整体的思路：$Ma = f_I + f_E$ K是刚度矩阵，

$$M\frac{\Delta v}{\Delta t} = K(x+\Delta x) + dp\cdot K(v+\Delta v)$$ dp是damping factor

$$\Rightarrow M\Delta v = \Delta t[K(x+\Delta x) + dp\cdot K(v+\Delta v)]$$

$$= \Delta t[Kx + K\Delta x + dp\cdot Kv + dp\cdot K\Delta v]$$

$$= \Delta t Kx + \Delta t K\Delta x + \Delta t\cdot dp\cdot Kv + \Delta t\cdot dp\cdot K\Delta v$$

$\Delta x = \Delta t(v+\Delta v)$，故有

$$= \Delta t Kx + \Delta t^2 Kv + \Delta t^2 K\Delta v + \Delta t\cdot dp\cdot Kv + \Delta t\cdot dp\cdot K\Delta v$$ 整理有

$$(M - \Delta t^2 K - \Delta t\cdot dp\cdot K)\Delta v$$

$$= \Delta t Kx + \Delta t^2 Kv + \Delta t\cdot dp\cdot Kv$$

$$= \Delta t f + \Delta t^2 Kv + \Delta t\cdot dp\cdot Kv$$

整理有 $Kx = f$ 为对应的force

$$[M - \Delta t(\Delta t + dp)K]\Delta v$$

$$= \Delta t\cdot[f + (\Delta t + dp)Kv]$$

故有 $A = M - \Delta t(\Delta t + dp)K$，

$b = f + (\Delta t + dp)Kv$， $x = \Delta v$ 为未知量的

线性方程组

由上面的推导可知，计算刚度矩阵K是解整个方程组的核心.

由力学原理可知，关键是知道对应力的Energy，记为E.

由虚功原理可知 $\frac{\partial E}{\partial x} = f \quad \frac{\partial E}{\partial x^2} = K$，求出对应的K矩阵即可.

对于内力，分为 stretch force 和 Bending force 来讨论，

stretch force 对应的Energy为 $E_s = \frac{1}{2}\int_A \varepsilon^T \sigma t \, dA$，A是有限元的面积，

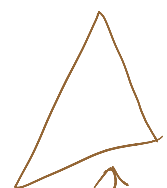
t是厚度 (cloth, t=1)，ε为应变张量，σ为应力张量，且有 $\sigma = C:\varepsilon$，

C是四阶弹性张量，ε和σ均为二阶张量，利用简单的线性各向异性

model，

$$C \Rightarrow \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \qquad \sigma = \begin{bmatrix} \sigma_{uu} \\ \sigma_{vv} \\ \sigma_{uv} \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} \varepsilon_{uu} \\ \varepsilon_{vv} \\ \varepsilon_{uv} \end{bmatrix}$$

$C_{13} = C_{23} = 0$，



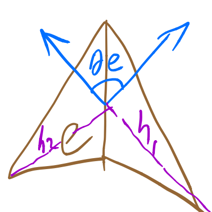
有限元

$$E_s = \frac{1}{2} \cdot A \cdot \varepsilon^T \cdot C \cdot \varepsilon$$

备注：

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \sigma_{ij} \qquad \varepsilon = \varepsilon_{kl} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}$$

$C = C_{ijkl}$ (9×9的矩阵)　　$\sigma = C:\varepsilon \quad E_s = \varepsilon : C : \varepsilon$ 也可以得到



e为公共边

Bending 的有限元，θ_e为二面角，

$$E_B = \frac{\|e\|}{\bar{h}_e} (\theta_e - \bar{\theta}_e)^2$$

$$\bar{h}_e = \frac{1}{3}(h_1 + h_2)$$

$\bar{\theta}_e$ = rest angle

θ_e = current angle

$\|e\|$ 是边e的长度

$ma = f_I + f_e = f$ $\Rightarrow$ 使用 Implicit backward Euler 可得

$$\begin{pmatrix} \Delta x \\ \Delta v \end{pmatrix} = h \begin{pmatrix} v + \Delta v \\ M^{-1} f(x+\Delta x, v+\Delta v) \end{pmatrix}$$

又有 $f(x+\Delta x, v+\Delta v) = K(x+\Delta x) + K \cdot dp(v+\Delta v)$,

故有 $M \frac{\Delta v}{\Delta t} = K(x+\Delta x) + K \cdot dp(v+\Delta v)$,

即得需要求解的方程.

