Problem Solving: Homework 3.3

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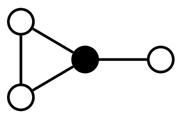
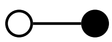
1 [GC] Problem 5.3

(a) No. Here is a counterexample.



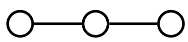
Consider the solid vertex. It lies on some cycle, however, it is a cut-vertex too.

(b) No. Here is a counterexample.



Cosider the solid vertex. It does not lie on any circle, but it is not a cut-vertex as well.

(c) No. Here is a counterexample.



It has two end-vertices but only one cut-vertex.

(d) No. Consider the counterexample in (c), it has two bridges but only one cut-vertex.

2 [GC] Problem 5.4

Let a, b be any two distinct vertices in $G-v$ that are connected in $\overline{G}$. We are going to prove that a and b are still connected in $\overline{G}-v$.

Case 1: a and b are in different components of $G-v$. This means that $ab \notin E(G)$ and $ab \in E(\overline{G})$, and thus $ab \in E(\overline{G}-v)$. Therefore, a and b are still connected in $\overline{G}-v$.

Case 2: a and b are in the same component of $G-v$. Since $G-v$ has at least two components, let c be any vertex in the component of G other than the one a and b are in. Therefore, $ac, cb \notin E(G)$, and $ac, cb \in E(\overline{G})$, and thus $ac, cb \in E(\overline{G})$.

Therefore, removing v in $\overline{G}$ does not make any two connected vertices $a, b (a, b \neq v)$ disconnected. This means that v is not a cut-vertex of $\overline{G}$.

3 [GC] Problem 5.6

The sufficiency is immediate by Corollary 5.2. We only have to prove the necessity.

Let v be any cut-vertex in G, and a, b, c be neighbors of v. a, b, c must in at least two different components of $G-v$, otherwise, v is not a cut-vertex. Furthermore, there exists a component of $G-v$ that contains exactly one of a, b, c, and without loss of generality, we assume the vertex is a. Suppose, to the contrary, that va is not a bridge of G, then va lies on some cycle by Theorem 4.1. Since a, b, c are only neighbors of v, either vb or vc lies on C. Without loss of generality, assume $vb \in C$. Therefore, $C-v$ is an $a-b$ path, which contradicts Theorem 5.3. Therefore, va is a bridge.

4 [GC] Problem 5.10

Note that a graph of size at least 2 contains at least 3 vertices, so Theorem 5.7 applies.

Sufficiency: let u, v be any two distinct vertices in G. Since G is connectd, there exists a $u-v$ path, say $(u, w_1, w_2, \cdots, w_n, v)$. Since edges uw_1, w_1w_2, edges w_1w_2, w_2w_3, $\cdots$, edges $w_{n-1}w_n, w_nv$ lie on common cycles respectively, by Theorem 5.8 uw_1, w_nv lie on common cycle, and thus u, v lie on same cycle. By Theorem 5.7 G is nonseparable.

Necessity: let uv, vw be any two adjacent edges in nonseperable graph G. By Theorem 5.7, u, w lie on common cycle C. If $v \in C$, then uv, vw lie on common cycle C; otherwise, let P be either $u-w$ path in cycle C, then $P \cup \{uv, vw\}$ forms a cycle where uv, vw lie on.

5 [GC] Problem 5.11

By Theorem 5.3, we only have to show that, for every three distinct vertices $u, v, w \in G$, there exists a $u-w$ path that does not contain v.

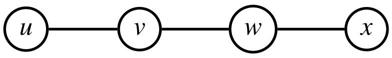
If $uw \in E(G)$, then the proof is done; otherwise, since $\deg u = \deg w \geq n/2$ and $|V(G-\{u, w\})| = n-2$, by pigeonhole principle, there exist two vertices

a,b, such that a,b are neighbors of both u and w. At least one of a and b is not v, that vertex, along with u and v, forms a $u-w$ path that does not contain v.

6 [GC] Problem 5.14

(a) Since G is connected, G_1 must be connected to some vertex in $G-G_1$. However, G_1 is not connected to $G-v-G_1$, because G_1 is a component of $G-v$. Therefore, only v is connected to G_1, and $G[V(G_1)\cup\{v\}]$ is connected.

(b) Consider the following graph G



v is a cut-vertex of G, and let G_1 be the component containing w and x. $G[V(G_1)\cup\{v\}]$ is not a block, because w is a cut-vertex.

7 [GC] Problem 5.15

Let B be a block of G.

(1) $\to$ (2): since B is a nontrivial connected graph, it is induced by its edge set $E(B)$. Then we are going to prove $E(B)$ is an equivalent class resulting from the equivalence relation R defined in Theorem 5.8. Since B is connected, by Problem 5.10, for every two edges e,f, we have $e\ R\ f$. Let e be any edge in $E(B)$, if $e\ R\ f$, then $f\in E(B)$. Otherwise, let C be the common cycle where e,f lie on. $G\cup C$ is a nonseparable graph by Problem 5.10, and G is a proper subgraph of $G\cup C$, which violates interpretation (1). Hence, $E(B)$ is an equivalent class of R.

(2) $\to$ (1): since $E(B)$ is an equivalent class of R, by Problem 5.10, B is nonseperable. B can't be a proper subgraph of any other nonseparable graph B' of G, otherwise, there exists $E(B)$ is a proper subset of $E(B')$ because B' contains no isolated vertex, and for every $e,f\in E(B')$, $e\ R\ f$, which means $E(B)$ is not an equivalent class.

