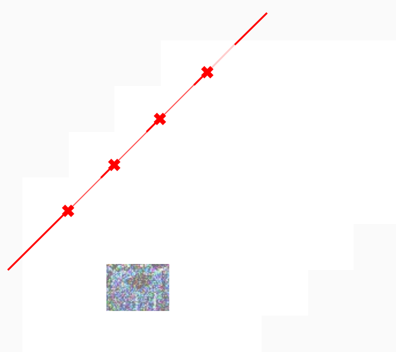
Digital optical imagery – Noise modeling

- Take several pictures of the same scene, and focus on one given pixel,
- There are always unwanted fluctuations around the “true” pixel value,
- This fluctuations are called noise,
- Usually described by a probability density or mass function (pdf/pmf),
- Stochastic process Y parametrized by a deterministic signal of interest x.



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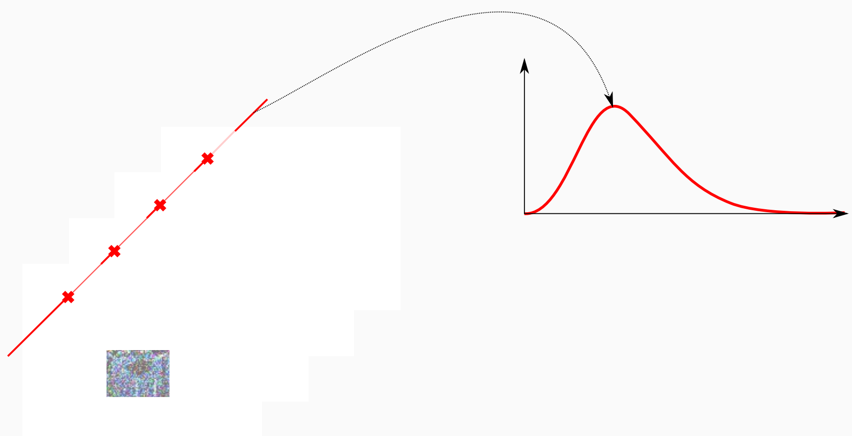

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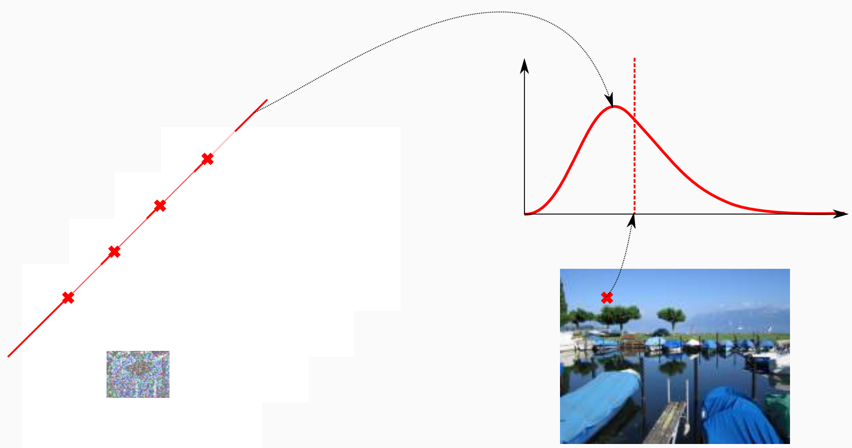

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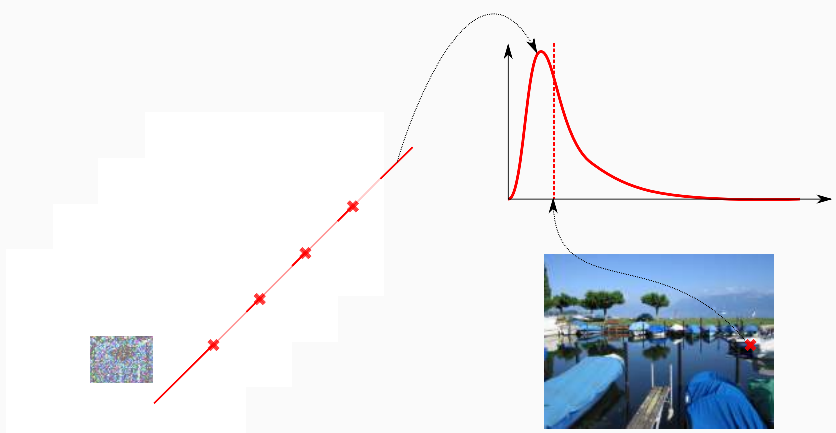

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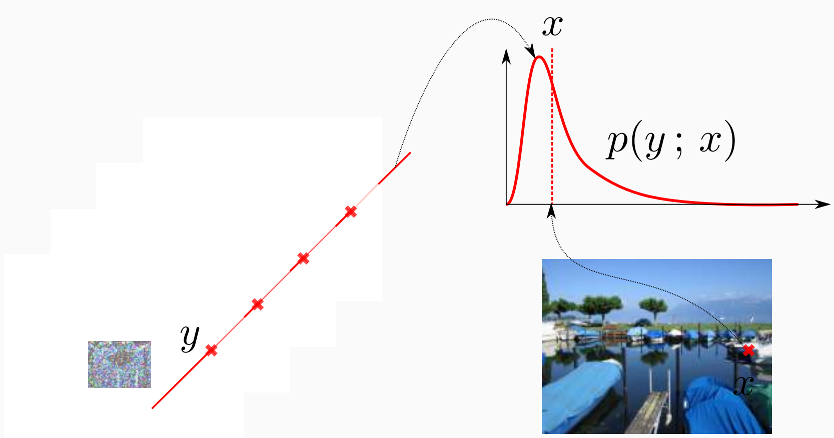


x true unknown pixel value, y noisy observed value (a realization of Y),
link: $p_Y(y\,;\,x)$ noise model

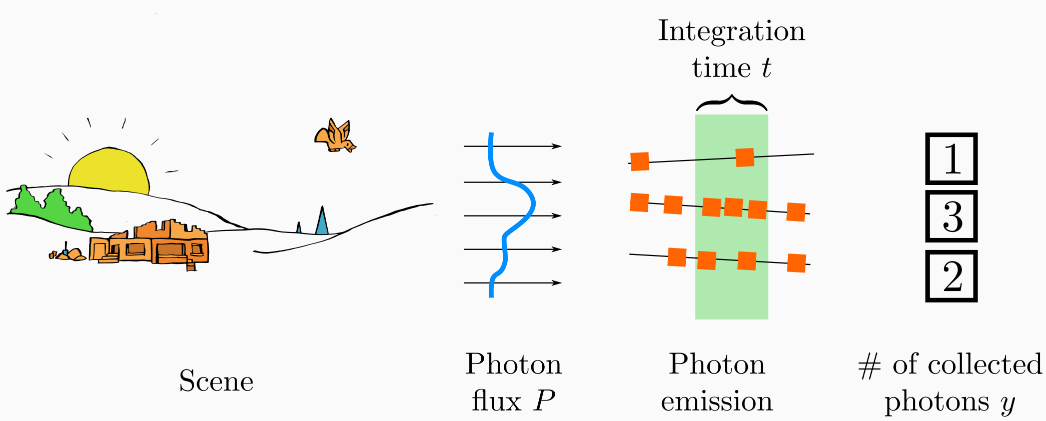
Digital optical imagery – Shot noise

Shot noise

- Number of captured photons $y \in \mathbb{N}$ fluctuates around the signal of interest

$$x = PQ_e t$$

  - x: expected quantity of light
  - Q_e: quantum efficiency (depends on wavelength)
  - P: photon flux (depends on light intensity and pixel size)
  - t: integration time
- Variations depends on exposure times and light conditions.