具体有限元上的 f 与 K 的推导:

① stretch force 推导: $E_s = \frac{1}{2} \cdot A \cdot \varepsilon^T C \cdot \varepsilon$

$\frac{\partial E_s}{\partial x}$ (9维向量) $\frac{\partial E_s}{\partial x^2}$ (9×9的二阶张量)

$\overset{2\times2}{\varepsilon} = \frac{1}{2}(F^T F - I)$ $F = (w_u \ \ w_v) \rightarrow$ 3×2的矩阵

$$E_s = \frac{1}{2} k_0 \varepsilon_{00}^2 + k_1 \varepsilon_{00} \varepsilon_{11} + \frac{1}{2} k_2 \varepsilon_{11}^2 + k_3 \varepsilon_{01}^2$$

9×1 向量

$$\frac{\partial E_s}{\partial x} = k_0 \varepsilon_{00} \frac{\partial \varepsilon_{00}}{\partial x} + k_1 \left(\varepsilon_{00} \frac{\partial \varepsilon_{11}}{\partial x} + \varepsilon_{11} \frac{\partial \varepsilon_{00}}{\partial x} \right) + k_2 \varepsilon_{11} \frac{\partial \varepsilon_{11}}{\partial x} + 2 k_3 \varepsilon_{01} \frac{\partial \varepsilon_{01}}{\partial x}$$

怎么计算 $\frac{\partial \varepsilon_{00}}{\partial x}$, $\frac{\partial \varepsilon_{11}}{\partial x}$, $\frac{\partial \varepsilon_{01}}{\partial x}$?

$$\underset{3\times2}{(w_u, w_v)} = (\Delta x_1 \ \ \Delta x_2) \begin{pmatrix} \Delta u_1 & \Delta u_2 \\ \Delta v_1 & \Delta v_2 \end{pmatrix}^{-1}$$

$$F^T F = \underset{2\times3}{(w_u, w_v)^T} \underset{3\times2}{(w_u, w_v)}$$

I 中的 1 可以忽略

$$\varepsilon = \frac{1}{2}(F^T F - I)$$

$$F^TF=\underset{2\times 2}{\begin{pmatrix}\Delta u_1 & \Delta u_2\\ \Delta v_1 & \Delta v_2\end{pmatrix}^{-T}}\ \underset{2\times 3}{\begin{pmatrix}\Delta x_1 & \Delta x_2\end{pmatrix}^{T}}\ \underset{3\times 2}{\begin{pmatrix}\Delta x_1 & \Delta x_2\end{pmatrix}}\ \underset{2\times 2}{\begin{pmatrix}\Delta u_1 & \Delta u_2\\ \Delta v_1 & \Delta v_2\end{pmatrix}^{-1}}$$

$$=\begin{pmatrix}\Delta u_1 & \Delta u_2\\ \Delta v_1 & \Delta v_2\end{pmatrix}^{-T}\begin{pmatrix}\Delta x_1^T\\ \Delta x_2^T\end{pmatrix}\begin{pmatrix}\Delta x_1 & \Delta x_2\end{pmatrix}\begin{pmatrix}\Delta u_1 & \Delta u_2\\ \Delta v_1 & \Delta v_2\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}\overset{A}{\Delta u_1} & \overset{B}{\Delta u_2}\\ \underset{C}{\Delta v_1} & \underset{D}{\Delta v_2}\end{pmatrix}^{-T}\begin{pmatrix}\Delta x_1^T\cdot\Delta x_1 & \Delta x_1^T\cdot\Delta x_2\\ \Delta x_2^T\cdot\Delta x_1 & \Delta x_2^T\cdot\Delta x_2\end{pmatrix}\begin{pmatrix}\Delta u_1 & \Delta u_2\\ \Delta v_1 & \Delta v_2\end{pmatrix}^{-1}$$

