Problem Solving: Homework 3.8

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1 [TC] Problem 29.1-4

$$
\begin{aligned}
\text{maximize} \quad & -2x_1' + 2x_1'' - 7x_2 + x_3' \\
\text{subject to} \quad & \\
& -x_1' + x_1'' - x_3' \le 7 \\
& x_1' - x_1'' + x_3' \le 7 \\
& -3x_1' + 3x_1'' - x_2 \le 24 \\
& x_1', x_1'', x_2, x_3' \ge 0 \\
\text{where} \quad & \\
& x_1' - x_1'' = x_1 \\
& x_3' = -x_3
\end{aligned}
$$

2 [TC] Problem 29.1-5

$$
\begin{aligned}
z &= 2x_1 - 6x_3 \\
x_4 &= 7 - x_1 - x_2 + x_3 \\
x_5 &= -8 + 3x_1 - x_2 \\
x_6 &= -x_1 + 2x_2 + 2x_3
\end{aligned}
$$

3 [TC] Problem 29.1-6

The second constraint implies $x_1 + x_2 \ge 5$, which contradicts the first one. So this linear program is infeasible.

4 [TC] Problem 29.1-7

Let $x_1 = 2t, x_2 = t$. It is easy to verify, that the solution (x_1, x_2) is feasible if $t > 1$. Therefore, the objective function $z = x_1 - x_2 = 2t - t = t$ can be arbitrarily large, and this linear program is unbounded.

5 [TC] Problem 29.1-9

$$
\begin{aligned}
\text{minimize} \quad & x_1 + x_2 \\
\text{subject to} \quad & \\
& x_1, x_2 \ge 0
\end{aligned}
$$

In this program, the feasible region is obviously unbounded, but it has finite optimal objective value 0.

6 [TC] Problem 29.2-2

$$
\begin{aligned}
\text{maximize} \quad & d_y \\
\text{subject to} \quad & \\
& d_s = 0 \\
& d_t - d_s \le 3 \\
& d_y - d_s \le 5 \\
& d_t - d_y \le 1 \\
& d_y - d_t \le 2 \\
& d_x - d_t \le 6 \\
& d_x - d_y \le 4 \\
& d_z - d_y \le 6 \\
& d_s - d_z \le 3 \\
& d_x - d_z \le 7 \\
& d_z - d_x \le 2
\end{aligned}
$$

7 [TC] Problem 29.2-3

$$
\begin{aligned}
\text{maximize} \quad & \sum_{v \in V} d_v \\
\text{subject to} \quad & \\
& d_s = 0 \\
& d_v - d_u \le w(u,v), \forall (u,v) \in E
\end{aligned}
$$

8 [TC] Problem 29.2-6

$$
\begin{aligned}
\text{maximize} \quad & \sum_{(u,v) \in E} f_{uv} \\
\text{subject to} \quad & \\
& f_{uv} \ge 0, \forall (u,v) \in E \\
& \sum_{u \in V} f_{uv} \le 1, \forall v \in P_2 \\
& \sum_{v \in V} f_{uv} \le 1, \forall u \in P_1
\end{aligned}
$$

9 [TC] Problem 29.3-2

Line 14 of PIVOT gives the new value of v: $\hat{v} = v + c_e\hat{b}_e$. Line 14 of SIMPLEX guarantees that c_e is positive, while $\hat{b}_e$ is defined as b_l/a_{le}. Note that the index l chosen in line 9 of SIMPLEX must satisfy that $\Delta_i \neq \infty$, which means a_{le} is positive. b_l is nonnegative, for otherwise the program is infeasible. Therefore, the call to PIVOT in SIMPLEX never decreases the value of v.

10 [TC] Problem 29.3-2

First, the procedure PIVOT do not changes the nonnegativity constraints, which are still $x_i > 0, \forall i \in N \cup B$. Second, it solves one equality for the entering variable, and substitutes the entering variable in the target function, which does not change the function. Then, it replaces the equality solved before with its solution form, and substitutes the solution for the entering variable in all other equalities and target functions. In the new equalities, if we solve the equality containing the original entering variable for the original leaving variable, and do substitutions like before, we will get the original equalities and target function. Therefore, the slack form given to PIVOT and the one it returns are equivalent.

11 [TC] Problem 29.3-5

First, convert it to slack form:

$$\begin{aligned} z &= 18x_1 + 12.5x_2 \\ x_3 &= 20 - x_1 - x_2 \\ x_4 &= 12 - x_1 \\ x_5 &= 16 - x_2 \end{aligned}$$

Choose x_1 as entering variable and x_4 as leaving variable, and perform a pivot:

$$\begin{aligned} z &= 216 - 18x_4 + 12.5x_2 \\ x_3 &= 8 - x_2 - x_4 \\ x_1 &= 12 - x_4 \\ x_5 &= 16 - x_2 \end{aligned}$$

Choose x_2 as entering variable and x_3 as leaving variable, and perform a pivot:

$$\begin{aligned} z &= 316 - 12.5x_3 - 30.5x_4 \\ x_2 &= 8 - x_3 - x_4 \\ x_1 &= 12 - x_4 \\ x_5 &= 24 - x_3 - x_4 \end{aligned}$$

Now, all coefficients of the target function is not positive, so the algorithm terminates. The optimal solution is $(x_1, x_2) = (12, 8)$, and the optimal value is 316.