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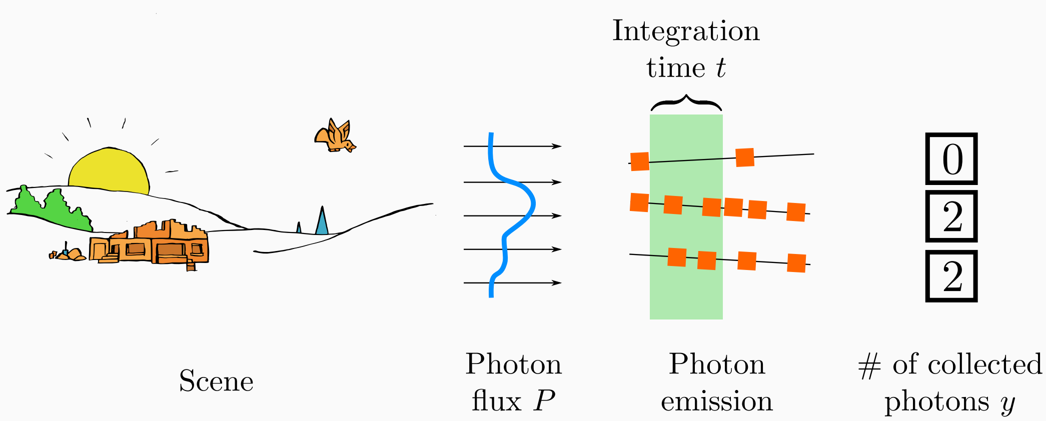


Digital optical imagery – Shot noise

Shot noise and Poisson distribution

- Distribution of Y modeled by the Poisson distribution

$$p_Y(y\,;\,x) = \frac{x^y e^{-x}}{y!}$$

- Number of photons $y \in \mathbb{N}$ fluctuates around the signal of interest $x \in \mathbb{R}$

$$\mathbb{E}[Y] = \sum_{y=0}^{\infty} y p_Y(y\,;\,x) = x$$

- Fluctuations proportional to $\mathrm{Std}[Y] = \sqrt{\mathrm{Var}[Y]} = \sqrt{x}$

$$\mathrm{Var}[Y] = \sum_{y=0}^{\infty} (y-x)^2 p_Y(y\,;\,x) = x$$

- Inherent when counting particles in a given time window

We write $Y \sim \mathcal{P}(x)$

Digital optical imagery – Shot noise

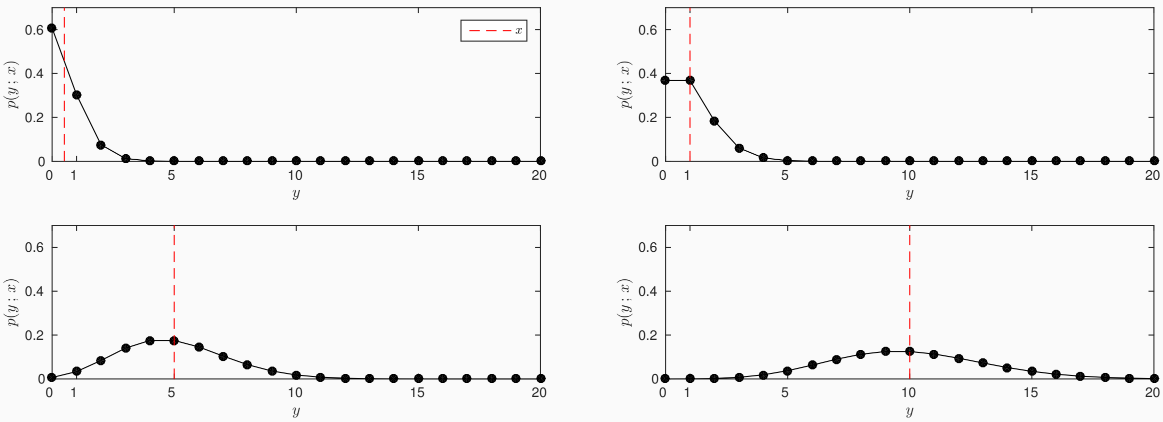
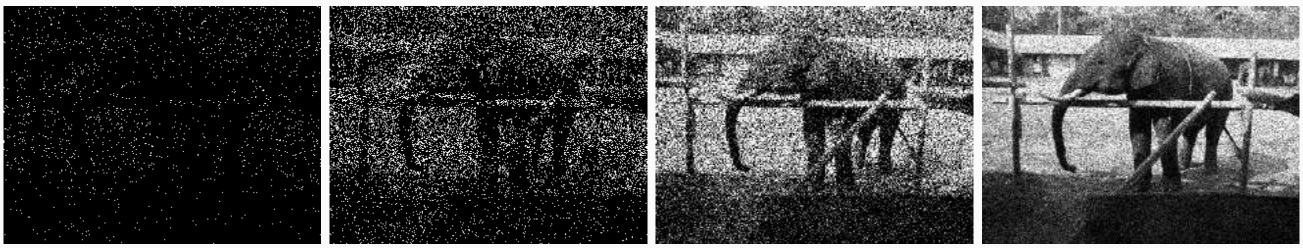


Figure 1 – Distribution of Y for a given quantity of light x

- For $x = 0.5$: mostly 0 photons, Spread ≈ 0.7
- For $x = 1$: mostly 0 or 1 photons, Spread $= 1$
- For $x \gg 1$: bell shape around x, Spread $= \sqrt{x}$

Spread is higher when $x = PQ_e t$ is large.
Should we prefer small exposure time t? and lower light conditions P?

