Problem Solving: Homework 3.7

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1 [GC] Problem 28.1-2

By using formula (28.8) repeatedly, we can get the LU decomposition of a matrix.

$$\begin{aligned} A &= \begin{pmatrix} 4 & -5 & 6 \\ 8 & -6 & 7 \\ 12 & -7 & 12 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & -5 & 6 \\ 0 & 4 & -5 \\ 0 & 8 & -6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & -5 & 6 \\ 0 & 4 & -5 \\ 0 & 0 & 4 \end{pmatrix} \end{aligned}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}, U = \begin{pmatrix} 4 & -5 & 6 \\ 0 & 4 & -5 \\ 0 & 0 & 4 \end{pmatrix}$$

2 [GC] Problem 28.1-3

We use the algorithm described in LUP-DECOMPOSITION to calculate the LUP decomposition.

For the first iteration, $p = 3$, $\pi = (3,2,1)$

$$A \to \begin{pmatrix} 5 & 8 & 2 \\ 2 & 0 & 3 \\ 1 & 5 & 4 \end{pmatrix} \to \begin{pmatrix} 5 & 8 & 2 \\ 2/5 & -16/5 & 11/5 \\ 1/5 & 17/5 & 18/5 \end{pmatrix}$$

For the second iteration, $p = 3$, $\pi = (3,1,2)$

$$A \to \begin{pmatrix} 5 & 8 & 2 \\ 1/5 & 17/5 & 18/5 \\ 2/5 & -16/5 & 11/5 \end{pmatrix} \to \begin{pmatrix} 5 & 8 & 2 \\ 1/5 & 17/5 & 18/5 \\ 2/5 & -16/17 & 95/17 \end{pmatrix}$$

Therefore, $\pi = (3,1,2)$,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1/5 & 1 & 0 \\ 2/5 & -16/17 & 1 \end{pmatrix}, U = \begin{pmatrix} 5 & 8 & 2 \\ 0 & 17/5 & 18/5 \\ 0 & 0 & 95/17 \end{pmatrix}$$

Multiplying P on both sides and substitute PA with LU, we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/5 & 1 & 0 \\ 2/5 & -16/17 & 1 \end{pmatrix} \begin{pmatrix} 5 & 8 & 2 \\ 0 & 17/5 & 18/5 \\ 0 & 0 & 95/17 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \\ 9 \end{pmatrix}$$

Eliminating L,

$$\begin{pmatrix} 5 & 8 & 2 \\ 0 & 17/5 & 18/5 \\ 0 & 0 & 95/17 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ 295/17 \end{pmatrix}$$

Eliminating U,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -3/19 \\ -1/19 \\ 59/19 \end{pmatrix}.$$

3 [GC] Problem 28.1-6

Since L is a unit lower-triangular matrix, it is non-singular. Therefore, if a matrix A has an LU decomposition is singular if and only if U is singular. Hence, let L be any $n \times n$ unit lower-triangular matrix, e.g. I_n, and U be any $n \times n$ singular upper-triangular matrix, e.g. $0_{n\times n}$, then $A = LU = 0_{n\times n}$ is an $n \times n$ singular matrix that has LU decomposition.

4 [GC] Problem 28.1-7

It is necessary in LU-DECOMPOSITION, because a_{nn} needs be assigned to u_{nn}. It is unnecessary in LUP-DECOMPOSITION, because when executing the **for** iteration with $k = n$, k' must be k, thus the statements in line 13–15 have no effect, and the statements in line 16–19 will not be executed at all.

5 [GC] Problem 28.2-1

The first half, an $M(n)$-time matrix-multiplication algorithm implies an $O(M(n))$-time squaring algorithm, is obvious, because to square a matrix is just to multiply two identical matrices.

For the second half, if $S(n) = \Omega(n^3)$, we can use $\Theta(n^3) = O(S(n))$ brute-force matrix-multiplication algorithm to compute the product of two $n \times n$ matrices. Otherwise, we can compute the product of two $n \times n$ matrices AB by squaring a $2n \times 2n$ matrix:

$$\begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} A^2 & AB \\ 0 & 0 \end{pmatrix}.$$

Since $S(n) = O(n^3)$ is polynomially bounded, we have the regularity property $S(2n) = O(S(n))$, and the total running times is $O(S(2n)) = O(S(n))$

6 [GC] Problem 28.2-2

(to be done)

7 [GC] Problem 28.2-3

If the LUP decomposition of matrix A is $PA = LU$, since $\det(P) = \det(L) = 1$ and U is triangular, the determinant of A is the product of diagonal entries of U. We have proved that computing LUP decomposition and multiplying two matrices have essentially the same difficulty, so we can compute the determinant in $O(M(n))$-time if $M(n)$-time matrix-multiplication algorithm exists.