$$=\frac{1}{\alpha}\begin{pmatrix}\Delta v_2 & -\Delta u_2\\ -\Delta v_1 & \Delta u_1\end{pmatrix}^{T}\begin{pmatrix}\overset{E}{\underline{\Delta x_1^T\cdot\Delta x_1}} & \overset{F}{\underline{\Delta x_1^T\cdot\Delta x_2}}\\ \underset{G}{\underline{\Delta x_2^T\cdot\Delta x_1}} & \underset{H}{\underline{\Delta x_2^T\cdot\Delta x_2}}\end{pmatrix}\begin{pmatrix}\Delta u_1 & \Delta u_2\\ \Delta v_1 & \Delta v_2\end{pmatrix}^{-1}$$

$$\alpha=\Delta u_1\Delta v_2-\Delta v_1\Delta u_2$$

$$=\frac{1}{\alpha}\begin{pmatrix}D & -B\\ -C & A\end{pmatrix}^{T}\begin{pmatrix}\Delta x_1^T\cdot\Delta x_1 & \Delta x_1^T\cdot\Delta x_2\\ \Delta x_2^T\cdot\Delta x_1 & \Delta x_2^T\cdot\Delta x_2\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}^{-1}$$

$$=\frac{1}{\alpha^2}\begin{pmatrix}D & -C\\ -B & A\end{pmatrix}\begin{pmatrix}\Delta x_1^T\cdot\Delta x_1 & \Delta x_1^T\cdot\Delta x_2\\ \Delta x_2^T\cdot\Delta x_1 & \Delta x_2^T\cdot\Delta x_2\end{pmatrix}\begin{pmatrix}D & -B\\ -C & A\end{pmatrix}$$

$$\varepsilon=\frac{1}{2}(F^TF-I)$$

$$\varepsilon=\frac{1}{2}\left\{\frac{1}{\alpha^2}\begin{pmatrix}D & -C\\ -B & A\end{pmatrix}\begin{pmatrix}\Delta x_1^T\cdot\Delta x_1 & \Delta x_1^T\cdot\Delta x_2\\ \Delta x_2^T\cdot\Delta x_1 & \Delta x_2^T\cdot\Delta x_2\end{pmatrix}\begin{pmatrix}D & -B\\ -C & A\end{pmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right.$$

$$\varepsilon = \frac{1}{2}\left\{ \frac{1}{\alpha^2}\begin{pmatrix} D & -C \\ -B & A \end{pmatrix}\begin{pmatrix} D\Delta x_1^T\Delta x_1 - C\Delta x_1^T\Delta x_2 & A\Delta x_1^T\Delta x_2 - B\Delta x_1^T\Delta x_1 \\ D\Delta x_2^T\Delta x_1 - C\Delta x_2^T\Delta x_2 & A\Delta x_2^T\Delta x_2 - B\Delta x_2^T\Delta x_1 \end{pmatrix} - I \right\}$$

以ε_{00}为例:

$$\varepsilon_{00} = \frac{1}{2\alpha^2}\left[D(D\cdot\Delta x_1^T\Delta x_1 - C\Delta x_1^T\Delta x_2) - C(D\Delta x_2^T\Delta x_1 - C\Delta x_2^T\Delta x_2) - 1 \right]$$

$$= \frac{1}{2\alpha^2}\left[D^2\Delta x_1^T\Delta x_1 - 2DC\Delta x_1^T\Delta x_2 + C^2\Delta x_2^T\Delta x_2 - 1 \right]$$

$$\frac{\partial\varepsilon_{00}}{\partial x} = \frac{1}{2\alpha^2}\left[D^2\cdot 2\cdot\Delta x_1^T\frac{\partial\Delta x_1}{\partial x} - 2DC\left[\Delta x_1^T\frac{\partial x_2}{\partial x} + \Delta x_2^T\frac{\partial\Delta x_1}{\partial x}\right] + 2C^2\frac{\partial\Delta x_2}{\partial x} \right]$$

$$= \frac{1}{\alpha^2}\left\{ D^2\Delta x_1^T\frac{\partial\Delta x_1}{\partial x} - DC\left[\Delta x_1^T\frac{\partial x_2}{\partial x} + \Delta x_2^T\frac{\partial\Delta x_1}{\partial x}\right] + C^2\frac{\partial\Delta x_2}{\partial x} \right\}$$

