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Consensus reaching in social network DeGroot Model: The roles of the Self-confidence and node degree

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A B S T R A C T

In this paper, we investigate how the agent's self-confidence level and the node degree influence the consensus opinion formation and the consensus convergence speed in the social network DeGroot model. We find that (1) the higher self-confidence will increase the agent's importance degree to determine the consensus opinion, but will also slow down the convergence speed for all agents to be able to obtain consensus, and (2) it is conducive to accelerating the convergence speed to be able to reach a consensus where all agents can manage to balance self-confidence levels and node degrees in the social network.

1. Introduction

Consensus is one of the most important aspects within an interacting group. Everyday life presents many situations in which it is necessary for an interacting group to reach a consensus. Hence consensus formation is an interesting problem which attracts a lot of attention in natural and social sciences [2,10,11,13,24,31,33-36,46,50,51].

Opinion dynamics is a very useful tool to investigate the consensus formation. Over the past few decades, many models of opinion dynamics have been developed, in which all agents change their opinions according to certain rules and either reach or do not reach a consensus. Opinion dynamics models can be classified as continuous opinion dynamics and discrete opinion dynamics, relying on the opinion space being a real interval or a set of discrete values. Discrete opinion dynamics models mainly include the Ising model [4,23], Sznajd model [3,38], Voter model [27,39], Majority rule model [7,32] and so on. Continuous opinion dynamics models contain linear models, such as the DeGroot model [10], Friedkin and Johnsen model [20], and nonlinear models, for instance, the bounded confidence model [9,26] and the social judgment based opinion dynamics model [16]. The earliest model of opinion dynamics is formulated by French [18], Friedkin further investigated public opinion dynamics in social influence networks based on the Friedkin and Johnsen model [19,21,22]. What is more, some interesting extended studies had been conducted at social networks [6,17,30,42,52,53], dynamic networks [35] and super-networks [40]. Some recent studies on opinion evolution/dynamics can be found in the review papers [14,41].

Opinion dynamics is a complex process with lots of interactions and relationships. The interactions and relationships are usually characterized by a social network. In real-life, each agent would have a certain degree self-confidence regarding their

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own opinion, and would simultaneously be influenced by others linked to them in a social network. And thus the opinion of an agent is updated partly due to their insistence of their own opinion and partly due to the weighted summation of neighbors in a social network. Taking this idea in the mind, Dong et al. [12] revisited the DeGroot model [10] in a social network, and presented the social network DeGroot (SNDG) model. Particularly, Dong et al. [12] highlighted that leadership plays an important role in opinion dynamics, showing that the consensus among all agents can be reached if and only if there are the agents which influence all other agent in the network. They also demonstrated that the collective consensus opinion can be calculated by a linear combination of all agents' original opinions (See Theorem 1, Corollaries 1 and 2 in Dong et al. [12]).

However, in the SNDG model there are still two gaps that need to be filled:

(1) Although the collective consensus opinion of all agents is a linear representation of all agents' original opinions, we do not know the linear combination coefficients of all agents, especially the linear combination coefficients of all opinion leaders, i.e., the weights of opinion leaders to determine the formation of the consensus opinion have not been investigated.
(2) Although the condition of consensus reaching in a social network has been presented, we also wonder how the consensus convergence speed is influenced by the agent's self-confidence level and the node degree in the network.

In this paper we want to fill in the above gaps. We use theoretical analysis and simulation experiments to investigate the effect of an agent's self-confidence level and the node degree on weights determination of opinion leaders to be able to reach a consensus and the consensus convergence speed.

Although this study is a continuation of the SNDG model, we present three new findings:

(1) The higher self-confidence will increase the opinion leader's weight determination to determine the consensus opinion in the directed social network, and the higher node degree also has this effect in the undirected social network,
(2) If there are no less than two agents whose self-confidence levels are higher than 0.5, the consensus convergence speed will be restrained, and this effect is further intensified for the larger $\beta_{(2)}$ which is the second largest self-confidence level in the group, and
(3) It is conducive to accelerate the consensus convergence speed when all agents manage to balance their level of self-confidence and the node degrees in the undirected social network.

The rest of this paper is organized as follows. First, Section 2 presents the SNDG model briefly. Next, Section 3 studies how the self-confidence level and the node degree affect the consensus opinion, and Section 4 analyzes the convergence speed to reach a consensus. Finally, Section 5 concludes this paper.