(to be done)

8 [GC] Problem 28.3-1

Let A be a symmetric positive-definite matrix. Suppose, to the contrary, that there exists i, such that $a_{ii} \le 0$. Let $x_j = \delta_{ij}$ (δ_{ij} is the Kronecker delta), then vector $x \ne 0$, and $x^{\mathrm{T}} A x = a_{ii} \le 0$, which contradicts the definition of positive-definite matrix.

9 [GC] Problem 28.3-3

Let A be a symmetric positive-definite matrix. Suppose, to the contrary that a_{ij} $(i \neq j)$p is the maximum element. Since $a_{ii} > 0$, let $x_k = -(a_{ij}/a_{ii})\delta_{ik} + \delta_{jk}$, then vector $x \neq 0$. Consider $x^{\mathrm{T}}Ax$, we have

$$\begin{aligned} x^{\mathrm{T}}Ax &= a_{ii}\left(-\frac{a_{ij}}{a_{ii}}\right)^2 + 2a_{ij}\left(-\frac{a_{ij}}{a_{ii}}\right) + a_{jj} \\ &= \frac{a_{ii}a_{jj} - a_{ij}^2}{a_{ii}}. \end{aligned}$$

Since a_{ii} is positive and $a_{ij} \geq a_{ii} > 0$, $a_{ij} \geq a_{ii} > 0$, we have $x^{\mathrm{T}}Ax \leq 0$, which leads to contradiction.

10 [GC] Problem 28-1

a. By using formula (28.8) repeatedly,

$$\begin{aligned} A &= \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix} \end{aligned}$$

$$L = \begin{pmatrix} 1 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix}.$$

b. Let $Ux = y$, then $Ly = (1,1,1,1,1)^{\mathrm{T}}$. By applying forward substitution, we get $y_1 = 1$, $y_2 = y_1 + 1 = 2$, $y_3 = y_2 + 1 = 3$, $y_4 = y_3 + 1 = 4$, $y_5 = y_4 + 1 = 5$. Hence $Ux = y = (1,2,3,4,5)^{\mathrm{T}}$. By applying backward substitution, we get $x_5 = 5$, $x_4 = x_5 + 4 = 9$, $x_3 = x_4 + 3 = 12$, $x_2 = x_3 + 2 = 14$, $x_1 = x_2 + 1 = 15$, i.e. $x = (15,14,12,9,5)^{\mathrm{T}}$.

c. The inverses of L and U are easy to find:

$$L^{-1} = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, U^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix},$$

and the inverse of A is

$$A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 4 & 3 & 2 & 1 \\ 3 & 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

d. Recall formula (28.8):

$$A = \begin{pmatrix} a_{11} & w^{\mathrm{T}} \\ v & A' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v/a_{11} & I_{n-1} \end{pmatrix} \begin{pmatrix} a_{11} & w^{\mathrm{T}} \\ 0 & A' - vw^{\mathrm{T}}/a_{11} \end{pmatrix}$$

Note that all elements except the first of v and w^{T} are zero, and all entries except the top-left one of vw^{T}/a_{11} are zero, thus $A' - vw^{\mathrm{T}}/a_{11}$ is still a tridiagonal matrix, which means that we can transform an $n \times n$ tridiagonal matrix LU decomposition problem to an $(n-1) \times (n-1)$ one in constant time, and thus computing LU decomposition in $O(n)$ time. Furthermore, we can easily conclude by mathematical induction, that L and U are lower bidiagonal and upper bidiagonal, respectively.

For a lower bidiagonal matrix L, we can solve $Ly = b$ in $O(n)$ time:

$$y_1 = \frac{b_1}{l_{11}}, \quad y_2 = \frac{b_2 - l_{21}b_1}{l_{22}}, \quad \cdots, \quad y_n = \frac{b_n - l_{n,n-1}b_{n-1}}{l_{nn}}.$$

Likewise, for upper bidiagonal matrix U, we can solve $Uy = b$ in $O(n)$ time. Therefore, we can solve $Ax = b$ in $O(n)$ time by performing LU decomposition if A is tridiagonal.

Since forming A^{-1} takes $\Omega(n^2)$ to compute the entries, any method based on the inverse is asymptotically more expensive in the worst case.

e. (to be done)