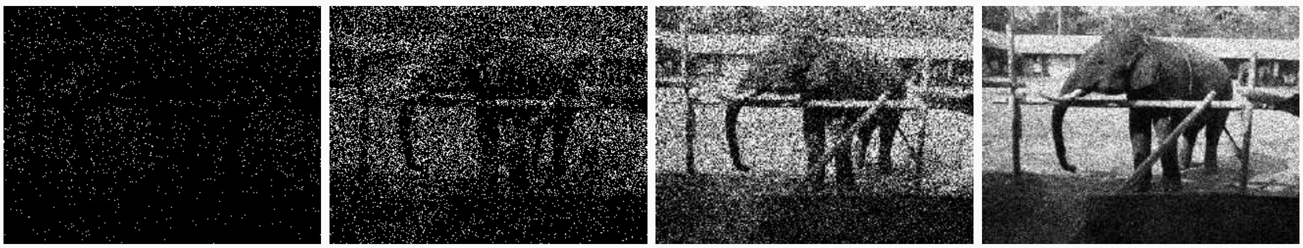
Digital optical imagery – Shot noise



(a) Peak $= 0.05$ (b) Peak $= 0.40$ (c) Peak $= 3.14$ (d) Peak $= 24.37$

Figure 2 – Aspect of shot noise under different light conditions. Peak $= \max_i x_i$.

Digital optical imagery – Shot noise



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Signal to Noise Ratio

$$\mathrm{SNR} = \frac{x}{\sqrt{\mathrm{Var}[Y]}}, \quad \text{for shot noise} \quad \mathrm{SNR} = \sqrt{x}$$

- Measure of difficulty/quality
- The higher the easier/better
- Rose criterion: an SNR of at least 5 is needed to be able to distinguish image features at 100% certainty.

The spread (variance) is not informative,
what matters is the spread relatively to the signal (SNR)

Digital optical imagery – Readout noise

Readout noise (a.k.a, electronic noise)

- Inherent to the process of converting CCD charges into voltage
- Measures $y \in \mathbb{R}$ fluctuate around a voltage $x \in \mathbb{R}$

$$\mathbb{E}[Y] = \int y p_Y(y\,;\,x)\,\mathrm{d}y = x$$

- Fluctuations are independent of x

$$\mathrm{Var}[Y] = \int (y-x)^2 p_Y(y\,;\,x)\,\mathrm{d}y = \sigma^2$$

- Described as Gaussian distributed

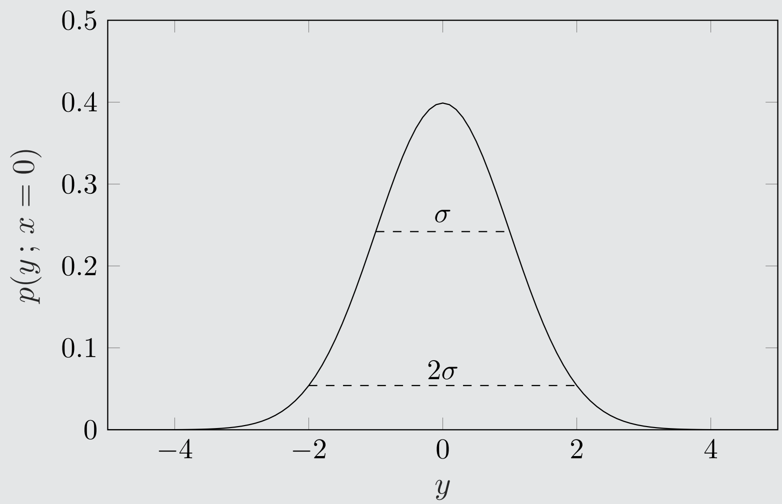
$$p_Y(y\,;\,x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right)$$

- Additive behavior: $Y = x + W, \quad W \sim \mathcal{N}(0, \sigma^2)$

We write $Y \sim \mathcal{N}(x, \sigma^2)$

Digital optical imagery – Readout noise

Gaussian/Normal distribution



- Symmetric with bell shape.
- Common to models $\pm\sigma$ uncertainties with very few outliers
  $\mathbb{P}[|Y - x| \leqslant \sigma] \approx 0.68$, $\mathbb{P}[|Y - x| \leqslant 2\sigma] \approx 0.95$, $\mathbb{P}[|Y - x| \leqslant 3\sigma] \approx 0.99$.
- Arises in many problems due to the Central Limit Theorem.
- Simple to manipulate: ease computation in many cases.

Digital optical imagery – Shot noise vs Readout noise

Shot noise is signal-dependent (Poisson noise)

Readout noise is signal-independent (Gaussian noise)

Digital optical imagery – Thermal and total noise

Thermal noise (a.k.a, dark noise)

- Number of generated electrons fluctuates with the CCD temperature
- Additive Poisson distributed: $Y = x + N \quad \text{with} \quad N \sim \mathcal{P}(\lambda)$
- Signal independent

Digital optical imagery – Thermal and total noise

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- Number of generated electrons fluctuates with the CCD temperature
- Additive Poisson distributed: $Y = x + N \quad \text{with} \quad N \sim \mathcal{P}(\lambda)$
- Signal independent

Total noise in CCD models

$$Y = Z + N + W$$

$$\text{with} \quad \begin{cases} Z \sim \mathcal{P}(x), \\ N \sim \mathcal{P}(\lambda), \\ W \sim \mathcal{N}(0, \sigma^2). \end{cases}$$

$$\text{SNR} = \frac{x}{\sqrt{x + \lambda + \sigma^2}}$$

where $\quad x = PQ_e t, \quad \lambda = Dt$

- t: exposure time
- P: photon flux per pixel *(depends on luminosity)*
- Q_e: quantum efficiency *(depends on wavelength)*
- D: dark current *(depends on temperature)*
- σ: readout noise *(depends on electronic design)*

Digital optical imagery – How to reduce noise?

$$\mathrm{SNR} = \frac{x}{\sqrt{x + \lambda + \sigma^2}} \quad \text{where} \quad x = PQ_e t, \quad \lambda = Dt$$

Photon noise

- Cannot be reduced via camera design
- Reduced by using a longer exposure time t
- Reduced by increasing the scene luminosity, higher P (e.g., using a flash)
- Reduced by increasing the aperture, higher P

Digital optical imagery – How to reduce noise?

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Thermal noise

- Reduced by cooling the CCD, *i.e.*, lower D $\Rightarrow$ More expensive cameras
- Or by using a longer exposure time t

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Thermal noise

- Reduced by cooling the CCD, *i.e.*, lower D $\Rightarrow$ More expensive cameras
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Readout noise

- Reduced by employing carefully designed electronics, *i.e.*, lower σ

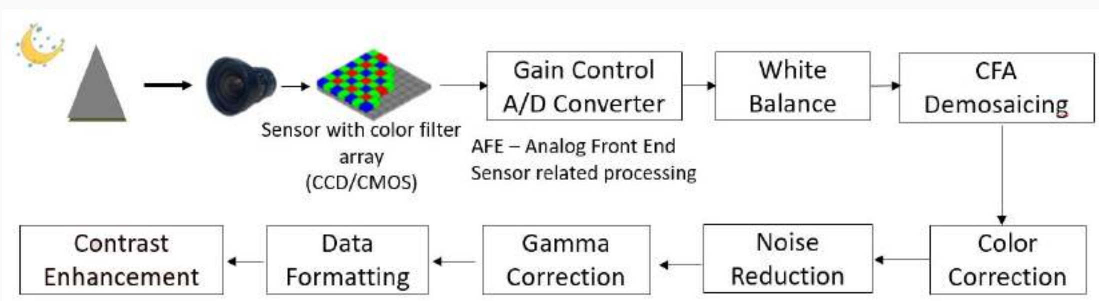
  $\Rightarrow$ More expensive cameras

Or, reduced by image restoration software. But only if such models are accurate

Digital optical imagery – Are these models accurate?

