Problem Solving: Homework 3.16

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1 [TJ] Problem 12.2

$O(n)$ is a subset of $GL_n(\mathbb{R})$. Hence, $O(n)$ is a group only if it is a subgroup of $GL_n(\mathbb{R})$. For every two orthogonal matrices M, N, we have $(MN)^{-1} = N^{-1}M^{-1} = N^tM^t = (MN)^t$ and $(M^{-1})^{-1} = M = (M^{-1})^t$, so MN and M^{-1} are also orthogonal matrices. Therefore, $O(n)$ is a subgroup of $GL_n(\mathbb{R})$, and thus $O(n)$ is a group.

2 [TJ] Problem 12.3

(a)

$$\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

(b)

$$\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

(c)

$$\begin{pmatrix} 4/\sqrt{5} & 0 & 3/\sqrt{5} \\ -3/\sqrt{5} & 0 & 4/\sqrt{5} \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 4/\sqrt{5} & -3/\sqrt{5} & 0 \\ 0 & 0 & -1 \\ 3/\sqrt{5} & 4/\sqrt{5} & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq I$$

(d)

$$\begin{pmatrix} 1/3 & 2/3 & -2/3 \\ -2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -4/9 \\ 0 & 1 & 8/9 \\ -4/9 & 8/9 & 1 \end{pmatrix} \neq I$$

So the first two matrices are orthogonal. Both of their determinants are 1, so they are in $SO(n)$.

3 [TJ] Problem 12.6

For every two elements of $E(n)$, say $(A, \boldsymbol{x})$ and $(B, \boldsymbol{y})$, their product $(A, \boldsymbol{x})(B, \boldsymbol{y}) = (AB, A\boldsymbol{y} + \boldsymbol{x})$ is still an element of $E(n)$. Also, we have

Identity

$$(I, \mathbf{0})(A, \boldsymbol{x}) = (A, \boldsymbol{x})(I, \mathbf{0}) = (A, \boldsymbol{x})$$

Invertibility

$$(A, \boldsymbol{x})(A^{-1}, -A^{-1}\boldsymbol{x}) = (A^{-1}, -A^{-1}\boldsymbol{x})(A, \boldsymbol{x}) = (I, \mathbf{0})$$

Associativity

$$\begin{aligned} &((A, \boldsymbol{x})(B, \boldsymbol{y}))(C, \boldsymbol{z}) \\ =&(AB, A\boldsymbol{y} + \boldsymbol{x})(C, \boldsymbol{z}) \\ =&(ABC, AB\boldsymbol{z} + A\boldsymbol{y} + \boldsymbol{x}) \\ &(A, \boldsymbol{x})((B, \boldsymbol{y})(C, \boldsymbol{z})) \\ =&(A, \boldsymbol{x})(BC, B\boldsymbol{z} + \boldsymbol{y}) \\ =&(ABC, AB\boldsymbol{z} + A\boldsymbol{y} + \boldsymbol{z}) \end{aligned}$$

$$\therefore ((A, \boldsymbol{x})(B, \boldsymbol{y}))(C, \boldsymbol{z}) = (A, \boldsymbol{x})((B, \boldsymbol{y})(C, \boldsymbol{z}))$$

So $E(n)$ is indeed a group.

4 [TJ] Problem 12.11

For $x, y \in \mathbb{R}^n$, if $f(x) = f(y)$, then $\|f(x) - f(y)\| = 0$, hence $\|x - y\| = 0$, i.e. $x = y$, so f is one-to-one.

5 [TJ] Problem 14.2

(a) $X_{(1)} = X$, $X_{(12)} = \{3\}$, $X_{(13)} = \{2\}$, $X_{(23)} = \{1\}$, $X_{(123)} = X_{(132)} = \varnothing$;

$G_1 = \{(1), (23)\}$, $G_2 = \{(1), (13)\}$, $G_3 = \{(1), (12)\}$.

(b) $X_{(1)} = X$, $X_{(12)} = \{3, 4, 5, 6\}$, $X_{(345)} = X_{(354)} = \{1, 2, 6\}$, $X_{(12)(345)} = X_{(12)(354)} = \{6\}$;

$G_1 = G_2 = \{(1), (345), (354)\}$, $G_3 = G_4 = G_5 = \{(1), (12)\}$, $G_6 = G$.

6 [TJ] Problem 14.3

(a) The G-equivalence class of X is $\{1, 2, 3\}$.

For every $x \in X$, $\mathcal{O}_x = X$, $|\mathcal{O}_x| = 3$, and $|G_x| = 2$, so $|\mathcal{O}_x| \cdot |G_x| = 6 = |G|$.

