354)} = \{1, 2, 6\}$ ,  $X_{(12)(345)} = X_{(12)(354)} = \{6\}$ ;

$G_1 = G_2 = \{(1), (345), (354)\}$ ,  $G_3 = G_4 = G_5 = \{(1), (12)\}$ ,  $G_6 = G$ .

## 6 [TJ] Problem 14.3

- (a) The  $G$ -equivalence class of  $X$  is  $\{1, 2, 3\}$ .

For every  $x \in X$ ,  $|\mathcal{O}_x| = 3$ , and  $|G_x| = 2$ , so  $|\mathcal{O}_x| \cdot |G_x| = 6 = |G|$ .

- (b) The  $G$ -equivalence classes of  $X$  are  $\{1, 2\}$ ,  $\{3, 4, 5\}$  and  $\{6\}$ .

$$|\mathcal{O}_1| \cdot |G_1| = 2 \cdot 3 = 6$$

$$|\mathcal{O}_2| \cdot |G_2| = 2 \cdot 3 = 6$$

$$|\mathcal{O}_3| \cdot |G_3| = 3 \cdot 2 = 6$$

$$|\mathcal{O}_4| \cdot |G_4| = 3 \cdot 2 = 6$$

$$|\mathcal{O}_5| \cdot |G_5| = 3 \cdot 2 = 6$$

$$|\mathcal{O}_6| \cdot |G_6| = 1 \cdot 6 = 6$$

## 7 [TJ] Problem 14.4

- (a) Obviously, rotating every point on the real plane  $\mathbb{R}^2$  counterclockwise about the origin through 0 radians yields the identical point. Also, rotating any point counterclockwise about the origin through  $x$  radians, then through  $y$  radians, yields the same point as rotating through  $x + y$  radians. So  $\mathbb{R}^2$  is a  $G$ -set.
- (b) The orbit containing  $P$  is the circle centered at the origin with radius  $|OP|$ .
- (c)  $G_P = \{0\}$  if  $P$  is not the origin; otherwise,  $G_P = G$ .

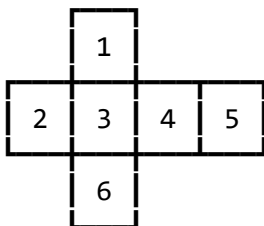
## 8 [TJ] Problem 14.8

Let 1, 2, 3, 4 denote the corners of the square in counterclockwise order. Then the symmetry group of the square is  $G = \{(1), (1234), (13)(24), (1432)\}$ . By Pólya enumeration theorem, the number of different ways to color the corners is

$$(3^4 + 3^1 + 3^2 + 3^1)/4 = 24.$$

## 9 [TJ] Problem 14.11

The surfaces are numbered as the following net.

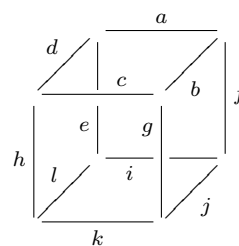


The symmetry operations of the cube can be classified into 6 types. The representatives and numbers of operations of these types are:  $(1) \times 1$ ,  $(2345) \times 6$ ,  $(24)(35) \times 3$ ,  $(134)(265) \times 8$ ,  $(12)(46)(35) \times 6$ . By Pólya enumeration theorem, the number of different ways to color the faces is

$$(3^6 + 6 \times 3^3 + 3 \times 3^4 + 8 \times 3^2 + 6 \times 3^3)/24 = 57.$$

## 10 [TJ] Problem 14.12

The edges are numbered as the following diagram.



The symmetry operations can be classified into 6 types. The representatives and numbers of operations of these types are: identity  $\times 1$ ,  $(abcd)(efgh)(ijkl) \times 6$ ,  $(ac)(bd)(eg)(fh)(li)(jk) \times 3$ ,  $(cbg)(dfk)(ajh)(eil) \times 8$ ,  $(dg)(bh)(ej)(fl)(ak) \times 6$ . By Pólya enumeration theorem, the number of different ways to color the faces is

$$(2^12 + 6 \times 2^3 + 3 \times 2^6 + 8 \times 2^4 + 6 \times 2^5)/24 = 194.$$

## 11 [TJ] Problem 14.16

- (a) Let 1, 2, 3, 4, 5, 6 denote the hydrogen atoms in clockwise order. The symmetry group of benzene is

$$\{(0), (123456), (135)(246), (14)(25)(36), (153)(264), (165432), (26)(35), (12)(36)(45), (13)(46), (14)(23)(56), (15)(24), (16)(25)(34)\}.$$

By Pólya enumeration theorem, the number of compounds that formed by replacing zero or more of the hydrogen atoms is

$$(2^6 + 2^1 + 2^2 + 2^3 + 2^2 + 2^1 + 3 \times 2^4 + 3 \times 2^3)/12 = 13$$

Excluding benzene itself, there are 12 different compounds.

- (b) There are 3. The three compounds are 1,2,3-trimethylbenzene, 1,2,4-trimethylbenzene and 1,3,5-trimethylbenzene.

## 12 [TJ] Problem 14.17

Let  $0, 1, 2, \dots, 7$  denote the input combinations  $(0, 0, 0), (0, 0, 1), (0, 1, 0), \dots, (1, 1, 1)$ , respectively. Then the symmetry group of the input combinations is

$$G = \{(0), (24)(35), (14)(36), (12)(56), (142)(356), (241)(653)\}.$$

By Pólya's enumeration theorem, the number of equivalence classes is

$$(2^8 + 3 \times 2^6 + 2 \times 2^4)/6 = 80.$$

When there are four input variables and they can be permuted by any permutation in  $S_4$ , the permutations of input combinations can be classified as By

representative	number
(0)	1
(48)(59)(6a)(7b)	6
(12)(48)(5a)(69)(7b)(de)	3
(842)(6ac)(953)(7bd)	8
(8421)(39c6)(4a)(7bde)	6

Pólya's enumeration theorem, the number of equivalence classes is

$$(2^{16} + 6 \times 2^{12} + 3 \times 2^{10} + 8 \times 2^8 + 6 \times 2^6)/24 = 3984.$$

## 13 [TJ] Problem 14.19

Since the bands of a necktie have no symmetry, there are  $4^12$  different-colored neckties.