Problem Solving: Homework 3.2

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1 [TC] Problem 25.1-4

The matrix 'multiplication' defined by EXTEND-SHORTEST-PATHS reads

$$C = A \cdot B$$
$$C_{ij} = \min_{1 \le k \le n} \{A_{ik} + B_{kj}\}$$

To prove the associativity of such 'multiplication', we only have to verify that $(A \cdot B) \cdot C = A \cdot (B \cdot C)$. Let D denote lhs, D' denote rhs:

$$\begin{aligned} D_{ij} &= \min_{1 \le l \le n} \{ \min_{1 \le k \le n} \{A_{ik} + B_{kl}\} + C_{lj}\} \\ &= \min_{1 \le l \le n} \{ \min_{1 \le k \le n} \{A_{ik} + B_{kl} + C_{lj}\}\} \\ &= \min_{1 \le k \le n} \{ \min_{1 \le l \le n} \{A_{ik} + B_{kl} + C_{lj}\}\} \\ &= \min_{1 \le k \le n} \{A_{ik} + \min_{1 \le l \le n} \{B_{kl} + C_{lj}\}\} = D'_{ij} \end{aligned}$$

this completes the proof of associativity.

2 [TC] Problem 25.1-5

Let W be the adjacency matrix, the single-source shortest-path problem is to calculate $V_i = W^{(\infty)} \cdot W_i$, where W_i denotes the i-th column of W. The index of source is i, and the j-th element of V_i is the weight of the shortest path from i to j. The product of two matrices $C = A \cdot B$ here is defined as

$$C_{ij} = \min_{1 \le k \le n} \{A_{ik} + B_{kj}\}$$

Since a shortest path contains at most $|V| - 1$ edges, we only have to find W raised to the power of $|V| - 1$. We may calculate that from right to left. Each multiplication takes $|V|^2$ time, and we performs such multiplication $|V| - 1$ times, therefore the total running time is $|V|^3$.

3 [TC] Problem 25.1-6

BUILD-PREDECESSOR-MATRIX(W, L, n)

let Π be a new $n \times n$ matrix initialized with NIL
for $i = 1$ to n
    for $j = 1$ to n
        for $k = 1$ to n
            if $L_{ij} + W_{jk} == L_{ik}$ and $i \neq k$
                $\Pi_{ik} = j$
return Π

Remark: we can prove that, providing only the completed matrix L is not sufficient to compute the predecessor matrix Π. Consider such two graphs G_1 and G_2, both containing n vertices($n \ge 3$) labeled from 1. G_1 contains directed edges $(1,2)$, $(2,3)$, $\cdots$, $(n-1,n)$ and $(n,1)$, while G_2 contains $(n,n-1)$, $\cdots$, $(2,1)$ and $(1,n)$. All edges in both graphs weigh 0. It is obvious that for both G_1 and G_2, $L = 0_{n \times n}$. Suppose, to the contrary, that there exits such algorithm that can compute predecessor matrix Π for any valid matrix L produced by all-pair shortest-paths algorithm. If we choose $0_{n \times n}$ as the input of such algorithm, and its output matrix is Π. Let $\Pi_{ij} = k$ be any non-nil element in Π, then (k, j) must be in the original graph. Note that $E(G_1) \cap E(G_2) = \varnothing$, which means at least one of G_1 and G_2 will make such algorithm produce wrong answer.

4 [TC] Problem 25.1-9

FASTER-ALL-PAIRS-SHORTEST-PATHS-MODIFIED(W, n)

$L^{(1)} = W$
$m = 1$
while $m < n$
    let $L^{(2m)}$ be a new $n \times n$ matrix
    $L^{(2m)} =$ EXTEND-SHORTEST-PATHS$(L^{(m)}, L^{(m)})$
    $m = 2m$
for $i = 1$ to n
    if $L^{(m)}_{ii} < 0$
        error contains negative cycle
return $L^{(m)}$

5 [TC] Problem 25.1-10

MINIMUN-LENGTH-NEGATIVE-CYCLE(W, n)

$L^{(1)} = W$
for $m = 2$ to n
  let $L^{(m)}$ be a new $n \times n$ matrix
  $L^{(m)} =$ EXTEND-SHORTEST-PATHS$(L^{(m-1)}, W)$
  for $i = 1$ to n
    if $L_{ii}^{(m)} < 0$
      return m
return -1 // does not contain negative cycle

The total running time is $O(|V|^2 \cdot ans)$ if the graph contains a negative cycle, or $O(|V|^3)$ if not.

