Problem Solving: Homework 3.4

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1 [GC] Problem 7.1

(a) For every ordered pair of vertices u, v in D, let w be any vertex of D other than u, v, then $D - w$ is strong and there exists a path from u to v. Hence D is strong.

(b) Let $\{a, b, c, d\}$ be the vertex set of D. Consider $D - d$ whose order is 3, since it is oriented and strong, it must be a directed triangle. Without loss of generality, assume that the arcs are ab, bc, ca. Likewise $D - c$ is also a directed triangle. Since ab has already been an arc of $D - c$, its arcs are ab, bd, da. Now consider $D - b$ containing arcs ca, da. Note that it is impossible to make $D - b$ both strong and oriented by adding arcs. Therefore there does not exist such graph satisfying the property described in the problem.

2 [GC] Problem 7.2

- If: since G is Eulerian, it must have an Eulerian cycle $(u_1, u_2, \cdots, u_n, u0)$. Let the directions of the edges be (u_1, u_2), (u_2, u_3), $\cdots$, (u_{n-1}, u_n), (u_n, u_0). For every vertex v of G, it may appear more than once in Eulerian cycle, but for each apperance u_i of v, (u_{i-1}, u_i) and (u_i, u_{i+1}) provide one indegree and one outdegree for v. Therefore, $\operatorname{id} v = \operatorname{od} v$ for every vertex v of the orientation. Hence G has an Eulerian orientation.

- Only if: if G has an Eulerian orientation, then for every vertex v of G, $\operatorname{id} v = \operatorname{od} v$. The underlying graph of the orientation, i.e. G, satisfies that $\deg v$ is odd for every v because $\deg v = \operatorname{id} v + \operatorname{od} v$ where $\operatorname{id} v = \operatorname{od} v$. Therefore G is Eulerian.

3 [GC] Problem 7.4

- If: for every ordered pair of vertices u, v, there exists a $v - u$ path in $\vec{D}$ because $\vec{D}$ is strong. By the definition of the converse of a graph, the $v - u$ path in $\vec{D}$ is simultaneously a $u - v$ path in D. Hence D is strong.

- Only if: note that the converse of the converse of a graph of is the graph itself. Interchanging D with $\vec{D}$ in the proof of 'if' gives the proof of 'only if'.

4 [GC] Problem 7.5

- If: Let's prove by contradiction. If D is not strong, then it must contain at least two strongly connected components. Let C_1 be one of the components, and $C_2 = D - C_1$ contains all other strongly conneted components. The edges connecting C_1 and C_2 in G form an edge cut of G, and thus there exists an arc from C_1 to C_2 and an arc from C_2 to C_1. This means that C_1 and C_2 are strongly connected, which leads to contradiction.

- Only if: Let u and v be any vertex of A and B, respectively. Since D is strong, there must exists a directed path from u to v. Futhermore, there must exist an arc (x, y), such that $x \in A$ and $y \in B$. Likewise there must exist an arc (z, w) such that $z \in B$ and $w \in A$. Therefore, there is an arc from A to B and an arc from B to A.

5 [GC] Problem 7.9

- If: for a tournament T of order n, it contains $n(n-1)/2$ arcs, and thus the sum of outdegrees over all verices of T is $n(n-1)/2$. Since every two vertices of T have distinct outdegrees, it must be the case that, for every integer i ($0 \le i \le n$), there exists exactly one vertex v_i such that $\operatorname{od} v_i = i$. Futhermore, there exist arcs from v_n to all other vertices, arcs from v_{n-1} to all other vertices except v_n, $\cdots$. In other words, (v_i, v_j) is an arc of T if and only if $i < j$. Since the relation '$<$' is transitive, the tournament T is also transitive.

- Only if: By Theorem 7.8, there exists a Hamiltonian path $P = (u_1, u_2, \cdots u_n)$ of T. By transitivity, (u_i, u_j) is an arc of T if and only if $i < j$. Therefore, $\text{od}\, u_i = n - i$, i.e. every two vertices of T have distinct outdegrees.

6 [GC] Problem 7.10

Let $(u = u_0, u_1, \cdots, u_k = v)$ be a shortest path from u to v. (u, u_2), (u, u_3), $\cdots$m (u, u_k) are not arcs of the tournament, for any one of them will make the shortest path even shorter. Therefore, (u_2, u), (u_3, u), $\cdots$, (u_k, u) are arcs of the tournament, which means that $\text{id}\, u \geq k - 1$.

7 [GC] Problem 7.13

Since u and v are vertices of a tournament, either arc (u, v) or arc (v, u) is in the tournament. Without loss of generality, assume (u, v) is in the tournament, then $\vec{d}(u, v) = 1$. However, (v, u) is not in the tournament, which makes $\vec{d}(v, u) > 1$. Therefore, $\vec{d}(u, v) \neq \vec{d}(v, u)$.

8 [GC] Problem 7.14

(a) We try to construct a tournament T of order n (n is odd) such that every vertex of T has the same indegree (or outdegree). Let $v_1, v_2, \cdots, v_n$ be vertices of K_n. We assign directions for edges of K_n according to the following rule: for every pair of vertices v_i, v_j ($i < j$), (v_i, v_j) is an arc of T if i and j have the same parity, and (v_j, v_i) is an arc of T if the parities of i and j differ. For every vertex v_i, $\text{id}\, v_i = \lfloor (i-1)/2 \rfloor + \lceil (n - i)/2 \rfloor = (n-1)/2$, which means that all teams tie for first place.

(b) Suppose, to the contrary, that all teams tie for first place. Let T denote the corresponding tournament of even order n. If all vertice of T has equal indegree d and outdegree $n - 1 - d$, then the sum of the indegrees and outdegrees over all vertices are nd and $n(n - 1 - d)$, respectively. Since d and $n - 1 - d$ have different parities, $nd \neq n(n - 1 - d)$, which violates Theorem 7.1. So it is impossible for all teams to tie for the first place.

9 [GC] Problem 7.15

We first prove by mathematical induction, that for every integer k with $3 \leq k \leq n$, a strong tournament T of order n has a strong tournament subgraph T_k of order k.

For the base step, $T = T_n$ it self is a strong tournament subgraph of T.

For the induction step, assume that T_n contains a tournament subgraph T_k of order k $(4 \leq k \leq n)$. By Theorem 7.11, there exists vertex v of T_k, such that $T_{k-1} = T_k - v$ is a strong tournament subgraph of T_n.

By mathematical induction, T has a strong tournament subgraph T_k of order k for all k with $3 \leq k \leq n$. Since T_k is Hamiltonian by Theorem 7.10, T contains a cycle of length k, i.e. the Hamiltonian cycle of T_k, for every integer k with $3 \leq k \leq n$.