2. Preliminaries: social network DeGroot model

In this section we introduce the SNDG model briefly, as this study is a continuation of the SNDG model.

First, the following basic concepts regarding graphs (see [5,28]) will be used as a basis in the SNDG model. A directed graph $G(V, E)$ consists of a set V of nodes and a set E of edges. The matrix $\mathbf{A} = (a_{ij})_{n\times n}$ is the adjacency matrix of directed graph $G(V, E)$, whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1, (v_i, v_j) \in E \\ 0, (v_i, v_j) \notin E \end{cases}.$$

The in-degree d_i^{in} of a node v_i in $G(V, E)$ is the number of edges with head v_i, and the out-degree d_i^{out} of a node v_i in $G(V, E)$ is the number of edges with tail v_i.

The social network is always described by a directed graph $G(V, E)$, where $V = \{v_1, v_2, ..., v_n\}$ is the set of all agents, E is the set of directional edges which mean directional relationships among all agents, and let $\mathbf{A} = (a_{ij})_{n\times n}$ be the adjacency matrix of the directed graph $G(V, E)$. We assume that $G(V, E)$ is a simple directed graph, i.e., $G(V, E)$ is without loops or parallel edges, and each agent v_i is influenced by at least one agent except himself.

Let $x_i^t \in R$ denote the opinion of the agent v_i at time t. In general, an agent will neither simply follow nor strictly disregard others' opinions. We assume the agent v_i has a certain level of self-confidence β_i and he is not a completely stubborn agent, and thus $\beta_i \in (0, 1)$. What is more, he distributes $(1 - \beta_i)$ across his neighbors. Let

$$w_{ij} = \frac{(1-\beta_i)a_{ij}}{\sum_{j=1, j\neq i}^{n} a_{ij}}. \tag{1}$$

so w_{ij} is the weight that the agent v_i puts on his neighbor v_j.

The opinion dynamics of agent v_i is as follows:

$$x_i^{t+1} = \beta_i x_i^t + \sum_{j=1, j\neq i}^{n} w_{ij}\times x_j^t. \tag{2}$$

Above all, the opinions evolution of all agents can be compactly written as

$$\mathbf{X}^{t+1} = \mathbf{B} \times \mathbf{X}^t,\ t = 0, 1, 2, \ldots \tag{3}$$

where $\mathbf{X}^t = (x_1^t, x_2^t, \ldots, x_n^t)^T \in R^n$,

$$\mathbf{B} = \begin{bmatrix} \beta_1 & w_{12} & \ldots & w_{1n} \\ w_{21} & \beta_2 & \ldots & w_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ w_{n1} & w_{n2} & \ldots & \beta_n \end{bmatrix}.$$

Firstly, we give two definitions in order to make the reader better understand this study regarding to the SNDG model.

Definition 1 [12]**.** Opinion leader of a social network $G(V, E)$ is the agent v_k, for$all v_i \in V/\{v_k\}$, there is a directed path in the social network from v_i to v_k, and in this study we let V_G^{leader} denote the set of opinion leaders in the directed graph $G(V, E)$.

In additional, the sequence of edges $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_{n-1}}, v_{i_n})$ in the social network $G(V, E)$ is called a directed path. Some basic concepts regarding graph can be found in [28].

A consensus among all agents can be reached when there is a $c \in R$ such that

$$\lim_{t\to\infty} x_i^t = c(i = 1, 2, \ldots, n)\,\text{forevery}\,\mathbf{X}^0 \in \mathbf{R}^n.$$

We let c denote the consensus opinion.

Some conclusions regarding the consensus in the SNDG model had been proposed in [12] as Lemmas 1 and 2.

Lemma 1 [12]**.** If $V_G^{leader} \neq \emptyset$, then a consensus among agents in $G(V, E)$ can be reached.

Lemma 2 [12]**.** The collective consensus opinion c is dependent on the opinion leaders' original opinions. i.e.,

$$\mathrm{c} = \sum_{v_i \in V_G^{leader}} \lambda_i x_i^0. \tag{4}$$

Lemmas 1 and 2 emphasize the importance of leadership when approaching a consensus in the SNDG model. Specifically, Lemma 1 shows the consensus condition in the social network, and Lemma 2 further reveals the relationship between the collective consensus opinion and the opinion leaders' original opinions.

In the following sections, we further reveal

(a) how the self-confidence level and the node degree determine the weights determination (i.e., λ in Lemma 2) of opinion leaders to form a consensus;
(b) how the self-confidence level and the node degree influence the consensus convergence speed.

