Simulation_summary

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Generate pseudorandom variable from the pdf

1. Inverse transformation

$$\mathrm{P}(X \leq x) = \mathrm{P}\left(F^{-1}(U) \leq x\right) = \mathrm{P}(U \leq F(x)) = F(x)$$

So $F(x)$ is the cdf of X and thus $f(x)$ is its density.

Basic algorithm:

Derive the inverse function $F^{-1}(\cdot)$ then

1. Generate a random number u from $U(0,1)$ distribution
2. Set $X = F^{-1}(U)$ Then X will have the cdf $F(\cdot)$

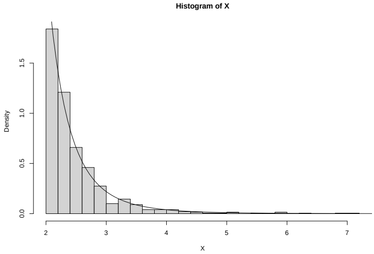
Example

$$\text{pdf:} \quad f(x;\alpha,\gamma) = \frac{\gamma\alpha^{\gamma}}{x^{\gamma+1}} I\{x \geq \alpha\} \quad \alpha > 0, \gamma > 0.$$

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{\alpha}^{x} \frac{\gamma\alpha^{\gamma}}{t^{\gamma+1}} dt = 1 - \alpha^{\gamma} x^{-\gamma}$$

$$x = F^{-1}(u) = \frac{\alpha}{(1-u)^{\frac{1}{\gamma}}}$$

```
set.seed(111)
ar_generator = function(n,alpha,gamma) {
  U <- runif(n);
  X <- (alpha/(1-U)^(1/gamma))
  return(X)}
X <- ar_generator(1000, alpha = 2, gamma = 5)
```



Generate pseudorandom variable from the pdf

2. Inverse transformation

Target cdf $F(\cdot)$ or pdf $f(\cdot)\big($ no closed from of $F^{-1}\big)$ Alternative cdf $G(\cdot)$ or pdf $g(x)$ (one that's easy to sample from), satisfying

$$\frac{f(x)}{g(x)} \leq M, \text{ for all } x$$

Choose $g(\cdot)$ and the value M

1. Generate y from the distribution with pdf $g(\cdot)$
2. Generate u from $U(0,1)$ distribution.
3. If $u \leq \frac{f(y)}{Mg(y)}$ then set $x = y$, if $u > \frac{f(y)}{Mg(y)}$ then return to step 1. Then x has pdf $f(\cdot)$.

Example

$$\text{pdf}: \quad f(x) = \frac{2}{\pi\beta^2}\sqrt{\beta^2 - x^2}, \; -\beta \le x \le \beta$$

$$g(x) = \frac{1}{2\beta}, \; -\beta \le x \le \beta$$

$$M = sup(\frac{f(x)}{g(x)}) = \frac{4}{\pi}$$

```r
accrej <- function(fdens, gdens, M, beta, x){
  ncand <- length(x) # Generate the uniforms
  u <- runif(ncand)
  accepted <- NULL           # Initialize the vector of accepted values
  for(i in 1:ncand) {
    if(u[i] <= fdens(x[i],beta) / (M * gdens(x[i],beta)))
      accepted <- c(accepted, x[i])}
  return(accepted[1:100])
}

unifbetadens = function(x,beta)  ## Convenient
  if(x> -beta & x < beta)
    return(1/(2*beta))
witchshatdens = function(x,beta) ## Target
  if(x> -beta & x < beta)
    return(2/(pi* beta^2) * sqrt(beta^2 - x^2))

set.seed(111)
beta = 2
x = 2*beta* runif(1000) - beta  # U(-beta,beta)
y = accrej(witchshatdens, unifbetadens, M = 4/pi, beta = 2, x)
```

Monte-Carlo Procedure

1. Generate K independent data sets under a condition of interest;
2. Compute the estimator or test statistics for each simulated data; $S_1, \ldots, S_K$
3. With sufficiently large K, the empirical distribution of $S_1, \ldots, S_K$ is a good approximation to the true sampling distribution of the target estimator / test statistics under the conditions of interest. e.g.
    - Sample mean $(S_1, \ldots, S_K) - \theta_0$ is a good approximation of bias
    - $\mathrm{Var}(S_1, \ldots, S_K)$ is a good approximation of standard errors

Mento Carlo Integration

$$\int_a^b g(x)dx = \int_a^b \frac{g(x)}{p(x)} p(x)dx$$

where $p(x)$ is a known probability density function on the support of $[a, b]$

- $g(x)-$ nominal distribution
- $p(x)-$ importance distribution (Common)

Then, generate $X_1, \cdots, X_n \sim p(x)$ and estimate the integral with

$$\frac{1}{n}\sum_{i=1}^{n}\frac{g(X_i)}{p(X_i)}$$

Example:

$$\theta = \int_0^1 e^{x^2} dx = \int_0^1 \frac{e^{x^2}}{1} * 1dx$$

$g(x) = e^{x^2}$, $p(x)$ is U[0,1], $x_1, ..., x_n \sim p(x)$. We just need to simulate random uniforms on [0,1] and compute $\frac{1}{n}\sum_{i=1}^{n}\frac{e^{x^2}}{1}$.