Problem Solving: Homework 3.5

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1 [GC] Problem 26.1-1

For every feasible flow f in G', by flow conservation and capacity constraint, $f(u,x) = f(x,v) \leq c(x,v)$. If we define $f' : V(G) \times V(G) \to \mathbb{R}$:

$$f'(a,b) = \begin{cases} f(u,x) & a = u, b = y \\ f(a,b) & \text{otherwise} \end{cases},$$

f' is still a feasible flow of G with equal value because it still satisfies capacity constraint and flow conservation, and the total flow out of the source do not change. Likewise, for every feasible flow in G, we can find out a corresponding flow in G' with equal value. Therefore, a maximum flow in G' has the same value as a maximum flow in G.

2 [GC] Problem 26.1-2

Let G denote the flow network with multiple sources and multiple sinks, and G' denote the corresponding single-source, single-sink flow network. For every feasible flow f in G, we define $f' : V(G') \times V(G') \to \mathbb{R}$:

$$f'(a,b) = \begin{cases} \sum\limits_x f(b,x) & a = s' \\ \sum\limits_x f(x,a) & b = t' \\ f(a,b) & \text{otherwise} \end{cases},$$

where s' is the supersource and t' is the supersink, we can easily verify that f' is a feasible flow of G' with equal value.

For every feasible flow f' in G', if we simply define a flow f of G as $f(a,b) = f'(a,b)$ for all $a, b \in V(G)$, we get a feasible flow of G with the same value as f'.

3 [GC] Problem 26.1-6

First, model the map of the town as a symmetric directed graph. Then, assign capacity 1 to each arc in the graph. Finally, let their home be the source, and the school be the sink, and compute the max-flow of the flow network. It is possible that both his children can go to the same school if and only if the max-flow is not less than 2.

4 [GC] Problem 26.1-7

We replace each vertex v in V by v_i and v_o, and replace each edge (u,v) by (u_o, v_i), with the capacity unchanged. Then we add an edge (v_i, v_o) for each v with capacity $l(v)$. The maximum flow of the new network is the same as the original one with vertex capacities. In the new graph, there are $2|V|$ vertices and $|V| + |E|$ edges.

5 [GC] Problem 26.2-2

The flow across the cut is 19, and the capacity of the cut is 31.

6 [GC] Problem 26.2-6

We add a supersource s and a supersink t to this network, and edges (s, s_i) with capacities p_i, (t_j, t) with capacities q_j. Then compute the maximum flow f of the new network. The flow obeying these constrains in multiple-source multiple-sink flow network exists if and only if the flow f we find obeys these constrains. Otherwise, there must exist an augmenting path from some source to some sink without violating the constrains, and the flow we find before is not a maximum flow.

7 [GC] Problem 26.2-8

The Ford-Fulkerson method repeatedly finds an augmenting path from s to t, and add it to current flow. Note that a path from s to t never contains an edge to s. Therefore, the procedure still correctly computes a maximum flow.

8 [GC] Problem 26.2-10

Assume that we have already found a maximum flow f. Let (u,v) be the edge such that $f(u,v)$ has minimum positive value. Then find a path containing this edge from s to t. Since for every edge e in the path, the flow passing through e is at least $f(u,v)$, we decrease the flows of these edges by $f(u,v)$; furthermore, we can remove edge (u,v). Repeatedly doing this will finally decompose the flow into at most $|E|$ augmenting paths.

9 [GC] Problem 26.2-12

By flow conservation property, there must exist a cycle, or a path from t to s, containing (v,s). Decreasing the flow of each edge in the cycle or path by 1 yields another maximum flow f' with $f'(v,s) = 0$. To find such cycle or path, we only have to perform DFS or BFS on f from the edge (v,s), which takes $O(E)$ time.

10 [GC] Problem 26.2-12

For every edge (u,v), we change its capacity from $c(u,v)$ to $c(u,v)(|E|+1)+1$. If the minimum cut of the new network is $c(S,T)$, then the minimum cut of the original network is $\lfloor c(S,T)/(|E|+1) \rfloor$, which contains $c(S,T) \bmod (|E|+1)$ edges.

11 [GC] Problem 26.3-3

The length of any augmenting path is at most $|V|-1$, because every vertex appears in the path at most once. We can show that the bound is sharp: consider graph P_{2n} (a graph of order $2n$ which only contains a chain), it is bipartite and the length of the only augmenting path is $2n-1$.

12 [GC] Problem 26-1

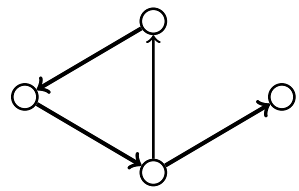
a. See Problem 26.1-7.

b. We model this grid as a flow network, for every vertex and edge in which, we assign unit capacity. We treat the m points to escape as sources, and the points in the boundary as sinks. Now the escape problem has been reduced to multi-source multi-sink maximum flow problem, with vertex capacities allowed. If we choose Ford-Fulkerson method as max-flow algorithm, the total running time is $O(mn^2)$. Note that there are only $4n-4$ sinks, so if $m > 4n-4$, the grid does not have an escape. So the running time can be optimized to $O(\min(n,m)n^2)$.

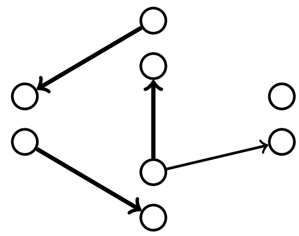
13 [GC] Problem 26-2

a. For every vertex v in the original graph, we split it into two vertices v_i, v_o in the new graph, and for every arc (u,v) in the original graph, we add edge (u_o, v_i) into the new graph. The new graph is bipartite, and we compute a maximum bipartite matching on the new graph. For every matched vertices in the maximum bipartite matching, we join the corresponding vertices in the original graph, forming a longer path. Finally, we will get a maximum path cover.

b. No. Consider the following graph and its minimum path cover



If we compute the maximum matching of the derived bipartite graph, one possible maximum matching is



according this matching, the original graph can be covered by a cycle and an isolated vertex, which is not minimum.