(b) The G-equivalence classes of X are $\{1, 2\}$, $\{3, 4, 5\}$ and $\{6\}$.

$$|\mathcal{O}_1| \cdot |G_1| = 2 \cdot 3 = 6$$
$$|\mathcal{O}_2| \cdot |G_2| = 2 \cdot 3 = 6$$
$$|\mathcal{O}_3| \cdot |G_3| = 3 \cdot 2 = 6$$
$$|\mathcal{O}_4| \cdot |G_4| = 3 \cdot 2 = 6$$
$$|\mathcal{O}_5| \cdot |G_5| = 3 \cdot 2 = 6$$
$$|\mathcal{O}_6| \cdot |G_6| = 1 \cdot 6 = 6$$

7 [TJ] Problem 14.4

(a) Obviously, rotating every point on the real plane $\mathbb{R}^2$ counterclockwise about the origin through 0 radians yields the identical point. Also, rotating any point counterclockwise about the origin through x radians, then through y radians, yields the same point as rotating through $x + y$ radians. So $\mathbb{R}^2$ is a G-set.

(b) The orbit containing P is the circle centered at the origin with radius $|OP|$.

(c) $G_P = \{0\}$ if P is not the origin; otherwise, $G_P = G$.

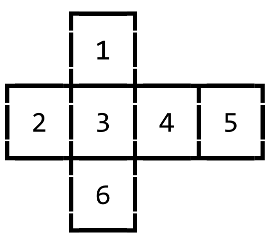
8 [TJ] Problem 14.8

Let 1, 2, 3, 4 denote the corners of the square in counterclockwise order. Then the symmetry group of the square is $G = \{(1), (1234), (13)(24), (1432)\}$. By Pólya enumeration theorem, the number of different ways to color the corners is

$$(3^4 + 3^1 + 3^2 + 3^1)/4 = 24.$$

9 [TJ] Problem 14.11

The surfaces are numbered as the following net.

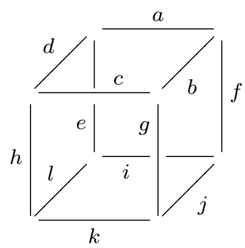


The symmetry operations of the cube can be classified into 6 types. The representatives and numbers of operations of these types are: $(1) \times 1$, $(2345) \times 6$, $(24)(35) \times 3$, $(134)(265) \times 8$, $(12)(46)(35) \times 6$. By Pólya enumeration theorem, the number of different ways to color the faces is

$$(3^6 + 6 \times 3^3 + 3 \times 3^4 + 8 \times 3^2 + 6 \times 3^3)/24 = 57.$$

10 [TJ] Problem 14.12

The edges are numbered as the following diagram.



The symmetry operations can be classified into 6 types. The representatives and numbers of operations of these types are: identity $\times$ 1, $(abcd)(efgh)(ijkl) \times 6$, $(ac)(bd)(eg)(fh)(lj)(ik) \times 3$, $(cbg)(dfk)(ajh)(eil) \times 8$, $(dg)(bh)(ej)(fl)(ak) \times 6$. By Pólya enumeration theorem, the number of different ways to color the faces is

$$(2^{1}2 + 6 \times 2^3 + 3 \times 2^6 + 8 \times 2^4 + 6 \times 2^5)/24 = 194.$$

11 [TJ] Problem 14.16

(a) Let 1, 2, 3, 4, 5, 6 denote the hydrogen atoms in clockwise order. The symmetry group of benzene is

$$\{(0), (123456), (135)(246), (14)(25)(36),$$
$$(153)(264), (165432), (26)(35), (12)(36)(45),$$
$$(13)(46), (14)(23)(56), (15)(24), (16)(25)(34)\}.$$

By Pólya enumeration theorem, the number of compounds that formed by replacing zero or more of the hydrogen atoms is

$$(2^6+2^1+2^2+2^3+2^2+2^1+3\times 2^4+3\times 2^3)/12 = 13$$

Excluding benzene itself, there are 12 different compounds.

(b) There are 3. The three compounds are 1,2,3-trimethylbenzene, 1,2,4-trimethylbenzene and 1,3,5-trimethylbenzene.

12 [TJ] Problem 14.17

Let $0, 1, 2, \cdots, 7$ denote the input combinations $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $\cdots$, $(1,1,1)$, respectively. Then the symmetry group of the input combinations is

$$G = \{(0), (24)(35), (14)(36), (12)(56), \\ (142)(356), (241)(653)\}.$$

By Pólya's enumeration theorem, the number of equivalence classes is

$$(2^8 + 3 \times 2^6 + 2 \times 2^4)/6 = 80.$$

When there are four input variables and they can be permuted by any permutation in S_4, the permutations of input combinations can be classified as

representative	number
(0)	1
(48)(59)(6a)(7b)	6
(12)(48)(5a)(69)(7b)(de)	3
(842)(6ac)(953)(7bd)	8
(8421)(39c6)(4a)(7bde)	6

By Pólya's enumeration theorem, the number of equivalence classes is

$$(2^{16} + 6 \times 2^{12} + 3 \times 2^{10} + 8 \times 2^8 + 6 \times 2^6)/24 = 3984.$$

13 [TJ] Problem 14.19

Since the bands of a necktie have no symmetry, there are 4^{12} different-colored neckties.