3. The weights determination of opinion leaders to form a consensus

In this section we present theoretical analysis to show the roles of the self-confidence level and the node degree in the weights determination of opinion leaders to approach a consensus in the SNDG model. The results are described in Propositions 1 and 2.

Proposition 1. *Let $G(V, E)$ be a directed graph/social network, and $V_G^{leader} \neq \emptyset$. Let $v_i \in V_G^{leader}$, β_i denotes as before, and λ_i denote the consensus weight of the opinion leader v_i as Eq. (4) in Lemma 2. And then we get*

$$\lambda_i = f(\beta_i), \tag{5}$$

and the function $f(\cdot)$ is continuous and strictly monotonically increasing when $\beta_i \in (0, 1)$.

Proof. According to Ref [12], if $V_G^{leader} \neq \emptyset$, all agents will reach a consensus in the end. Further, if the agent is the opinion leader, his consensus weight is strictly greater than zero; otherwise, his consensus weight is zero. Based on this, we only consider all opinion leaders, without loss of generality, they are $v_1, \ldots, v_m (m \geq 2)$, respectively.

And thus we can obtain consensus weights of opinion leaders to solve the equations as follows:

$$\mathbf{XC} = \mathbf{X}, \tag{6}$$

where $\mathbf{C}_{m\times m}$ is a sub matrix which is formed by row i and column j of the matrix $\mathbf{B}$.

There are the following facts: first of all, $v_1, \ldots, v_m$ are opinion leaders, and thus $\mathbf{C}_{m\times m}$ is still nonnegative and stochastic. The next, as we know, all opinion leaders are strongly connected in the social network, hence $\mathbf{C}_{m\times m}$ is irreducible. Since $0 < \beta_i < 1, i = 1, \ldots, m$, we further acknowledge that the matrix $\mathbf{C}_{m\times m}$ is primitive.

We change the Eq. (6) as follows:

$$(\mathbf{C}^T - \mathbf{I})\mathbf{x}^T = 0. \tag{7}$$

The matrix $\mathbf{C}_{m\times m}$ is primitive and the spectral radius $\rho(\mathbf{C}) = 1$, then we have

$$rank(\mathbf{C}^T - \mathbf{I}) = m - 1.$$

According to Perron-Frobenius Theorem the solution of Eq. (7) can be positive. Without loss of generality, let x_i be a free variable and $x_i > 0$.

For convenience of explanation, let $\mathbf{D} = (d_{ij})_{m\times m}$ denote $(\mathbf{C}^T - \mathbf{I})$ and $\mathbf{E} = (e_{ij})_{(m-1)\times(m-1)}$ be a sub matrix which is formed by throwing away row i and column i in$(\mathbf{C}^T - \mathbf{I})$.

By Cramer's Rule, we can obtain $x_k(k \neq i)$ by $(\mathbf{C}^T - \mathbf{I})\mathbf{x}^T = 0$. i.e.,

$$x_k = \frac{\sum_{j\neq i, j=1}^{m} e_{ji} Q_{jk}}{\det(\mathbf{D})} x_i,$$

where Q_{jk} is Cofactor of e_{jk}. What is more, if $a_{ij} = 1$, then $e_{ji} = \frac{1-\beta_i}{d_{out}^i}$; otherwise, $e_{ji} = 0$, where d_{out}^i is the out degree of agent v_i. To emphasize an important point, when we use the Cramer's Rule, the right-hand-sides of the equations are free variables but they are constants in the system of linear equations.

Since $\mathbf{x} = [x_1, ..., x_m]$ is positive, and thus we have

$$\frac{1-\beta_i}{d_{out}^i} \frac{\sum_{j\neq i, j=1, a_{ij}=1}^{m} Q_{jk}}{\det(\mathbf{D})} > 0.$$

Let a_k denote $\frac{\sum_{j\neq i, j=1, a_{ij}=1}^{m} Q_{jk}}{\det(\mathbf{D})}$, then $a_k > 0$; And thus

$$\lambda_i = \frac{x_i}{\sum_{i=1}^{m} x_i} = \frac{1}{\frac{1-\beta_i}{d_{out}^i}\sum_{j=1, j\neq i}^{m} a_j + 1}.$$

Let a denote $\frac{1}{d_{out}^i}\sum_{j=1, j\neq i}^{m} a_j$, then $a > 0$. We have

$$\lambda_i = f(\beta_i),$$

where $f(\beta_i) = \frac{1}{a(1-\beta_i)+1}$.

