Problem Solving: Homework 3.6

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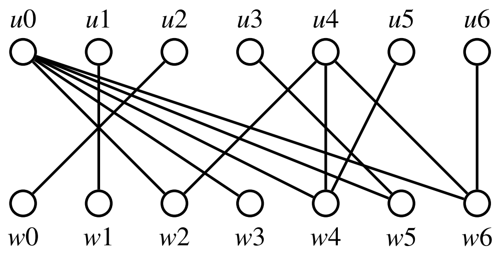
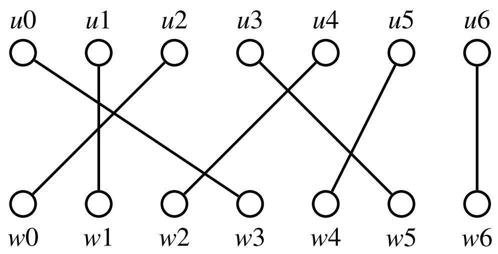
1 [GC] Problem 8.1

(a) G:



(b) Yes. A perfect matching is



This means that there exists a permutation π of $\{0,1,2,3,4,5,6\}$, such that $w_{\pi(i)}$ is a correct response to u_i, for $0 \le i \le 6$.

2 [GC] Problem 8.3

For graph G_1, U can be matched to W. One possible matching is $\{(a,x),(b,w),(c,v),(d,z),(e,y)\}$.

For graph G_2, U can't be matched to W. Consider vertex set $\{b,d,e\}$, the cardinality of its neighborhood is only 2, which violates Hall's condition.

3 [GC] Problem 8.4

For all subset U' of U, since every two vertices in U' have distinct degrees, the maximum degree of all vertices of U' is at least $|U'|$, and thus the cardinality of the neighborhood of U' is at least $|U'|$. Therefore, the graph G satisfies Hall's condition, which means G contains a perfect matching.

4 [GC] Problem 9.6

(a) True. It is obvious that if a graph does not contain a subdivision of K_5 or $K_{3,3}$, neither does its subgraph. Therefore, by Kuratowski's Theorem, every subgraph of a planar graph is planar.

(b) False. For any nonplanar graph, if we take one of its vertices as a trivial subgraph, then it is obviously a planar subgraph.

(c) False. Consider K_5, removing any of its edges or vertices will make the resulting graph not contain a subdivision of K_5 or $K_{3,3}$, so K_5 is a counterexample to this statement.

(d) False. If we insert an vertex to any edge of K_5, the resulting graph does not contain K_5 or $K_{3,3}$ as a subgraph, however, it is still nonplanar.

(e) False. Consider the union of K_5 and C_3, with order $n = 8$ and size $m = 13$, which satisfies $m \le 3n - 6$. However, one of its components is nonplanar, and thus the graph is nonplanar.

(f) False. Consider the union of $K_{3,3}$ and C_3, it has a triangle and contains no subdivision of K_5 as a subgraph, however, it is nonplanar.

5 [GC] Problem 9.13

(a) Since G contains no triangle, the boundary of every region has at least 4 edges. Because every edge belongs to at most two of the boundaries, we have $2m \ge 4r$, i.e. $2r \le m$. By Euler's Identity, we have $r = 2 + m - n$. Hence, $4 + 2m - 2n \le m$, i.e. $m \le 2n - 4$.

(b) For $K_{3,3}$, $n = 6$, $m = 9$, $m > 2n - 4$. Note that $K_{3,3}$ contains no triangle, so $K_{3,3}$ is nonplanar.

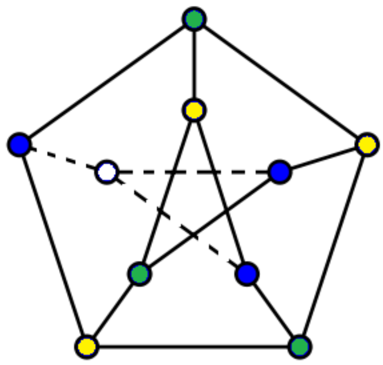
(c) This is true. First, G contains no triangle because it is bipartite. Suppose that every vertex has degree 4 or more, then $2m \ge 4n$, i.e. $m \ge 2n$, which violates the inequality we proved in (a). Therefore, G has a vertex of degree 3 or less.

6 [GC] Problem 9.14

(a) Since the length of a smallest cycle in G is 5, the boundary of every region has at least 5 edges. Because every edge belongs to at most two of the boundaries, we have $2m \geq 5r$. By Euler's Identity, we have $r = 2+m-n$. Hence $2+m-n \leq \frac{2}{5}m$, i.e. $m \leq \frac{5}{3}(n-2)$.

(b) Petersen graph has 15 edges and 10 vertices, so $m > \frac{5}{3}(n-2)$. Since the length of a smallest cycle in Petersen graph is 5, it is nonplanar.

(c) Removing any vertex of the Petersen graph yields a subdivision of $K_{3,3}$, so the Peterson graph is nonplanar.



(d) Suppose, to the contrary that every vertex of G has a degree of 3 or more, then $2m \geq 3n$. Therefore, $\frac{3}{2}n \leq m \leq \frac{5}{3}(n-2)$. After some algebra we get $n \geq 20$, which leads to contradiction.