Processing pipeline

- There are always some pre-processing steps such as
  - white balance: to make sure neutral colors appear neutral,
  - demosaicing: to create a color image from incomplete color samples,
  - γ-correction: to optimize the usage of bits, and fit human perception of brightness,
  - compression: to improve memory usage (*e.g.*, JPEG).
- Technical details often hidden by the camera vendors
- The noise in the resulting becomes much harder to model



Source: Y. Gong and Y. Lee

Digital optical imagery – Noise models and post-processing

Example (γ-correction)

$$y^{(\text{new})} = A y^{\gamma}$$

(a) Non corrected

(b) γ-corrected

(c) Zoom $\times 8$

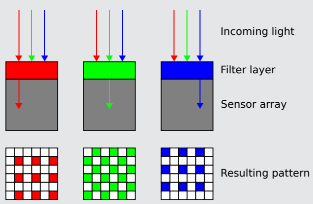
(d) Zoom $\times 30$

What is the influence on the noise?

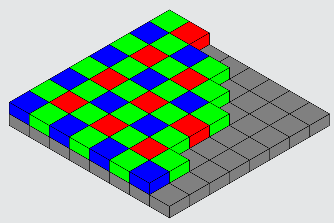
- CCD model: $\mathbb{E}[Y] = x + \lambda$ and $\text{Var}[Y] = x + \lambda + \sigma^2$
- Delta method: $\text{Var}[f(Y)] \approx f'(\mathbb{E}[Y])^2 \text{Var}[Y]$
- Resulting model: $\text{Var}[Y^{(\text{new})}] \approx A^2 \gamma^2 (x + \lambda)^{2(\gamma - 1)} (x + \lambda + \sigma^2)$
- But A and γ are usually not known ☹

Digital optical imagery – Noise models and post-processing

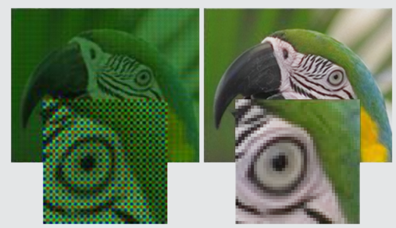
Example (Demosaicing)



(a) Bayer filter



(b) Bayer pattern



(c) Demosaicing

Basic idea:

- Use interpolation techniques.
- Bilinear interpolation: the red value of a non-red pixel is computed as the average of the two or four adjacent red pixels, and similarly for blue and green.

What is the influence on the noise?

- noise is no longer independent from one pixel to another,
- noise becomes spatially correlated.

Compression also create spatial correlations.

Digital optical imagery – Noise models and correlations

Reminder of basic statistics

- X and Y two real random variables
- Independence: $p_{X,Y}(x,y) = p_X(x)p_Y(y)$
- Decorrelation: $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$
- Covariance:
$$\mathrm{Cov}(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$
$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] + \mathrm{Cov}(X,Y)$$
$$\mathrm{Var}(X) = \mathrm{Cov}(X,X)$$
- Correlation:
$$\mathrm{Corr}(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}[X]\mathrm{Var}[Y]}}$$
$$\mathrm{Corr}(X,X) = 1$$

❶	Independence	$\Leftrightarrow / \Rightarrow / \Leftarrow$	Decorrelation	?
❷	$\mathrm{Corr}(X,Y) = 1$	$\Leftrightarrow / \Rightarrow / \Leftarrow$	$X = Y$	?
❸	$\mathrm{Corr}(X,Y) = -1$	$\Leftrightarrow / \Rightarrow / \Leftarrow$	$X = aY + b,\ a < 0$	?

Reminder of multivariate statistics

- $X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$ and $Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{pmatrix}$ two real random vectors
- Entries are independent: $p_X(x) = \prod_k p_{X_k}(x_k)$
- Covariance matrix: $\mathrm{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])(X - \mathbb{E}[X])^T] \in \mathbb{R}^{n \times n}$

  $\mathrm{Var}(X)_{ij} = \mathrm{Cov}(X_i, X_j)$
- Correlation matrix $\mathrm{Corr}(X)_{ij} = \mathrm{Corr}(X_i, X_j)$
- Cross-covariance matrix: $\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])^T] \in \mathbb{R}^{m \times n}$
- Cross-correlation matrix: $\mathrm{Corr}(X, Y)_{ij} = \mathrm{Corr}(X_i, Y_j)$

NB: cross-correlation definition is slightly different in signal processing (in few slides)

Digital optical imagery – Noise models and correlations

- See an image x as a vector of $\mathbb{R}^n$,
- Its observation y is a realization of a random vector

$$Y = x + W \quad \text{and} \quad p_Y(y; x) = p_W(y - x; x),$$

- In general, noise is assumed to be zero-mean $\mathbb{E}[W] = 0$, then

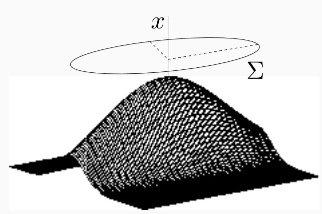
$$\mathbb{E}[Y] = x \quad \text{and} \quad \mathrm{Var}[Y] = \mathrm{Var}[W] = \mathbb{E}[WW^T] = \Sigma.$$

- Σ encodes variances and correlations (may depend on x).
- p_Y is often modeled with a multivariate Gaussian/normal distribution

$$p_Y(y; x) \approx \frac{1}{\sqrt{2\pi}^n |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(y - x)^T \Sigma^{-1} (y - x)\right).$$

Underlying noise distribution

$\approx$



Gaussian approximation $Y \sim \mathcal{N}(x; \Sigma)$

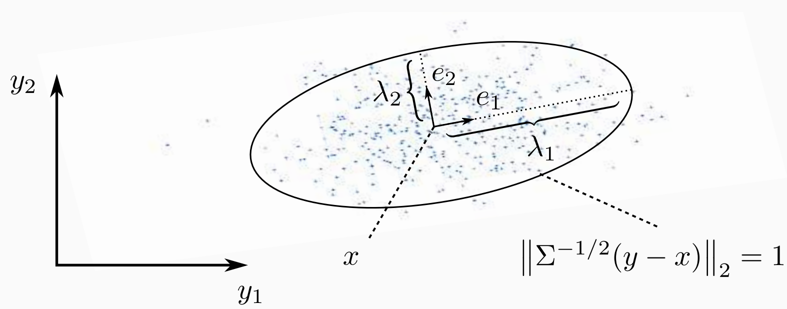
Digital optical imagery – Noise models and correlations