When $\beta_i \in (0,\ 1)$, the function $f(\beta_i)$ is an elementary function. Therefore the function $f(\cdot)$ is continuous and strictly monotonically increasing as β_i also increases. □

From Proposition 1, in a directed social network we find that a higher self-confidence level will yield more weight allocation to influence the consensus opinion. It is a pity that we could not obtain the more information to analytically define the function $f(\cdot)$ in Proposition 1. However, if any two agents in V influence each other, and then the social network $G(V, E)$ is an undirected graph. In this case, we can accurately define $f(\cdot)$ in an undirected network as Proposition 2. We firstly give some symbolic instructions in an undirected social network as follows:

(1) Considering $d_i^{in} = d_i^{out}$ for any agent v_i, and thus we simply denote the node degree of agent v_i as $d_i = d_i^{in} = d_i^{out}$.
(2) If $G(V, E)$ is a connected undirected graph, according to Definition 1, every agent should be seen as an opinion leader, and thus we reach the conclusion: $V_G^{leader} = V$.

Proposition 2. *Let $G(V, E)$ be a connected undirected graph/social network and d_i denote the node degree of the agent v_i. Likewise, β_i and λ_i denote as before. Then we have*

$$\lambda_i = \frac{\frac{d_i}{1-\beta_i}}{\sum_{i=1}^{n} \frac{d_i}{1-\beta_i}}. \tag{8}$$

Proof. According to the proof of Proposition 1 and the social network being a connected undirected graph, we can obtain consensus weights to solve the equations as follows:

$$\mathbf{XB} = \mathbf{X},\ \mathbf{X} = [x_1, \ldots, x_n].$$

More specially, according to SNDG model, we have

$$\begin{cases} \beta_1 x_1 + \frac{1-\beta_2}{d_2} a_{12} x_2 + ... + \frac{1-\beta_n}{d_n} a_{1n} x_n = x_1 \\ \frac{1-\beta_1}{d_1} a_{21} x_1 + \beta_2 x_2 + ... + \frac{1-\beta_n}{d_n} a_{2n} x_n = x_2 \\ ...... \\ \frac{1-\beta_1}{d_1} a_{n1} x_1 + \frac{1-\beta_2}{d_2} a_{n2} x_2 + ... + \beta_n x_n = x_n \end{cases}.$$

Since $a_{i1} + a_{i2} + ... + a_{in} = d_i$ for any agent v_i, then we get the solution of the above equation set as follows:

$$x_i = \frac{d_i}{1-\beta_i}.$$

Additional, $\mathbf{B}^T$ is the nonnegative and stochastic matrix, and thus the spectral radius of $\mathbf{B}^T$ is 1. According to Perron-Frobenius Theorem [28], the solutions of the equation set are positive and unique after being normalized.

Hence, the consensus weight after being normalized is as follows:

$$\lambda_i = \frac{\frac{d_i}{1-\beta_i}}{\sum_{i=1}^{n} \frac{d_i}{1-\beta_i}}. \quad \square$$

Proposition 2 further shows that both the self-confidence level and the node degree will influence the weights to form a consensus. The higher self-confidence level and the node degree in the undirected social network the larger the consensus weights that will be yielded according to Eq. (8).

4. The convergence speed to form a consensus

As is well-known, the transition matrix **B** is a stochastic matrix, and thus the second largest eigenvalue $|\tau_2|$ of transition matrix **B** is less than or equal to one. If $|\tau_2|$ equals one, all agents cannot form a consensus; if not, all agents can approach a consensus and the consensus convergence time of the power method in Eq. (3) is generally governed by $|\tau_2|$ theoretically [25,37]. Precisely, the consensus convergence time T^* is essentially proportional to $\frac{-1}{\log|\tau_2|}$, i.e.,

$$T^* \propto \frac{-1}{\log|\tau_2|}. \tag{9}$$

According to Eq. (9), if $|\tau_2|$ is less than and near to 1, all agents reach a consensus very slowly. In other words, the smaller $|\tau_2|$, the faster the convergence speed to form a consensus. In the following, we try to unfold the relationships between $|\tau_2|$ and the self-confidence level and the node degree. Specifically speaking, we reveal the effect of the second largest self-confidence level at first and then study the effect of a balance between the self-confidence level and the node degree on the consensus convergence speed.