$$\Delta x_1 = \begin{pmatrix} x_{1x} - x_{0x} \\ x_{1y} - x_{0y} \\ x_{1z} - x_{0z} \end{pmatrix} \quad \Delta x_2 = \begin{pmatrix} x_{2x} - x_{0x} \\ x_{2y} - x_{0y} \\ x_{2z} - x_{0z} \end{pmatrix} \quad \frac{\partial\Delta x_1}{\partial x} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial\Delta x_2}{\partial x} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial\Delta x_1}{\partial x} = [-I \quad I \quad 0]$$

$$\frac{\partial\Delta x_2}{\partial x} = [-I \quad 0 \quad I]$$

$$\frac{\partial\varepsilon_{00}}{\partial x} = \frac{1}{\alpha^2}\Big\{ D^2\Delta x_1^T[-I \quad I \quad 0] - DC\left[\Delta x_1^T[-I \quad 0 \quad I] + \Delta x_2^T[-I \quad I \quad 0]\right] + C^2[-I \quad 0 \quad I] \Big\}$$

$$\frac{\partial\varepsilon_{00}}{\partial x} = \frac{1}{\alpha^2}\begin{bmatrix} (C-D)D\Delta x_1^T I - (C-D)C\Delta x_2^T I \\ D^2\Delta x_1^T I - DC\Delta x_2^T I \\ C^2\Delta x_2^T I - DC\Delta x_1^T I \end{bmatrix}$$

同理可计算

$$\frac{\partial\varepsilon_{01}}{\partial x},\ \frac{\partial\varepsilon_{10}}{\partial x},\ \frac{\partial\varepsilon_{11}}{\partial x}$$

$$= \left[\frac{1}{\alpha}\left((C-D) \quad D \quad -C\right) \otimes I\right] \cdot \left[\frac{1}{\alpha}\left(D\Delta x_1 - C\Delta x_2\right)\right]$$

注：若设 $D = D_m^{-T} \cdot \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ $\quad D_m = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \Delta u_1 & \Delta u_2 \\ \Delta v_1 & \Delta v_2 \end{pmatrix}$

$du = D.row(0)$

$dv = D.Col(0)$ $\quad \Rightarrow \quad D_u = du \otimes I \quad D_v = dv \otimes I$

X.row / X.col 即取X的第几行或第几列。

$x_u = F.Col(0) \quad x_v = F.Col(1)$

$F = (w_u \ w_v) = (\Delta x_1 \ \Delta x_2)\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}$

$\frac{\partial \varepsilon_{00}}{\partial x} = D_u^T \cdot x_u \qquad \frac{\partial \varepsilon_{11}}{\partial x} = D_v^T \cdot x_v \qquad \frac{\partial \varepsilon_{10}}{\partial x} = \frac{1}{2}\left(D_u^T \cdot x_v + D_v^T \cdot x_u\right)$

若求 $\frac{\partial E_s}{\partial x}$，同样以 ε_{00} 为例

$$E_s = k_0 \varepsilon_{00} \frac{\partial \varepsilon_{00}}{\partial x} + \cdots + \cdots$$

$$\frac{\partial E_s}{\partial x} = k_0\left[\frac{\partial \varepsilon_{00}}{\partial x} \otimes \frac{\partial \varepsilon_{00}}{\partial x} + \varepsilon_{00}\frac{\partial^2 \varepsilon_{00}}{\partial x^2}\right] + \cdots +$$

又已知 $\frac{\partial \varepsilon_{00}}{\partial x} = D_u^T \cdot x_u$，注意到 D_u^T 不是 x 的函数，故有

$$\frac{\partial^2 \varepsilon_{00}}{\partial x^2} = D_u^T \frac{\partial x_u}{\partial x}, \quad 又 \quad \frac{\partial x_u}{\partial x} = \frac{1}{\alpha}\left(D\frac{\partial \Delta x_1}{\partial x} - C\frac{\partial \Delta x_2}{\partial x}\right) = D_u$$

即 $\frac{\partial^2 \varepsilon_{00}}{\partial x^2} = D_u^T D_u$

即 $\frac{\partial E_s}{\partial x} = k_0\left(\frac{\partial \varepsilon_{00}}{\partial x} \otimes \frac{\partial \varepsilon_{00}}{\partial x} + \varepsilon_{00} \cdot D_u^T D_u\right) + \cdots +$ → 实现时希望 $\varepsilon_{00} \geq 0$

其他项同理，由此得 $\frac{\partial E_s}{\partial x} = f \quad \frac{\partial E_s}{\partial x^2} = H$.