Properties of covariance matrices

- $\Sigma = \mathrm{Var}[Y]$ is square, symmetric and non-negative definite:

$$x^T \Sigma x \geqslant 0, \quad \text{for all } x \neq 0 \text{ (eigenvalues } \lambda_i \geqslant 0).$$

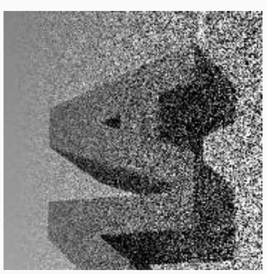
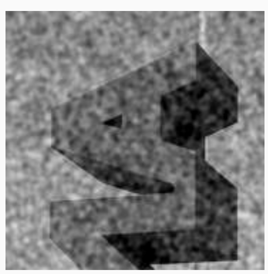
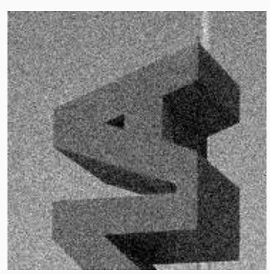
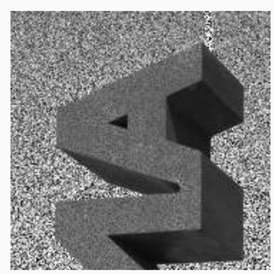
- If all Y_k are linearly independent, then
  - Σ is positive definite: $x^T \Sigma x > 0$, for all $x \neq 0$ ($\lambda_i > 0$),
  - Σ is invertible and Σ^{-1} is also symmetric positive definite,
  - Mahanalobis distance: $\sqrt{(y-x)^T \Sigma^{-1} (y-x)} = \|\Sigma^{-1/2}(y-x)\|_2$,
  - Its isoline $\left\{x \,;\, \|\Sigma^{-1/2}(y-x)\|_2 = c, c > 0\right\}$ describes an ellipsoid of center x and semi-axes the eigenvectors e_i with length $c\lambda_i$.



Digital optical imagery – Noise dictionary

Vocabulary in signal processing

- White noise: zero-mean noise + no correlations
- Stationary noise: identically distributed whatever the location
- Colored noise: stationary with pixels influencing their neighborhood
- Signal dependent: noise statistics depends on the signal intensity
- Space dependent: noise statistics depends on the location
- AWGN: Additive White Gaussian Noise: $Y \sim \mathcal{N}(x; \sigma^2 \mathrm{Id}_n)$

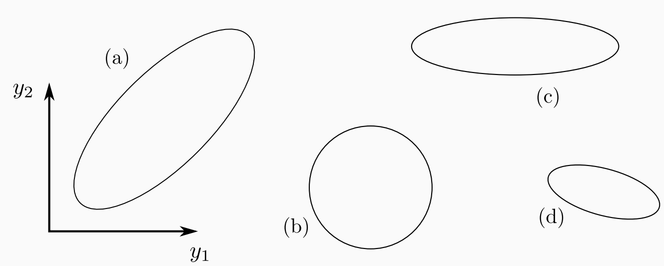
Digital optical imagery – Noise models and correlations

How is it encoded in Σ?

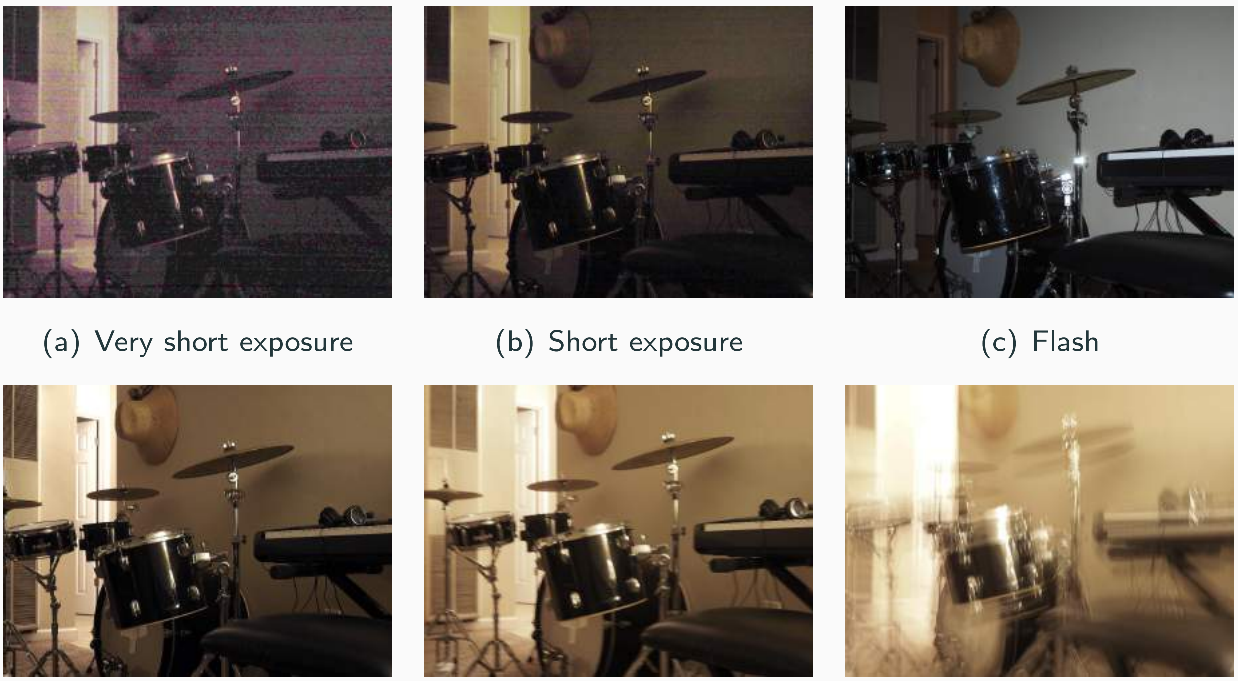
1. Σ diagonal: noise is uncorrelated *– white*
2. $\Sigma_{i,i} = f(s_i)$: variance depends on pixel location s_i *– space dependent*
3. $\Sigma_{i,i} = f(x_i)$: variance depends on pixel value x_i *– signal dependent*
4. $\Sigma_{i,j} = f(s_i - s_j)$: correlations depends on the shift *– stationary*

   For 1d signals, Σ is Toeplitz: $\Sigma = \begin{pmatrix} a & b & \dots & c \\ d & a & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ e & \dots & d & a \end{pmatrix}$

5. $\Sigma = \underbrace{\begin{pmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{pmatrix}}_{=\sigma^2 \mathrm{Id}_n}$: noise is homoscedastic ($\neq$ heteroscedastic) *– white+stationary*



Digital optical imagery – Settings to avoid noise



(a) Very short exposure (b) Short exposure (c) Flash

(d) Normal exposure (e) Long exposure (f) Long + hand shaking

- Short exposure: too much noise
- Using a flash: change the aspect of the scene
- Long exposure: subject to blur and saturation (use a tripod)

What is blur?