4.1. The effect of the second largest self-confidence on the consensus convergence speed

Let β_i denote the self-confidence level of the agent v_i as before, and then the larger β_i indicates that the agent v_i holds more stubbornly to their own opinion. If there are many such agents in the group, intuitively, all agents will reach a consensus very slowly. In addition, β_i is located diagonally in the transition matrix **B**. If β_i is higher than 0.5, and then the ith row of the transition matrix **B** is diagonally dominant. Inspired by the above reasons, we discover the relationship between the second largest eigenvalue of the transition matrix **B**(i.e.,$|\tau_2|$) and the second largest self-confidence level of all agents in Proposition 3.

Proposition 3. *Let β_i denote as before. If there are no less than two agents whose self-confidence levels are larger than 0.5, then denote them as* $0.5 < \beta_{l_k} \le \dots \le \beta_{l_2} \le \beta_{l_1} < 1 (k \ge 2)$. *We get*

$$|\tau_2| \ge 2\beta_{l_2} - 1. \tag{10}$$

Proof. According to the SNDG model in this study and the Gersgorin's disk theorem [28], and thus we have

$$|\tau_i - \beta_i| \le 1 - \beta_i,$$

where τ_i is the eigenvalue of the transition matrix **B**.

And then

$$|\tau_i| - \beta_i \le 1 - \beta_i \text{ and } \beta_i - |\tau_i| \le 1 - \beta_i.$$

Further, $|\tau_i| \le 1$ and $|\tau_i| \ge 2\beta_i - 1$.

According to $\beta_{l_1}, \dots, \beta_{l_k} > 0.5 (k \ge 2)$, we have

$$0 < 2\beta_{l_k} - 1 \le \dots \le 2\beta_{l_2} - 1 \le 2\beta_{l_1} - 1,$$

$$\text{and} \quad \left|\tau_{l_1}\right| \ge 2\beta_{l_1} - 1, \dots, \left|\tau_{l_k}\right| \ge 2\beta_{l_k} - 1.$$

And then we induce

$$|\tau_2| \ge \left|\tau_{l_2}\right| \ge 2\beta_{l_2} - 1. \quad \square$$

From Proposition 3**,** if the second largest self-confidence level in the group is close to 1, then $|\tau_2|$ is also close to 1, which means that all agents will reach a consensus very slowly due to the arbitrary network structure. In other words, the second largest self-confidence level in the group determines the lower bound of consensus convergence time according to Eq. (9).

Let's give a simple example so that the reader may understand Proposition 3. In order to reduce interference from the size of agents, the network structure and the initial opinion as much as possible, we consider $n = 10$ agents in a complete network (i.e., where everybody talks to everybody else), and then the opinion space [0,1] is divided into 10 intervals on

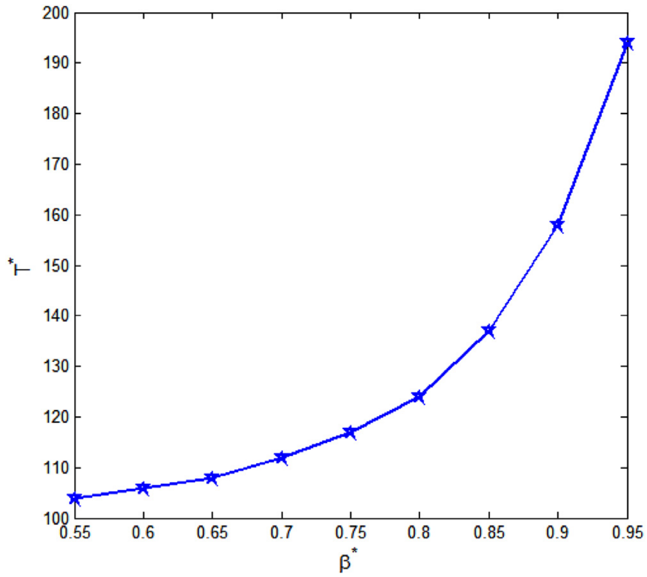

Fig. 1. The relationship between the convergence time T^* and the second largest self-confidence level can be seen in the simple case.

