Problem Solving: Homework 3.2

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1 [TC] Problem 25.1-4

The matrix 'multiplication' defined by EXTEND-SHORTEST-PATHS reads

$$C = A \cdot B$$
$$C_{ij} = \min_{1 \le k \le n} \{A_{ik} + B_{kj}\}$$

To prove the associativity of such 'multiplication', we only have to verify that $(A \cdot B) \cdot C = A \cdot (B \cdot C)$. Let D denote lhs, D' denote rhs:

$$\begin{aligned} D_{ij} &= \min_{1 \le l \le n} \{ \min_{1 \le k \le n} \{A_{ik} + B_{kl}\} + C_{lj} \} \\ &= \min_{1 \le l \le n} \{ \min_{1 \le k \le n} \{A_{ik} + B_{kl} + C_{lj}\} \} \\ &= \min_{1 \le k \le n} \{ \min_{1 \le l \le n} \{A_{ik} + B_{kl} + C_{lj}\} \} \\ &= \min_{1 \le k \le n} \{A_{ik} + \min_{1 \le l \le n} \{B_{kl} + C_{lj}\} \} = D'_{ij} \end{aligned}$$

this completes the proof of associativity.

2 [TC] Problem 25.1-5

Let W be the adjacency matrix, the single-source shortest-path problem is to calculate $V_i = W^{(\infty)} \cdot W_i$, where W_i denotes the i-th column of W. The index of source is i, and the j-th element of V_i is the weight of the shortest path from i to j. The product of two matrices $C = A \cdot B$ here is defined as

$$C_{ij} = \min_{1 \le k \le n} \{A_{ik} + B_{kj}\}$$

Since a shortest path contains at most $|V| - 1$ edges, we only have to find W raised to the power of $|V| - 1$. We may calculate that from right to left. Each multiplication takes $|V|^2$ time, and we performs such multiplication $|V| - 1$ times, therefore the total running time is $|V|^3$.

3 [TC] Problem 25.1-6

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BUILD-PREDECESSOR-MATRIX(W, L, n)
Let Π be a new n × n matrix initialized with NIL
for i = 1 to n
    for j = 1 to n
        for k = 1 to n
            if L_ij + W_jk == L_ik and i ≠ k
                Π_ik = j
return Π
```

4 [TC] Problem 25.1-9

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FASTER-ALL-PAIRS-SHORTEST-PATHS-MODIFIED(W, n)
L^(1) = W
m = 1
while m < 2(n − 1)
    let L^(2m) be a new n × n matrix
    L^(2m) = EXTEND-SHORTEST-PATHS(L^(m), L^(m))
    m = 2m
if L^(m) ≠ L^(m/2)
    error contains negative cycle
return L^(m)
```

5 [TC] Problem 25.1-9

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MINIMUN-LENGTH-NEGATIVE-CYCLE(W, n)
L^(1) = W
for m = 2 to n − 1
    let L^(m) be a new n × n matrix
    L^(m) = EXTEND-SHORTEST-PATHS(L^(m−1), W)
    for i = 1 to n
        if L_ii^(m) < 0
            return m
return −1 // does not contain negative cycle
```

The total running time is $O(|V|^2 \cdot ans)$ if the graph contains a negative cycle, or $O(|V|^3)$ if not.

6 [TC] Problem 25.1-9

Let W be the adjacency matrix, where $W_{ij} = 1$ if there exists a directed edge from i to j, or 0 if not. The method of computing transitive closure is the same as 'multiplying matrices' technique described in Section 25.1, except the definition of 'multiplication' of matrices A and B should be changed to

$$C_{ij} = \bigvee_{k=1}^{n} (A_{ik} \wedge B_{kj}).$$