8 [GC] Problem 5.20

(a) Since G is not k-connected ($k\le n-2$), $\kappa(G)\le k-1$, and there exists a vertex cut U_0 with $|U_0|=\kappa(G)$. Let C_1,C_2 be any two components of $G-U_0$, and let u,v be two vertices such that $u\in C_1$ and $v\in C_2$. Take any $k-1-\kappa(G)$ (possibly zero) vertices from $G-U_0-\{u,v\}$ and add then to U_0, we obtain a vertex-cut U with $|U|=k-1$.

(b) Since G is not k-edge-connected, $\lambda(G)\le k-1$, and there exists an edge cut X_0 with $|X_0|=\lambda(X_0)$. Since removing arbitrary number of edges does not decrease the number of components, we can take any $k-1-\lambda(G)$ (possibly zero) edges from $E(G)\backslash X$ and add them to X_0, and thus we obtain an edge cut X with $|X|=k-1$.

9 [GC] Problem 5.22

(a) When $k=1$, the proof is done. Otherwise, let $e=uv$. Consider every vertex subset U of $G-e$ with $|U|=k-2$. Since G is k-connected, if $u\in|U|$, $G-e-|U|$ is still connected; if $u\notin|U|$, $G-e-|U\cup\{u\}|=G-|U\cup\{u\}|$ is still connected. Therefore, $G-e$ is $(k-1)$-connected.

(b) When $k=1$, the proof is done. Otherwise, consider every edge subset X of $G-e$ with $|X|=k-2$, since $G-e-X=iG-(\{e\}\cup X)$ and $\{e\}\cup X$ is not an edge cut of G because G is k-edge-connected, X is neither an edge cut of $G-e$. Therefore, $G-e$ is $(k-1)$-edge-connected.

10 [GC] Problem 5.30

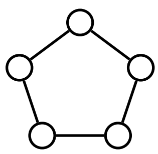
$\overline{\kappa}(G)\ge\kappa(G)$, $\overline{\lambda}(G)\ge\lambda(G)$, $\overline{\kappa}(G)\le\overline{\lambda}(G)$.

The first two inequalities are what the definitions of $\overline{\kappa}$ and $\overline{\lambda}$ imply. Let's prove the third one.

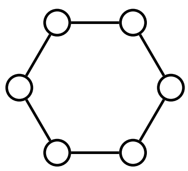
Let H' be the subgraph of G such that $\kappa(H')=\overline{\kappa}(G)$. By Theorem 5.11, $\overline{\kappa}(G)=\kappa(H')\le\lambda(H')$. Since $\lambda(H')\le\overline{\lambda}(G)$, we have $\overline{\kappa}(G)\le\overline{\lambda}(G)$.

11 [GC] Problem 6.4

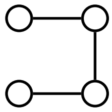
(a) C_5



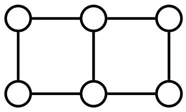
(b) C_6



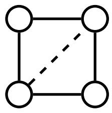
(c) P_4



(d) $P_2 \times P_3$



(e) (the dashed edge is e)



12 [GC] Problem 6.5

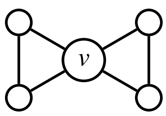
It is K_5 with any of its edges removed.

13 [GC] Problem 6.6

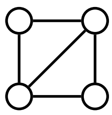
Assume G is r-regular. Since G is not Eulerian, by Theorem 6.1, r is odd, and thus n is even. Therefore, $\overline{G}$ is $(n-1-r)$-regular graph, where $n-1-r$ is even. By Theorem 6.1, $\overline{G}$ is Eulerian.

14 [GC] Problem 6.13

(a) The following graph is not Hamiltonian, because it violates a necessary condition for a graph to be Hamiltonian (Theorem 6.5), if we take $S = \{v\}$.

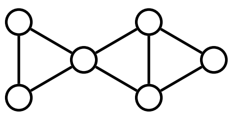


(b) The following graph is not Eulerian, because it has two odd vertices.



(c) The same one as (b).

(d)



15 [GC] Problem 6.16

(a) We have proved in Problem 6.6 that if G is not Eulerian, then $\overline{G}$ is Eulerian. If G is Eulerian, then r is even. Hence, $\overline{G}$ is $(n-1-r)$-regular, where $n-1-r$ is odd, which means $\overline{G}$ is not Eulerian. Therefore, either G or $\overline{G}$ is Eulerian.

(b) If G is not Hamiltonian, by Corollary 6.7, there exists a vertices u, such that

$$\deg u = r < n/2.$$

Therefore, for each vertex u' in $\overline{G}$,

$$\deg u' = n-1-r > n/2 - 1 \geq n/2$$

by Corollary 6.7, $\overline{G}$ is Hamiltonian.

It is possible that both G and $\overline{G}$ are Hamiltonian. (consider C_6)

16 [GC] Problem 6.21

Add a new vertex u along with edges $uv(v \in G)$ to G, and let G' denote the the resulting graph. It is obvious that for every pair of non- adjacent vertices $u, v \in G'$, $\deg_{G'} u + \deg_{G'} v \geq n+1$. By Theorem 6.6, G' contains a Hamiltonian cycle C. Removing v and edges incident with v from C gives a Hamiltonian path of G.