average to conduct the initial opinion $\mathbf{X}^0$. To meet the condition of Proposition 3, their self-confidence levels are set by $\beta = (0.1, 0.1, ..., \beta^*, 0.95)$. Naturally, if $0.5 < \beta^* <= 0.95$, $\beta_{(2)} = \beta^*$ is the second largest self-confidence level in the group. Finally, we make use of the agent-based simulation to obtain the convergence time T^*. The details of the agent-based simulation can be described as follows:

The opinions of all agents reach a stable state when

$$\left\|\mathbf{X}^{t+1} - \mathbf{X}^t\right\| \leq \delta,$$

where $\|\mathbf{X}\| = \underset{1\leq i\leq n}{sum} |x_i|$, and we set $\delta = 10^{-7}$ in this study. Here, in the simulation, the convergence time T^* can be defined as the minimum time to approach a consensus. Based on this, the relationship between the second largest self-confidence level and the consensus convergence time is described in Fig. 1.

It should be stressed that $|\tau_2|$ doesn't necessarily increase when $\beta_{(2)}$ increases, where $\beta_{(2)}$ is the second largest self-confidence level in the group. Therefore, the trend regarding the consensus convergence time T^* increases with $\beta_{(2)}$ increasing, as is illustrated in Fig. 1, and does not always appear. However, $|\tau_2| \geq 2\beta_{(2)} - 1$ has been proven by the theory in Proposition 3, and thus the larger $\beta_{(2)}$ will yields the larger $|\tau_2|$.

4.2. The effect of a balance between the self-confidence and the node degree on the consensus convergence speed

In the following, we study how the agents' self-confidence levels and the agents' node degrees in the undirected social network simultaneously affect the consensus convergence speed.

We first consider the simple case: n agents in the complete network. In this case, the balance between the self-confidence level and the node degree in a social network can make the consensus convergence speed faster. The simple result is described in Proposition 4.

Proposition 4. *Let β_i,d_i denote as before. Considering n agents in the complete network, for any agent v_i, if the relationship between the agent's self-confidence level and the node degree in the social network satisfy Eq. (11):*

$$\beta_i = \frac{d_i}{\sum_{i=1}^{n} d_i} = \frac{1}{n}. \tag{11}$$

And then we have the consensus convergence time $T^* = 1$ for every $\mathbf{X}^0 \in \mathbf{R}^n$.

Proof. For any agent v_i, if

$$\beta_i = \frac{d_i}{\sum_{i=1}^{n} d_i} = \frac{1}{n},$$

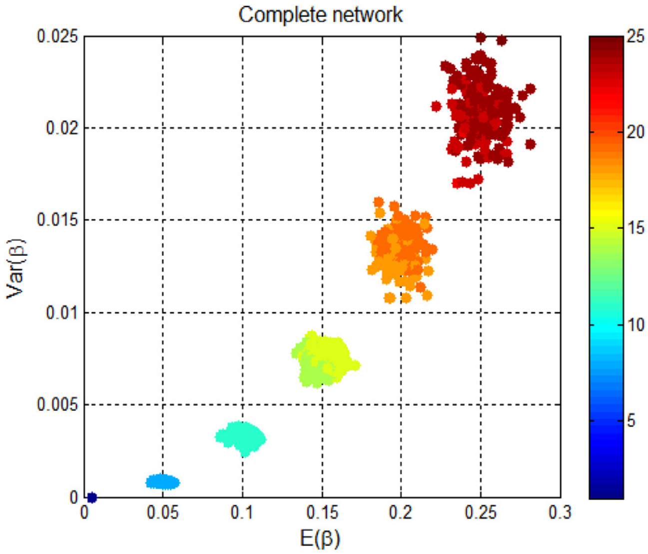

Fig. 2. The consensus convergence time in the complete network, and the color-bar represents the convergence time, where $n = 200$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

and then we have the transition matrix **B** as follows in this case according to the SNDG model:

$$\mathbf{B} = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}.$$

$$\text{Naturally, } \mathbf{X}^1 = \mathbf{B} \times \mathbf{X}^0 = \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \\ \cdots \\ \cdots \\ \cdots \\ x_n^0 \end{bmatrix} = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} x_i^0 \\ \frac{1}{n}\sum_{i=1}^{n} x_i^0 \\ \cdots \\ \frac{1}{n}\sum_{i=1}^{n} x_i^0 \end{bmatrix}.$$

And thus all agents only need one time step to reach a consensus for every $\mathbf{X}^0 \in \mathbf{R}^n$. □