Blur: The best of, 2000

Digital optical imagery – Blur

Motion blur

- Moving object
- Camera shake
- Atmospheric turbulence
- Long exposure time

Camera blur

- Limited resolution
- Diffraction
- Bad focus
- Wrong optical design

Bokeh

- Out-of-focus parts
- Often for artistic purpose

Motion blur

London (UK) Munich (Germany)

Camera blur

Hubble Space Telescope (NASA)

Bokeh

Christmas tree Mulholand drive (2001)

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Bokeh

Christmas tree

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How to model blur?

Digital optical imagery – Blur – Linear property

Blur () = Blur ( + )

= Blur () + Blur ()

=  +  = 

Blur is linear

Digital optical imagery – Linear blur

Linear model of blur

- Observed pixel values are a mixture of the underlying ones

$$y_{i,j} = \sum_{k=1}^{n} \sum_{l=1}^{n} h_{i,j,k,l} x_{k,l} \quad \text{where} \quad h_{k,l} \geqslant 0 \text{ and } \sum_{l=1}^{n} h_{k,l} = 1$$

- Matrix/vector representation: $y = Hx \qquad y \in \mathbb{R}^n$, $x \in \mathbb{R}^n$, $H \in \mathbb{R}^{n \times n}$

Digital optical imagery – Linear blur

Linear model of blur

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$$y_{i,j} = \sum_{k=1}^{n} \sum_{l=1}^{n} h_{i,j,k,l} x_{k,l} \quad \text{where} \quad h_{k,l} \geqslant 0 \text{ and } \sum_{l=1}^{n} h_{k,l} = 1$$

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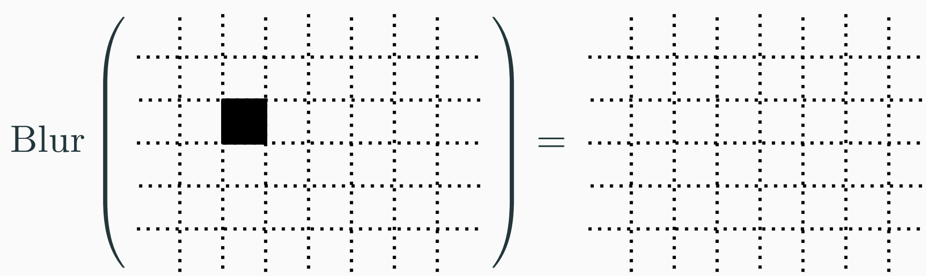
$$y = \begin{array}{c}
\begin{array}{ccc}
\overbrace{\hspace{6cm}}^{\text{First line}} & \hspace{1.5cm} & \overbrace{\hspace{6cm}}^{\text{Last line}}
\end{array} \\
\left(\begin{array}{ccc|c|ccc}
h_{1,1,1,1} & \cdots & h_{1,1,1,n_2} & \cdots & h_{1,1,n_1,1} & \cdots & h_{1,1,n_1,n_2} \\
\vdots & & \vdots & & \vdots & & \vdots \\
h_{1,n_2,1,1} & \cdots & h_{1,n_2,1,n_2} & \cdots & h_{1,n_2,n_1,1} & \cdots & h_{1,n_2,n_1,n_2} \\
\hline
\vdots & & \vdots & & \vdots & & \vdots \\
\hline
h_{n_1,1,1,1} & \cdots & h_{n_1,1,1,n_2} & \cdots & h_{n_1,1,n_1,1} & \cdots & h_{n_1,1,n_1,n_2} \\
\vdots & & \vdots & & \vdots & & \vdots \\
h_{n_1,n_2,1,1} & \cdots & h_{n_1,n_2,1,n_2} & \cdots & h_{n_1,n_2,n_1,1} & \cdots & h_{n_1,n_2,n_1,n_2}
\end{array}\right)
\end{array}
\left(\begin{array}{c}
x_{1,1} \\ \vdots \\ x_{1,n_2} \\ \hline \vdots \\ \hline x_{n_1,1} \\ \vdots \\ x_{n_1,n_2}
\end{array}\right)$$

Digital optical imagery – Point Spread Function (PSF)

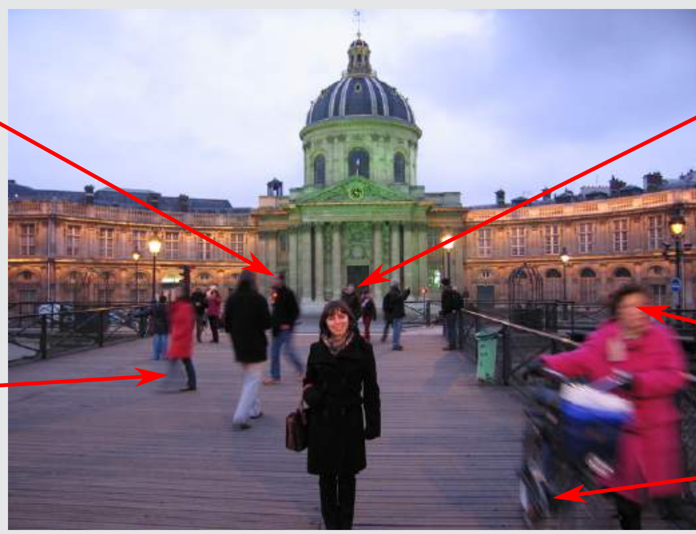
$$H \times \underbrace{\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix}}_{\substack{\text{Only one 1 for}\\ \text{some pixel } (i,\, j)}} = \underbrace{\begin{pmatrix} h_{1,1,i,j}\\ \vdots \\ h_{i,j-1,i,j}\\ h_{i,j,i,j}\\ h_{i,j+1,i,j}\\ \vdots\\ \vdots\\ h_{n_1,n_2,i,j}\end{pmatrix}}_{\text{One column of } H} \xrightarrow{\text{“reshape”}} \underbrace{\begin{pmatrix} h_{1,1,i,j} & h_{2,1,i,j} & \dots & h_{n_1,1,i,j}\\ \vdots & \vdots & & \vdots\\ h_{1,n_2,i,j} & h_{2,n_2,i,j} & \dots & h_{n_1,n_2,i,j}\end{pmatrix}}_{\substack{\text{System's impulse response at location } (i,\, j)\\ \text{called, Point spread function}}}$$



Digital optical imagery – Point Spread Function (PSF)

Spatially varying PSF – non-stationary blur

[redacted]



[redacted]

[redacted]

Digital optical imagery – Stationary blur

Stationary blur

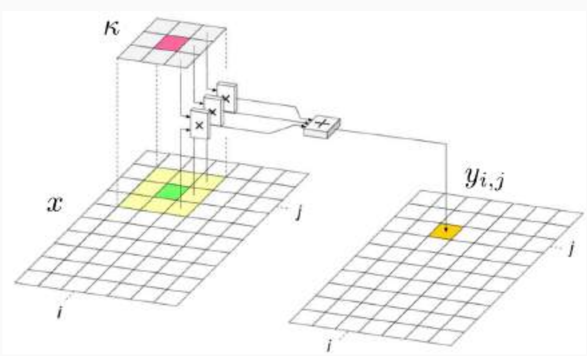
- Shift invariant: blurring depends only on the relative position:

$$h_{i,j,k,l} = \kappa_{k-i,l-j},$$

i.e., same PSF everywhere.