12 [TC] Problem 29.4-2

Consider a linear program which only contains greater-than-or-equal-to constraints:

$$\text{max./min.} \quad \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i, \text{ for } 0 < i \leq m,$$

We first convert it to standard form:

$$\text{max./min.} \quad \sum_{j=1}^{n} c_j(x_j - x'_j)$$

subject to

$$\begin{aligned} \sum_{j=1}^{n} a_{ij}(x_j - x'_j) \geq b_i, &\text{ for } 0 < i \leq m, \\ x_j, x'_j &\geq 0 \end{aligned}$$

Then, take the dual of the program:

$$\text{min./max.} \quad \sum_{i=1}^{m} b_i y_i$$

subject to

$$\begin{aligned} \sum_{i=1}^{m} a_{ij}y_i \leq 0, &\text{ for } 0 < j \leq n \\ \sum_{i=1}^{m} -a_{ij}y_i \leq 0, &\text{ for } 0 < j \leq n \\ y_i \geq 0, &\text{ for } 0 < i \leq m \end{aligned}$$

which can be rewritten as

$$\text{min./max.} \quad \sum_{i=1}^{n} b_i y_i$$

subject to

$$\begin{aligned} \sum_{i=1}^{m} a_{ij}y_i = 0, &\text{ for } 0 < j \leq n \\ y_i \geq 0, &\text{ for } 0 < i \leq m \end{aligned}$$

Likewise, the dual of a linear program which only contains greater-than-or-equal-to constraints:

$$\text{max./min.} \quad \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i, \text{for } 0 < i \leq m$$

is

$$\text{min./max.} \quad \sum_{i=1}^{n} b_i y_i$$

subject to

$$\sum_{i=1}^{m} a_{ij}y_i = 0, \text{ for } 0 < j \leq n$$
$$y_i \leq 0, \text{ for } 0 < i \leq m$$

If a linear program only contains equality constraints,

$$\text{max./min.} \quad \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \text{for } 0 < i \leq m$$

its dual program is

$$\text{min./max.} \quad \sum_{i=1}^{n} b_i(y_i + y_i')$$

subject to

$$\sum_{i=1}^{m} a_{ij}(y_i + y_i') = 0, \text{ for } 0 < j \leq n$$
$$y_i \geq 0, y_i' \leq 0, \text{ for } 0 < i \leq m$$

or, equivalently,

$$\text{min./max.} \quad \sum_{i=1}^{n} b_i y_i$$

subject to

$$\sum_{i=1}^{m} a_{ij}y_i = 0, \text{ for } 0 < j \leq n$$

Combine these three cases, we obtain the method of taking the dual of arbitrary linear program directly. For arbitrary linear program

$$\text{max./min.} \quad \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \text{ for } 0 < i \leq m_1,$$
$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i, \text{ for } m_1 < i \leq m_2,$$
$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \text{ for } m_2 < i \leq m_3$$

its dual program is

$$\text{min./max.} \quad \sum_{i=1}^{m_3} b_i y_i$$

subject to

$$\sum_{i=1}^{m_3} a_{ij}y_i = 0, \text{ for } 0 < j \leq n$$
$$y_i \geq 0, \text{ for } 0 < i \leq m_1$$
$$y_i \leq 0, \text{ for } m_1 < i \leq m_2$$

13 [TC] Problem 29-1

a. We can arbitrarily define a target function and run the algorithm for linear programming. If it returns an optimal solution or reports the program is unbounded, then the inequalities are feasible; otherwise, they are infeasible.

b. First, we take the dual of the program, and use the algorithm to test whether the primal and the dual are satisfiable or not. If the primal is unsatisfiable, then it is infeasible. If it is satisfiable, but its dual is unsatisfiable, then the program is unbounded. If both the primal and the dual are satisfiable, then the program is feasible and bounded. In this case, we combine the constraints of the primal and the dual, and add a new constraint that let the target functions in primal and dual be equal. By strong duality theorem, the exists exactly one set of values that satisfies these constraints, which is the optimal solution of the primal.

If the algorithm for linear-inequality feasibility problem would return a satisfiable set of values, the proof is done. But we can still get the optimal solution if not. We modify one equality in the primal to an equation, and run the algorithm to test whether the optimal solution lies on the hyperplane. Repeatedly doing this, we get a set of hyperplanes represented by equations, where the optimal solution lies. By solving these linear equations, we get the optimal solution. Note that the equations may have more than one solution, which means the optimal solutions form a polytope of positive dimension, rather than a point.