The agents can reach a consensus quickly if all agents manage to balance between the self-confidence level and the node degree in the social network which is defined by

$$\beta_i = \frac{d_i}{\sum_{i=1}^{n} d_i}. \tag{12}$$

In other words, the convergence speed is always fast when the self-confidence level and the node degree in the social network satisfy Eq. (12). We make use of the agent-based simulation to verify the above conclusion, and the details of the agent-based simulation can be described as follows:

The definitions of the stable state and the convergence time T^* in the simulation are mentioned above. What is more, we investigate the convergence time for four different networks: Complete networks, Erdős & Rényi (ER) random networks [15], Barabási & Albert (BA) scale-free networks [1] and Watts & Strogatz (WS) small-world networks [43]. To have better comparability, four different networks are the same size, of which the last three different networks have a near-equal average degree. The above four networks have connected undirected graphs to make all agents approach a consensus.

The self-confidence degrees are chosen in (0, 1). Further, and $E(\boldsymbol{\beta})$ and $Var(\boldsymbol{\beta})$ are the expected value and variance of the vector $\beta = [\beta_1, \beta_2, \ldots, \beta_n]$, respectively.

The initial opinion vector $\mathbf{X}^0$ is generated using a uniform random distribution in[0, 1], and then we obtain the convergence time to reach a consensus under different parameter combinations: $\mathbf{X}^0$, $\beta = [\beta_1, \beta_2, \ldots, \beta_n]$ and obtain the network structure.

The results regarding to the convergence time taken to reach a consensus are displayed in Figs. 2–5. As we know, when all agents manage to balance between the self-confidence level and the node degree in the social network just like Eq. (12),

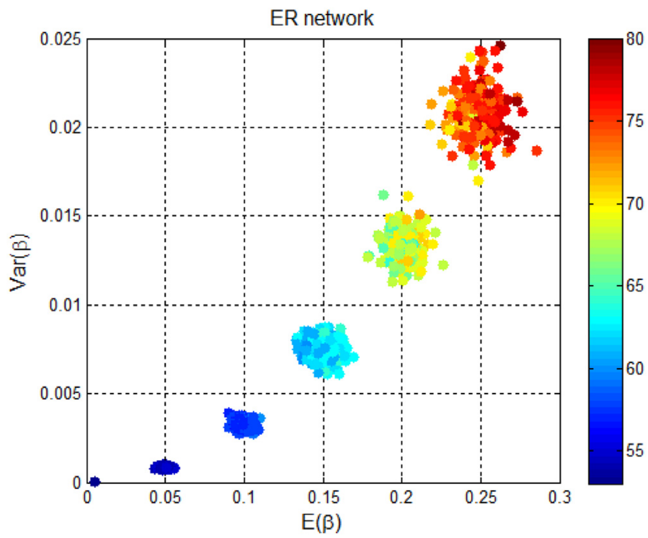

Fig. 3. The consensus convergence time in the ER random networks, and the color-bar represents the convergence time, where $n = 200$, $p = 0.03$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

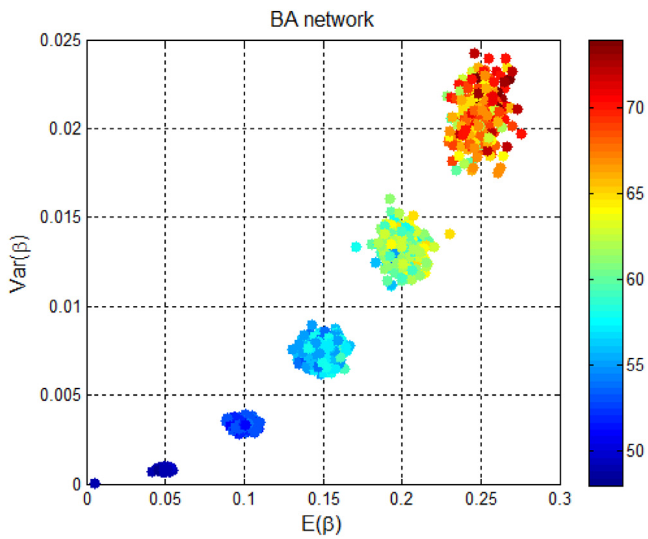

Fig. 4. The consensus convergence time in the BA scale-free networks, and the color-bar represents the convergence time, where $n = 200, m_0 = 6, m_1 = 3$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

let $\overline{\beta} = [\overline{\beta_1}, \overline{\beta_2}, ..., \overline{\beta_n}]$ denote the self-confidence levels of all agents in this case, and then we have

$$\sum_{i=1}^{n} \overline{\beta_i} = \sum_{i=1}^{n} \frac{d_i}{\sum_{i=1}^{n} d_i} = 1.$$