- Corresponds to the (discrete) cross-correlation *(not the same as in statistics)*

$$y = \kappa \star x \quad \Leftrightarrow \quad y_{i,j} = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \kappa_{k,l} x_{i+k,j+l}$$



Digital optical imagery – Stationary blur

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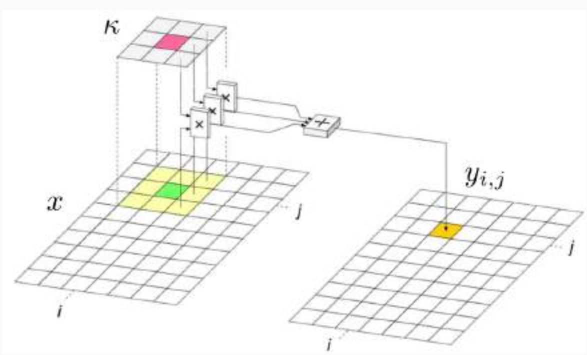


Here κ has a $q = 3 \times 3$ support

$$\Rightarrow \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \equiv \sum_{k=-1}^{+1} \sum_{l=-1}^{+1}$$

q called window size.

Direct computation requires $O(nq)$.

$$\Rightarrow q \ll n$$

Digital optical imagery – Stationary blur

Cross-correlation vs Convolution product

- If κ is complex then the cross-correlation becomes

$$y = \kappa \star x \quad \Leftrightarrow \quad y_{i,j} = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \kappa^*_{k,l} x_{i+k,j+l}.$$

- Complex conjugate: $(a + ib)^* = a - ib$.
- $y = \kappa \star x$ can be re-written as the (discrete) convolution product

$$y = \nu * x \quad \Leftrightarrow \quad y_{i,j} = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} \nu_{k,l} x_{i-k,j-l} \quad \text{with} \quad \nu_{k,l} = \kappa^*_{-k,-l}.$$

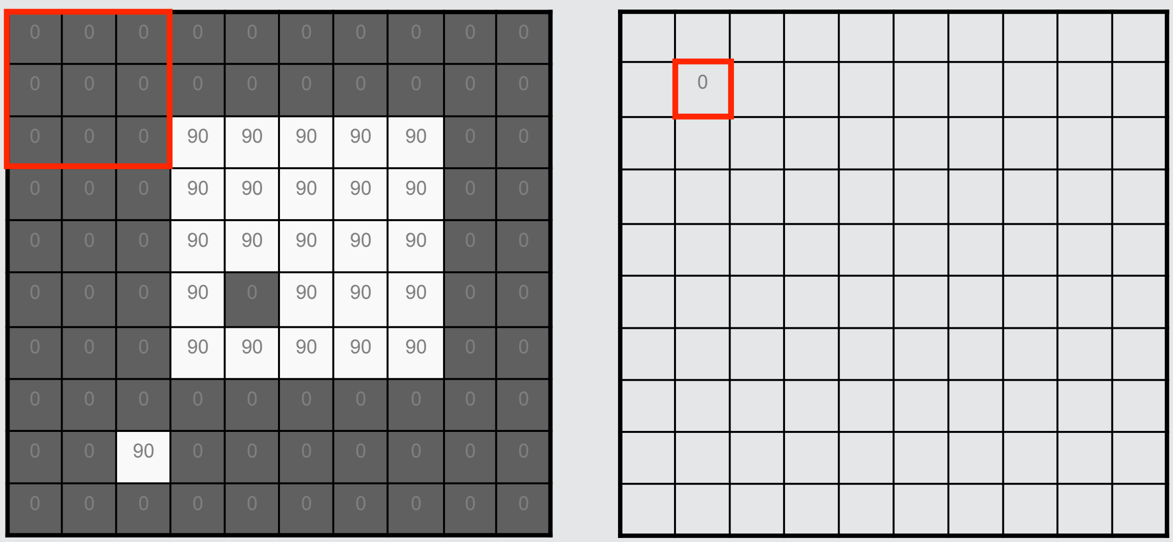
- ν called convolution kernel.

Why convolution instead of cross-correlation?

- **Associative:** $(f * g) * h = f * (g * h)$
- **Commutative:** $f * g = g * f$

For cross-correlation, only true if the signal is Hermitian, i.e., if $f_{k,l} = f^*_{-k,-l}$.

3×3 box convolution



Source: Steven Seitz

Digital optical imagery – Stationary blur

3×3 box convolution

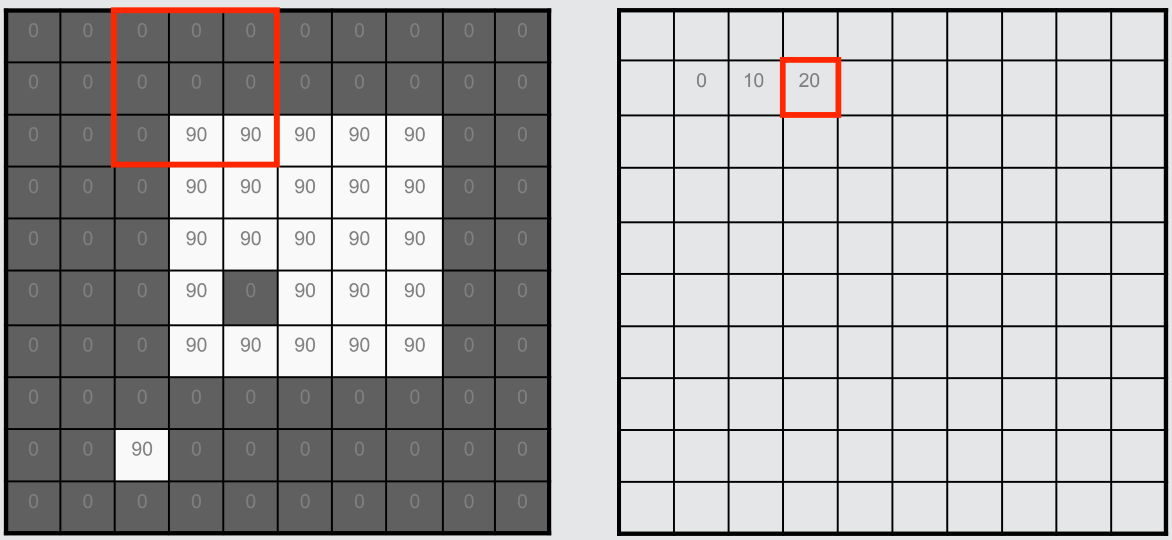
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