6 [TC] Problem 25.2-2

Let W be the adjacency matrix, where $W_{ij} = 1$ if there exists a directed edge from i to j, or 0 if not. The method of computing transitive closure is the same as 'multiplying matrices' technique described in Section 25.1, except the definition of 'multiplication' of matrices A and B should be changed to

$$C_{ij} = \bigvee_{k=1}^{n} (A_{ik} \wedge B_{kj}).$$

7 [TC] Problem 25.2-4

The only difference between these two implementations of Floyd-Warshall algorithm is that, when updating d_{ij}, which version of d_{ik} and d_{kj} is used. Note that, during an iteration of outermost loop, for all index i, d_{ik} and d_{ki} are not changed, because $d_{kk} = 0$ and thus $\min(d_{ik}, d_{ik} + dkk) = d_{ik}$ and $\min(d_{ki}, d_{kk} + dki) = d_{ki}$. Therefore, the version of d_{ik} and d_{kj} used for updating d_{ij} does not matter, and the implementation remains correct.

8 [TC] Problem 25.2-6

Let d be the matrix produced by Floyd-Warshall algorithm.If vertex i lies in some negative-weight cycle, then d_{ii} must be negative. So we only have to inspect the diagonal of d. If negative number exists in the diagonal, then the graph must contains a negative cycle.

9 [TC] Problem 25.2-6

TRANSITIVE-CLOSURE(G)

let T be a new $|V(G)| \times |V(G)|$ matrix filled with 0
let $Vis[1..|V(G)|]$ be a new boolean array
for $i = 1$ to $|V(G)|$
  for $j = 1$ to $|V(G)|$
    $V[j] =$ FALSE
  DFS(T, Vis, G, i, i)
return T

DFS(T, Vis, G, x, p)

if $Vis[x]$
  return
$Vis[x] =$ TRUE
$T_{px} = 1$
for each vertex v in $G.Adj[x]$
  DFS(T, vis, G, v, p)

The procedure described above performs depth-first search from each vertex. Every round of search takes $O(|V| + |E|) = O(|E|)$ time, and it is performed $|V|$ times, so the total running time is $O(|V||E|)$.

10 [TC] Problem 25.3-2

The purpose of adding s to V, is to determining the function $h : V \to \mathbb{R}$, such that after the reweighting described in Lemma 25.1, all edges are non-negative, therefore Dijkstra's algorithm applies. If define $h(v) = \delta(s, v)$, the triangle inequality guarantees the non-negativity of the edges in the new graph.

11 [TC] Problem 25.3-3

If all edge all non-negative in the original graph G, consider the vertex v added to the original graph, since the v is connected to every vertex in G by an edge weighted 0, it is obvious that $\delta(v, u) = 0$ for all u in $V(G)$, i.e. $h(u) = 0$. Therefore, $w = \hat{w}$.

12 [TC] Problem 25-2

a. By referring to Problem 6-2, the asymptotic running times for INSERT, EXTRACT-MIN, DECREASE-KEY are $O(\log_d n)$, $O(d \log_d n)$, $O(\log_d n)$, respectively. If we choose $d = \Theta(n^\alpha)$, their running times are $O(1/\alpha)$, $O(n^\alpha/\alpha)$, $O(1/\alpha)$, respectively. Since the running times for those of a Fibonacci heap are $O(1)$, $O(\log n)$, $O(1)$, respectively, the running times for INSERT and DECREASE-KEY of a d-ary heap are equal to those of a Fibonacci heap, while d-ary heap is slower in EXTRACT-MIN.

b. Since the graph does not contain negative-weight edge, Dijkstra's algorithm applies here. If we use V^α-ary heap and run Dijkstra's algorithm, the total running time will be $O(V/\alpha + V \cdot V^\alpha/\alpha + V^{1+\varepsilon}/\alpha)$. If we take $\alpha = \varepsilon$, i.e. $d = V^\varepsilon$, since ε is a constant, the total running time is $O(V^{1+\varepsilon}) = O(E)$.

c. For each vertex in the graph, we take it as the source and run Dijkstra's algorithm described above. The total running time is $O(VE)$.

d. We can perform Johnson's algorithm on the graph, and use the above version of Dijkstra's algorithm in Johnson's algorithm. The total running time is $O(VE + VE) = O(VE)$.