Consequently, the expected value of $\overline{\beta}$ is as follows:

$$E(\overline{\beta}) = \frac{1}{n}. \tag{13}$$

Meanwhile the value of $Var(\overline{\beta})$ depends on the social networks. As $E(\overline{\beta}) = \frac{1}{n} = 0.005$ in the simulation, the convergence time can be seen in the lower left corner of Figs. 2–5. We find that the consensus convergence time is always short when Eq. (12) holds in different network structures.

However, when the expected value of $E(\boldsymbol{\beta})$ is large, the consensus convergence time is always longer in Figs. 2–5. As we know, on the one hand, the trace of the transition matrix **B** (i.e.,$tr(\mathbf{B})$) is the summation of the self-confidence levels, in other words,

$$tr(\mathbf{B}) = sum(\beta) = nE(\beta). \tag{14}$$

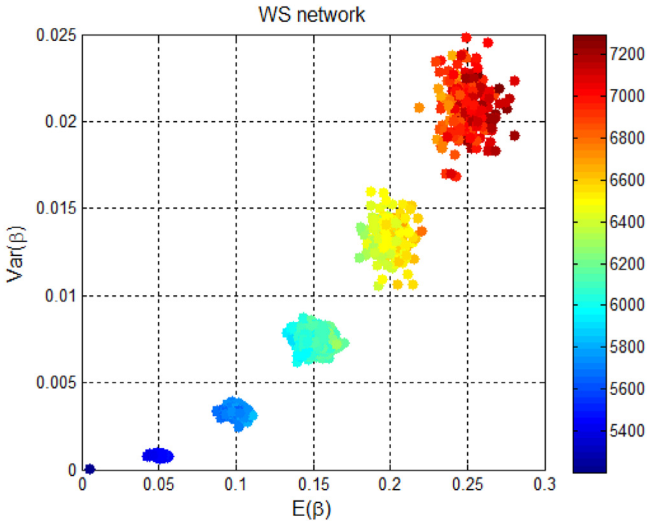

Fig. 5. The consensus convergence time in the WS small-world networks, and the color-bar represents the convergence time, where $n = 200, 2K = 6, p = 0.01$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

On the other hand, $tr(\mathbf{B})$ is also the summation of the eigenvalues of the matrix $\mathbf{B}$ [28], and thus we have

$$tr(\mathbf{B}) = \tau_1 + \tau_2 + \ldots + \tau_n, \tag{15}$$

where τ_i is the eigenvalue of the matrix $\mathbf{B}$.

Considering one of the eigenvalues of matrix $\mathbf{B}$ is one and $|\tau_2|$ is the second largest eigenvalue of transition matrix $\mathbf{B}$, Hence we have

$$nE(\beta) = \tau_1 + \tau_2 + \ldots + \tau_n.$$

Further,

$$\begin{aligned} nE(\beta) &= \tau_1 + \tau_2 + \ldots + \tau_n \le 1 + |\tau_2 + \ldots + \tau_n| \le 1 + |\tau_2| + \ldots + |\tau_n| \\ &\le 1 + (n-1)|\tau_2| \end{aligned}$$

$$|\tau_2| \ge \frac{nE(\beta) - 1}{n - 1}. \tag{16}$$

Based on this, $|\tau_2|$ is large when the expected value of $E(\boldsymbol{\beta})$ is large. Consequently, the consensus convergence time is long according to Eq. (9).

5. Conclusion

In this paper, we highlight the key role of the agent's self-confidence level and the agent's node degree on the consensus reaching in the SNDG model. The main findings are as follows:

(1) A higher self-confidence level increases the opinion leader's degree of importance when determining the consensus opinion, but agents with higher self-confidence levels will slow down the consensus convergence speed, and
(2) Consensus convergence speed can be maximized so that all agents that all agents manage to balance self-confidence levels and node degrees in the undirected social network.

Consensus reaching is a classical issue in group decision making, and has been widely studied [8,24,29,44,45,47-49]. Particularly, opinion evolution is a natural phenomenon in group decision making, and thus we argue that it is very necessary to further develop a theoretical basis to build a robust bridge between these different disciplines (opinion dynamics and group decision making) in the future.

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