	0	10							

Source: Steven Seitz

Digital optical imagery – Stationary blur

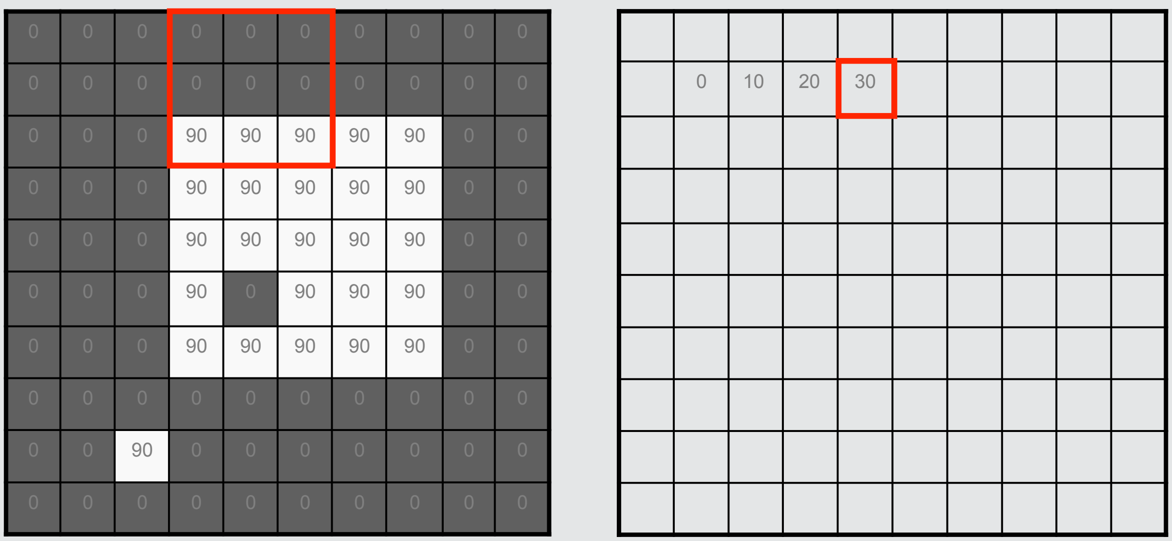
3×3 box convolution



Source: Steven Seitz

Digital optical imagery – Stationary blur

3×3 box convolution



Source: Steven Seitz

Digital optical imagery – Stationary blur

3×3 box convolution

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

	0	10	20	30	30				

Source: Steven Seitz

Digital optical imagery – Stationary blur

3×3 box convolution

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

Source: Steven Seitz

Digital optical imagery – Convolution kernels

Classical kernels

- Box kernel:

$$\kappa_{i,j} = \frac{1}{Z} \begin{cases} 1 & \text{if} \quad \max(|i|, |j|) \leqslant \tau \\ 0 & \text{otherwise} \end{cases}$$

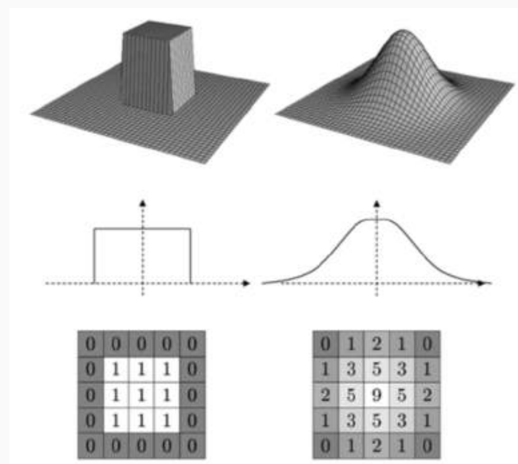
- Gaussian kernel:

$$\kappa_{i,j} = \frac{1}{Z} \exp\left(-\frac{i^2 + j^2}{2\tau^2} \right)$$

- Exponential kernel:

$$\kappa_{i,j} = \frac{1}{Z} \exp\left(-\frac{\sqrt{i^2 + j^2}}{\tau} \right)$$

- Z normalization constant: $Z = \sum_{i,j} \kappa_{i,j}$



Digital optical imagery – Gaussian kernel

$$\kappa_{i,j} = \frac{1}{Z} \exp\left(-\frac{i^2 + j^2}{2\tau^2}\right)$$

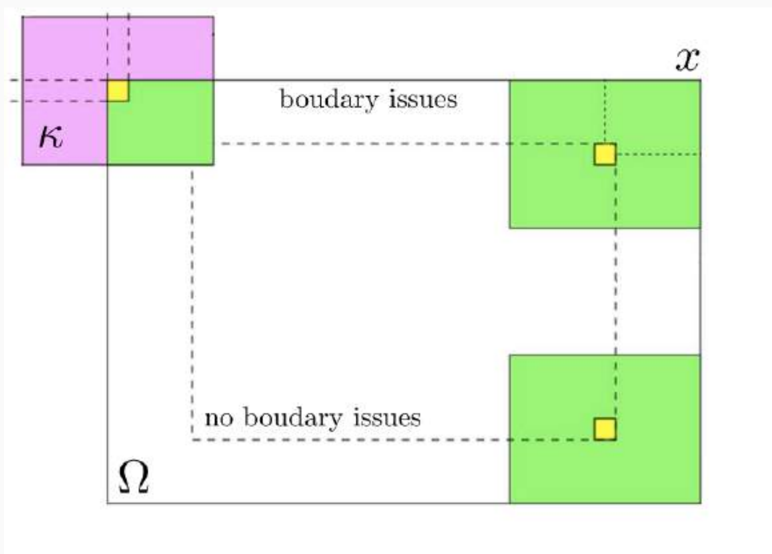
Influence of τ

- $\sqrt{i^2 + j^2}$: distance to the central pixel,
- τ: controls the influence of neighbor pixels, *i.e.*, the strength of the blur

Small τ | Medium τ | Large τ

Digital optical imagery – Boundary conditions

How to deal when the kernel window overlaps outside the image domain?



i.e., how to evaluate $y_{i,j} = \sum_{k,l} \kappa_{k,l} x_{i+k,j+l}$ when $(i+k, j+l) \notin \Omega$?

Digital optical imagery – Boundary conditions

Standard techniques:


zero-padding


extension


mirror


periodical